# Characterizing Omega-Regularity through FiniteMemory Determinacy of Games on Infinite Graphs 

Patricia Bouyer ©<br>Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France<br>Mickael Randour (0)<br>F.R.S.-FNRS \& UMONS - Université de Mons, Mons, Belgium<br>Pierre Vandenhove ©<br>F.R.S.-FNRS \& UMONS - Université de Mons, Mons, Belgium<br>Université Paris-Saclay, CNRS, ENS Paris-Saclay,<br>Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France


#### Abstract

We consider zero-sum games on infinite graphs, with objectives specified as sets of infinite words over some alphabet of colors. A well-studied class of objectives is the one of $\omega$-regular objectives, due to its relation to many natural problems in theoretical computer science. We focus on the strategy complexity question: given an objective, how much memory does each player require to play as well as possible? A classical result is that finite-memory strategies suffice for both players when the objective is $\omega$-regular. We show a reciprocal of that statement: when both players can play optimally with a chromatic finite-memory structure (i.e., whose updates can only observe colors) in all infinite game graphs, then the objective must be $\omega$-regular. This provides a game-theoretic characterization of $\omega$-regular objectives, and this characterization can help in obtaining memory bounds. Moreover, a by-product of our characterization is a new one-to-two-player lift: to show that chromatic finite-memory structures suffice to play optimally in two-player games on infinite graphs, it suffices to show it in the simpler case of one-player games on infinite graphs. We illustrate our results with the family of discounted-sum objectives, for which $\omega$-regularity depends on the value of some parameters.


2012 ACM Subject Classification Theory of computation $\rightarrow$ Formal languages and automata theory

Keywords and phrases two-player games on graphs, infinite arenas, finite-memory determinacy, optimal strategies, $\omega$-regular languages

Funding This work has been partially supported by the ANR Project MAVeriQ (ANR-ANR-20-CE25-0012). Mickael Randour is an F.R.S.-FNRS Research Associate and a member of the TRAIL Institute. Pierre Vandenhove is an F.R.S.-FNRS Research Fellow.

## 1 Introduction

Games on graphs and synthesis. We study zero-sum turn-based games on infinite graphs. In such games, two players, $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$, interact for an infinite duration on a graph, called an arena, whose state space is partitioned into states controlled by $\mathcal{P}_{1}$ and states controlled by $\mathcal{P}_{2}$. The game starts in some state of the arena, and the player controlling the current state may choose the next state following an edge of the arena. Moves of the players in the game are prescribed by their strategy, which can use information about the past of the play. Edges of the arena are labeled with a (possibly infinite) alphabet of colors, and the interaction of the players in the arena generates an infinite word over this alphabet of colors. These infinite words can be used to specify the players' objectives: a winning condition is a set of infinite words, and $\mathcal{P}_{1}$ wins a game on a graph if the infinite word generated by its interaction with $\mathcal{P}_{2}$ on the game graph belongs to this winning condition - otherwise, $\mathcal{P}_{2}$ wins.

This game-theoretic model has applications to the reactive synthesis problem [4]: a system (modeled as $\mathcal{P}_{1}$ ) wants to guarantee some specification (the winning condition) against an uncontrollable environment (modeled as $\mathcal{P}_{2}$ ). Finding a winning strategy in the game for $\mathcal{P}_{1}$ corresponds to building a controller for the system that achieves the specification against all possible behaviors of the environment.

Strategy complexity. We are interested in the strategy complexity question: given a winning condition, how complex must winning strategies be, and how simple can they be? We are interested in establishing the sufficient and necessary amount of memory to play optimally. We consider in this work that an optimal strategy in an arena must be winning from any state from which winning is possible (a property sometimes called uniformity in the literature). The amount of memory relates to how much information about the past is needed to play in an optimal way. With regard to reactive synthesis, this has an impact in practice on the resources required for an optimal controller.

Three classes of strategies are often distinguished, depending on the number of states of memory they use: memoryless, finite-memory, and infinite-memory strategies. A notable subclass of finite-memory strategies is the class of strategies that can be implemented with finite-memory structures that only observe the sequences of colors (and not the sequences of states nor edges). Such memory structures are called chromatic [30]. By contrast, finitememory structures that have access to the states and edges of arenas are called general. Chromatic memory structures are syntactically less powerful and may require more states than general ones [11, but have the benefit that they can be defined independently of arenas.

We seek to characterize the winning conditions for which chromatic-finite-memory strategies suffice to play optimally against arbitrarily complex strategies, for both players, in all finite and infinite arenas. We call this property chromatic-finite-memory determinacy. This property generalizes memoryless determinacy, which describes winning conditions for which memoryless strategies suffice to play optimally for both players in all arenas. Our work follows a line of research [6, 8] giving various characterizations of chromatic-finite-memory determinacy for games on finite arenas (see Remark 2 for more details).
$\omega$-regular languages. A class of winning conditions commonly arising as natural specifications for reactive systems (it encompasses, e.g., linear temporal logic specifications 38]) consists of the $\omega$-regular languages. They are, among other characterizations, the languages of infinite words that can be described by a finite parity automaton [36]. It is known that all $\omega$-regular languages are chromatic-finite-memory determined, which is due to the facts that an $\omega$-regular language is expressible with a parity automaton, and that parity conditions admit memoryless optimal strategies [27, 43]. Multiple works study the strategy complexity of $\omega$-regular languages, giving, e.g., precise general memory requirements for all Muller conditions [18] or a characterization of the chromatic memory requirements of Muller conditions [11, Theorem 28].

A result in the other direction is given by Colcombet and Niwinski 17): they showed that if a prefix-independent winning condition is memoryless-determined in infinite arenas, then this winning condition must be a parity condition. As parity conditions are memorylessdetermined, this provides an elegant characterization of parity conditions from a strategic perspective, under prefix-independence assumption.

Congruence. A well-known tool to study a language $L$ of finite (resp. infinite) words is its right congruence relation $\sim_{L}$ : for two finite words $w_{1}$ and $w_{2}$, we write $w_{1} \sim_{L} w_{2}$ if
for all finite (resp. infinite) words $w, w_{1} w \in L$ if and only if $w_{2} w \in L$. There is a natural deterministic (potentially infinite) automaton recognizing the equivalence classes of the right congruence, called the minimal-state automaton of $\sim_{L}$ [42, 35].

The relation between a regular language of finite words and its right congruence is given by the Myhill-Nerode theorem [37], which provides a natural bijection between the states of the minimal deterministic automaton recognizing a regular language and the equivalence classes of its right congruence relation. Consequences of this theorem are that a language is regular if and only if its right congruence has finitely many equivalence classes, and a regular language can be recognized by the minimal-state automaton of its right congruence.

For the theory of languages of infinite words, the situation is not so simple: $\omega$-regular languages have a right congruence with finitely many equivalence classes, but having finitely many equivalence classes does not guarantee $\omega$-regularity (for example, a language is prefixindependent if and only if its right congruence has exactly one equivalence class, but this does not imply $\omega$-regularity). Moreover, $\omega$-regular languages cannot necessarily be recognized by adding a natural acceptance condition (parity, Rabin, Muller...) to the minimal-state automaton of their right congruence [1]. There has been multiple works about the links between a language of infinite words and the minimal-state automaton of its right congruence; one relevant question is to understand when a language can be recognized by this minimalstate automaton 42, 35, 1.

Contributions. We characterize the $\omega$-regularity of a language of infinite words $W$ through the strategy complexity of the zero-sum turn-based games on infinite graphs with winning condition $W$ : the $\omega$-regular languages are exactly the chromatic-finite-memory determined languages (seen as winning conditions) (Theorem 11). As discussed earlier, it is well-known that $\omega$-regular languages admit chromatic-finite-memory optimal strategies [36, 43, 11] our results yield the other implication. This therefore provides a characterization of $\omega$-regular languages through a game-theoretic and strategic lens.

Our technical arguments consist in providing a precise connection between the representation of $W$ as a parity automaton and a chromatic memory structure sufficient to play optimally. If strategies based on a chromatic finite-memory structure are sufficient to play optimally for both players, then $W$ is recognized by a parity automaton built on top of the direct product of the minimal-state automaton of the right congruence and this chromatic memory structure (Theorem 10). This result generalizes the work from Colcombet and Niwiński [17] in two ways: by relaxing the prefix-independence assumption about the winning condition, and by generalizing the class of strategies considered from memoryless to chromatic-finite-memory strategies. We recover their result as a special case.

Moreover, we actually show that chromatic-finite-memory determinacy in one-player games of both players is sufficient to show $\omega$-regularity of a language. As $\omega$-regular languages are chromatic-finite-memory determined in two-player games, we can reduce the problem of chromatic-finite-memory determinacy of a winning condition in two-player games to the easier chromatic-finite-memory determinacy in one-player games (Theorem 12). Such a one-to-two-player lift holds in multiple classes of zero-sum games, such as deterministic games on finite arenas [23, 6, 31] and stochastic games on finite arenas [24, 8]. The proofs for finite arenas all rely on an edge-induction technique (also used in other works about strategy complexity in finite arenas [28, 21, 13]) that appears unfit to deal with infinite arenas. Although not mentioned by Colcombet and Niwiński, it was already noticed [30 that for prefix-independent winning conditions in games on infinite graphs, a one-to-two-player lift for memoryless determinacy follows from [17].

