Counting the number of non-equivalent vertex colorings of a graph

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Abstract.
We give some extremal properties on the number $\mathcal{B}(G)$ of non-equivalent ways of coloring a given graph G, also known as the (graphical) Bell number of G. In particular, we study bounds on $\mathcal{B}(G)$ for graphs with a maximum degree constrained. First, an upper bound on $\mathcal{B}(G)$ is given for graphs with fixed order n and maximum degree Δ. Then, we give lower bounds on $\mathcal{B}(G)$ for fixed order n and maximum degree 1, 2, $n-2$ and $n-1$. In each case, the bound is tight and we describe all graphs that reach the bound with equality.

Keywords: non-equivalent colorings, graphical Bell numbers, chromatic polynomial.

1 Introduction

A question which probably sounds familiar for many researchers in graph theory is: what is the number of ways of coloring a given graph G? In the literature, the common answer to this question is related to the notion of chromatic polynomial which was introduced by Birkhoff [1] in 1912 in an attempt to prove the four-color theorem. The *chromatic polynomial* of a graph G of order n is the (unique) polynomial of degree n passing by points $(k, \Pi(G, k))$ for $k = 0, 1, \ldots, n$, where $\Pi(G, k)$ is the number of ways of coloring G with at most k colors, counting two colorings as distinct when they are obtained by a permutation from the other. For example, for the path P_3 we have

$$\Pi(\mathsf{P}_3, k) = k(k-1)^2.$$

Indeed, $\Pi(\mathsf{P}_3, 0) = \Pi(\mathsf{P}_3, 1) = 0$; $\Pi(\mathsf{P}_3, 2) = 2$ (take for instance the first two colorings in the left column of Figure 1) and $\Pi(\mathsf{P}_3, 3) = 12$ as shown in Figure 1. The number of vertex colorings of a graph G is nowadays commonly interpreted as $\Pi(G, n)$, where n is the number of vertices in G, meaning that P_3 has 12 colorings according to this interpretation. However, for P_3, another answer to the above question that makes sense is *two* as depicted in Figure 2: there is only one coloring with two colors (the two extremities share the same color while the central vertex has its own color), and only one coloring with three colors (each vertex has its own color).

More generally, we are interested in this paper by the number $\mathcal{B}(G)$ of *non-equivalent* colorings of a graph G. This way of counting colorings is especially meaningful when a set

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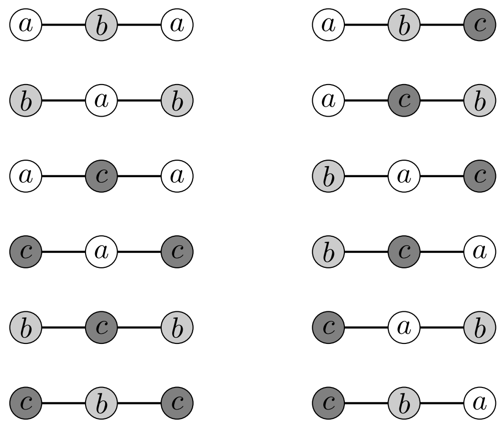

Figure 1: The 12 colorings of P_3 (using 3 colors) as defined by the chromatic polynomial.

Figure 2: The 2 non-equivalent colorings of P_3 (using any number of colors).

of elements has to be partitioned into a given number of non-empty subsets, subject to some constraints. Indeed, $\mathcal{B}(G)$ is the number of partitions of the vertex set of G whose blocks are stable sets. This invariant has been studied by several authors in the last few years [7–10] under the name of *(graphical) Bell number* of G. However, historically, this invariant is related to the σ-polynomial introduced by Korfhage [11] in 1978. Indeed, the *σ-polynomial* of a graph G is a polynomial in x such that the coefficient of x^k is the number of non-equivalent ways of properly coloring G using exactly k colors. It follows from that definition that $\mathcal{B}(G)$ is the value of the σ-polynomial at $x = 1$. The σ-polynomial was studied intensively by Brenti [3] and Brenti *et al.* [4] in the early nineties. It appears that several results on $\mathcal{B}(G)$ published later (including results in [7–9]) are special cases of results from [3, 4].

It is interesting to note that while $\mathcal{B}(G)$ and $\Pi(G, n)$ might appear as similar concepts (since they both count colorings with at most n colors), they differ in several ways. We have already mentioned that only non-equivalent colorings are counted in $\mathcal{B}(G)$, which means that $\mathcal{B}(G)$ corresponds to the number of partitions of the vertex set of G, taking into account constraints that prevent some pairs of vertices of belonging to the same subset of the partition. Also, observe that if $\Pi(G, n) < \Pi(H, n)$ for two graphs G and H of order n, this does not necessarily imply that $\mathcal{B}(G) < \mathcal{B}(H)$ (and conversely) as shown in Figure 3.

Similarly, there exist pairs of graphs $(G,\ H)$ such that $\mathcal{B}(G) = \mathcal{B}(H)$ but $\Pi(G, n) \neq \Pi(H, n)$, and conversely (see examples in Figure 4).

In the next section we fix some notations, give a formal definition of the number $\mathcal{B}(G)$ of non-equivalent vertex colorings of a graph G and recall some basic properties of $\mathcal{B}(G)$. Then, in Sections 3 and 4, we prove some bounds on $\mathcal{B}(G)$ for graphs of bounded maximum degree and let other bounds as open problems.

$\mathcal{B}(G) = 18$ $\qquad$ $\mathcal{B}(H) = 17$

$\Pi(G, 6) = 8520$ $\qquad$ $\Pi(H, 6) = 9000$

Figure 3: Two graphs G and H with 6 vertices such that $\Pi(G, 6) < \Pi(H, 6)$ and $\mathcal{B}(G) > \mathcal{B}(H)$.

