Radial and non-radial positive solutions to a system with critical growth on $\mathbb{R}^N$

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First days of Nonlinear Elliptic PDE
in Hauts-de-France

A straightforward generalization...

$$\begin{cases} -\Delta u_i = \sum_{j=1}^{k} a_{ij} u_j^{2^*-1} & \text{in } \mathbb{R}^N, \\ u_i > 0 & \text{in } \mathbb{R}^N, \\ u_i \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where $i = 1, \dots, k$ and $N \geqslant 3$. As usual, $2^* = \frac{2N}{N-2}$ denotes the critical exponent and $D^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*}(\mathbb{R}^N) \;\middle|\; |\nabla u| \in L^2(\mathbb{R}^N) \right\}$.

- $(a_{ij}) \in \mathbb{R}^{k \times k}$ is symmetric;
- $\sum_{j=1}^{k} a_{ij} = 1$ for any $i = 1, \dots, k$.

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- $(a_{ij}) \in \mathbb{R}^{k\times k}$ is symmetric;
- $\sum_{j=1}^{k} a_{ij} = 1$ for any $i = 1, \dots, k$.

Characteristics:

- translation and dilation invariance;
- family of *trivial* (radial) solutions $u = (U, \dots, U)$.

… of the critical Sobolev equation

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases}$$

possesses the $(N+1)$-parameter family of solutions:

$$U_{\delta,y}(x) := \frac{\left[N(N-2)\delta^2\right]^{\frac{N-2}{4}}}{(\delta^2 + |x-y|^2)^{\frac{N-2}{2}}}$$

Let

$$U(x) := U_{1,0}(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}.$$

The case of 2 equations

For $k = 2$, parametrize $(a_{ij}) = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}$. So

$$\begin{cases} -\Delta u_1 = \alpha u_1^{2^*-1} + (1-\alpha) u_2^{2^*-1} & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = (1-\alpha) u_1^{2^*-1} + \alpha u_2^{2^*-1} & \text{in } \mathbb{R}^N, \\ u_1 > 0,\ u_2 > 0 & \text{in } \mathbb{R}^N, \\ u_1, u_2 \in D^{1,2}(\mathbb{R}^N). \end{cases}$$

Question : Does there exist *non-trivial* solutions, possibly non-radial, for some $\alpha \in \mathbb{R}$?

Gross-Pitaevskii System

$$\begin{cases} -\Delta u_1 = \alpha u_1^{2^*-1} + (1-\alpha) u_1^{\frac{2}{N-2}} u_2^{\frac{N}{N-2}} & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = \alpha u_2^{2^*-1} + (1-\alpha) u_2^{\frac{2}{N-2}} u_1^{\frac{N}{N-2}} & \text{in } \mathbb{R}^N, \\ u_1 > 0,\ u_2 > 0, \quad u_1, u_2 \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

For this system, Y. Guo, B. Li, and J. Wei proved in 2014 via a perturbative argument, that, for $N \in \{3,4\}$ and $\alpha > 1$ (non-cooperative case), the system possesses infinitely many non-radial solutions.

The General System

$$\begin{cases} -\Delta u_1 = F_1(\alpha, u_1, u_2) & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = F_2(\alpha, u_1, u_2) & \text{in } \mathbb{R}^N, \\ u_1 > 0, \ u_2 > 0 & \text{in } \mathbb{R}^N, \\ u_1, u_2 \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where $F_i : \mathbb{R} \times (0, +\infty)^2 \to \mathbb{R} : (\alpha, u) \mapsto F_i(\alpha, u)$, $i = 1, 2$ satisfy

- smoothness and integrability assumptions;
- $F_i(\alpha, 1, 1) = 1$;
- $F_i(\alpha, \lambda u_1, \lambda u_2) = \lambda^{2^*-1} F_i(\alpha, u_1, u_2)$ for all $\lambda > 0$
- $F_1(\alpha, u_1, u_2) = F_2(\alpha, u_2, u_1)$ for all $(u_1, u_2) \in (0, +\infty)^2$;
- for all α, $\partial_\alpha \beta(\alpha) > 0$ where $\beta(\alpha) := \partial_{u_1} F_1(\alpha, 1, 1) - \partial_{u_2} F_1(\alpha, 1, 1)$.

Existence of non-trivial *radial* solutions (1/4)

Theorem (F. Gladiali, M. Grossi, C. T.)

