Multiplicity and symmetry of positive solutions to semi-linear elliptic problems with Neumann boundary conditions

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(Joint work with D. Bonheure & C. Grumiau)

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The Lane-Emden problem

Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded, $N \geqslant 2$, and $2 < p < 2^* := \frac{2N}{N-2}$. We consider

$$(\mathcal{P}_p) \begin{cases} -\Delta u + u = |u|^{p-2} u, & \text{in } \Omega \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

Solutions are critical points of the functional

$$\mathcal{E}_p : H^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2} \int_\Omega |\nabla u|^2 + u^2 - \frac{1}{p} \int_\Omega |u|^p$$

$$\partial\mathcal{E}_p(u) : H^1(\Omega) \to \mathbb{R} : v \mapsto \int_\Omega \nabla u \nabla v + uv - \int_\Omega |u|^{p-2} uv$$

Notation: $1 = \lambda_1 < \lambda_2 < \cdots$ denote the eigenvalues of $-\Delta + 1$
E_i denote the corresponding eigenspaces

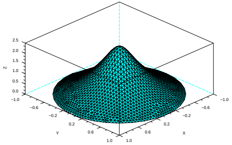
Remark: 0 is always a (trivial) solution.

Outline

1. $p \approx 2$: ground state solutions
2. $p \approx 2$: positive solutions
3. p large: symmetry breaking of the ground state
4. p large: bifurcations from 1
5. p large: multiplicity results (radial domains)
6. Numerics

Dirichlet boundary conditions

$$\begin{cases} -\Delta u + u = |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$



- The ground state solution is positive and is even w.r.t. any hyperplane leaving Ω invariant (when Ω is convex). In particular, it is radially symmetric on a ball.
- Uniqueness of the positive solution when Ω is a ball.
- If Ω is strictly starshaped and $p \geqslant 2^*$, no solution exist.

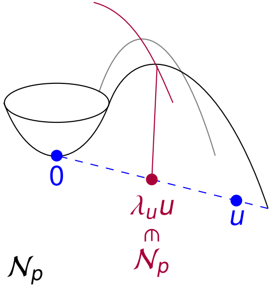
Existence of ground state solutions ($p < 2^*$)

Theorem (Z. Nehari, A. Ambrosetti, P.H. Rabinowitz)

For any $p \in]2, 2^*[$, there exists a ground state solution to $(\mathcal{P}_p)$. It is a one-signed function.

Sketch of the proof.

- The functional $\mathcal{E}_p$ possesses a mountain pass structure.
- $\exists u_0 \neq 0$, $\mathcal{E}_p(u_0) = \inf_{u \neq 0} \max_{\lambda > 0} \mathcal{E}_p(\lambda u)$
  $= \inf_{u \in \mathcal{N}_p} \mathcal{E}_p(u)$
  where $\mathcal{N}_p$ is the Nehari manifold of $\mathcal{E}_p$.
- For any sign-changing solution u: if $u^\pm \neq 0$, $u^\pm \in \mathcal{N}_p$ and $\mathcal{E}_p(u^\pm) < \mathcal{E}_p(u)$, where $u^\pm := \pm\max\{\pm u, 0\}$.



$p \approx 2$: symmetry of ground state solutions

Theorem (D. Bonheure, V. Bouchez, C. Grumiau, C. T., J. Van Schaftingen, '08)

For p close to 2 and any $R \in O(N)$ s.t. $R(\Omega) = \Omega$, ground state solutions to $(\mathcal{P}_p)$ are symmetric w.r.t. R.

E.g. if Ω is radially symmetric, so must the the ground state solution be.

Remark that the seminal method of moving planes is not applicable.

Uniqueness of the positive solution

Theorem

1 *is the unique positive solution for p small.*

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Let $v := P_{E_1} u_p$ (constant function) and $w := P_{E_1^\perp} u_p$.

$$\begin{aligned}
\lambda_2 \int_\Omega w^2 &\leqslant \int_\Omega |\nabla w|^2 + w^2 \\
&= \int_\Omega |u_p|^{p-1} w = \int_\Omega \Bigl((v+w)^{p-1} - v^{p-1}\Bigr) w \\
&= \int_\Omega (p-1)(v + \vartheta_p w)^{p-2} w^2 \qquad\qquad (\vartheta_p \in \,]0,1[) \\
&\leqslant (p-1)(|v| + \|w\|_\infty)^{p-2} \int_\Omega w^2 \leqslant (p-1) K^{p-2} \int_\Omega w^2.
\end{aligned}$$

As $\lambda_1 = 1 < \lambda_2$, for $p \approx 2$, $w = 0$ and then $u_p = v = 1$.

A priori bounds for positive solutions

Lemma

Positive solutions (u_p) are bounded in L^∞ as $p \approx 2$.

- Integration & Hölder: $\displaystyle\int_\Omega u_p^{p-1} = \int_\Omega u_p \leqslant |\Omega|$ (recall $u_p > 0$).
- Brezis-Strauss: from the bound on $\displaystyle\int_\Omega u_p^{p-1}$, we deduce a bound on $\|u_p\|_{W^{1,q}(\Omega)}$, $1 \leqslant q < N/(N-1)$.
- Sobolev embedding: (u_p) bounded in $L^r(\Omega)$, $1 < r < N/(N-2)$.
- Bootstrap: $\|u_p\|_{W^{2,r}(\Omega)}$ is bounded for some $r > N/2$ when $p \approx 2$.