$\mathcal{B}(G) = 4$ $\qquad$ $\mathcal{B}(H) = 4$

$\Pi(G, 5) = 420$ $\qquad$ $\Pi(H, 5) = 480$

$\mathcal{B}(G') = 6$ $\qquad$ $\mathcal{B}(H') = 5$

$\Pi(G', 5) = 600$ $\qquad$ $\Pi(H', 5) = 600$

Figure 4: Two pairs of graphs with 5 vertices showing that equality for one way to counts the colorings does not imply equality for the other.

2 Basics and notations

For basic notions of graph theory that are not defined here, we refer to Diestel [5]. Let $G = (V, E)$ be a simple undirected graph. We denote by $n = |V|$ the order of G and by $m = |E|$ its size. Also, $\bar{G}$ is the complement of G and we write $G \simeq H$ if G and H are two isomorphic graphs.

Let K_n (resp. C_n and P_n) be the *complete graph* (resp. the *cycle* and the *path*) of order n. The *wheel* W_n is the graph of order n obtained by connecting a vertex to all vertices of C_{n-1}. Also, we write $\mathsf{K}_{a,b}$ for the complete bipartite graph where a and b are the cardinalities of the two sets of vertices of the bipartition. Finally, S_n denotes the *star* on n vertices, that is $\mathsf{K}_{1,n-1}$.

Let $N(v)$ denote the neighbors of a vertex v in G. The *degree* of a vertex v is denoted $d(v)$ (i.e., $d(v) = |N(v)|$). A vertex v is *isolated* if $d(v) = 0$ and is *dominating* if $d(v) = n - 1$. The *maximum degree* of G is denoted $\Delta(G)$.

Let u and v be two vertices in a graph G of order n, we denote $G \setminus uv$ the graph (of order $n-1$) obtained by identifying (merging) the vertices u and v and, if $uv \in E(G)$, by removing edge uv. Also, if $uv \in E(G)$, we note $G - uv$ the graph obtained from G by removing edge uv, while if $uv \notin E(G)$, the graph $G + uv$ is the graph obtained by adding uv in G. For a vertex v of G, we denote $G - v$ the graph obtained from G by removing v and all its incident edges.

A *vertex coloring* (or simply a coloring in the sequel) is an assignment of colors to the vertices of G. A *proper* coloring is a coloring such that adjacent vertices have different colors. The *chromatic number* $\chi(\mathrm{G})$ of a graph G is the minimum numbers of colors in a proper coloring of G. Two colorings are *equivalent* if they induce the same partition of the vertex set. One defines $S(G,k)$ as the number of proper non-equivalent colorings of a graph G that use *exactly* k colors. The total *number* $\mathcal{B}(G)$ *of non-equivalent colorings* of a graph G is then defined as:

$$\mathcal{B}(G) = \sum_{k=\chi(G)}^{n} S(G,k). \tag{1}$$

Given a graph G with a dominating vertex v, the following property (stated in a more general form in [8]) states that the computation of $\mathcal{B}(G)$ can be reduced to that of $\mathcal{B}(G-v)$.

Property 1. *If a graph G has a dominating vertex v, then $\mathcal{B}(G) = \mathcal{B}(G-v)$.*

The following property shows how to compute the number of colorings of the disjoint union of two graphs.

Property 2 ([7])**.** *Let $G = G_1 \cup G_2$ be a graph that is the disjoint union of two graphs G_1 and G_2. Then,*

$$\mathcal{B}(G) = \sum_{k=1}^{n}\sum_{i=1}^{k}\sum_{j=0}^{i} S(G_1,i)S(G_2,k-j)\binom{i}{i-j}\binom{k-j}{i-j}(i-j)!$$

Given two graphs G and H of order n, we note $G >_S H$ and say that G *strictly dominates* H for the number of non-equivalent colorings if $S(G,k) \geq S(H,k)$ for all $k = 1,2,\ldots,n$, and there exists some integer k such that $S(G,k) > S(H,k)$. By Property 2, the following corollary is straightforward.

Corollary 3. *Let G, G' and H be three graphs such that G and G' have the same order. If $G >_S G'$, then, $\mathcal{B}(G \cup H) > \mathcal{B}(G' \cup H)$.*

As for several other algorithms in graph coloring, the *deletion-contraction* rule (also often called the *Fundamental Reduction Theorem*[6]) is a well known method to compute $\mathcal{B}(G)$ ([8, 10]). More precisely, let u and v be any pair of distinct vertices of G, we have,

$$S(G,k) = S(G-uv,k) - S(G \setminus uv,k), \tag{2}$$

if $uv \in E(G)$, and

$$S(G,k) = S(G+uv,k) + S(G \setminus uv,k), \tag{3}$$

if $uv \notin E(G)$. Similarly, if $uv \in E(G)$, we have,

$$\mathcal{B}(G) = \mathcal{B}(G-uv) - \mathcal{B}(G \setminus uv), \tag{4}$$

and

$$\mathcal{B}(G) = \mathcal{B}(G+uv) + \mathcal{B}(G \setminus uv), \tag{5}$$

if $uv \notin E(G)$.