Let $n \geqslant 2$ and α_n^* be the solution to

$$\beta(\alpha^*) = \frac{(2n+N)(2n+N-2)}{N(N-2)}.$$

Then there exists a $\mathcal{C}^1$ curve $\varepsilon \mapsto \big(\alpha(\varepsilon), u_1(\varepsilon), u_2(\varepsilon)\big)$: $(-\varepsilon_0, \varepsilon_0) \to \mathbb{R} \times \big(D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)\big)^2$ such that, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\big(u_1(\varepsilon), u_2(\varepsilon)\big)$ is a radial solution to

$$\begin{cases} -\Delta u_1 = F_1(\alpha, u_1, u_2) & \text{in } \mathbb{R}^N,\\ -\Delta u_2 = F_2(\alpha, u_1, u_2) & \text{in } \mathbb{R}^N,\\ u_1 > 0,\ u_2 > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

with $\alpha = \alpha(\varepsilon)$. Moreover,

Existence of non-trivial *radial* solutions (2/4)

Theorem (cont'd)

$$\begin{cases} u_1(\varepsilon) = U + \varepsilon\, W_{n,0}(|x|) + \varepsilon\varphi_{1,\varepsilon}(|x|), \\ u_2(\varepsilon) = U - \varepsilon\, W_{n,0}(|x|) + \varepsilon\varphi_{2,\varepsilon}(|x|), \end{cases}$$

with $W_{n,0}$ being the function

$$W_{n,0}(|x|) := \frac{1}{\left(1+|x|^2\right)^{\frac{N-2}{2}}} P_n^{\left(\frac{N-2}{2},\frac{N-2}{2}\right)}\left(\frac{1-|x|^2}{1+|x|^2}\right)$$

where $\varphi_{1,\varepsilon}, \varphi_{2,\varepsilon}$ are functions uniformly bounded in $D^{1,2}(\mathbb{R}^N)$ with respect to $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, and such that $\varphi_{i,0} = 0$ for $i = 1, 2$. Finally the bifurcation is global and the Rabinowitz alternative holds.

Existence of non-trivial *radial* solutions (3/4)

For the system

$$\begin{cases} -\Delta u_1 = \alpha u_1^{2^*-1} + (1-\alpha) u_2^{2^*-1} & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = (1-\alpha) u_1^{2^*-1} + \alpha u_2^{2^*-1} & \text{in } \mathbb{R}^N, \\ u_1 > 0,\ u_2 > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

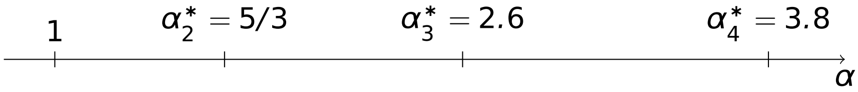
the bifurcations occur at (α_n^*, U, U) where

$$\alpha_n^* = \frac{2n^2 + 2(N-1)n + N^2}{N(N+2)} \qquad (n \geqslant 2).$$

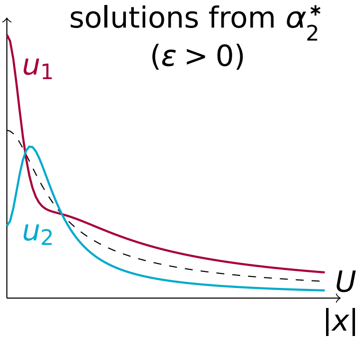
Note that $1 < \alpha_2^* < \alpha_3^* < \cdots < \alpha_n^* \xrightarrow[n\to\infty]{} +\infty$.

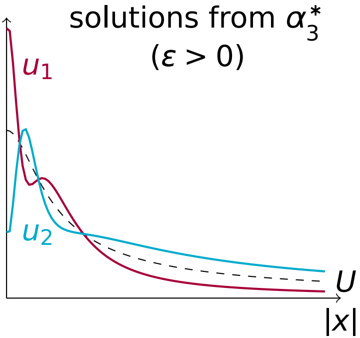
Existence of non-trivial *radial* solutions (4/4)

For $N = 3$.



Trivial solution $(u_1, u_2) = (U, U)$





Existence of non-trivial *non-radial* solutions (1/4)

Theorem (F. Gladiali, M. Grossi, C. T.)

The point (α_n^*, U, U), $n \geqslant 2$, is a non-radial bifurcation point — meaning there is a continuum $\mathcal{C}$ of nontrivial non-radial solutions emanating from (α_n^*, U, U) — if $n \in \mathcal{N}$ where $\mathcal{N} \subseteq \mathbb{N}$ is infinite.
Moreover, for any sequence of $(\alpha_k, u_{1,k}, u_{2,k}) \in \mathcal{C}$ converging to (α_n^*, U, U), one has (up to a subsequence):

$$\begin{cases} u_{1,k} = U + \varepsilon_k Z_n(x) + o(\varepsilon_k), \\ u_{2,k} = U - \varepsilon_k Z_n(x) + o(\varepsilon_k), \end{cases}$$

as $k \to \infty$, where $\varepsilon_k \to 0$ and $Z_n \not\equiv 0$ is non-radial.