A priori bounds for positive solutions

Proposition

Let $2 < \bar{p} < 2^*$. *There exists* $C_{\bar{p}} > 0$ *such that any positive solution to* $(\mathcal{P}_p)$ *with* $2 < p \leqslant \bar{p}$ *satisfies* $\max\left\{\|u\|_{H^1}, \|u\|_{L^\infty}\right\} \leqslant C_{\bar{p}}$.

A priori bounds for positive solutions

Proposition

Let $2 < \bar{p} < 2^*$. There exists $C_{\bar{p}} > 0$ such that any positive solution to $(\mathcal{P}_p)$ with $2 < p \leqslant \bar{p}$ satisfies $\max\left\{\|u\|_{H^1}, \|u\|_{L^\infty}\right\} \leqslant C_{\bar{p}}$.

It remains to obtain a bound for $2 < \underline{p} < \bar{p} < 2^*$ in L^∞. Blow up argument (Gidas-Spruck). Suppose on the contrary that there is a sequence $(p_n) \subseteq [\underline{p}, \bar{p}]$ and (u_{p_n}) s.t.

$$u_{p_n}(x_{p_n}) := \|u_{p_n}\|_{L^\infty} \to +\infty \qquad \text{and} \qquad p_n \to p^* \in [\underline{p}, \bar{p}].$$

(Drop index n.) Define

$$v_p(y) := \mu_p u_p\left(\mu_p^{(p-2)/2} y + x_p\right) \qquad \text{where } \mu_p := 1/\|u_p\|_{L^\infty} \to 0.$$

Note: $v_p(0) = \|v_p\|_{L^\infty} = 1$.

A priori bounds for positive solutions

The rescaled function v_p satisfies

$$-\Delta v_p + \mu_p^{p-2} v_p = v_p^{p-1} \quad \text{on } \Omega_p := (\Omega - x_p)/\mu_p^{(p-2)/2}$$

with NBC. By elliptic regularity, (v_p) is bounded in $W^{2,r}$ and $C^{1,\alpha}$, $0 < \alpha < 1$ on any compact set. Thus, taking if necessary a subsequence,

$$v_n \to v^* \quad \text{in } W^{2,r} \text{ and } C^{1,\alpha} \text{ on compact sets of } \Omega^* = \mathbb{R}^N \text{ or } \mathbb{R}^{N-1} \times \mathbb{R}_{>a}.$$

One has $v^* \geqslant 0$, $v^*(0) = 1 = \|v\|_{L^\infty}$ and v^* satisfies

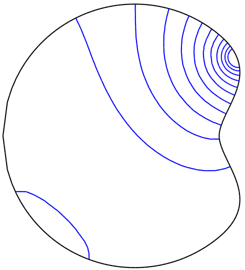
$$-\Delta v^* = (v^*)^{p^*-1} \quad \text{in } \mathbb{R}^N \qquad \text{or} \qquad \begin{cases} -\Delta v^* = (v^*)^{p^*-1} & \text{in } \mathbb{R}^{N-1} \times \mathbb{R}_{>a} \\ \partial_N v^* = 0 & \text{when } x_N = a \end{cases}$$

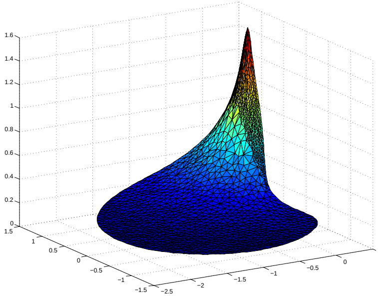
Liouville theorems imply $v^* = 0$. □

Symmetry breaking of the ground state

Theorem (W.-M. Ni and I. Takagi, ’93)

When R is sufficiently large, ground state solutions possess a unique maximum point $P_R \in \partial(R\Omega)$. Moreover, $u_R \to 0$ outside a small neighborhood of P_R. P_R is situated at the “most curved” part of $\partial(R\Omega)$.





p large: symmetry breaking of the ground state

Corollary

1 cannot remain the ground state for all p on "large" domains.

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Lemma

1 cannot remain the ground state solution for $p > 1 + \lambda_2$.

Proof. The Morse index of 1 is the sum of the dimension of the eigenspaces corresponding to negative eigenvalues λ of

$$\begin{cases} -\Delta v + v = (p-1)v + \lambda v, & \text{in } \Omega, \\ \partial_\nu v = 0, & \text{on } \partial\Omega. \end{cases}$$

i.e. eigenvalues of $-\Delta + 1$ less than $p-1$. When $p > 1 + \lambda_2$, the Morse index of the solution 1 is > 1.

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Proposition (Lopez, '96)

On radial domains, the ground state is either constant or (e.g. when $p > 1 + \lambda_2$) not radially symmetric.

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Proposition

When Ω is a ball or an annulus, the Morse index of a non-constant positive radial solution is at least $N + 1$.

Based on: A. Aftalion, F. Pacella, Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains, CRAS, 339(5), '04.

Let u be non-constant positive radial solution of $(\mathcal{P}_p)$. We have to show that

$$L v := -\Delta v + v - (p-1)|u|^{p-2} v$$

with NBC possesses $N + 1$ negative eigenvalues.

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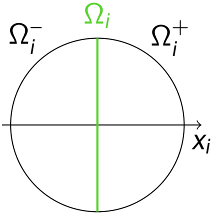
u radial $\Rightarrow \partial_{x_i} u = 0$ on $\partial\Omega$ and on Ω_i.