Since there is only one possible coloring for K_n (using exactly n colors), we have

$$S(\mathsf{K}_n, k) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathcal{B}(\mathsf{K}_n) = 1$. This constitutes a base case for a straightforward recursive algorithm to compute $\mathcal{B}(G)$ for any graph G using relation (5). Another recursive procedure can be obtained from (4) using the empty graph $\overline{\mathsf{K}_n}$ to define the base case. Indeed, we have ([7,8])

$$S(\overline{\mathsf{K}_n}, k) = \begin{cases} \left\{ {n \atop k} \right\} & \forall k \leq n, \\ 0 & \forall k > n, \end{cases}$$

where

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

is a *Stirling number of the second kind*, that is the number of ways to partition a set of n elements into k non-empty subsets. It follows that

$$\mathcal{B}(\overline{\mathsf{K}_n}) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} = \mathsf{B}_n,$$

where B_n is the n^{th} *Bell number* (sequence A000110 in OEIS [13]). This is not surprising since B_n represents the number of partitions of a set of n elements which is obviously the same as the number of non-equivalent colorings in a graph without any edge. This is also the reason why $\mathcal{B}(n)$ is known as the *(graphical) Bell number* of G, while $S(G, k)$ is called the *(graphical) Stirling number of* G (see for example [8–10]).

Generalized Stirling and Bell numbers have been defined and studied in [2] and are also linked to $\mathcal{B}(G)$. More precisely, let

$$S_r(n, k) = \frac{1}{k!} \sum_{j=r}^{k} (-1)^{k-j} \binom{k}{j} \left(\frac{j!}{(j-r)!} \right)^n.$$

Consider n sets $E_1, E_2, \ldots, E_n$ of r elements. The *generalized Stirling number* $S_r(n, k)$ is the number of different partitions of these nr elements into k non-empty subsets such that each subset contains at most one element of each E_i. In other words, $S_r(n, k) = S(n\mathsf{K}_r, k)$. The *Generalized Bell numbers* $\mathsf{B}_{r,n}$ are then defined as follows:

$$\mathsf{B}_{r,n} = \sum_{k=r}^{rn} S_r(n, k).$$

They represent the number of partitions of the nr elements so that each subset contains at most one element of each E_i. Hence, $\mathsf{B}_{r,n} = \mathcal{B}(n\mathsf{K}_r)$.

The number of non-equivalent colorings is known for several classes of graphs. We mention some of these values from [8–10]:

- Let T be a tree of order $n \geq 1$. Then, $\mathcal{B}(T) = \mathsf{B}_{n-1}$.

- Let C_n be a cycle of order $n \geq 3$. Then,

$$\mathcal{B}(\mathsf{C}_n) = \sum_{j=1}^{n-1} (-1)^{j+1} \mathsf{B}_{n-j}.$$

- Let W_n be a wheel of order $n \geq 4$. Then,

$$\mathcal{B}(\mathsf{W}_n) = \sum_{j=1}^{n-2} (-1)^{j+1} \mathsf{B}_{n-j-1}.$$

- Let $\bar{\mathsf{P}}_n$ be the complement of a path of order $n \geq 1$. Then, $\mathcal{B}(\bar{\mathsf{P}}_n) = \mathsf{F}_{n+1}$ where F_n denotes the nth Fibonacci number.

- Let $\bar{\mathsf{C}}_n$ be the complement of a cycle of order $n \geq 4$. Then, $\mathcal{B}(\bar{\mathsf{C}}_n) = \mathsf{L}_n$ where L_n denotes the nth Lucas number.

However, there are not many results in the literature about the *extremal* properties of $\mathcal{B}(G)$ and this is the subject of the two next sections where we study upper and lower bounds on $\mathcal{B}(G)$ for graphs G with bounded maximum degree. We note that the following results were first conjectured with the help of the conjecture-making system GraPHedron [12].

3 Upper bound on the number of colorings of graphs with fixed maximum degree

The upper bound on $\mathcal{B}(G)$ for graphs G with bounded maximum degree is straightforward. We define $G^{>}_{n,\Delta}$ to be the graph of order n and with a maximum degree Δ that is composed of a star $\mathsf{S}_{\Delta+1}$ and $n - \Delta - 1$ isolated vertices (see Figure 5 for an example).

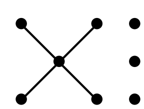

Figure 5: The graph $G^{>}_{8,4}$.

Theorem 4. *Let G be a graph of order n and maximum degree Δ. Then,*

$$\mathcal{B}(G) \leq \sum_{i=0}^{\Delta} (-1)^i \binom{\Delta}{i} \mathsf{B}_{n-i},$$

with equality if and only if G is isomorphic to $G^{>}_{n,\Delta}$.

Proof. The graph $G^{>}_{n,\Delta}$ is clearly the graph minimizing the number of edges among all graphs of order n with maximum degree Δ. Adding edges to $G^{>}_{n,\Delta}$ (in such a way that the maximum degree is not increased) will add new constraints between pairs of vertices, and this will therefore strictly decrease the number of colorings. Hence $\mathcal{B}(G) \le \mathcal{B}(G^{>}_{n,\Delta})$, with equality if and only G is isomorphic to $G^{>}_{n,\Delta}$. It remains to prove that

$$\mathcal{B}(G^{>}_{n,\Delta}) = \sum_{i=0}^{\Delta}(-1)^i\binom{\Delta}{i}\mathsf{B}_{n-i} \qquad \text{for all } n \text{ and } \Delta.$$

The equality holds for $\Delta = 0$ since $\mathcal{B}(G^{>}_{n,\Delta})$ is then isomorphic to $\overline{\mathsf{K}_n}$ and we have already recalled that $\mathcal{B}(\overline{\mathsf{K}_n}) = \mathsf{B}_n$. For larger values of Δ, we proceed by induction using the following equality obtained from (4):

$$\mathcal{B}(G^{>}_{n,\Delta}) = \mathcal{B}(G^{>}_{n,\Delta-1}) - \mathcal{B}(G^{>}_{n-1,\Delta-1}).$$