Existence of non-trivial *non-radial* solutions (2/4)

For example, $\mathcal{N} = \left\{n \in \mathbb{N}^{\geqslant 2} \;\middle|\; n \bmod 4 \in \{1, 2\}\right\}$ and

$$Z_n(x) = \sum_{h=1,\ h \text{ odd}}^{n} a_h \frac{r^h}{(1+r^2)^{h+\frac{N-2}{2}}} P_{n-h}^{\left(h+\frac{N-2}{2},\, h+\frac{N-2}{2}\right)}\left(\frac{1-r^2}{1+r^2}\right) \cdot P_h^{\left(\frac{N-3}{2},\, \frac{N-3}{2}\right)}(\cos\theta_{N-2})$$

for some coefficients $a_h \in \mathbb{R}$, where $(r, \varphi, \theta_1, \dots, \theta_{N-2}) \in [0, +\infty) \times [0, 2\pi) \times [0, \pi)^{N-2}$ are the spherical coordinates.

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Note that $P_h^{\left(\frac{N-3}{2},\, \frac{N-3}{2}\right)}(\cos\theta_{N-2})$ are the spherical harmonics that are $O(N-1)$-invariant and is odd w.r.t. x_N iff h is odd.

Existence of non-trivial *non-radial* solutions (3/4)

Another example: $\mathcal{N} = \mathbb{N}^{\geqslant 2}$ and

$$Z_n(r, \varphi, \theta_1, \dots, \theta_{N-2}) = \frac{r^n}{\left(1 + r^2\right)^{n + \frac{N-2}{2}}} \sin(n\varphi)(\sin\theta_1)^n \cdots (\sin\theta_{N-2})^n.$$

Thus there exist at least a non-radial bifurcation branch for each $n \geqslant 2$.

Existence of non-trivial *non-radial* solutions (4/4)

Putting our results together, we have the following multiplicity of non-trivial solutions (1 radial, the other ones non-radial):

	$N = 3$	$N = 4$	$N = 5$
$n = 2$	4	4	4
$n = 3$	4	4	4
$n = 4$	4	5	5
$n = 5$	4	5	6
$n = 6$	3	4	5
$n = 7$	2	3	3

Cooperative system & radial solutions (1/3)

We believe that, if all entries of (a_{ij}) are positive, all positive solutions are radial.

Theorem (M. Chipot, I. Shafrir, G. Wolansky, '97)

All entire solutions u to

$$-\Delta u_i = \mu_i \exp\Bigl(\sum_{j=1}^{k} a_{ij} u_j\Bigr), \quad \text{in } \mathbb{R}^2, \qquad 1 \leqslant i \leqslant k,$$

where $\mu_i > 0$, (a_{ij}) is invertible and all $a_{ij} \geqslant 0$, then all u_i are necessarily radially symmetric. If (a_{ij}) is irreducible, the u_i are radially symmetric around the same point.

Cooperative system & radial solutions (2/3)

Theorem (Y. Guo, J. Liu, '08)

If $\forall i, j \in \{1, 2\}$, $a_{ij} > 0$ and $a_{12} = a_{21}$, then solutions to the Gross-Pitaevskii's system

$$\begin{cases} -\Delta u_1 = a_{11} u_1^{2^*-1} + a_{12} u_1^{\frac{2}{N-2}} u_2^{\frac{N}{N-2}} & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = a_{22} u_2^{2^*-1} + a_{21} u_2^{\frac{2}{N-2}} u_1^{\frac{N}{N-2}} & \text{in } \mathbb{R}^N, \\ u_1 > 0,\ u_2 > 0, \quad u_1, u_2 \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

are radially symmetric around the same point (they actually are multiples of the same $U_{\delta,y}$).

Cooperative system & radial solutions (3/3)

Theorem (O. Druet, E. Hebey, '09)

When all $a_{ij} = 1$, the components u_i of any nonnegative entire solution u to

$$-\Delta u_i = \Bigl(\sum_{j=1}^{k} a_{ij} u_j^2\Bigr)^{\frac{2^*-2}{2}} u_i \quad \text{on } \mathbb{R}^N, \qquad i = 1, \dots, k,$$

are all radially symmetric around the same point (actually, all u_i multiples of the same $U_{\delta,y}$).