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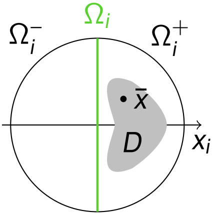
u radial $\Rightarrow \partial_{x_i} u = 0$ on $\partial\Omega$ and on Ω_i.
Let $\bar{x} \in \Omega_i^+$ s.t. $\partial_{x_i} u(\bar{x}) \neq 0$. Let D be the connected component of $\{\partial_{x_i} u(\bar{x}) \neq 0\}$ containing $\bar{x}$. $D \subseteq \Omega_i^+$.



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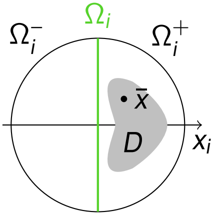

$$L(\partial_{x_i} u) = 0, \quad \text{on } D; \qquad \partial_{x_i} u = 0, \quad \text{on } \partial D.$$

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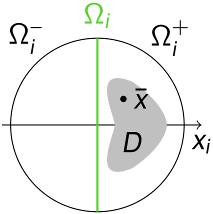

$$L(\partial_{x_i} u) = 0, \quad \text{on } D; \qquad \partial_{x_i} u = 0, \quad \text{on } \partial D.$$

$\Rightarrow \lambda_1(L, D, \text{DBC}) = 0$

$\Rightarrow \lambda_1(L, \Omega_i^+, \text{DBC}) \leqslant 0$

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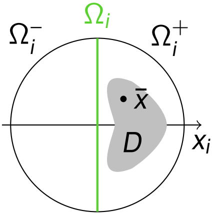

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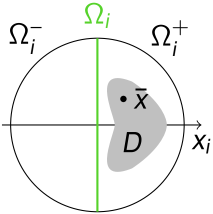

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If $\psi_i > 0$ is the first eigenfunction of L on Ω_i^+ with DBC on Ω_i and NBC on $\partial\Omega_i^+ \setminus \Omega_i$, its odd extension ψ_i^* to Ω satisfies

$$L(\psi_i^*) = \mu_i \psi_i^*, \quad \text{on } \Omega, \qquad \partial_\nu \psi_i^* = 0, \quad \text{on } \partial\Omega.$$

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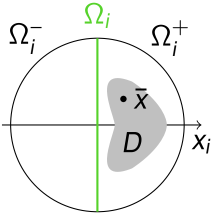

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All ψ_j^*, $j \neq i$ vanish on the axis $x_i \Rightarrow$ the family $(\psi_j^*)_{j=1}^N$ is lin. indep.

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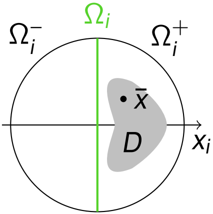

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All ψ_j^*, $j \neq i$ vanish on the axis $x_i \Rightarrow$ the family $(\psi_j^*)_{j=1}^N$ is lin. indep.
None of the $(\psi_j^*)_{j=1}^N$ is a first eigenfunction.

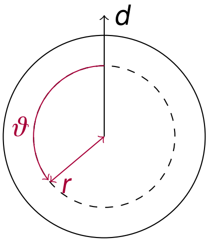
p large: symmetry breaking of the ground state

Theorem (Lopes, '96)

On radial domains, ground state solutions are symmetric w.r.t. any hyperplane containing a line L passing through the origin.

Theorem (J. Van Schaftingen, '04)

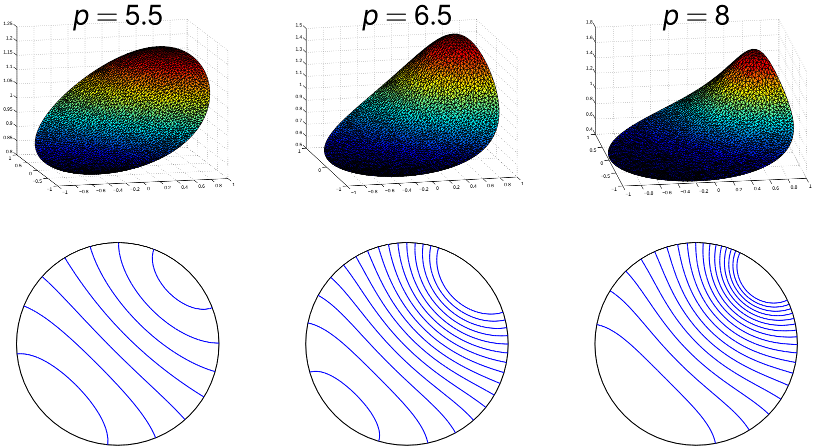
On radial domains, ground state solutions are foliated Schwarz symmetric.



There exists a unit vector d s.t. u depends only on $r = |x|$ and $\vartheta = \arccos(\frac{x}{|x|} \cdot d)$ and is non-increasing in ϑ.

p large: non radially symmetric ground state

$$\Omega = B_1 \subseteq \mathbb{R}^2 \;\Rightarrow\; 1 + \lambda_2 \approx 5.39$$



Symmetry breaking at exactly $p = 1 + \lambda_2$?

The linearisation of the equation around $u = 1$,

$$Lv := -\Delta v + v - (p-1)v$$

is not invertible iff $p = 1 + \lambda_i$, $i \geqslant 2$.