Related works. We have already mentioned [18, 43, 17, 29, 11 for fundamental results on the memory requirements of $\omega$-regular conditions, [23, 24, 6, 8] for characterizations of "low" memory requirements in finite (deterministic and stochastic) arenas, and 42, 35, 1] for links between an $\omega$-regular language and the minimal-state automaton of its right congruence.

One stance of our work is that our assumptions about strategy complexity affect both players. Another intriguing question is to understand when the memory requirements of only one player are finite. In finite arenas, a few results in this direction are sufficient conditions for the existence of memoryless optimal strategies for one player [28, 3], and a proof by Kopczyński that the chromatic memory requirements of prefix-independent $\omega$-regular conditions are computable [29, 30].

Other articles study the strategy complexity of (non-necessarily $\omega$-regular) winning conditions in infinite arenas; see, e.g., [20, 25, 16]. In such non- $\omega$-regular examples, as can be expected given our main result, at least one player needs infinite memory to play optimally, or the arena model is different from ours (e.g., only allowing finite branching - we discuss such differences in more depth after Theorem 10. A particularly interesting example w.r.t. our results is considered by Chatterjee and Fijalkow [15]. They study the strategy complexity of finitary Büchi and parity conditions, and show that $\mathcal{P}_{1}$ has memoryless optimal strategies for finitary Büchi and chromatic-finite-memory optimal strategies for finitary parity ${ }^{1}$ However, for these (non- $\omega$-regular) winning conditions, $\mathcal{P}_{2}$ needs infinite memory. This example illustrates that our main result would not hold if we just focused on the strategy complexity of one player.

We mention other works on finite-memory determinacy in different contexts: finite arenas [34, non-zero-sum games [33, countable one-player stochastic games [26], concurrent games 32, 7.

This paper extends and complements a preceding conference version 9 with additional details and complete proofs of all the statements.

Structure. We fix definitions and notations in Section 2 Our main results are discussed in Section 3, and their proofs lie in Sections 4 and 5 We provide applications of our results to discounted-sum, mean-payoff, and total-payoff winning conditions in Section 6

## 2 Preliminaries

Let $C$ be an arbitrary non-empty set of colors. Given a set $A$, we write $A^{*}$ for the set of finite sequences of elements of $A$ and $A^{\omega}$ for the set of infinite sequences of elements of $A$.

Arenas. We consider two players $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$. An arena is a tuple $\mathcal{A}=\left(S, S_{1}, S_{2}, E\right)$ such that $S=S_{1} \uplus S_{2}$ (disjoint union) is a non-empty set of states (of any cardinality) and $E \subseteq S \times C \times S$ is a set of edges. States in $S_{1}$ are controlled by $\mathcal{P}_{1}$ and states in $S_{2}$ are controlled by $\mathcal{P}_{2}$. We allow arenas with infinite branching. Given $e \in E$, we denote by in, col, and out the projections to its first, second, and third component, respectively (i.e., $e=(\operatorname{in}(e), \operatorname{col}(e), \operatorname{out}(e)))$. We assume arenas to be non-blocking: for all $s \in S$, there exists $e \in E$ such that $\operatorname{in}(e)=s$.

Let $\mathcal{A}=\left(S, S_{1}, S_{2}, E\right)$ be an arena with $s \in S$. We denote by $\operatorname{Plays}(\mathcal{A}, s)$ the set of plays of $\mathcal{A}$ from $s$, that is, infinite sequences of edges $\rho=e_{1} e_{2} \ldots \in E^{\omega}$ such that in $\left(e_{1}\right)=s$ and for all $i \geq 1$, out $\left(e_{i}\right)=\operatorname{in}\left(e_{i+1}\right)$. For $\rho \in \operatorname{Plays}(\mathcal{A}, s)$, we denote by $\operatorname{col}^{\omega}(\rho)$ the infinite sequence

[^0]of colors obtained from applying the col function to each edge in $\rho$. We denote by $\operatorname{Hists}(\mathcal{A}, s)$ the set of histories of $\mathcal{A}$ from $s$, which are all finite prefixes of plays of $\mathcal{A}$ from $s$. We write $\operatorname{Plays}(\mathcal{A})$ and $\operatorname{Hists}(\mathcal{A})$ for the sets of all plays of $\mathcal{A}$ and all histories of $\mathcal{A}$ (from any state), respectively. If $h=e_{1} \ldots e_{k}$ is a history of $\mathcal{A}$, we define $\operatorname{in}(h)=\operatorname{in}\left(e_{1}\right)$ and $\operatorname{out}(h)=\operatorname{out}\left(e_{k}\right)$. For convenience, for every $s \in S$, we also consider the empty history $\lambda_{s}$ from $s$, and we set $\operatorname{in}\left(\lambda_{s}\right)=\operatorname{out}\left(\lambda_{s}\right)=s$. For $i \in\{1,2\}$, we denote by $\operatorname{Hists}_{i}(\mathcal{A})$ the set of histories $h$ such that $\operatorname{out}(h) \in S_{i}$.

An arena $\mathcal{A}=\left(S, S_{1}, S_{2}, E\right)$ is a one-player arena of $\mathcal{P}_{1}\left(\right.$ resp. $\left.\mathcal{P}_{2}\right)$ if $S_{2}=\emptyset\left(\right.$ resp. $\left.S_{1}=\emptyset\right)$.

Skeletons. A skeleton is a tuple $\mathcal{M}=\left(M, m_{\text {init }}, \alpha_{\text {upd }}\right)$ such that $M$ is a finite set of states, $m_{\text {init }} \in M$ is an initial state, and $\alpha_{\text {upd }}: M \times C \rightarrow M$ is an update function. We denote by $\alpha_{\text {upd }}^{*}$ the natural extension of $\alpha_{\text {upd }}$ to finite sequences of colors. We always assume that all states of skeletons are reachable from their initial state. We define the trivial skeleton $\mathcal{M}_{\text {triv }}$ as the only skeleton with a single state. Notice that although we require skeletons to have finitely many states, we allow them to have infinitely many transitions (which happens when $C$ is infinite).

We say that a non-empty sequence $\pi=\left(m_{1}, c_{1}\right) \ldots\left(m_{k}, c_{k}\right) \in(M \times C)^{+}$is a path of $\mathcal{M}$ (from $m_{1}$ to $\alpha_{\text {upd }}\left(m_{k}, c_{k}\right)$ ) if for all $i \in\{1, \ldots, k-1\}, \alpha_{\text {upd }}\left(m_{i}, c_{i}\right)=m_{i+1}$. For convenience, we also consider every element $(m, \perp)$ for $m \in M$ and $\perp \notin C$ to be an empty path of $\mathcal{M}$ (from $m$ to $m$ ). A non-empty path of $\mathcal{M}$ from $m$ to $m^{\prime}$ is a cycle of $\mathcal{M}$ (on $m$ ) if $m=m^{\prime}$. Cycles of $\mathcal{M}$ are usually denoted by letter $\gamma$. For $\pi=\left(m_{1}, c_{1}\right) \ldots\left(m_{k}, c_{k}\right)$ a path of $\mathcal{M}$, we define $\operatorname{st}(\pi)$ to be the set $\left\{m_{1}, \ldots, m_{k}\right\}$, and $\operatorname{col}^{*}(\pi)$ to be the sequence $c_{1} \ldots c_{k} \in C^{*}$. For an infinite sequence $\left(m_{1}, c_{1}\right)\left(m_{2}, c_{2}\right) \ldots \in(M \times C)^{\omega}$, we also write $\operatorname{col}^{\omega}\left(\left(m_{1}, c_{1}\right)\left(m_{2}, c_{2}\right) \ldots\right)$ for the infinite sequence $c_{1} c_{2} \ldots \in C^{\omega}$. If $(m, c) \in M \times C$ occurs in a path $\pi$ of $\mathcal{M}$, we call $(m, c)$ a transition of $\pi$ and we write $(m, c) \in \pi$.

For $m, m^{\prime} \in M$, we write $\Pi_{m, m^{\prime}}$ for the set of paths of $\mathcal{M}$ from $m$ to $m^{\prime}, \Gamma_{m}$ for the set of cycles of $\mathcal{M}$ on $m$, and $\Gamma_{\mathcal{M}}$ for the set of all cycles of $\mathcal{M}$ (on any skeleton state). When considering sets of paths or cycles of $\mathcal{M}$, we add a c in front of the set to denote the projections of the corresponding paths or cycles to colors (e.g., $\mathrm{c}_{\mathcal{M}}=\left\{\operatorname{col}^{*}(\gamma) \in C^{+} \mid \gamma \in \Gamma_{\mathcal{M}}\right\}$ ).

For $w=c_{1} c_{2} \ldots \in C^{\omega}$, we define skel $(w)$ as the infinite sequence $\left(m_{1}, c_{1}\right)\left(m_{2}, c_{2}\right) \ldots \in$ $(M \times C)^{\omega}$ that $w$ induces in the skeleton $\left(m_{1}=m_{\text {init }}\right.$ and for all $\left.i \geq 1, \alpha_{\text {upd }}\left(m_{i}, c_{i}\right)=m_{i+1}\right)$.