The critical Sobolev equation

The equation

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases}$$

is invariant under translations and dilations:

$$u \mapsto u(\cdot - x_0), \qquad x_0 \in \mathbb{R}^N,$$

$$u \mapsto \delta^{-\frac{N-2}{2}} u\Big(\frac{\cdot}{\delta}\Big), \qquad \delta > 0.$$

Linearization of the critical Sobolev equation

Thus the linearization at U,

$$-\Delta w = \lambda U^{2^*-2} w, \qquad w \in D^{1,2}(\mathbb{R}^N).$$

has the eigenvalue

$$\lambda_1 := 2^* - 1 = \frac{N+2}{N-2}$$

with the $N+1$-dim. eigenfunction space generated by

$$\frac{\partial U}{\partial x_i}, \qquad i = 1, \dots, N,$$

$$W(|x|) := \text{const.}\left(x \cdot \nabla U + \frac{N-2}{2} U\right) = \frac{1 - |x|^2}{(1+|x|^2)^{N/2}}.$$

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Also, $\lambda_0 := 1$ with eigenfunction U.

Spectrum (1/2)

Theorem (F. Gladiali, M. Grossi, C. T.)

The eigenvalues of

$$-\Delta w = \lambda U^{2^*-2} w, \qquad w \in D^{1,2}(\mathbb{R}^N).$$

are the numbers

$$\lambda_n = \frac{(2n+N-2)(2n+N)}{N(N-2)}, \quad n \geqslant 0.$$

Each eigenvalue λ_n has multiplicity

$$m(\lambda_n) = \frac{(N+2n-1)(N+n-2)!}{(N-1)!\,n!}$$

...

Spectrum (2/2)

Theorem (cont'd)

and the corresponding eigenfunctions are, in radial coordinates (r, Θ), *linear combinations of*

$$W_{n,h}(r)\, Y_h(\Theta) \quad \textit{for } h = 0, \dots, n,$$

where

$$W_{n,h}(r) := \frac{r^h}{(1+r^2)^{h+\frac{N-2}{2}}} P_{n-h}^{\left(h+\frac{N-2}{2},\, h+\frac{N-2}{2}\right)}\left(\frac{1-r^2}{1+r^2}\right),$$

$Y_h(\theta)$ *are spherical harmonics related to the eigenvalue* $h(h+N-2)$ *and* $P_j^{(\beta,\gamma)}$ *are the Jacobi polynomials.*

Note: $W_{0,0} = \text{const.}\, U$ and $W_{1,0} = \frac{N}{2} W$.

Spectrum: sketch of the proof (1/2)

Let $\Pi : S^N \to \mathbb{R}^N$ be the stereographic projection and define $\Phi : S^N \to \mathbb{R}^N : y \mapsto \Phi(y)$ as

$$\Phi(y) := w(\Pi(y)) \cdot \left(\frac{2}{1 + |\Pi(y)|^2}\right)^{-\frac{N-2}{2}}$$

Then

$$-\Delta_{\mathbb{S}^N} \Phi = (\lambda - 1)\frac{N(N-2)}{4}\Phi$$

The eigenvalues of the Laplace-Beltrami operator on S^N are well known:

$$(\lambda - 1)\frac{N(N-2)}{4} = n(N-1+n), \qquad \text{for some } n \in \mathbb{N}.$$

Spectrum: sketch of the proof (2/2)

For the eigenfunctions w, express the eigenfunctions Φ in cylindrical coordinates

$$y = \left(\Theta\sqrt{1-z^2}, z\right) \in \mathbb{S}^N$$

where $\Theta \in \mathbb{S}^{N-1}$ and $z \in [-1, 1]$. This yields

$$\Phi(y) = (1-z^2)^{h/2} P_{n-h}^{\left(h+\frac{N-2}{2}, h+\frac{N-2}{2}\right)}(z)\, Y_h(\Theta), \qquad h = 0, \dots, n.$$

To go back to $w(x)$ with $x = \Pi(y)$, remark that

$$r\Theta = \Pi\left(\Theta\sqrt{1-z^2}, z\right) \quad \Rightarrow \quad z = \frac{r^2-1}{r^2+1} \text{ and } \sqrt{1-z^2} = \frac{2r}{r^2+1}.$$

Let’s go back to the system...

Change of variables

$$\begin{cases} -\Delta u_1 = F_1(\alpha, u_1, u_2) & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = F_2(\alpha, u_1, u_2) & \text{in } \mathbb{R}^N, \\ u_1, u_2 \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

$$\Bigg\downarrow \quad \begin{cases} z_1 = u_1 + u_2 - 2U, \\ z_2 = u_1 - u_2, \end{cases}$$

$$\begin{cases} -\Delta z_1 = f_1(|x|, z_1, z_2) & \text{in } \mathbb{R}^N, \\ -\Delta z_2 = f_2(|x|, z_1, z_2) & \text{in } \mathbb{R}^N, \\ z_1, z_2 \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

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trivial sol $(u_1, u_2) = (U, U)$