We then have

$$\begin{aligned}
\mathcal{B}(G^{>}_{n,\Delta}) &= \textstyle\sum_{i=0}^{\Delta-1}(-1)^i\binom{\Delta-1}{i}\mathsf{B}_{n-i} - \sum_{i=0}^{\Delta-1}(-1)^i\binom{\Delta-1}{i}\mathsf{B}_{n-1-i}\\
&= \textstyle\sum_{i=0}^{\Delta-1}(-1)^i\binom{\Delta-1}{i}\mathsf{B}_{n-i} + \sum_{i=1}^{\Delta}(-1)^i\binom{\Delta-1}{i-1}\mathsf{B}_{n-i}\\
&= \textstyle\mathsf{B}_n + \sum_{i=1}^{\Delta-1}(-1)^i(\binom{\Delta-1}{i} + \binom{\Delta-1}{i-1}) + (-1)^{\Delta}\mathsf{B}_{n-\Delta}\\
&= \textstyle\sum_{i=0}^{\Delta}(-1)^i\binom{\Delta}{i}\mathsf{B}_{n-i}.
\end{aligned}$$

□

4 Lower bound on the number of colorings of graphs with fixed maximum degree

A lower bound on $\mathcal{B}(G)$ for graphs of order n and bounded maximum degree Δ is easy to obtain for some values of Δ, but more intricate or still open for the other ones. In the rest of this section, we say that a graph G^* is *extremal* if $\mathcal{B}(G^*) \le \mathcal{B}(G)$ for all graphs G of order n such that $\Delta(G) = \Delta(G^*)$. The following property will be used intensively in the ongoing proofs.

Property 5. *Let G be a graph with two vertices v and w such that $vw \notin E$ and*

$$\max(d(v), d(w)) < \Delta(G).$$

Then, G is not extremal.

Proof. Adding the edge vw will not change the value of $\Delta(G)$ but will strictly decrease the number of colorings of G. □

We start by defining a graph of order n and with maximum degree equal to 1. If n is even, then $G^{<}_{n,\Delta=1}$ is the disjoint union of $\frac{n}{2}$ copies of K_2; if n is odd, it is the disjoint union of $G^{<}_{n-1,\Delta=1}$ and an isolated vertex. The graph $G^{<}_{7,\Delta=1}$ is drawn on the left-hand side of Figure 6.

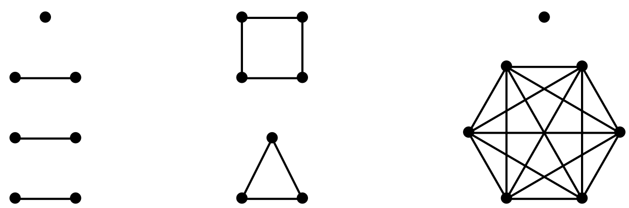

Figure 6: The graphs $G^{<}_{7,\Delta=1}$, $G^{<}_{7,\Delta=2}$ and $\mathsf{K}_6 \cup \mathsf{K}_1$ (from left to right).

Theorem 6. *Let G be a graph of order n such that $\Delta(G) = 1$. Then,*

$$\mathcal{B}(G) \geq \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{\lfloor n/2 \rfloor}{i} \mathsf{B}_{n-i},$$

with equality if and only if G is isomorphic to $G^{<}_{n,\Delta=1}$.

Proof. Since $\Delta(G) = 1$, G is a disjoint union of several copies of K_2 and isolated vertices. If G has at least two isolated vertices v and w, we know from Property 5 that it cannot be extremal. Thus, if G is extremal it must be isomorphic to $G^{<}_{n,\Delta=1}$.
Consider now the disjoint union of p K_2 and q K_1. We prove that

$$\mathcal{B}(p\mathsf{K}_2 \cup q\mathsf{K}_1) = \sum_{i=0}^{p} (-1)^i \binom{p}{i} \mathsf{B}_{2p+q-i}.$$

The equality holds for $p = 0$ since the graph is then isomorphic to $\overline{\mathsf{K}_q}$ and we have $\mathcal{B}(\overline{\mathsf{K}_q}) = \mathsf{B}_q$. For larger values of p, we proceed by induction using the following equality obtained from (4):

$$\mathcal{B}(p\mathsf{K}_2 \cup q\mathsf{K}_1) = \mathcal{B}((p-1)\mathsf{K}_2 \cup (q+2)\mathsf{K}_1) - \mathcal{B}((p-1)\mathsf{K}_2 \cup (q+1)\mathsf{K}_1).$$

We then have

$$\begin{aligned}
\mathcal{B}(p\mathsf{K}_2 \cup q\mathsf{K}_1) &= \textstyle\sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} \mathsf{B}_{2p+q-i} - \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} \mathsf{B}_{2p+q-1-i} \\
&= \textstyle\sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} \mathsf{B}_{2p+q-i} + \sum_{i=1}^{p} (-1)^i \binom{p-1}{i-1} \mathsf{B}_{2p+q-i} \\
&= \textstyle\mathsf{B}_{2p+q} + \sum_{i=1}^{p-1} (-1)^i (\binom{p-1}{i} + \binom{p-1}{i-1}) + (-1)^p \mathsf{B}_{p+q} \\
&= \textstyle\sum_{i=0}^{p} (-1)^i \binom{p}{i} \mathsf{B}_{2p+q-i}.
\end{aligned}$$

To conclude, it is sufficient to observe that $G^{<}_{n,\Delta=1}$ is isomorphic to $p\mathsf{K}_2 \cup q\mathsf{K}_1$ with $p = \lfloor n/2 \rfloor$ and $q = n - 2p$.