Let $\mathcal{M}_{1}=\left(M_{1}, m_{\text {init }}^{1}, \alpha_{\text {upd }}^{1}\right)$ and $\mathcal{M}_{2}=\left(M_{2}, m_{\text {init }}^{2}, \alpha_{\text {upd }}^{2}\right)$ be two skeletons. Their (direct) product $\mathcal{M}_{1} \otimes \mathcal{M}_{2}$ is the skeleton $\left(M, m_{\text {init }}, \alpha_{\text {upd }}\right)$ where $M=M_{1} \times M_{2}, m_{\text {init }}=\left(m_{\text {init }}^{1}, m_{\text {init }}^{2}\right)$, and, for all $m_{1} \in M_{1}, m_{2} \in M_{2}, c \in C, \alpha_{\text {upd }}\left(\left(m_{1}, m_{2}\right), c\right)=\left(\alpha_{\text {upd }}^{1}\left(m_{1}, c\right), \alpha_{\text {upd }}^{2}\left(m_{2}, c\right)\right)$.

Strategies. Let $\mathcal{A}=\left(S, S_{1}, S_{2}, E\right)$ be an arena and $i \in\{1,2\}$. A strategy of $\mathcal{P}_{i}$ on $\mathcal{A}$ is a function $\sigma_{i}: \operatorname{Hists}_{i}(\mathcal{A}) \rightarrow E$ such that for all $h \in \operatorname{Hists}_{i}(\mathcal{A})$,out $(h)=\operatorname{in}\left(\sigma_{i}(h)\right)$. We denote by $\Sigma_{i}(\mathcal{A})$ the set of strategies of $\mathcal{P}_{i}$ on $\mathcal{A}$. Given a strategy $\sigma_{i}$ of $\mathcal{P}_{i}$, we say that a play $\rho$ is consistent with $\sigma_{i}$ if for all finite prefixes $h=e_{1} \ldots e_{i}$ of $\rho$ such that out $(h) \in S_{i}, \sigma_{i}(h)=e_{i+1}$. For $s \in S$, we denote by $\operatorname{Plays}\left(\mathcal{A}, s, \sigma_{i}\right)$ the set of plays from $s$ that are consistent with $\sigma_{i}$.

For $\mathcal{M}=\left(M, m_{\text {init }}, \alpha_{\text {upd }}\right)$ a skeleton, a strategy $\sigma_{i} \in \Sigma_{i}(\mathcal{A})$ is based on (memory) $\mathcal{M}$ if there exists a function $\alpha_{\mathrm{nxt}}: S \times M \rightarrow E$ such that for all $s \in S_{i}, \sigma_{i}\left(\lambda_{s}\right)=\alpha_{\mathrm{nxt}}\left(s, m_{\text {init }}\right)$, and for all non-empty paths $h \in \operatorname{Hists}_{i}(\mathcal{A}), \sigma_{i}(h)=\alpha_{\text {nxt }}\left(\operatorname{out}(h), \alpha_{\text {upd }}^{*}\left(m_{\text {init }}, \operatorname{col}^{*}(h)\right)\right)$. A strategy is memoryless if it is based on $\mathcal{M}_{\text {triv }}$.

- Remark 1. Our memory model is chromatic [30], i.e., it observes the sequences of colors and not the sequences of edges of arenas, due to the fact that the argument of the update function of a skeleton is in $M \times C$. It was recently shown that the amount of memory states required to play optimally for a winning condition using chromatic skeletons may be strictly larger
than using general memory structures (i.e., using memory structures observing edges) [11, Proposition 32], disproving a conjecture by Kopczyński [30]. The example provided is a Muller condition (hence an $\omega$-regular condition), in which both kinds of memory requirements are still finite. A result in this direction is also provided by Le Roux [32, Corollary 1] for games on finite arenas: it shows that in many games, a strategy using general finite memory can be swapped for a (larger) chromatic finite memory.

For games on infinite arenas, which we consider in this article, we do not know whether there exists a winning condition with finite general memory requirements, but infinite chromatic memory requirements. Our results focus on chromatic memory requirements. $\lrcorner$

Winning conditions. A (winning) condition is a set $W \subseteq C^{\omega}$. When a winning condition $W$ is clear in the context, we say that an infinite word $w \in C^{\omega}$ is winning if $w \in W$, and losing if $w \notin W$. For a winning condition $W$ and a word $w \in C^{*}$, we write $w^{-1} W=\left\{w^{\prime} \in\right.$ $\left.C^{\omega} \mid w w^{\prime} \in W\right\}$ for the set of winning continuations of $w$. We write $\bar{W}$ for the complement $C^{\omega} \backslash W$ of a winning condition $W$.

A game is a tuple $\mathcal{G}=(\mathcal{A}, W)$ where $\mathcal{A}$ is an arena and $W$ is a winning condition.

Optimality and determinacy. Let $\mathcal{G}=\left(\mathcal{A}=\left(S, S_{1}, S_{2}, E\right), W\right)$ be a game, and $s \in S$. We say that $\sigma_{1} \in \Sigma_{1}(\mathcal{A})$ is winning from $s$ if $\operatorname{col}^{\omega}\left(\operatorname{Plays}\left(\mathcal{A}, s, \sigma_{1}\right)\right) \subseteq W$, and we say that $\sigma_{2} \in \Sigma_{2}(\mathcal{A})$ is winning from $s$ if $\operatorname{col}^{\omega}\left(\operatorname{Plays}\left(\mathcal{A}, s, \sigma_{2}\right)\right) \subseteq \bar{W}$.

A strategy of $\mathcal{P}_{i}$ is optimal for $\mathcal{P}_{i}$ in $(\mathcal{A}, W)$ if it is winning from all the states from which $\mathcal{P}_{i}$ has a winning strategy. We often write optimal for $\mathcal{P}_{i}$ in $\mathcal{A}$ if the winning condition $W$ is clear from the context. We stress that this notion of optimality requires a single strategy to be winning from all the winning states (a property sometimes called uniformity).

A winning condition $W$ is determined if for all games $\mathcal{G}=\left(\mathcal{A}=\left(S, S_{1}, S_{2}, E\right), W\right)$, for all $s \in S$, either $\mathcal{P}_{1}$ or $\mathcal{P}_{2}$ has a winning strategy from $s$. Let $\mathcal{M}$ be a skeleton. We say that a winning condition $W$ is $\mathcal{M}$-determined if $(i) W$ is determined and (ii) in all arenas $\mathcal{A}$, both players have an optimal strategy based on $\mathcal{M}$. A winning condition $W$ is one-player $\mathcal{M}$-determined if in all one-player arenas $\mathcal{A}$ of $\mathcal{P}_{1}, \mathcal{P}_{1}$ has an optimal strategy based on $\mathcal{M}$ and in all one-player arenas $\mathcal{A}$ of $\mathcal{P}_{2}, \mathcal{P}_{2}$ has an optimal strategy based on $\mathcal{M}$. A winning condition $W$ is (one-player) memoryless-determined if it is (one-player) $\mathcal{M}_{\text {triv-determined. A }}$ winning condition $W$ is (one-player) chromatic-finite-memory determined if there exists a skeleton $\mathcal{M}$ such that it is (one-player) $\mathcal{M}$-determined.

- Remark 2. It might seem surprising that for chromatic-finite-memory determinacy, we require the existence of a single skeleton that suffices to play optimally in all arenas, rather than the seemingly weaker existence, for each arena, of a finite skeleton (which may depend on the arena) that suffices to play optimally. In infinite arenas, it turns out that these notions are equivalent.
- Lemma 3. Let $W \subseteq C^{\omega}$ be a winning condition. The following are equivalent:

1. for all arenas $\mathcal{A}$, there exists a skeleton $\mathcal{M}^{\mathcal{A}}$ such that both players have an optimal strategy based on $\mathcal{M}^{\mathcal{A}}$ in $\mathcal{A}$;
2. $W$ is chromatic-finite-memory determined.

Proof. It is clear that $2 \Longrightarrow 1$, as 2 means that there is a (fixed) skeleton that suffices in each arena. We now show $1, \Longrightarrow 2$ We proceed by contraposition. Assume that 2 does not hold, i.e., that for all skeletons $\mathcal{M}$, there exists $\mathcal{A}^{\mathcal{M}}=\left(S^{\mathcal{M}}, S_{1}^{\mathcal{M}}, S_{2}^{\mathcal{M}}, E^{\mathcal{M}}\right)$ such that at least one player does not have an optimal strategy based on $\mathcal{M}$ in $\mathcal{A}^{\mathcal{M}}$. We consider the arena $\mathcal{A}=\left(\biguplus_{\mathcal{M}} S^{\mathcal{M}}, \biguplus_{\mathcal{M}} S_{1}^{\mathcal{M}}, \biguplus_{\mathcal{M}} S_{2}^{\mathcal{M}}, \biguplus_{\mathcal{M}} E^{\mathcal{M}}\right)$ consisting in the "disjoint union" over all
skeletons $\mathcal{M}$ of the arenas $\mathcal{A}^{\mathcal{M}}$. Clearly, no strategy based on a skeleton suffices to play optimally in $\mathcal{A}$; this shows that 1 does not hold.