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Change of variables

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$$F_1(\alpha, u_1, u_2) = F_2(\alpha, u_2, u_1)$$

$$\Downarrow$$

$$f_1(|x|, z_1, -z_2) = f_1(|x|, z_1, z_2),$$
$$f_2(|x|, z_1, -z_2) = -f_2(|x|, z_1, z_2).$$

$$\begin{cases} -\Delta z_1 = f_1(|x|, z_1, z_2) & \text{in } \mathbb{R}^N, \\ -\Delta z_2 = f_2(|x|, z_1, z_2) & \text{in } \mathbb{R}^N, \\ z_1, z_2 \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

trivial sol $(z_1, z_2) = (0, 0)$

Linearization of the system (1/3)

Solutions are zeros of

$$T(\alpha, z_1, z_2) := \begin{pmatrix} z_1 - (-\Delta)^{-1}\big(f_1(|x|, z_1, z_2)\big) \\ z_2 - (-\Delta)^{-1}\big(f_2(|x|, z_1, z_2)\big) \end{pmatrix}.$$

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Look at the kernel of the linearization at $(z_1, z_2) = (0, 0)$: $\partial_{(z_1,z_2)} T(\alpha, 0, 0)[(w_1, w_2)] = 0$ is equivalent to

$$\begin{cases} -\Delta w_1 = \frac{N+2}{N-2} U^{2^*-2} w_1 & \text{in } \mathbb{R}^N, \\ -\Delta w_2 = \beta(\alpha) U^{2^*-2} w_2 & \text{in } \mathbb{R}^N, \\ w_1, w_2 \in D^{1,2}(\mathbb{R}^N). \end{cases}$$

Linearization of the system (2/3)

Theorem (F. Gladiali, M. Grossi, C. T.)

Recall that the eigenvalues of the single equation are

$$\lambda_n = \frac{(2n+N-2)(2n+N)}{N(N-2)}, \quad n \geqslant 0$$

- *When $\beta(\alpha) \neq \lambda_n$ for all $n \in \mathbb{N}$, all solutions in the kernel are given by*

$$(w_1, w_2) = \left(\sum_{i=1}^{N} a_i \frac{\partial U}{\partial x_i} + bW,\ 0 \right)$$

  for some real constants $a_1, \dots, a_N, b$, where W is the radial function defined above.

...

Linearization of the system (3/3)

Theorem (cont'd)

- *When $\beta(\alpha) = \lambda_n$ for some $n \in \mathbb{N}$, all solutions in the kernel are given by*

$$(w_1, w_2) = \left(\sum_{i=1}^{N} a_i \frac{\partial U}{\partial x_i} + bW,\ \sum_{h=0}^{n} A_h W_{n,h}(r) Y_h(\Theta) \right)$$

for some real constants $a_1, \dots, a_N, b, A_0, \dots, A_n$, where $W_{n,h}$ are defined above.

Problems to apply bifurcation theorems

We would like to apply bifurcation results to

$$T : \mathbb{R} \times \left(D^{1,2}(\mathbb{R}^N)\right)^2 \to \left(D^{1,2}(\mathbb{R}^N)\right)^2$$

$$T(\alpha, z_1, z_2) := \begin{pmatrix} z_1 - (-\Delta)^{-1}\big(f_1(|x|, z_1, z_2)\big) \\ z_2 - (-\Delta)^{-1}\big(f_2(|x|, z_1, z_2)\big) \end{pmatrix}.$$

- When (z_1, z_2) belongs to a continuum emanating from $(0, 0)$, we want the $u_1 > 0$ and $u_2 > 0$ where

$$\begin{cases} u_1 = U + \frac{z_1 + z_2}{2}, \\ u_2 = U + \frac{z_1 - z_2}{2}. \end{cases}$$

- The problem is degenerate for all α.
- Lack of compactness to apply degree theory.

Positiveness of solutions (1/3)

- The $D^{1,2}$ topology is not strong enough.
- The trick $u_i \mapsto u_i^+$ does not work. For example:

$$-\Delta u_i = \sum_{j=1}^{k} a_{ij} (u_j^+)^{2^*-1}$$

In the non-cooperative regime, no maximum principle is expected.

Positiveness of solutions (2/3)

Define

$$D := \left\{ u \in L^\infty(\mathbb{R}^N) \;\middle|\; \|u\|_D < \infty \right\} \qquad \text{where } \|u\|_D := \sup_{x \in \mathbb{R}^N} \frac{|u(x)|}{U(x)}$$

and

$$X := D^{1,2}(\mathbb{R}^N) \cap D, \qquad \|u\|_X := \max\{\|u\|_{D^{1,2}}, \|u\|_D\}.$$

and let

$$\mathcal{X} := \left\{ (z_1, z_2) \in X^2 \;\middle|\; \exists \delta > 0,\ |z_2| \leqslant (2-\delta)U + z_1 \right\}$$

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Consequences:

- $(z_1, z_2) \in \mathcal{X} \Rightarrow u_i > \frac{\delta}{2} U$ for $i = 1, 2$,
- $\mathcal{X}$ is an open neighborhood of $(0, 0)$ in X^2.