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Eigenfunctions of $-\Delta + 1$ with NBC have the form:

$$u(x) = r^{-\frac{N-2}{2}} J_\nu(\sqrt{\mu} r) P_k\left(\frac{x}{|x|}\right), \qquad \text{where } \nu = k + \frac{N-2}{2},$$

$r = |x|$, and $P_k : \mathbb{R}^N \to \mathbb{R}$ is an harmonic homogenous polynomial of degree k for some $k \in \mathbb{N}$. To satisfy the boundary conditions:

$$\sqrt{\mu} R \text{ is a root of } z \mapsto (k-\nu)J_\nu(z) + z\partial J_\nu(z) = kJ_\nu(z) - zJ_{\nu+1}(z).$$

$\Rightarrow \lambda_i = 1 + \mu$

Symmetry breaking at exactly $p = 1 + \lambda_2$?

In particular, a basis of E_2 is

$$x \mapsto r^{-\frac{N-2}{2}} J_{N/2}(\sqrt{\mu} r) \frac{x_j}{|x|}, \qquad j = 1, \dots, N.$$

There is single function (up to a multiple) that is invariant under rotation in $(x_2, \dots, x_N)$.

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Theorem (Crandall-Rabinowitz)

Let X and Y two Banach spaces, $u^* \in X$, and a function $F : \mathbb{R} \times X \to Y : (p, u) \mapsto F(p, u)$ such that $\forall p \in \mathbb{R},\ F(p, u^*) = 0$. Let $p^* \in \mathbb{R}$ be such that $\ker(\partial_u F(p^*, u^*)) = \operatorname{span}\{\varphi^*\}$ has a dimension 1 and $\operatorname{codim}\bigl(\operatorname{Im}(\partial_u F(p^*, u^*))\bigr) = 1$. Let $\psi : Y \to \mathbb{R}$ be a continuous linear map such that $\operatorname{Im}(\partial_u F(p^*, u^*)) = \{y \in Y : \langle \psi, y \rangle = 0\}$.

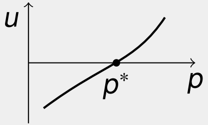
Symmetry breaking at exactly $p = 1 + \lambda_2$?

Theorem (Crandall-Rabinowitz (cont'd))

If $a := \langle \psi, \partial_{p,u} F(p^*, u^*)[\varphi^*] \rangle \neq 0$*, then* (p^*, u^*) *is a bifurcation point for* F*. In addition, the set of non-trivial solutions of* $F = 0$ *around* (p^*, u^*) *is given by a unique* $\mathcal{C}^1$ *curve* $p \mapsto u_p$*. The local behavior of the branch* (p, u_p) *for* p *close to* p^* *is as follows.*

- *If* $b := -\frac{1}{2a} \Big\langle \psi, \partial_u^2 F(p^*, u^*)[\varphi^*, \varphi^*] \Big\rangle \neq 0$ *then the branch is transcritical and*

$$u_p = u^* + \frac{p - p^*}{b} \varphi^* + o(p - p^*).$$



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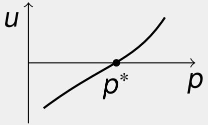


In our case,

$$a = -\int_\Omega \varphi_2^2 = -1 \qquad \text{and} \qquad b = -\tfrac{1}{2} \lambda_2 (\lambda_2 - 1) \int_\Omega \varphi_2^3 = 0.$$

Symmetry breaking at exactly $p = 1 + \lambda_2$?

Theorem (Crandall-Rabinowitz (cont'd))

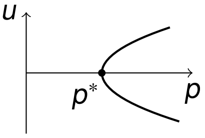
- *If $b = 0$, let us define*

$$c := -\frac{1}{6a}\Big(\big\langle \psi, \partial_u^3 F(p^*, u^*)[\varphi^*, \varphi^*, \varphi^*]\big\rangle + 3\big\langle \psi, \partial_u^2 F(p^*, u^*)[\varphi^*, w]\big\rangle\Big)$$

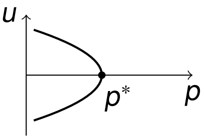
where $w \in X$ is any solution of the equation $\partial_u F(p^*, u^*)[w] = -\partial_u^2 F(p^*, u^*)[\varphi^*, \varphi^*]$. If $c \neq 0$ then

$$u_p = u^* \pm \Big(\frac{p - p^*}{c}\Big)^{1/2} \varphi^* + o\big(|p - p^*|^{1/2}\big).$$

In particular, the branch is supercritical if $c > 0$ and subcritical if $c < 0$.



Supercritical



Subcritical

Symmetry breaking at exactly $p = 1 + \lambda_2$?

In our case,

$$c = \frac{1}{6}\lambda_2(\lambda_2 - 1)\Big(-(\lambda_2 - 2)\int_{B_R} \varphi_2^4 - 3\lambda_2(\lambda_2 - 1)\int_{B_R} \varphi_2^2 w\Big)$$

where $(-\Delta + 1 - \lambda_2)w = \varphi_2^2$ with NBC on B_R.

Symmetry breaking at exactly $p = 1 + \lambda_2$?