When restricted to finite arenas, we do not have an equivalence between these two notions (hence the distinction between finite-memory determinacy and arena-independent finitememory determinacy [6, [8]). Our proof of Lemma 3 exploits that an infinite union of arenas is still an arena, which is not true when restricted to finite arenas.
$\omega$-regular languages. We define a parity automaton as a pair $(\mathcal{M}, p)$ where $\mathcal{M}$ is a skeleton and $p: M \times C \rightarrow\{0, \ldots, n\}$; function $p$ assigns priorities to every transition of $\mathcal{M}$. This definition implies that we consider deterministic and complete parity automata (i.e., in every state, reading a color leads to exactly one state). Following [12], if $\mathcal{M}$ is a skeleton, we say that a parity automaton $\left(\mathcal{M}^{\prime}, p\right)$ is defined on top of $\mathcal{M}$ if $\mathcal{M}^{\prime}=\mathcal{M}$.

A parity automaton $(\mathcal{M}, p)$ defines a language $L_{(\mathcal{M}, p)}$ of all the infinite words $w \in C^{\omega}$ such that, for $\operatorname{skel}(w)=\left(m_{1}, c_{1}\right)\left(m_{2}, c_{2}\right) \ldots, \lim \sup _{i \geq 1} p\left(m_{i}, c_{i}\right)$ is even. We say that $W \subseteq C^{\omega}$ is recognized by $(\mathcal{M}, p)$ if $W=L_{(\mathcal{M}, p)}$. We emphasize that we consider here transition-based parity acceptance conditions: we assign priorities to transitions, and not to states of $\mathcal{M}$. For further information on links between state-based and transition-based acceptance conditions, we refer to [11. If a language of infinite words can be recognized by a parity automaton, it is called $\omega$-regular.

- Remark 4. Formally, a deterministic parity automaton should be defined on a finite set of colors, whereas here our set of colors $C$ can have any cardinality. However, given a parity automaton $\left(\mathcal{M}=\left(M, m_{\text {init }}, \alpha_{\text {upd }}\right), p\right)$, as there are finitely many states in $\mathcal{M}$ and finitely many priorities, there are in practice only finitely many "truly different" classes of colors: two colors $c_{1}, c_{2} \in C$ can be assumed to be equal (w.r.t. $(\mathcal{M}, p)$ ) if $\alpha_{\text {upd }}\left(\cdot, c_{1}\right)=\alpha_{\text {upd }}\left(\cdot, c_{2}\right)$ and $p\left(\cdot, c_{1}\right)=p\left(\cdot, c_{2}\right)$.

Right congruence. For $\sim$ an equivalence relation, we call the index of $\sim$ the number of equivalence classes of $\sim$. We denote by $[a]_{\sim}$ the equivalence class of an element $a$ for $\sim$.

Let $W$ be a winning condition. We define the right congruence $\sim_{W} \subseteq C^{*} \times C^{*}$ of $W$ as $w_{1} \sim_{W} w_{2}$ if $w_{1}^{-1} W=w_{2}^{-1} W$ (meaning that $w_{1}$ and $w_{2}$ have the same winning continuations). Relation $\sim_{W}$ is an equivalence relation. When $W$ is clear from the context, we write $\sim$ for $\sim_{W}$. We denote by $\varepsilon$ the empty word. When $\sim$ has finite index, we can associate a natural skeleton $\mathcal{M}_{\sim}=\left(M_{\sim}, m_{\text {init }}^{\sim}, \alpha_{\text {upd }}^{\sim}\right)$ to $\sim$ such that $M_{\sim}$ is the set of equivalence classes of $\sim, m_{\text {init }}^{\sim}=[\varepsilon]_{\sim}$, and $\alpha_{\text {upd }}^{\sim}\left([w]_{\sim}, c\right)=[w c]_{\sim}$. This transition function is well-defined since it follows from the definition of $\sim$ that if $w_{1} \sim w_{2}$, then for all $c \in C, w_{1} c \sim w_{2} c$. Hence, the choice of representatives for the equivalence classes does not have an impact in this definition. We call skeleton $\mathcal{M}_{\sim}$ the minimal-state automaton of $\sim 42$, 35.

## 3 Concepts and characterization

We define two concepts at the core of our characterization, one of them dealing with prefixes and the other one dealing with cycles. Let $W \subseteq C^{\omega}$ be a winning condition and $\mathcal{M}=\left(M, m_{\text {init }}, \alpha_{\text {upd }}\right)$ be a skeleton.

Prefix-independence. Let $\sim$ be the right congruence of $W$.

- Definition 5. Condition $W$ is $\mathcal{M}$-prefix-independent if for all $m \in M$, for all $w_{1}, w_{2} \in$ $\mathrm{c} \Pi_{m_{\text {init }}, m}, w_{1} \sim w_{2}$.


Figure 1 Skeleton $\mathcal{M}$ such that $W=\operatorname{Büchi}(a) \cap \operatorname{Büchi}(b)$ is $\mathcal{M}$-cycle-consistent (Example 7. In figures, we use rhombuses (resp. circles, squares) to depict skeleton states (resp. arena states controlled by $\mathcal{P}_{1}$, arena states controlled by $\mathcal{P}_{2}$ ).

In other words, $W$ is $\mathcal{M}$-prefix-independent if finite words reaching the same state of $\mathcal{M}$ from its initial state have the same winning continuations. The classical notion of prefixindependence is equivalent to $\mathcal{M}_{\text {triv }}$-prefix-independence (as all finite words have the exact same set of winning continuations, which is $W$ ). If $\sim$ has finite index, $W$ is in particular $\mathcal{M}_{\sim}$-prefix-independent: indeed, two finite words reach the same state of $\mathcal{M}_{\sim}$ (if and) only if they are equivalent for $\sim$. Any skeleton $\mathcal{M}$ such that $W$ is $\mathcal{M}$-prefix-independent must have at least one state for each equivalence class of $\sim$, but multiple states may partition the same equivalence class.

Cycle-consistency. For $w \in C^{*}$, we define

$$
\Gamma_{\mathcal{M}}^{\text {win }, w}=\left\{\gamma \in \Gamma_{m} \mid m=\alpha_{\text {upd }}^{*}\left(m_{\text {init }}, w\right) \text { and }\left(\operatorname{col}^{*}(\gamma)\right)^{\omega} \in w^{-1} W\right\}
$$

as the cycles on the skeleton state reached by $w$ in $\mathcal{M}$ that induce winning words when repeated infinitely many times after $w$. We define

$$
\Gamma_{\mathcal{M}}^{\text {lose }, w}=\left\{\gamma \in \Gamma_{m} \mid m=\alpha_{\mathrm{upd}}^{*}\left(m_{\mathrm{init}}, w\right) \text { and }\left(\operatorname{col}^{*}(\gamma)\right)^{\omega} \in w^{-1} \bar{W}\right\}
$$

as their losing counterparts. We emphasize that cycles are allowed to go through the same edge multiple times.

- Definition 6. Condition $W$ is $\mathcal{M}$-cycle-consistent if for all $w \in C^{*},\left(c \Gamma_{\mathcal{M}}^{\text {win } w}\right)^{\omega} \subseteq w^{-1} W$ and $\left(c \Gamma_{\mathcal{M}}^{\text {lose }, w}\right)^{\omega} \subseteq w^{-1} \bar{W}$.

What this says is that after any finite word, if we concatenate infinitely many winning (resp. losing) cycles on the skeleton state reached by that word, then it only produces winning (resp. losing) infinite words.

- Example 7. For $c^{\prime} \in C$, let Büchi $\left(c^{\prime}\right)$ be the set of infinite words on $C$ that see color $c^{\prime}$ infinitely often. Let $C=\{a, b, c\}$. Condition $W=\operatorname{Büchi}(a) \cap \operatorname{Büchi}(b)$ is $\mathcal{M}_{\text {triv-prefix- }}$ independent, but not $\mathcal{M}_{\text {triv }}$-cycle-consistent: for any $w \in C^{*}, a$ and $b$ are both in $c \Gamma_{\mathcal{M}_{\text {triv }}}^{\text {lose }, w}$ (as $w a^{\omega}$ and $w b^{\omega}$ are losing), but word $w(a b)^{\omega}$ is winning. However, $W$ is $\mathcal{M}$-cycle-consistent for the skeleton $\mathcal{M}$ with two states $m_{\text {init }}$ and $m_{2}$ represented in Figure 1 For finite words reaching $m_{\text {init }}$, the losing cycles only see $a$ and $c$, and combining infinitely many of them gives an infinite word without $b$, which is a losing continuation of any finite word. The winning cycles are the ones that go to $m_{2}$ and then go back to $m_{\text {init }}$, as they must see both $a$ and $b$; combining infinitely many of them guarantees a winning continuation after any finite word. A similar reasoning applies to state $m_{2}$. Notice that $W$ is also $\mathcal{M}$-prefix-independent. With regard to memory requirements, condition $W$ is not $\mathcal{M}_{\text {triv }}$-determined but is $\mathcal{M}$-determined.

Properties of these concepts. Both $\mathcal{M}$-prefix-independence and $\mathcal{M}$-cycle-consistency hold symmetrically for a winning condition and its complement, and are stable by product with an arbitrary skeleton (as products generate even smaller sets of prefixes and cycles to consider).