Positiveness of solutions (3/3)

Lemma

The operator $T : \mathbb{R} \times \mathcal{X} \to X^2$ *is well defined and continuous. Moreover,* $\partial_\alpha T$, $\partial_z T$ *and* $\partial_{\alpha z} T$ *exist and are continuous.*

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Idea of the proof. $\big(T(\alpha, z_1, z_2)\big)_i = z_i - (-\Delta)^{-1}(f_i(|x|, z_1, z_2))$.

$$\begin{aligned}(z_1, z_2) \in \mathcal{X} \subseteq D^2 &\Rightarrow |z_i| \leqslant CU \\ &\Rightarrow |f_i| \leqslant CU^{2^*-1} \\ &\Rightarrow |(-\Delta)^{-1} f_i| \leqslant C(-\Delta)^{-1} U^{2^*-1} = CU.\end{aligned}$$

□

Compactness (1/2)

Lemma

For all α, the operator

$$\mathcal{X} \to X^2 : (z_1, z_2) \mapsto \begin{pmatrix} (-\Delta)^{-1} f_1(|x|, z_1, z_2) \\ (-\Delta)^{-1} f_2(|x|, z_1, z_2) \end{pmatrix}$$

is compact.

Relies on some decay estimates.

Lemma (D. Siegel, E. Talvila, '99)

If $0 < p < N$ and $h \geqslant 0$, radial function belonging to $L^1(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} \frac{h(y)}{|x-y|^p} \,\mathrm{d}y = O\Big(\frac{1}{|x|^p}\Big) \quad \text{as } |x| \to +\infty.$$

Compactness (2/2)

Consequence

The operator

$$X \to X : w \mapsto (-\Delta)^{-1}\left(\frac{w}{(1+|x|^2)^2}\right)$$

is compact.

Consequence: $\partial_z T(\alpha, 0, 0) : X^2 \to X^2$ is a compact perturbation of the identity. Thus, it is a Fredholm operator of index 0.

Degenerate solution for all α

Use the Kelvin transform $\mathbf{k}(z)$ of z:

$$\mathbf{k}(z)(x) := \frac{1}{|x|^{N-2}} z\left(\frac{x}{|x|^2}\right)$$

Define

$$X_{\mathbf{k}}^{+} := \{z \in X \mid \mathbf{k}(z) = z\} \quad \text{and} \quad X_{\mathbf{k}}^{-} := \{z \in X \mid \mathbf{k}(z) = -z\}.$$

- $U \in X_{\mathbf{k}}^{+}$
- $W \in X_{\mathbf{k}}^{-}$, $\frac{\partial U}{\partial x_i} \in X_{\mathbf{k}}^{+}$

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- $U \in X^+_{\mathbf{k}}$
- $W \in X^-_{\mathbf{k}}$, $\frac{\partial U}{\partial x_i} \in X^+_{\mathbf{k}}$
- in general
  - $W_{n,h} \in X^+_{\mathbf{k}}$ if $n - h$ is even;
  - $W_{n,h} \in X^-_{\mathbf{k}}$ if $n - h$ is odd.

$$W_{n,h}(r) := \frac{r^h}{(1+r^2)^{h+\frac{N-2}{2}}} P_{n-h}^{\left(h+\frac{N-2}{2}, h+\frac{N-2}{2}\right)}\left(\frac{1-r^2}{1+r^2}\right)$$

Invariance of T under Kelvin transform

Lemma

The operator $T : \mathbb{R} \times \mathcal{X} \to X^2$ *maps* $\mathbb{R} \times \big(\mathcal{X} \cap (X_{\mathbf{k}}^{+} \times X_{\mathbf{k}}^{\pm})\big)$ *to* $X_{\mathbf{k}}^{+} \times X_{\mathbf{k}}^{\pm}$.