□

We now consider graphs G with maximum degree $\Delta(G) = 2$. Before giving a lower bound on $\mathcal{B}(G)$ for such graphs, we prove some useful lemmas.

Lemma 7. *Consider a cycle C_n of order $n \geq 6$. Then,*

$$\begin{aligned}
S(\mathsf{C}_n, k) &> S(\mathsf{C}_{n-3} \cup \mathsf{C}_3, k) \quad \textit{for } k = 3, 4, \ldots, n-2; \\
S(\mathsf{C}_n, k) &= S(\mathsf{C}_{n-3} \cup \mathsf{C}_3, k) \quad \textit{for } k = n-1, n.
\end{aligned}$$

Proof. The values in the following table show that the result holds for $n = 6$.

k	3	4	5	6
$S(\mathsf{C}_6, k)$	10	20	9	1
$S(2\mathsf{C}_3, k)$	6	18	9	1

For larger values or n, the following equalities are obtained from (2)and (3):

$$\begin{aligned} S(\mathsf{C}_{n-3} \cup \mathsf{C}_3, k) &= S(\mathsf{P}_{n-3} \cup \mathsf{C}_3, k) - S(\mathsf{C}_{n-4} \cup \mathsf{C}_3, k) \\ &= S(\mathsf{P}_{n-3} \cup \mathsf{P}_3, k) - S(\mathsf{P}_{n-3} \cup \mathsf{P}_2, k) - S(\mathsf{C}_{n-4} \cup \mathsf{C}_3, k) \\ &= (S(\mathsf{P}_n, k) + S(\mathsf{P}_{n-1}, k)) - (S(\mathsf{P}_{n-1}, k) + S(\mathsf{P}_{n-2}, k)) \\ &\quad - S(\mathsf{C}_{n-4} \cup \mathsf{C}_3, k) \\ &= S(\mathsf{P}_n, k) - S(\mathsf{P}_{n-2}, k) - S(\mathsf{C}_{n-4} \cup \mathsf{C}_3, k) \end{aligned}$$

Clearly, $S(\mathsf{P}_{n-2}, k) > 0$ for $k = 3, 4, \ldots, n-2$ and $S(\mathsf{P}_{n-2}, k) = 0$ for $k = n-1, n$. Also, by induction, we have $S(\mathsf{C}_{n-4} \cup \mathsf{C}_3, k) < S(\mathsf{C}_{n-1}, k)$ for $k = 3, 4, \ldots, n-3$, and $S(\mathsf{C}_{n-4} \cup \mathsf{C}_3, k) = S(\mathsf{C}_{n-1}, k)$ for $k = n-2, n-1$. Moreover, $S(\mathsf{C}_{n-4} \cup \mathsf{C}_3, n) = S(\mathsf{C}_{n-1}, n) = 0$. Hence, $S(\mathsf{C}_{n-3} \cup \mathsf{C}_3, k) \le S(\mathsf{P}_n, k) - S(\mathsf{C}_{n-1}, k)$, with equality only if $k = n-1, n$. To conclude, we observe from (2) that $S(\mathsf{P}_n, k) - S(\mathsf{C}_{n-1}, k) = S(\mathsf{C}_n, k)$. □

Since $S(\mathsf{C}_n, 2) \ge 0$ while $S(\mathsf{C}_{n-3} \cup \mathsf{C}_3, 2) = 0$ for $n \ge 6$, the following corollary is straightforward.

Corollary 8. *Consider a cycle C_n of order $n \ge 6$. Then $\mathsf{C}_n >_S \mathsf{C}_{n-3} \cup \mathsf{C}_3$.*

Lemma 9. *Consider a cycle C_n of order $n \ge 3$. Then,*

$$S(\mathsf{C}_n \cup \mathsf{K}_1, k) = S(\mathsf{P}_{n+1}, k) \qquad \textit{for } k = 3, 4, \ldots, n+1.$$

Proof. The result is valid for $n = 3$ since $S(\mathsf{C}_3 \cup \mathsf{K}_1, 3) = S(\mathsf{P}_4, 3) = 3$ and $S(\mathsf{C}_3 \cup \mathsf{K}_1, 4) = S(\mathsf{P}_4, 4) = 1$. For larger values or n and $k \ge 3$, we proceed by induction and apply (2) and (3) to obtain:

$$\begin{aligned} S(\mathsf{C}_n \cup \mathsf{K}_1, k) &= S(\mathsf{P}_n \cup \mathsf{K}_1, k) - S(\mathsf{C}_{n-1} \cup \mathsf{K}_1, k) \\ &= S(\mathsf{P}_{n+1}, k) + S(\mathsf{P}_n, k) - S(\mathsf{C}_{n-1} \cup \mathsf{K}_1, k) \\ &= S(\mathsf{P}_{n+1}, k) \end{aligned}$$

□

Corollary 10. *Consider a cycle C_n of order $n \ge 4$. Then*

$$\begin{array}{ll} \mathsf{C}_n \cup \mathsf{K}_1 >_S \mathsf{C}_{n+1} & \textit{if } n \textit{ is even;} \\ \mathsf{C}_n \cup \mathsf{K}_1 >_S \mathsf{C}_{n-2} \cup \mathsf{C}_3 & \textit{if } n \textit{ is odd.} \end{array}$$

Proof. Since $S(\mathsf{P}_{n+1}, k) > S(\mathsf{C}_{n+1}, k)$ for $k = 3, 4, \ldots, n$, it follows from Lemma 9 that $S(\mathsf{C}_n \cup \mathsf{K}_1, k) > S(\mathsf{C}_{n+1}, k)$ for $k = 3, 4, \ldots, n$.