- Lemma 8. Let $W \subseteq C^{\omega}$ be a winning condition and $\mathcal{M}$ be a skeleton. Then, $W$ is $\mathcal{M}$-prefix-independent (resp. $\mathcal{M}$-cycle-consistent) if and only if $\bar{W}$ is $\mathcal{M}$-prefix-independent (resp. $\mathcal{M}$-cycle-consistent). If $W$ is $\mathcal{M}$-prefix-independent (resp. $\mathcal{M}$-cycle-consistent), then for all skeletons $\mathcal{M}^{\prime}, W$ is $\left(\mathcal{M} \otimes \mathcal{M}^{\prime}\right)$-prefix-independent (resp. $\left(\mathcal{M} \otimes \mathcal{M}^{\prime}\right)$-cycle-consistent)

Proof. Let $\mathcal{M}=\left(M, m_{\text {init }}, \alpha_{\text {upd }}\right)$.
We assume that $W$ is $\mathcal{M}$-prefix-independent. Thus, for all $m \in M$, for all $w_{1}, w_{2} \in$ $\subset \Pi_{m_{\text {init }}, m}, w_{1} \sim w_{2}$, i.e., $w_{1}^{-1} W=w_{2}^{-1} W$. This last equality is equivalent to $\overline{w_{1}^{-1} W}=\overline{w_{2}^{-1} W}$, which can be rewritten as $w_{1}^{-1} \bar{W}=w_{2}^{-1} \bar{W}$. This shows that $\bar{W}$ is $\mathcal{M}$-prefix-independent.

To show that $W$ is $\mathcal{M}$-cycle-consistent if and only if $\bar{W}$ is $\mathcal{M}$-cycle-consistent, notice that the winning cycles for $W$ are exactly the losing cycles for $\bar{W}$, and vice versa.

Let $\mathcal{M}^{\prime}=\left(M^{\prime}, m_{\text {init }}^{\prime}, \alpha_{\text {upd }}^{\prime}\right)$ be a skeleton. We assume that $W$ is $\mathcal{M}$-prefix-independent and we show that $W$ is $\left(\mathcal{M} \otimes \mathcal{M}^{\prime}\right)$-prefix-independent. The sets of prefixes to consider are smaller in $\mathcal{M} \otimes \mathcal{M}^{\prime}$ than in $\mathcal{M}$ : for all $\left(m, m^{\prime}\right) \in M \times M^{\prime}, \mathrm{c} \Pi_{\left(m_{\text {init }}, m_{\text {init }}^{\prime}\right),\left(m, m^{\prime}\right)} \subseteq \mathrm{c} \Pi_{m_{\text {init }}, m}$. Therefore, for all $w_{1}, w_{2} \in \mathrm{c} \Pi_{\left(m_{\text {init }}, m_{\text {init }}^{\prime}\right),\left(m, m^{\prime}\right)}$, we also have $w_{1}, w_{2} \in \mathrm{c} \Pi_{m_{\text {init }}, m}$, so by $\mathcal{M}-$ prefix-independence, $w_{1} \sim w_{2}$.

We now assume that $W$ is $\mathcal{M}$-cycle-consistent and we show that $W$ is $\left(\mathcal{M} \otimes \mathcal{M}^{\prime}\right)$-cycleconsistent. The sets of winning and losing cycles to consider are smaller in $\mathcal{M} \otimes \mathcal{M}^{\prime}$ than in $\mathcal{M}$ : for all $w \in C^{*}, c \Gamma_{\mathcal{M} \otimes \mathcal{M}^{\prime}}^{\text {win }, w} \subseteq c \Gamma_{\mathcal{M}}^{\text {win }, w}$ and $c \Gamma_{\mathcal{M} \otimes \mathcal{M}^{\prime}}^{\text {lose }, w} \subseteq c \Gamma_{\mathcal{M}}^{\text {lose }, w}$. By $\mathcal{M}$-cycle-consistency, for all $w \in C^{*}$, we have $\left(c \Gamma_{\mathcal{M}}^{\text {win }, w}\right)^{\omega} \subseteq w^{-1} W$ and $\left(c \Gamma_{\mathcal{M}}^{\text {lose }, w}\right)^{\omega} \subseteq w^{-1} \bar{W}$, so we also have $\left(\mathrm{c} \Gamma_{\mathcal{M} \otimes \mathcal{M}^{\prime}}^{\mathrm{win}, w}\right)^{\omega} \subseteq w^{-1} W$ and $\left(\mathrm{c} \Gamma_{\mathcal{M} \otimes \mathcal{M}^{\prime}}^{\text {lose }, w}\right)^{\omega} \subseteq w^{-1} \bar{W}$.

An interesting property of languages defined by a parity automaton $(\mathcal{M}, p)$ is that they satisfy both aforementioned concepts with skeleton $\mathcal{M}$.

- Lemma 9. Let $W \subseteq C^{\omega}$ be a winning condition and $(\mathcal{M}, p)$ be a parity automaton. If $W$ is recognized by $(\mathcal{M}, p)$, then $W$ is $\mathcal{M}$-prefix-independent and $\mathcal{M}$-cycle-consistent.

Proof. By definition of the parity acceptance condition, any two finite words reaching the same state of the skeleton have the same winning continuations. Therefore, $W$ is $\mathcal{M}$-prefix-independent.

Also, the winning (resp. losing) cycles of $\mathcal{M}$ after any finite word are exactly the ones that have an even (resp. odd) maximal priority. Therefore, combining infinitely many winning (resp. losing) cycles can only produce a winning (resp. losing) infinite word.

Main results. We state our main technical tool. We recall that one-player $\mathcal{M}$-determinacy of a winning condition $W$ is both about one-player arenas of $\mathcal{P}_{1}$ (trying to achieve a word in $W)$ and of $\mathcal{P}_{2}$ (trying to achieve a word in $\bar{W}$ ).

- Theorem 10. Let $W \subseteq C^{\omega}$ be a winning condition and $\sim$ be its right congruence.

1. If there exists a skeleton $\mathcal{M}$ such that $W$ is one-player $\mathcal{M}$-determined, then $\sim$ has finite index (in particular, $W$ is $\mathcal{M}_{\sim}$-prefix-independent) and $W$ is $\mathcal{M}$-cycle-consistent.
2. If there exists a skeleton $\mathcal{M}$ such that $W$ is $\mathcal{M}$-prefix-independent and $\mathcal{M}$-cycle-consistent, then $W$ is $\omega$-regular and can be recognized by a deterministic parity automaton defined on top of $\mathcal{M}$.

We prove this theorem in Sections 4 and 5 . We state two consequences of this result that were already mentioned in the introduction: a strategic characterization of $\omega$-regular languages, and a novel one-to-two-player-lift.

Theorem 11 (Characterization). Let $W \subseteq C^{\omega}$ be a language of infinite words. Language $W$ is $\omega$-regular if and only if it is chromatic-finite-memory determined (in infinite arenas).

Proof. One implication is well-known [36, 43]: if $W$ is $\omega$-regular, then it can be recognized by a deterministic parity automaton whose skeleton can be used as a memory that suffices to play optimally for both players, in arenas of any cardinality. The other direction is given by Theorem 10. if $W$ is chromatic-finite-memory determined, then there exists in particular a skeleton $\mathcal{M}$ such that $W$ is one-player $\mathcal{M}$-determined, so $\sim$ has finite index and $W$ is $\mathcal{M}$-cycle-consistent. In particular, by Lemma $8, W$ is $\left(\mathcal{M}_{\sim} \otimes \mathcal{M}\right)$-prefix-independent and $\left(\mathcal{M}_{\sim} \otimes \mathcal{M}\right)$-cycle-consistent, so $W$ is $\omega$-regular and can be recognized by a deterministic parity automaton defined on top of $\mathcal{M}_{\sim} \otimes \mathcal{M}$.

- Theorem 12 (One-to-two-player lift). Let $W \subseteq C^{\omega}$ be a winning condition. Language $W$ is one-player chromatic-finite-memory determined if and only if it is chromatic-finite-memory determined.

Proof. The implication from two-player to one-player determinacy is trivial. The other implication is given by Theorem 10 if $W$ is one-player $\mathcal{M}$-determined, then $\sim$ has finite index and $W$ is $\mathcal{M}$-cycle-consistent. Again by Lemma 8 and Theorem 10, as $W$ can be recognized by a parity automaton defined on top of $\mathcal{M} \otimes \otimes \mathcal{M}, W$ is determined and strategies based on $\mathcal{M} \sim \otimes \mathcal{M}$ suffice to play optimally in all two-player arenas.

We discuss two specific situations in which we can easily derive interesting consequences using our results: the prefix-independent case, and the case where the minimal-state automaton suffices to play optimally.

Prefix-independent case. If a condition $W$ is prefix-independent (i.e., $\sim$ has index 1 and $\left.\mathcal{M}_{\sim}=\mathcal{M}_{\text {triv }}\right)$, and skeleton $\mathcal{M}$ suffices to play optimally in one-player games, then $W$ is recognized by a parity automaton defined on top of $\mathcal{M}_{\text {triv }} \otimes \mathcal{M}$, which is isomorphic to $\mathcal{M}$. This implies that the exact same memory $\mathcal{M}$ can be used by both players to play optimally in two-player arenas, with no increase in memory. Note that, in general, when $\mathcal{M}$ suffices to play optimally in one-player arenas, we do not know whether taking the product of $\mathcal{M}$ with $\mathcal{M}_{\sim}$ is necessary to play optimally in two-player arenas. Still, the question is automatically solved for prefix-independent conditions.