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where $(-\Delta + 1 - \lambda_2)w = \varphi_2^2$ with NBC on B_R.

$$= \frac{1}{6}\bar{\mu}_2 R^{-(N+2)}\Big(1 + \frac{\bar{\mu}_2}{R^2}\Big)\Big((\beta - \alpha)\frac{\bar{\mu}_2}{R^2} + \beta + \alpha\Big)$$

where $\alpha := \int_{B_1}\bar{\varphi}_2^4$, $\quad \beta := -3\bar{\mu}_2\int_{B_1}\bar{\varphi}_2^2\bar{w}$,

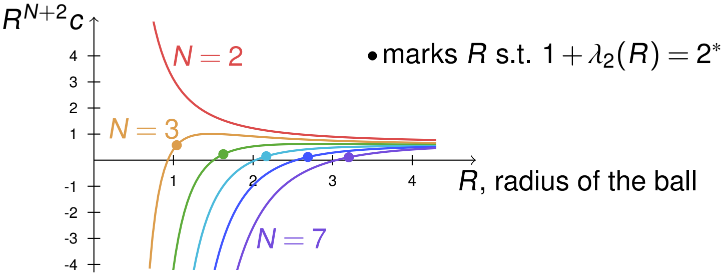
$(-\Delta - \bar{\mu}_2)\bar{w} = \bar{\varphi}_2^2$ with NBC on B_1,

$\bar{\varphi}_2$ and $\bar{\mu}_2 > 0$ are “the” second eigenfunction and eigenvalue of $-\Delta$ with NBC on B_1 s.t. $|\bar{\varphi}_2|_{L^2} = 1$.

Symmetry breaking at exactly $p = 1 + \lambda_2$?

We numerically have

N	α	β	$\beta - \alpha$	$\beta + \alpha$
2	0.5577	0.5884	0.0306	1.1461
3	0.4632	0.3096	−0.1536	0.7728
4	0.4222	0.1694	−0.2528	0.5916
5	0.4171	0.0858	−0.3313	0.5029
6	0.4421	0.0250	−0.4171	0.4671



Symmetry breaking at exactly $p = 1 + \lambda_2$?

Conjecture

Let $p \in \,]2, 2^*]$. The constant function 1 is the ground state of

$$(\mathcal{P}_p) \begin{cases} -\Delta u + u = |u|^{p-2} u, & \text{in } B_R \\ \partial_\nu u = 0, & \text{on } \partial B_R. \end{cases}$$

iff $p \leqslant 1 + \lambda_2(B_R)$.

Symmetry breaking at exactly $p = 1 + \lambda_2$?

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Conjecture

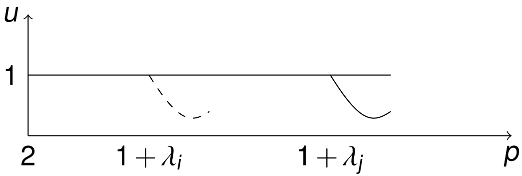
For $p > 1 + \lambda_2$, the gound state lives on the branch emanating from 1 at $p = 1 + \lambda_2$.

p large: bifurcations from 1

Lemma

When $p > 2$ is increasing,

1. *a bifurcation **sequence** start from 1 **iff** p crosses $1 + \lambda_i$;*
2. *this is actually a continuum if λ_i has **odd** multiplicity.*



Skip KMB theorem

Krasnoselskii-Boehme-Marino theorem (1/2)

Theorem (Krasnoselskii-Boehme-Marino)

Let $F : I \times H \to K : (t,u) \mapsto F(t,u)$ be a continuous function, where $I \subseteq \mathbb{R}$ is an interval, and H and K are Banach spaces, such that $F(\lambda, 0) = 0$ for any $\lambda \in I$.

- *If F is of class C^1 in a neighborhood of $(\lambda, 0)$ and $(\lambda, 0)$ is a bifurcation point of F then $\partial_u F(\lambda, 0)$ is not invertible.*
- *Let assume that for each $(\lambda, u) \in I \times H$,*

  $$F(\lambda,u) = L(\lambda,u) - N(\lambda,u), \quad L(\lambda,\cdot) = \lambda\mathbb{1} - T \quad \text{and} \quad N(\lambda,u) = o(\|u\|),$$

  with T linear, T and N compact, and the last equality being uniform on each compact set of λ.
  If λ_* is an eigenvalue of T with odd multiplicity, then $(\lambda_*, 0)$ is a global bifurcation point for $F(t,u) = 0$.

Krasnoselskii-Boehme-Marino theorem (2/2)

Theorem (Krasnoselskii-Boehme-Marino (cont'd))

- *Let assume that H is a Hilbert space and that for each $(\lambda, u) \in I \times \mathbb{R}$, $F(\lambda, u) = \nabla_u h(\lambda, u)$ where*

$$h(\lambda, u) = \tfrac{1}{2}\langle L(\lambda, u), u\rangle - g(\lambda, u),$$
$$L(\lambda, \cdot) = \lambda \mathbb{1} - T, \quad \text{and} \quad \nabla g(\lambda, u) = o(\|u\|),$$

with T linear and symmetric, $g(\lambda, \cdot) \in \mathcal{C}^2$ for all λ, and the last equality being uniform on each compact set of λ.
If λ_* is an eigenvalue of T with finite multiplicity and $h(\lambda, \cdot)$ verifies the Palais-Smale condition for each λ, then $(\lambda_*, 0)$ is a bifurcation point for $F(t, u) = 0$.

p large: transcritical radial bifurcations

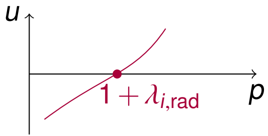
$\lambda_{i,\text{rad}}$ eigenvalues that possess a radial eigenfunction (simple in H^1_{rad}).