- If n is even, then $S(\mathsf{C}_n \cup \mathsf{K}_1, 2) = 2 > 0 = S(\mathsf{C}_{n+1}, 2)$ and $S(\mathsf{C}_n \cup \mathsf{K}_1, n+1) = S(\mathsf{C}_{n+1}, n+1) = 1$, which implies $\mathsf{C}_n \cup \mathsf{K}_1 >_S \mathsf{C}_{n+1}$.

- If n is odd, then we know from Lemma 7 that $S(\mathsf{C}_{n+1},k) \geq S(\mathsf{C}_{n-2} \cup \mathsf{C}_3, k)$ for $k = 3,4,\ldots,n+1$. Since $S(\mathsf{C}_n \cup \mathsf{K}_1, 2) = S(\mathsf{C}_{n-2} \cup \mathsf{C}_3, 2) = 0$. □

Lemma 11. *Consider a cycle C_n of order $n \geq 5$. Then, $\mathsf{C}_{n-2} \cup \mathsf{K}_2 >_S \mathsf{C}_n$.*

Proof. By applying (2) and (3), we obtain the following equalities which are valid for all $k \geq 2$:

$$\begin{aligned}
S(\mathsf{C}_{n-2} \cup \mathsf{K}_2, k) &= S(\mathsf{P}_{n-2} \cup \mathsf{K}_2, k) - S(\mathsf{C}_{n-3} \cup \mathsf{K}_2, k) \\
&= S(\mathsf{P}_n, k) + S(\mathsf{P}_{n-1}, k) - S(\mathsf{P}_{n-3} \cup \mathsf{K}_2, k) \\
&\quad + S(\mathsf{C}_{n-4} \cup \mathsf{K}_2, k) \\
&= S(\mathsf{P}_n, k) + S(\mathsf{P}_{n-1}, k) - S(\mathsf{P}_{n-1}, k) - S(\mathsf{P}_{n-2}, k) \\
&\quad + S(\mathsf{C}_{n-4} \cup \mathsf{K}_2, k) \\
&= S(\mathsf{P}_n, k) - S(\mathsf{P}_{n-2}, k) + S(\mathsf{C}_{n-4} \cup \mathsf{K}_2, k). \qquad (6)
\end{aligned}$$

We now analyse three different cases.

- If $k \geq 4$, we first show that $S(\mathsf{C}_{n-2} \cup \mathsf{K}_2, k) = S(\mathsf{P}_n, k)$. This is true for $n = 5, 6$ since $S(\mathsf{C}_3 \cup \mathsf{K}_2, 4) = S(\mathsf{P}_5, 4) = 6$, $S(\mathsf{C}_3 \cup \mathsf{K}_2, 5) = S(\mathsf{P}_5, 5) = 1$, $S(\mathsf{C}_4 \cup \mathsf{K}_2, 4) = S(\mathsf{P}_6, 4) = 25$, $S(\mathsf{C}_4 \cup \mathsf{K}_2, 5) = S(\mathsf{P}_6, 5) = 10$, and $S(\mathsf{C}_4 \cup \mathsf{K}_2, 6) = S(\mathsf{P}_6, 6) = 1$. For larger values or n, the equality is obtained by induction, using equation (6), since $S(\mathsf{C}_{n-4} \cup \mathsf{K}_2, k)$ is then equal to $S(\mathsf{P}_{n-2}, k)$.
  Since $S(\mathsf{C}_n, k) \leq S(\mathsf{P}_n, k)$ for all k, we have $S(\mathsf{C}_n, k) \leq S(\mathsf{C}_{n-2} \cup \mathsf{K}_2, k)$ for $k = 4, 5, \ldots, n$.
- If $k = 3$, we first show that $S(\mathsf{C}_{n-2} \cup \mathsf{K}_2, 3) = S(\mathsf{P}_n, 3) + (-1)^n$. This is true for $n = 5, 6$ since $S(\mathsf{C}_3 \cup \mathsf{K}_2, 3) = 6$, $S(\mathsf{P}_5, 3) = 7$, $S(\mathsf{C}_4 \cup \mathsf{K}_2, 3) = 16$, and $S(\mathsf{P}_6, 3) = 15$. For larger values or n, the equality is obtained by induction, using equation (6), since $S(\mathsf{C}_{n-4} \cup \mathsf{K}_2, k)$ is then equal to $S(\mathsf{P}_{n-2}, 3) + (-1)^{n-2} = S(\mathsf{P}_{n-2}, 3) + (-1)^n$.
  Since $S(\mathsf{C}_{n-1}, 3) > 1$ for all $n \geq 5$, we conclude that $S(\mathsf{C}_n, 3) = S(\mathsf{P}_n, 3) - S(\mathsf{C}_{n-1}, 3) \leq S(\mathsf{P}_n, 3) - 2 < S(\mathsf{P}_n, 3) + (-1)^n = S(\mathsf{C}_{n-2} \cup \mathsf{K}_2, 3)$.
- If $k = 2$ then $S(\mathsf{C}_n, 2) \leq S(\mathsf{C}_{n-2} \cup \mathsf{K}_2, 2)$ since both $S(\mathsf{C}_n, 2)$ and $S(\mathsf{C}_{n-2} \cup \mathsf{K}_2, 2)$ equal 0 if n is odd, while $S(\mathsf{C}_n, 2) = 1 < 2 = S(\mathsf{C}_{n-2} \cup \mathsf{K}_2, 2)$ if n is even. □

The graph $G^{<}_{n,\Delta=2}$ is defined as follows:

- it is the disjoint union of $\frac{n}{3}$ copies of K_3 if $n \equiv 0 \pmod 3$;
- it is the disjoint union of $G^{<}_{n-4,\Delta=2}$ and C_4 if $n \equiv 1 \pmod 3$;
- it is the disjoint union of $G^{<}_{n-5,\Delta=2}$ and C_5 if $n \equiv 2 \pmod 3$.