If, moreover, $\mathcal{M}=\mathcal{M}_{\text {triv }}$ (i.e., memoryless strategies suffice to play optimally in one-player arenas), we recover exactly the result from Colcombet and Niwiński [17]: $W$ can be recognized by a parity automaton defined on top of $\mathcal{M}_{\text {triv }}$, so we can directly assign a priority to each color with a function $p: C \rightarrow\{0, \ldots, n\}$ such that an infinite word $w=c_{1} c_{2} \ldots \in C^{\omega}$ is in $W$ if and only if $\lim \sup _{i \geq 1} p\left(c_{i}\right)$ is even.
$\mathcal{M}_{\sim}$-determined case. An interesting property of some $\omega$-regular languages is that they can be recognized by defining an acceptance condition on top of the minimal-state automaton of their right congruence [35], which is a useful property for the learning of languages 11. Here, Theorem 10 implies that $W$ can be recognized by defining a transition-based parity acceptance condition on top of the minimal-state automaton $\mathcal{M}_{\sim}$ if and only if $W$ is $\mathcal{M}_{\sim}$-determined (and more precisely, if and only if $W$ is $\mathcal{M}_{\sim}$-cycle-consistent). The transition-based parity acceptance condition was not considered in the cited results [35, 1].

- Corollary 13. Let $W \subseteq C^{\omega}$ be an $\omega$-regular language and $\mathcal{M}_{\sim}$ be the minimal-state automaton of its right congruence. The following are equivalent:

1. $W$ is recognized by defining a transition-based parity acceptance condition on top of $\mathcal{M}_{\sim}$;
2. $W$ is $\mathcal{M}_{\sim}$-determined;
3. $W$ is $\mathcal{M}_{\sim}$-cycle-consistent.

Proof. Implication $1 . \Longrightarrow 2$, follows from the memoryless determinacy of parity games 43]. Implication 2, $\Longrightarrow 3$ follows from the first item of Theorem 10 Implication 3 $\Longrightarrow 1$ follows from the second item of Theorem 10, we have by definition that $W$ is $\mathcal{M}_{\sim}$-prefix-independent, so if it is additionally $\mathcal{M}_{\sim}$-cycle-consistent, then $W$ can be recognized by a parity automaton defined on top of $\mathcal{M}_{\sim}$.

Classes of arenas. We discuss the sensitivity of Theorem 10 with regard to our model of arenas.

There are multiple conditions that are chromatic-finite-memory determined if we only consider finite arenas (finitely many states and edges), but which are not in infinite arenas. A few examples are discounted-sum games [41, mean-payoff games [19], total-payoff games [22], one-counter games [10] which are all memoryless-determined in finite arenas but which require infinite memory to play optimally in some infinite arenas (we discuss some of these in Section 66. In particular, Theorem 11 tells us that the derived winning conditions are not $\omega$-regular.

Strangely, the fact that our arenas have colors on edges and not on states is crucial for the result. Indeed, there exists a winning condition (a generalization of a parity condition with infinitely many priorities [25]) that is memoryless-determined in state-labeled infinite arenas, but not in edge-labeled infinite arenas (as we consider here). This particularity was already discussed [17], and it was also shown that the same condition is memoryless-determined in edge-labeled arenas with finite branching. Therefore, the fact that we allow infinite branching in our arenas is also necessary for Theorem 11 Another example of a winning condition with finite memory requirements in finitely branching arenas for one player but infinite memory requirements in infinitely branching arenas is presented in [16, Section 4].

In Sections 4 and 5 we prove respectively the first item and the second item of Theorem 10.

## 4 Two properties of one-player chromatic-finite-memory determinacy

Let $W \subseteq C^{\omega}$ be a winning condition, $\sim$ be the right congruence of $W$, and $\mathcal{M}=$ $\left(M, m_{\text {init }}, \alpha_{\text {upd }}\right)$ be a skeleton, fixed for this section. We aim to show the first item of Theorem 10, which is that for a skeleton $\mathcal{M}$, one-player $\mathcal{M}$-determinacy of $W$ implies that $\sim$ has finite index and that $W$ is $\mathcal{M}$-cycle-consistent.

Finite index of $\sim$. For $w_{1}, w_{2} \in C^{*}$, we define $w_{1} \preceq w_{2}$ if $w_{1}^{-1} W \subseteq w_{2}^{-1} W$ (meaning that any continuation that is winning after $w_{1}$ is also winning after $w_{2}$ ). Relation $\preceq \subseteq C^{*} \times C^{*}$ is a preorder. Notice that $\sim$ is equal to $\preceq \cap \succeq$. We also define the strict preorder $\prec=\preceq \backslash \sim$.

We will use preorder $\preceq$ to deduce that $\sim$ has finite index by showing that under hypotheses about the optimality of strategies based on $\mathcal{M}$ in one-player arenas, $(i)$ on each subset $\mathrm{c} \Pi_{m_{\text {init }}, m}$ of $C^{*}$ for $m \in M$, preorder $\preceq$ is total (Lemma 14) (ii) on each subset $c \Pi_{m_{\text {init }}, m}$ of $C^{*}$ for $m \in M$, preorder $\preceq$ has no infinite increasing nor decreasing sequence (Lemma 15 .

- Lemma 14. Assume $\mathcal{P}_{1}$ has optimal strategies based on $\mathcal{M}$ on all its one-player arenas. Then, for all $m \in M$, preorder $\preceq$ is total on $\subset \Pi_{m_{\text {init }}, m}$.

Proof. Let $m \in M$. Let $w_{1}, w_{2} \in \mathrm{c} \Pi_{m_{\text {init }}, m}$; we show that $w_{1} \npreceq w_{2}$ implies $w_{2} \preceq w_{1}$. If $w_{1} \npreceq w_{2}$, then there exists $w_{1}^{\prime} \in C^{\omega}$ such that $w_{1} w_{1}^{\prime} \in W$ and $w_{2} w_{1}^{\prime} \notin W$. We show that $w_{2} \preceq w_{1}$, i.e., that $w_{2}^{-1} W \subseteq w_{1}^{-1} W$. Let $w_{2}^{\prime} \in w_{2}^{-1} W$. We build an infinite one-player arena of $\mathcal{P}_{1}$, depicted in Figure 2, that merges the ends of finite chains for $w_{1}$ and $w_{2}$ and the starts of the infinite chains for $w_{1}^{\prime}$ and for $w_{2}^{\prime}$ in a state $s$.


Figure 2 Arena built in the proof of Lemma 14 . Squiggly arrows indicate a sequence of edges.

It is possible to win after seeing $w_{1}$ or $w_{2}$, by choosing respectively $w_{1}^{\prime}$ or $w_{2}^{\prime}$ in the merged state $s$. Moreover, there must be a strategy based on $\mathcal{M}$ that wins from the starts of the chains of both $w_{1}$ and $w_{2}$, which means that in both cases the same choice has to be made in $s$ (as memory state $m$ is reached in both cases). Continuing to $w_{1}^{\prime}$ in $s$ would be losing after $w_{2}$, so $w_{2}^{\prime}$ must be winning after $w_{1}$. Therefore, $w_{2}^{\prime} \in w_{1}^{-1} W$.

- Lemma 15. Assume $\mathcal{P}_{1}$ has optimal strategies based on $\mathcal{M}$ in all its one-player arenas. For all $m \in M$, there is no infinitely decreasing sequence of finite words for $\preceq$ in $\subset \Pi_{m_{\text {init }}, m}$.

Proof. Let $m \in M$. Assume by contraposition that there is an infinitely decreasing sequence of finite words $w_{1} \succ w_{2} \succ w_{3} \succ \ldots$, with $w_{i} \in \mathrm{c}_{m_{\text {init }}, m}$ for $i \geq 1$. Then for all $i \geq 1$, there exists $w_{i}^{\prime} \in C^{\omega}$ such that $w_{i} w_{i}^{\prime} \in W$ and $w_{i+1} w_{i}^{\prime} \notin W$. If we create an infinite one-player arena of $\mathcal{P}_{1}$ in which we merge the ends of chains for all $w_{i}$ to the starts of chains for all $w_{i}^{\prime}$, for all $i \geq 1$, it is always possible to win from the start of the chain for $w_{i}$, but there is no strategy based on $\mathcal{M}$ winning from all the starts of the chains simultaneously. Therefore, $\mathcal{M}$ is not sufficient to play optimally in all one-player arenas of $\mathcal{P}_{1}$.

We will also use this last lemma from the point of view of $\mathcal{P}_{2}$. If we were to define a preorder $\preceq^{\prime}$ for $\mathcal{P}_{2}$ (using winning condition $\bar{W}$ ), symmetrically to $\preceq$ for $\mathcal{P}_{1}$, we would obtain $w_{1} \preceq^{\prime} w_{2}$ if and only if $w_{2} \preceq w_{1}$ because for any finite word $w \in C^{*}, w^{-1} \bar{W}=\overline{w^{-1} W}$.

We can now combine the results of Lemmas 14 and 15 to find that $\sim$ has finite index if $W$ is one-player $\mathcal{M}$-determined.

- Lemma 16. If both $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ have optimal strategies based on $\mathcal{M}$ in their one-player arenas (i.e., if $W$ is one-player $\mathcal{M}$-determined), then the right congruence $\sim$ has finite index.