Proposition

On balls, two branches radial solutions in $C^{2,\alpha}(\Omega)$ of

$$(\mathcal{P}_p)\begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \Omega \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

start from each $(p,u) = (1+\lambda_{i,rad}, 1)$, $i > 1$. Locally, these branches form a unique C^1-curve. Moreover, for i large enough independent of the measure of Ω, the bifurcation is transcritical.



p large: transcritical radial bifurcations

Proof. $\Omega = B_R$. Using Crandall-Rabinowitz' theorem, one has to show

$$b = -\tfrac{1}{2}\lambda_i(\lambda_i - 1)\int_{B_R} \varphi_{i,\mathrm{rad}}^3 \neq 0.$$

Given that radial eigenfunctions are given by constant spherical harmonics ($k = 0$, $\nu = (N-2)/2$), this amounts to

$$\int_0^R \Bigl(r^{-\frac{N-2}{2}} J_\nu\bigl(r\sqrt{\bar\mu_{i,\mathrm{rad}}}/R\bigr)\Bigr)^3 r^{N-1}\,\mathrm{d}r \neq 0 \quad \text{i.e.} \quad \int_0^{\sqrt{\bar\mu_{i,\mathrm{rad}}}} t^{1-\nu} J_\nu^3(t)\,\mathrm{d}t \neq 0$$

where $\lambda_{i,\mathrm{rad}} = 1 + \bar\mu_{i,\mathrm{rad}}/R^2$. This is true for large i because

$$\int_0^\infty t^{1-\nu} J_\nu^3(t)\,\mathrm{d}t = \frac{2^{\nu-1}(3/16)^{\nu-1/2}}{\pi^{1/2}\Gamma(\nu+1/2)} > 0.$$

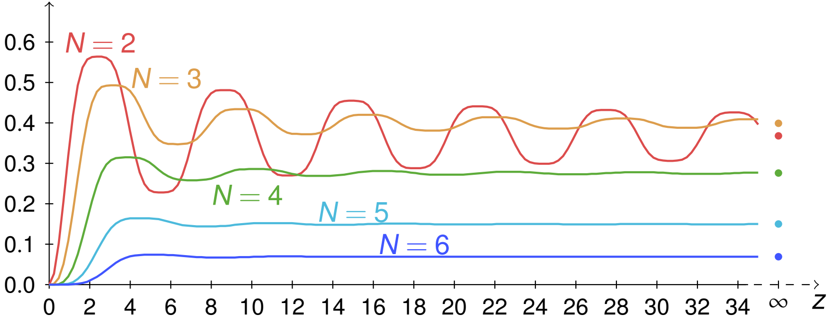
Thus $b < 0$.

p large: transcritical radial bifurcations

Numerical computations indicate that

$$\forall z \in]0, +\infty[, \quad \int_0^z t^{1-\nu} J_\nu^3(t)\,\mathrm{d}t > 0, \qquad \nu = (N-2)/2,$$

and therefore that radial bifurcations are transcritical for all i.

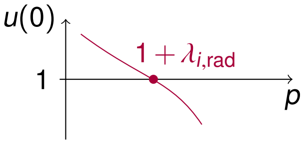


Shape of transcritical radial bifurcations

$$u_p = 1 + \frac{p-(1+\lambda_i)}{b}\varphi_i + o\big(p-(1+\lambda_i)\big)$$

where $\varphi_i(x) = |x|^{-\nu} J_\nu(\sqrt{\lambda_{i,\mathrm{rad}} - 1}\,|x|)$. Thus

- $u_p(0) > 1$ if $p < 1 + \lambda_i$
- $u_p(0) < 1$ if $p > 1 + \lambda_i$



These facts remain true along the whole banches.

p large: positive transcritical radial bifurcations

Corollary

The branches consist of positive functions.

Sketch: If it was not the case, there would be a point solution along the branch with a double root, hence $= 0$. There is no bifurcation from 0. □

p large: positive transcritical radial bifurcations

Corollary

The branches consist of positive functions.

SKETCH: If it was not the case, there would be a point solution along the branch with a double root, hence $= 0$. There is no bifurcation from 0. □

Theorem

Radial bifurcations obtained for the $C^{2,\alpha}(\Omega)$-norm are unbounded and do not intersect each other. Moreover, along bifurcations starting from $(1 + \lambda_{i,rad}, 1)$, the solutions always possess the same number of intersections with 1.

SKETCH: The number of crossings with 1 stays constant because otherwise a non-constant radial solution u s.t. $u - 1$ has a double root would exists. Since the branches do not intersect each other, Rabinowitz's principle says they must be undounded.

p large: multiplicity results (radial domains)

Theorem

Assume Ω is a ball.

- *In dimension 2, for any $n \in \mathbb{N}_0$, there exists $p_n > 2$ such that, for any $p > p_n$, at least n positive solutions exist*
- *In dimension $\geqslant 3$, for any $2 < p < 2^*$ and $n \in \mathbb{N}_0$, at least n different positive solutions exist if the measure of the ball Ω is large enough.*

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Conjecture

$p_n = 1 + \lambda_{n,rad}$.

The conjecture is proved as soon as $\forall z \in \,]0, +\infty[$, $\int_0^z t^{1-\nu} J_\nu^3(t)\,\mathrm{d}t > 0$, with $\nu \in \frac{1}{2}\mathbb{N}$, is established.

p large: degeneracy results (radial domains)

Theorem

On balls, there exists a degenerate positive radial solution for some p provided that the measure of Ω is large enough.

$p \geqslant 2^*$

Theorem (X-J. Wang, '91)

When $p = 2^*$ and R is large enough, $(\mathcal{P}_p)$ possesses at least one non-constant positive solution.

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When $p = 2^*$ and R is large enough, $(\mathcal{P}_p)$ possesses at least one non-constant positive solution.

Theorem (E. Serra & P. Tilli, '11)

Assume $a \in L^1(]0,R[)$ is increasing, not constant and satisfies $a > 0$ in $]0,R[$, then for any $p \in]2,+\infty[$, $-\Delta u + u = a(|x|)|u|^{p-2}u$ with NBC possesses a positive radially increasing solution.