The graph $G^{<}_{7,\Delta=2}$ is illustrated in the middle of Figure 6. We now give a lower bound on $\mathcal{B}(G)$ for graphs G with maximum degree $\Delta(G) = 2$ and order $n \geq 5$. This is not restrictive because if $n \leq 4$ and $\Delta = 2$, then $\Delta = n-2$ or $\Delta = n-1$ and these cases are treated later.

Theorem 12. *Let G be a graph of order $n \geq 5$ such that $\Delta(G) = 2$. Then,*

$$\mathcal{B}(G^{<}_{n,\Delta=2}) \leq \mathcal{B}(G),$$

with equality if and only if G is isomorphic to $G^{<}_{n,\Delta=2}$.

Proof. Suppose G is extremal. Since $\Delta(G) = 2$, G is the disjoint union of cycles and paths. It follows from Property 5 that at most one connected component of G is a path, and such a path can only be K_1 or K_2.

Case 1 : K_1 is a connected component of G.
Let C_r ($r \geq 3$) be a longest cycle of G. If $r = 3$, then G is the disjoint union of K_1 and at least two copies of C_3 (because $n \geq 5$). Thus, $G = 2\mathsf{C}_3 \cup \mathsf{K}_1 \cup H$ where H is a (possibly empty) disjoint union of C_3. The following table shows that G is not extreme since $2\mathsf{C}_3 \cup \mathsf{K}_1 >_S \mathsf{C}_3 \cup \mathsf{C}_4$, a contradiction.

k	2	3	4	5	6	7
$S(2\mathsf{C}_3 \cup \mathsf{K}_1, k)$	0	18	78	63	15	1
$S(\mathsf{C}_3 \cup \mathsf{C}_4, k)$	0	18	66	55	14	1

If $r \geq 4$, then we know from Corollary 10 that either C_{r+1} (if r is even) or $\mathsf{C}_{r-2} \cup \mathsf{C}_3$ (if r is odd) is strictly dominated by $\mathsf{C}_r \cup \mathsf{K}_1$. Hence, G is not extremal, a contradiction.

Case 2 : K_2 is a connected component of G.
Let C_r be any cycle in G. We know from Lemma 11 that $\mathsf{C}_r \cup \mathsf{K}_2 >_S \mathsf{C}_{r+2}$, which means that G is not extremal, a contradiction.

Case 3 : G is the disjoint union of cycles.
Since G is extremal, we know from Corollary 8 that these cycles are copies of C_3, C_4 or C_5. The following tables show that $2\mathsf{C}_5 >_S 2\mathsf{C}_3 \cup \mathsf{C}_4$, $\mathsf{C}_5 \cup \mathsf{C}_4 >_S 3\mathsf{C}_3$, and $2\mathsf{C}_4 >_S 2\mathsf{C}_5 \cup \mathsf{C}_3$. Hence, since G is extremal, it contains no more than one C_4 or one C_5, which means that G is isomorphic to $G^{<}_{n,\Delta=2}$.

k	2	3	4	5	6	7	8	9	10
$S(2\mathsf{C}_5, k)$	0	150	2250	6345	6025	2400	435	35	1
$S(2\mathsf{C}_3 \cup \mathsf{C}_4, k)$	0	108	1908	5838	5790	2361	433	35	1

k	2	3	4	5	6	7	8	9
$S(\mathsf{C}_5 \cup \mathsf{C}_4, k)$	0	90	750	1415	925	246	27	1
$S(3\mathsf{C}_3, k)$	0	36	540	1242	882	243	27	1

k	2	3	4	5	6	7	8
$S(2\mathsf{C}_4, k)$	2	52	241	296	126	20	1
$S(\mathsf{C}_5 \cup \mathsf{C}_3, k)$	0	30	210	285	125	20	1

□

Since $\mathsf{C}_3 = \mathsf{K}_3$, we can link the above result with the generalized Bell numbers mentioned in Section 2.

Corollary 13. *Let G be a graph of order n such that $n \equiv 0 \pmod 3$ and $\Delta(G) = 2$. Then*

$$\mathcal{B}(G) \geq \mathsf{B}_{3,\frac{n}{3}}$$

We now give a lower bound on $\mathcal{B}(G)$ for graphs G of order n and maximum degree $n - 2$.

Theorem 14. *Let G be a graph of order $n \geq 2$ such that $\Delta(G) = n - 2$. Then,*

$$\mathcal{B}(G) \geq n$$

with equality if and only if G is isomorphic to $\mathsf{K}_{n-1} \cup \mathsf{K}_1$ when $n \neq 4$, and G is isomorphic to $\mathsf{K}_3 \cup \mathsf{K}_1$ or C_4 otherwise.

Proof. The proof is by induction on n and the result is clearly valid for $n = 2$. Notice first that $\mathcal{B}(\mathsf{K}_{n-1} \cup \mathsf{K}_1) = n$ because either the isolated vertex of K_1 has its own color, or it uses one of the $n - 1$ colors in K_{n-1}. So let G be an extremal graph of order $n > 2$ with $\Delta(G) = n - 2$. We then have $\mathcal{B}(G) \leq \mathcal{B}(\mathsf{K}_{n-1} \cup \mathsf{K}_1) = n$. Let x be any vertex of degree $n - 2$, and let y be the unique vertex that is not adjacent to x. It follows from Property 5 that if two vertices v and w distinct from x and y are non-adjacent, then they are both adjacent to y. Hence, if y is an isolated vertex in G, then G is isomorphic to $\mathsf{K}_{n-1} \cup \mathsf{K}_1$.