Proof. Using Lemma 15 along with the hypothesis about $\mathcal{P}_{1}$, we have that for all $m \in M$, there are no infinitely decreasing sequence of words in $c \Pi_{m_{\text {init }}, m}$ for $\preceq$. Using the same result replacing $\mathcal{P}_{1}$ with $\mathcal{P}_{2}$, we obtain that there is no infinitely decreasing sequence for $\preceq^{\prime}$, or in other words, that there is no infinitely increasing sequence for $\preceq$. For $m \in M$, as $\preceq$ is total in $c \Pi_{m_{\text {init }}, m}$ (Lemma 14), we conclude that there are only finitely many equivalence classes of $\sim$ in $c \Pi_{m_{\text {init }}, m}$. As $M$ is finite, there are only finitely many equivalence classes of $\sim$ in $\bigcup_{m \in M} \mathrm{c} \Pi_{m_{\text {init }}, m}=C^{*}$.

Under one-player chromatic-finite-memory determinacy of $W$, we can therefore consider the minimal-state automaton $\mathcal{M}_{\sim}$ of $\sim$.
$\mathcal{M}$-cycle-consistency of $W$. We now prove in a straightforward way that one-player $\mathcal{M}$ determinacy of $W$ implies $\mathcal{M}$-cycle-consistency of $W$.

- Lemma 17. If both $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ have optimal strategies based on $\mathcal{M}$ in their one-player arenas (i.e., if $W$ is one-player $\mathcal{M}$-determined), then winning condition $W$ is $\mathcal{M}$-cycleconsistent.


Figure 3 Infinite one-player arena of $\mathcal{P}_{2}$ used in the proof of Lemma 17 The thick squiggly arrow indicates a choice between sequences of edges for any word in $\left\{w_{1}, w_{2}, \ldots\right\}$.

Proof. Let $w \in C^{*}$ and $m=\alpha_{\text {upd }}^{*}\left(m_{\text {init }}, w\right)$. We show that $\left(c \Gamma_{\mathcal{M}}^{\text {win }, w}\right)^{\omega} \subseteq w^{-1} W$. If $\mathrm{c} \Gamma_{\mathcal{M}}^{\text {win }, w}$ is empty, this is true. If not, let $w_{1}, w_{2}, \ldots$ be an infinite sequence of finite words in $c \Gamma_{\mathcal{M}}^{\text {win }, w}$ - we show that the infinite word $w_{1} w_{2} \ldots$ is in $w^{-1} W$. We consider the infinite one-player arena of $\mathcal{P}_{2}$ starting with a chain for $w$ from a state $s_{1}$, followed by a choice in a state $s_{2}$ between a chain for each finite word in $\left\{w_{1}, w_{2}, \ldots\right\}$, all coming back to $s_{2}$ (depicted in Figure 3). In this arena, $\mathcal{P}_{2}$ has no winning strategy based on $\mathcal{M}$ from $s_{1}$, since the same memory state $m$ is always reached in $s_{2}$ (hence the same choice must always be made in $s_{2}$ ), and repeating any cycle in $c \Gamma_{\mathcal{M}}^{w i n, w}$ forever after $w$ is winning for $\mathcal{P}_{1}$ by definition of $\mathrm{c} \Gamma_{\mathcal{M}}^{\text {win }, w}$. Therefore, $\mathcal{P}_{2}$ also has no winning strategy at all, which means in particular that the infinite word $w_{1} w_{2} \ldots$ must be a winning continuation of $w-w_{1} w_{2} \ldots$ is in $w^{-1} W$. Hence, $\left(c \Gamma_{\mathcal{M}}^{w^{\text {in }}, w}\right)^{\omega} \subseteq w^{-1} W$.

Using a similar one-player arena of $\mathcal{P}_{1}$, we can show in a symmetric way that $\left(\mathrm{c} \Gamma_{\mathcal{M}}^{\text {lose }, w}\right)^{\omega} \subseteq$ $w^{-1} \bar{W}$ for all $w \in C^{*}$.

The reciprocal of this result is false, as shown in the following example.

- Example 18. Let $C=\{a, b\}$ and $W=a b C^{\omega}$. If we consider the skeleton $\mathcal{M}$ in Figure 4 , then $W$ is $\mathcal{M}$-cycle-consistent: for all finite words $w$ except for $\varepsilon$ and $a$, either all continuations are winning (if $w \in a b C^{*}$ ) or all continuations are losing. If $w$ is $\varepsilon$ or $a$, then it reaches state $m_{\text {init }}$ of $\mathcal{M}$, and the only cycles on $m_{\text {init }}$ are in $a^{+}$, are losing, and are losing when infinitely many of them are combined into an infinite word. But this automaton does not suffice to play optimally in arena $\mathcal{A}$ in Figure 4 , as seeing $a$ does not change the state.

Notice that the minimal-state automaton $\mathcal{M}_{\sim}$ has four states (corresponding to equivalence classes $[\varepsilon]_{\sim},[a]_{\sim},[a b]_{\sim}$, and $\left.[b]_{\sim}\right)$ and suffices to play optimally.




Figure 4 Skeleton $\mathcal{M}$ (left), arena $\mathcal{A}$ (center) and skeleton $\mathcal{M}_{\sim}$ (right) used in Example 18

- Remark 19. As discussed in Section 3 Theorem 10 does not hold if we assume chromatic-finite-memory determinacy in arenas in which states rather than edges are labeled with colors. Lemma 17 is an example of a step in the proof of Theorem 10 that would not work with state-labeled arenas: the construction in Figure 3 would not work (there would have to be a color labeling $s$ seen at the start of every cycle, but words $w_{i}$ cannot all start with the same color in general). There is a winning condition that is memoryless-determined in state-labeled arenas [25] for which it is straightforward to show that it is not $\mathcal{M}_{\text {triv-cycle-consistent. }}$.

We will often use a weaker implication of $\mathcal{M}$-cycle-consistency, which is that a finite combination of winning cycles is still a winning cycle (i.e., if $\gamma, \gamma^{\prime} \in \Gamma_{\mathcal{M}}^{\text {win } w}$, then $\gamma \gamma^{\prime} \in \Gamma_{\mathcal{M}}^{\text {win }, w}$ ).

Wrap-up of the section. Thanks to the results from this section, we deduce the first item of Theorem 10 .

- Corollary 20 (First item of Theorem 10). If there exists a skeleton $\mathcal{M}$ such that $W$ is oneplayer $\mathcal{M}$-determined, then $\sim$ has finite index (in particular, $W$ is $\mathcal{M}_{\sim}$-prefix-independent) and $W$ is $\mathcal{M}$-cycle-consistent.

Proof. Follows from Lemmas 16 and 17
In particular, we obtain from the previous result that if both players have optimal strategies based on $\mathcal{M}$ in their one-player arenas, then $W$ is both $\left(\mathcal{M}_{\sim} \otimes \mathcal{M}\right)$-prefix-independent and $\left(\mathcal{M}_{\sim} \otimes \mathcal{M}\right)$-cycle-consistent (using Lemma 8).

- Remark 21. If we compare Example $7(W=\operatorname{Büchi}(a) \cap \operatorname{Büchi}(b))$ and Example 18 $\left(W=a b C^{\omega}\right)$, we see that we can easily classify the prefixes of the former, but that information is not sufficient to play optimally: we need some more information to classify cycles. For the latter, it is possible to find a skeleton classifying cycles that is insufficient to play optimally, but a good classification of the prefixes suffices to play optimally. In general, in order to understand $W$, we need to have information about prefixes and about cycles, which is why, intuitively, skeleton $\mathcal{M}_{\sim} \otimes \mathcal{M}$ turns out to be useful.
- Remark 22. In the proofs of this section, we only ever used arenas with countably many states and edges. This implies that we can actually formulate a slightly stronger version of Theorem 12 (one-to-two-player lift): chromatic-finite-memory determinacy in one-player countable arenas is equivalent to chromatic-finite-memory determinacy in arenas of any cardinality.


## 5 From properties of a winning condition to $\omega$-regularity

In this section, we fix a language $W \subseteq C^{\omega}$ and a skeleton $\mathcal{M}=\left(M, m_{\text {init }}, \alpha_{\text {upd }}\right)$, and we assume that $W$ is $\mathcal{M}$-prefix-independent and $\mathcal{M}$-cycle-consistent. Our goal is to show that $W$ can be recognized by a parity automaton defined on top of $\mathcal{M}$ and is thus $\omega$-regular. To do that, we show in multiple steps how to assign a priority to each transition of $\mathcal{M}$ through a function $p: M \times C \rightarrow\{0, \ldots, n\}$ so that $W$ is recognized by the parity automaton $(\mathcal{M}, p)$.

Simplified notations. In this section, as we have $\mathcal{M}$-prefix-independence and $\mathcal{M}$-cycleconsistency assumptions about $W$, we extend some notations from Section 2 for conciseness.