Trick: work on the space of radially increasing functions.

$p \geqslant 2^*$

Proposition

Assume Ω *is a ball of radius* R. *If* u *is a radial solution of* $(\mathcal{P}_p)$ *such that* $u(0) < 1$, *then* $\|u\|_{L^\infty} \leqslant \exp(1/2)$.

$p \geqslant 2^*$

Proposition

Assume Ω is a ball of radius R. If u is a radial solution of $(\mathcal{P}_p)$ such that $u(0) < 1$, then $\|u\|_{L^\infty} \leqslant \exp(1/2)$.

Proof. In radial coordinates, the equation writes

$$-u'' - \frac{N-1}{r} u' + u = u^{p-1}.$$

Multiplying by u', we get

$$\frac{\mathrm{d}}{\mathrm{d}r} h(r) = -\frac{N-1}{r} u'^2(r) \leqslant 0,$$

where

$$h(r) := \frac{u'^2(r)}{2} + \frac{u^p(r)}{p} - \frac{u^2(r)}{2}.$$

In particular, this means that $h(r) \leqslant h(0)$ for any r.

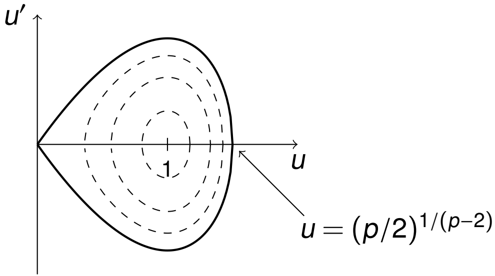
$p \geqslant 2^*$

Proof (cont'd). The assumption $u(0) < 1$ implies

$$h(0) = \frac{u^p(0)}{p} - \frac{u^2(0)}{2} = u^2(0)\left(\frac{u^{p-2}(0)}{p} - \frac{1}{2}\right) \leqslant 0.$$

Thus

$$\|u\|_{L^\infty} \leqslant \left(\frac{p}{2}\right)^{1/(p-2)} \leqslant \exp(1/2).$$ □



$p \geqslant 2^*$

Theorem

Assume Ω is a ball. Then, for any $n \in \mathbb{N}_0$, there exists p_n s.t., for any $p \in [p_n, +\infty[$, $(\mathcal{P}_p)$ has at least n positive radially symmetric solutions.

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Sketch: As we saw, radial bifurcations are transcritical and along the right branch (starting with $p > 1 + \lambda_{i,\mathrm{rad}}$) $u_p(0) < 1$. Thus all u belonging to that branch must satisfy $\|u\|_{L^\infty} \leqslant \exp(1/2)$. Since 1 is the only solution for $p \approx 2$, the branch must exist for all p large. □

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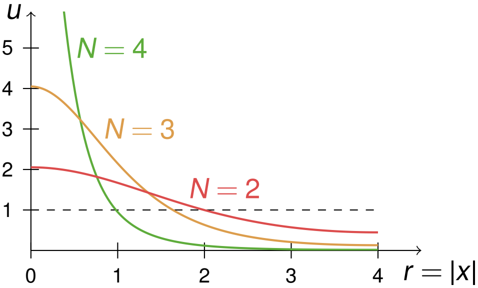
Conjecture

$p_n = 1 + \lambda_{n,rad}$.

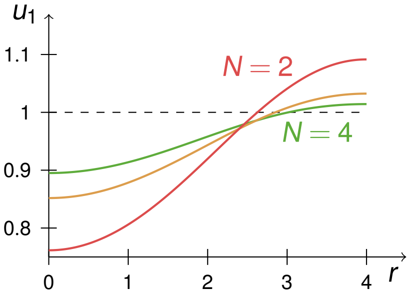
Radial ground state for $p = 0.95 + \lambda_{2,\mathrm{rad}}$ on B_4

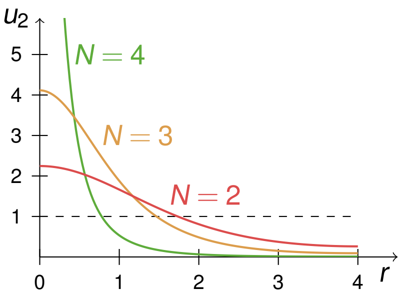
Using the Moutain Pass Algorithm:



N	R	$1 + \lambda_{2,\mathrm{rad}}$	$\mathcal{E}(1)$	$\min u$	$\max u$	$\mathcal{E}(u)$
2	4	2.92	7.60	0.447	2.05	7.45
3	4	3.26	50.58	0.130	4.05	34.85
4	4	3.65	280.58	0.016	13.31	66.39

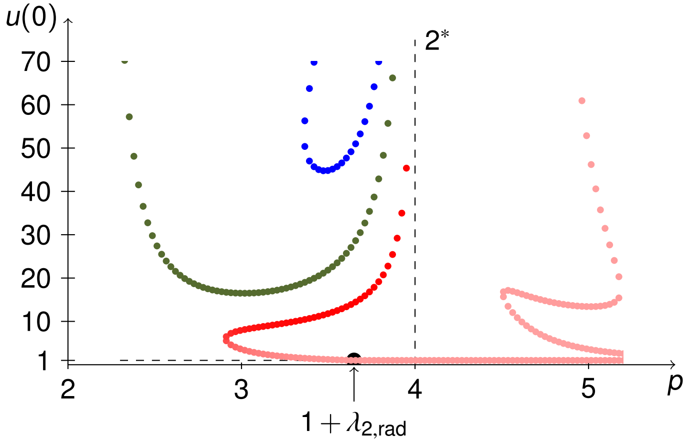
Radial ground state for $p = 1.1 + \lambda_{2,\mathrm{rad}}$ on B_4