Need to show

$$\mathbf{k}(z_1) = z_1,\ \mathbf{k}(z_2) = \pm z_2 \Rightarrow \begin{cases} g_1 := (-\Delta)^{-1}(f_1(|x|, z_1, z_2)) \in X_{\mathbf{k}}^{+} \\ g_2 := (-\Delta)^{-1}(f_2(|x|, z_1, z_2)) \in X_{\mathbf{k}}^{\pm} \end{cases}$$

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This stems from

- $-\Delta \mathbf{k}(g) = -\frac{1}{|x|^{N+2}} \Delta g\Big(\frac{x}{|x|^2}\Big)$;
- $\mathbf{k}(U) = U$;
- Critical growth and $\begin{cases} f_1(|x|, z_1, -z_2) = f_1(|x|, z_1, z_2), \\ f_2(|x|, z_1, -z_2) = -f_2(|x|, z_1, z_2). \end{cases}$

Radial solutions (1/3)

Restrict $T : \mathbb{R} \times (\mathcal{X} \cap \mathcal{Z}^{\pm}_{\text{rad}}) \to \mathcal{Z}^{\pm}_{\text{rad}}$ where

$$\mathcal{Z}^{\pm}_{\text{rad}} := \left\{ z \in X^{+}_{\mathbf{k}} \times X^{\pm}_{\mathbf{k}} \;\middle|\; \forall x \in \mathbb{R}^N,\; z(x) = z(|x|) \right\}.$$

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Kernel $\partial_z T(\alpha, 0, 0)[(w_1, w_2)] = 0$.

$$w_1 = \sum_{i=1}^{N} a_i \frac{\partial U}{\partial x_i} + bW \in X^{+}_{\mathbf{k},\text{rad}}$$

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Conclusion

$\partial_z T(\alpha, 0, 0)$ is invertible if $\alpha \neq \alpha_n^*$ for all n.

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Conclusion

$\partial_z T(\alpha, 0, 0)$ has a 1-dim kernel when $\alpha = \alpha^*_n$ for some n.

Radial solutions (3/3)

Consequences:

- There is a continuum in $\mathbb{R} \times (\mathcal{X} \cap \mathcal{Z}^{\pm}_{\mathrm{rad}})$ bifurcating from each $(\alpha^*_n, 0, 0)$ and Rabinowitz alternative holds.
- Crandall-Rabinowitz transversality condition can be checked so in a neighborhood of $(\alpha^*_n, 0, 0)$, the continuum is a C^1-curve.

Non-radial solutions: general idea

Let $\mathcal{S} \leqslant O(N)$ and $\sigma : \mathcal{S} \to \{-1, 1\}$ be a group morphism. Define

$$\mathcal{Z} := \Bigl\{ z = (z_1, z_2) \in X_{\mathbf{k}}^{+} \times X_{\mathbf{k}}^{\pm} \Bigm| \forall s \in \mathcal{S},\ z_1(s^{-1}(x)) = z_1(x) \text{ and } \sigma(s)\, z_2(s^{-1}(x)) = z_2(x) \Bigr\}.$$

Idea: $\sigma \not\equiv 1 \Rightarrow$ non-radial solutions.

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Idea: $\sigma \not\equiv 1 \Rightarrow$ non-radial solutions.

Lemma

The operator T maps $\mathbb{R} \times (\mathcal{X} \cap \mathcal{Z})$ into $\mathcal{Z}$.

Again, the relations

$$f_1(|x|, z_1, -z_2) = f_1(|x|, z_1, z_2),$$
$$f_2(|x|, z_1, -z_2) = -f_2(|x|, z_1, z_2).$$

are essential.

Non-radial solutions: first case

$\mathcal{S}_1 := \langle O(N-1), h_N \rangle$ where

$$h_N(x', x_N) := (x', -x_N), \qquad \text{where } x' := (x_1, \dots, x_{N-1})$$

and $\sigma_1 : \mathcal{S}_1 \to \{-1, 1\}$ is the group morphism s.t. $\sigma_1(s) := 1$ if $s \in O(N-1)$ and $\sigma_1(h_N) := -1$. Thus

$$\mathcal{Z}_1^{\pm} = \Big\{ z \in X_k^+ \times X_k^{\pm} \;\Big|\; z_1(x', x_N) = z_1(|x'|, -x_N) \text{ and } z_2(x', x_N) = -z_2(|x'|, -x_N) \Big\}.$$

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Kernel $\partial_z T(\alpha, 0, 0)[(w_1, w_2)] = 0$.

$$w_1 = \sum_{i=1}^{N-1} a_i \frac{\partial U}{\partial x_i} + a_N \frac{\partial U}{\partial x_N} + bW \quad \text{in } X_{\mathbf{k}}^+ \text{ and } \begin{cases} O(N-1)\text{-invariant} \\ \text{even w.r.t. } x_N \end{cases}$$

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Conclusion

$\partial_z T(\alpha, 0, 0)$ is invertible if $\alpha \neq \alpha_n^*$ for all n.