So suppose $d(y) \geq 1$ and let v be one of its neighbors. Since v is not dominating, there exists at least one vertex w not adjacent to v. As observed above, w is necessarily adjacent to y. Let W be the set of vertices adjacent to y. We therefore have $|W| \geq 2$ and, by Property 5, every vertex non-adjacent to y has degree $n - 2$. Let G' be the graph induced by W. No vertex of G' is dominating (else it would also be dominating in G), and since at least one of v and w has degree $n - 2$ in G (and thus has degree $|W| - 2$ in G'), we have $\Delta(G') = |W| - 2$. By induction, $\mathcal{B}(G') \geq |W|$.

Given any coloring of G', we can construct $n - |W|$ non-equivalent colorings of G by copying the colors on the vertices of W, assigning new colors to all vertices non-adjacent to y, and either assigning one of these $n - |W| - 1$ new colors to y, or a new one not shared by any other vertex. Hence,

$$n \geq \mathcal{B}(G) \geq \mathcal{B}(G')(n - |W|) \geq |W|(n - |W|). \tag{7}$$

Then, $n - |W| \geq |W|(n - |W| - 1) \geq 2(n - |W| - 1)$, which implies $n - |W| \leq 2$. Since x and y do not belong to W, we have $n - |W| = 2$. Hence, equation (7) becomes $|W| + 2 \geq 2|W|$, which is equivalent to $|W| \leq 2$. Since v and w belong to W, we have $|W| = 2$. In summary, $\mathcal{B}(G) = n = 4$ and G is isomorphic to C_4.

□

Finally, notice that the lower bound on $\mathcal{B}(G)$ for graphs G with $\Delta(G) = n - 1$ is trivial since K_n has clearly the minimum number of colorings among all graph of order n.

5 Concluding remarks

We have studied some properties of the Bell number of a graph that corresponds to the number of non-equivalent proper vertex colorings. We have shown similarities and differences between this invariant and the famous chromatic polynomial. We have given lower and upper bounds on its value for graphs with bounded maximum degree.

It would be interesting to determine a lower bound on $\mathcal{B}(G)$ for graphs G of order n and with maximum degree in $\{3, 4, \ldots, n - 3\}$. The extremal graphs in this case do not seem to have a simple structure, as was the case for $\Delta(G) = 1, 2, n - 2, n - 1$. We have determined some of them by exhaustive enumeration. For example, we have drawn in Figure 7 the only graphs G of order $n = 6, 7, 8, 9$ with minimum value $\mathcal{B}(G)$ when $\Delta(G) = 3, 4, 5$.

Notice also that several graphs with minimum value $\mathcal{B}(G)$ are non-connected. It would be interesting to determine these extremal graphs with the additional constraint that G must be connected. Also, it could be interesting to characterize the graphs G that minimize or maximize $\mathcal{B}(G)$ when the order and the size of G are fixed.

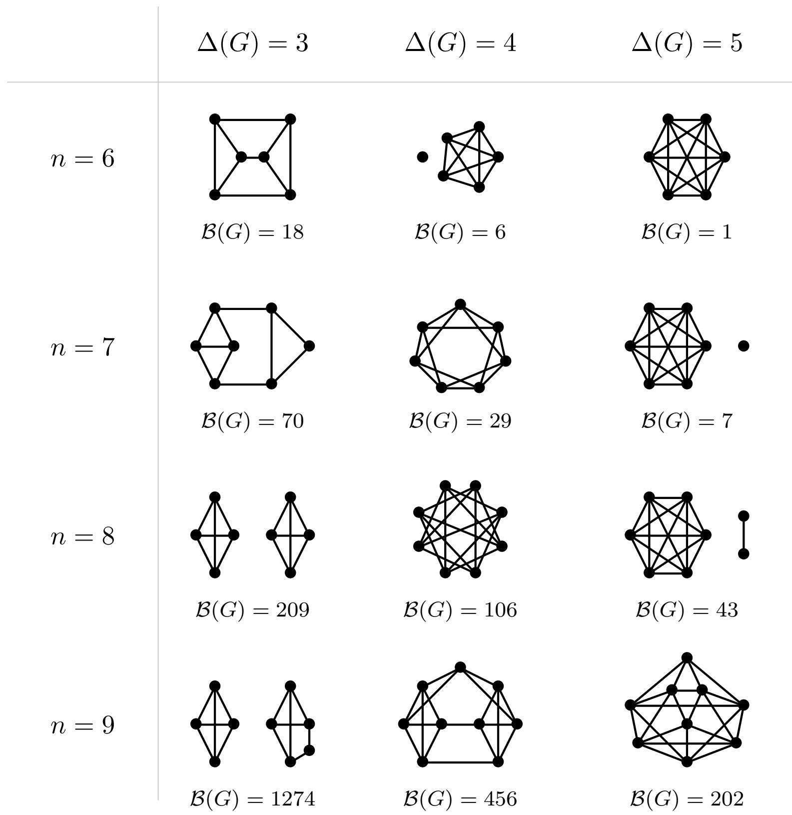

Figure 7: Unique graphs G of order $n = 6, 7, 8, 9$ and maximum degree $\Delta(G) = 3, 4, 5$ with minimum value for $\mathcal{B}(G)$.

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