N	$1+\lambda_{2,\mathrm{rad}}$	$\mathcal{E}(1)$	$\min u_1$	$\max u_1$	$\mathcal{E}(u_1)$	$\min u_2$	$\max u_2$	$\mathcal{E}(u_2)$
2	2.92	8.48	0.76	1.09	8.47	0.261	2.25	7.39
3	3.26	54.30	0.85	1.03	54.29	0.092	4.12	30.74
4	3.65	294.63	0.90	1.01	294.62	0.008	17.25	49.61

Bifurcation diagram $N = 4$, $R = 4$



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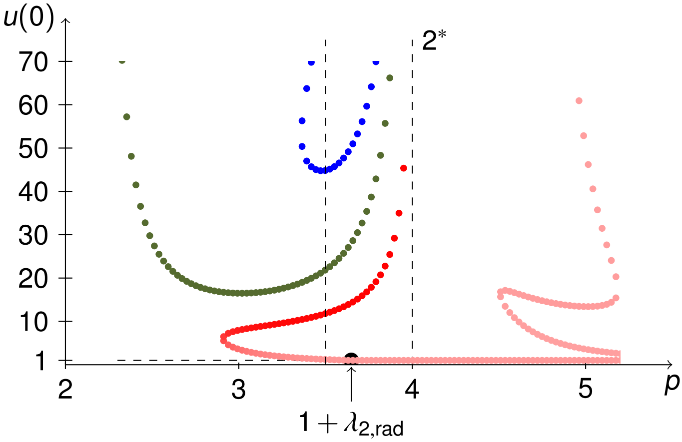
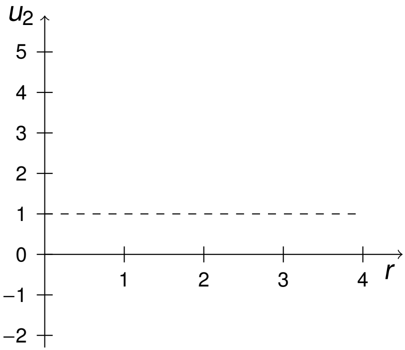


Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

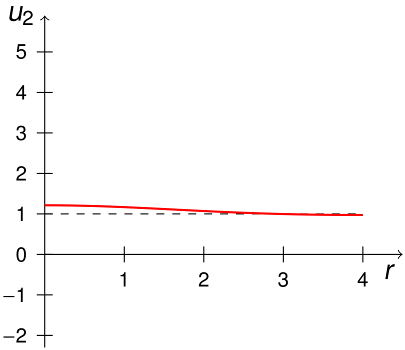
Shape of the solutions for $p = 3.5$.



$\mathcal{E}(1) = 270.709$

Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

Shape of the solutions for $p = 3.5$.



$$\mathcal{E}(1) = 270.709$$

$$u(0) \approx 1.213 \quad \Rightarrow \mathcal{E} = 270.753$$

Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

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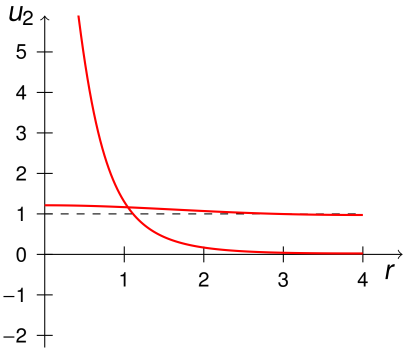


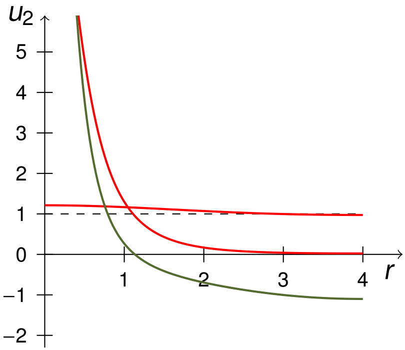
$$\mathcal{E}(1) = 270.709$$

$$u(0) \approx 1.213 \Rightarrow \mathcal{E} = 270.753$$

$$u(0) \approx 11.803 \Rightarrow \mathcal{E} = 79.730$$

Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

Shape of the solutions for $p = 3.5$.



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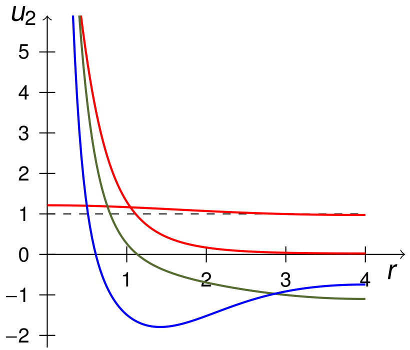
$$u(0) \approx 11.803 \Rightarrow \mathcal{E} = 79.730$$
$$u(0) \approx 21.887 \Rightarrow \mathcal{E} = 390.387$$

Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

Shape of the solutions for $p = 3.5$.



$$\mathcal{E}(1) = 270.709$$

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$$u(0) \approx 11.803 \Rightarrow \mathcal{E} = 79.730$$

$$u(0) \approx 21.887 \Rightarrow \mathcal{E} = 390.387$$

$$u(0) \approx 44.830 \Rightarrow \mathcal{E} = 436.267$$

Thank you for your attention.