Non-radial solutions: first case

For the second component:

$$w_2 = \sum_{h=0}^{n} A_h W_{n,h}(r) Y_h(\Theta) \quad \text{in } X_{\mathbf{k}}^{\pm} \text{ and } \begin{cases} O(N-1)\text{-invariant} \\ \text{odd w.r.t. } x_N \end{cases}$$

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For all h, there is a single (up to a multiple) spherical harmonic that is $O(N-1)$-invariant: $Y_h(\Theta) = P_h^{(\frac{N-3}{2}, \frac{N-3}{2})}(\cos \theta_{N-2})$ where $(r, \varphi, \theta_1, \dots, \theta_{N-2})$ are the spherical coordinates.

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Moreover, this Y_h is odd w.r.t. x_N iff h is odd.

Non-radial solutions: first case

$$w_2 = \sum_{h=0,\ h \text{ odd}}^{n} A_h \, W_{n,h}(r) \, Y_h(\Theta) \quad \text{in } X_{\mathbf{k}}^{\pm}$$

Non-radial solutions: first case

$$w_2 = \sum_{h=0,\ h \text{ odd}}^{n} A_h\, W_{n,h}(r)\, Y_h(\Theta) \quad \text{in } X_{\mathbf{k}}^{\pm}$$

Recall $W_{n,h} \in X_{\mathbf{k}}^{+}$(resp. $X_{\mathbf{k}}^{-}$) $\Leftrightarrow n - h$ is even (resp. odd). Thus

- if n is even, choose $X_{\mathbf{k}}^{-}$;
- if n is odd, choose $X_{\mathbf{k}}^{+}$.

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- if n is even, choose $X_{\mathbf{k}}^{-}$;
- if n is odd, choose $X_{\mathbf{k}}^{+}$.

$$\text{multiplicity} = \begin{cases} \sum_{h=0,\ h \text{ odd}}^{n} 1 = n/2 & \text{if } n \text{ is even,} \\ \sum_{h=0,\ h \text{ odd}}^{n} 1 = n \operatorname{div} 2 + 1 & \text{if } n \text{ is odd.} \end{cases}$$

The multiplicity is odd iff $n \bmod 4 \in \{1, 2\}$.

Non-radial solutions: more odd symmetries

Let $1 \leqslant m \leqslant N-1$, $\mathcal{S}_m = \langle O(N-m), h_{N-m+1}, \ldots, h_N \rangle$ where

$$h_m \text{ is the reflection w.r.t. } x_m = 0$$

and $\sigma_m : \mathcal{S}_m \to \{-1, 1\}$ be the group morphism defined by $\sigma_m(s) = 1$ for $s \in O(N-m)$ and $\sigma_m(h_i) = -1$.

Proposition

Bifurcation with these symmetries occur from (α_n^*, U, U) if

$$\binom{m + \lfloor \frac{n-m}{2} \rfloor}{m} \text{ is an odd integer.}$$

Non-radial solutions: highly oscillating

As before, let $(r, \varphi, \theta_1, \dots, \theta_{N-2})$ be the spherical coordinates.

For $m \geqslant 1$, let R_m be the rotation of angle $\frac{2\pi}{m}$ in φ, and take $\mathcal{S}_m = \langle R_m, h_2, h_3, \dots, h_N \rangle$ and $\sigma_m : \mathcal{S}_m \to \{-1, 1\}$ be the group morphism defined by $\sigma_m(R_m) = 1$, $\sigma_m(h_2) = -1$, and $\sigma_m(h_i) = 1$ for $i = 3, \dots, N$.

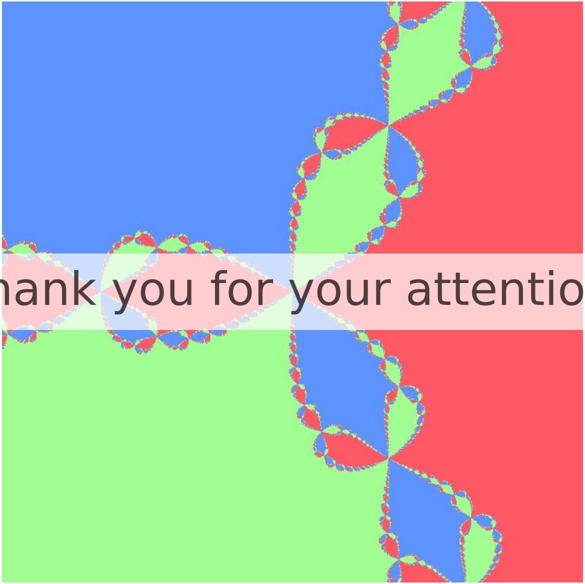
Among the function in the second component of the kernel at α_m^*

$$w_2 = \sum_{h=0}^{m} A_h W_{m,h}(r) Y_h(\varphi, \theta_1, \dots, \theta_{N-2}),$$

these symmetries select

$$W_{m,m}(r) Y_m(\Theta)$$

1-dim $\Rightarrow$ bifurcation.



Thank you for your attention.