As $W$ is $\mathcal{M}$-prefix-independent, for $m \in M$, we write $m^{-1} W$ for the set of infinite words that equals $w^{-1} W$ for any $w \in \mathrm{c} \Pi_{m_{\text {init }}, m}$. Notice in particular that $m_{\text {init }}^{-1} W=\varepsilon^{-1} W=$ $W$. Moreover, as we consider the property of $\mathcal{M}$-cycle-consistency along with $\mathcal{M}$-prefixindependence, the definition of $\mathcal{M}$-cycle-consistency can be written by only quantifying over states of $\mathcal{M}$ and not over all finite words. The reason is that there are then only finitely many classes of finite words that matter, which correspond to the states of $\mathcal{M}$. We define a few more notations that only make sense under the $\mathcal{M}$-prefix-independent hypothesis. Let

$$
\Gamma_{m}^{\operatorname{win}}=\left\{\gamma \in \Gamma_{m} \mid\left(\operatorname{col}^{*}(\gamma)\right)^{\omega} \in m^{-1} W\right\}
$$

be the cycles on $m$ that induce winning words when repeated infinitely many times from $m$, and $\Gamma_{m}^{\text {lose }}$ be their losing counterparts. In this case, $W$ is $\mathcal{M}$-cycle-consistent if and only if for
all $m \in M,\left(c \Gamma_{m}^{\text {win }}\right)^{\omega} \subseteq m^{-1} W$ and $\left(c \Gamma_{m}^{\text {lose }}\right)^{\omega} \subseteq m^{-1} \bar{W}$. We call elements of $\Gamma_{m}^{\text {win }}\left(\right.$ resp. $\left.\Gamma_{m}^{\text {lose }}\right)$ winning (resp. losing) cycles on $m$. The set of winning (resp. losing) cycles of $\mathcal{M}$ (on any state) is denoted $\Gamma_{\mathcal{M}}^{\mathrm{win}}\left(\right.$ resp. $\left.\Gamma_{\mathcal{M}}^{\text {lose }}\right)$. We write $\operatorname{val}(\gamma)$ for the value of a cycle: win if $\gamma \in \Gamma_{\mathcal{M}}^{\text {win }}$, and lose if $\gamma \in \Gamma_{\mathcal{M}}^{\text {lose }}$.

Proof ideas. Our intermediate technical lemmas will focus on cycles of $\mathcal{M}$, how they relate to each other, and what happens when we combine them. Our main tool is to define a preorder on cycles, which will help assign priorities to transitions of $\mathcal{M}$ - the aim being to define a parity condition on top of $\mathcal{M}$ that recognizes $W$. Intuitively, for some state $m$ of $\mathcal{M}, \gamma \in \Gamma_{m}^{\text {win }}$, and $\gamma^{\prime} \in \Gamma_{m}^{\text {lose }}$, we look at which cycle "dominates" the other, that is whether the combined cycle $\gamma \gamma^{\prime}$ is in $\Gamma_{m}^{\text {win }}$ (in which case $\gamma$ dominates $\gamma^{\prime}$ ) or in $\Gamma_{m}^{\text {lose }}$ (in which case $\gamma^{\prime}$ dominates $\gamma$ ). We will formalize this and show how to extend this idea to cycles that may not share any common state.

- Remark 23. One may wonder why we seek to define a parity condition on top of $\mathcal{M}$ to prove that $W$ is $\omega$-regular, rather than a more general Muller condition which would achieve the same goal. Indeed, using $\mathcal{M}$-cycle-consistency and a recent result by Casares et al. [12, Section 5], it is straightforward to show that we could relabel such a Muller automaton with a parity condition defining the same language.

One of the obstacles in our context is that we may start with infinitely many colors; in order to prove $\omega$-regularity of $W$, we need to show at some point that many colors can be assumed to be equal (for $W$ ) in order to get finitely many classes of "equivalent" colors. The way we manage that, using the aforementioned idea of ordering cycles, actually brings us very close to directly defining a relevant parity condition on top of $\mathcal{M}$ - it does not appear that our proof technique can be easily simplified by trying to obtain a Muller condition. $\lrcorner$

Combining cycles on the same skeleton state. We first prove that "shifting" the start of a cycle does not alter its value.

- Lemma 24 (Shift independence). Let $m_{1}, m_{2} \in M$ be two states of $\mathcal{M}$. Let $\pi_{1} \in \Pi_{m_{1}, m_{2}}$ and $\pi_{2} \in \Pi_{m_{2}, m_{1}} ; \pi_{1} \pi_{2}$ is a cycle on $m_{1}$ and $\pi_{2} \pi_{1}$ is a cycle on $m_{2}$. Then, $\operatorname{val}\left(\pi_{1} \pi_{2}\right)=\operatorname{val}\left(\pi_{2} \pi_{1}\right)$.
Proof. For all $w_{1} \in \mathrm{c} \Pi_{m_{1}, m_{2}}$ and $w_{2} \in C^{\omega}$, notice that

$$
\begin{aligned}
w_{1} w_{2} \in m_{1}^{-1} W & \Longleftrightarrow \exists w \in \mathrm{c} \Pi_{m_{\text {init }}, m_{1}}, w_{1} w_{2} \in w^{-1} W \\
& \Longleftrightarrow \exists w \in \mathrm{c} \Pi_{m_{\text {init }}, m_{1}}, w_{2} \in\left(w w_{1}\right)^{-1} W \\
& \Longleftrightarrow \exists w^{\prime} \in \mathrm{c} \Pi_{m_{\text {init }}, m_{2}}, w_{2} \in\left(w^{\prime}\right)^{-1} W \\
& \Longleftrightarrow w_{2} \in m_{2}^{-1} W .
\end{aligned}
$$

The third equivalence is due to the fact that $w w_{1}$ is in $\mathrm{c} \Pi_{m_{\text {init }}, m_{2}}$ for the left-to-right implication, and to $\mathcal{M}$-prefix-independence for the right-to-left implication; if there exists $w^{\prime} \in \mathrm{c} \Pi_{m_{\text {init }}, m_{2}}$ such that $w_{2} \in\left(w^{\prime}\right)^{-1} W$, then the same is true for any word in $\mathrm{c} \Pi_{m_{\text {init }}, m_{2}}$. Going back to the statement of the lemma, we have that

$$
\begin{aligned}
\pi_{1} \pi_{2} & \in \Gamma_{m_{1}}^{\operatorname{win}} \\
& \Longleftrightarrow\left(\operatorname{col}^{*}\left(\pi_{1} \pi_{2}\right)\right)^{\omega} \in m_{1}^{-1} W \\
& \Longleftrightarrow \operatorname{col}^{*}\left(\pi_{1}\right)\left(\operatorname{col}^{*}\left(\pi_{2} \pi_{1}\right)\right)^{\omega} \in m_{1}^{-1} W \\
& \text { as }\left(\operatorname{col}^{*}\left(\pi_{1} \pi_{2}\right)\right)^{\omega}=\operatorname{col}^{*}\left(\pi_{1}\right)\left(\operatorname{col}^{*}\left(\pi_{2} \pi_{1}\right)\right)^{\omega} \\
& \Longleftrightarrow\left(\operatorname{col}^{*}\left(\pi_{2} \pi_{1}\right)\right)^{\omega} \in m_{2}^{-1} W
\end{aligned} \quad \text { by the above property as } \operatorname{col}^{*}\left(\pi_{1}\right) \in \operatorname{c\Pi }_{m_{1}, m_{2}} .
$$

Hence, the values of $\pi_{1} \pi_{2}$ and $\pi_{2} \pi_{1}$ are always the same.

In particular, this result implies that swapping two cycles on the same skeleton state does not alter the value: if $\gamma, \gamma^{\prime} \in \Gamma_{m}$, then $\operatorname{val}\left(\gamma \gamma^{\prime}\right)=\operatorname{val}\left(\gamma^{\prime} \gamma\right)$.

The next two lemmas are used to show that although cycles of $\mathcal{M}$ that are taken infinitely often might have an impact on the result of a play, their relative number of repetitions is not relevant (i.e., $\operatorname{val}\left(\gamma \gamma^{\prime}\right)=\operatorname{val}\left(\gamma^{k}\left(\gamma^{\prime}\right)^{l}\right)$ for any $k, l \geq 1$ ). These two proofs and statements are very close to [17, Lemmas 9,10, and 11] and are a direct generalization to a larger class of winning conditions.

- Lemma 25. Let $m \in M$. Let $\Lambda, \Lambda^{\prime} \subseteq \Gamma_{m}$ be non-empty sets of cycles on $m$. We have

$$
\forall \gamma^{\prime} \in \Lambda^{\prime}, \exists \gamma \in \Lambda, \gamma \gamma^{\prime} \in \Gamma_{m}^{\text {win }} \Longrightarrow \exists \gamma \in \Lambda, \forall \gamma^{\prime} \in \Lambda^{\prime}, \gamma \gamma^{\prime} \in \Gamma_{m}^{\text {win }}
$$

This lemma says that if all cycles from $\Lambda^{\prime}$ can be made winning by adjoining them a cycle from $\Lambda$, then we can actually find a single cycle from $\Lambda$ that makes all cycles from $\Lambda^{\prime}$ winning.

Proof. We assume the premise of the implication, and by contradiction, we assume that the conclusion is false. We therefore assume that

$$
\forall \gamma^{\prime} \in \Lambda^{\prime}, \exists \gamma \in \Lambda, \gamma \gamma^{\prime} \in \Gamma_{m}^{\text {win }} \quad \text { and } \quad \forall \gamma \in \Lambda, \exists \gamma^{\prime} \in \Lambda^{\prime}, \gamma \gamma^{\prime} \in \Gamma_{m}^{\text {lose }}
$$

Let $\gamma_{1}$ be any word in $\Lambda$. We build inductively an infinite sequence starting with $\gamma_{1}$ by alternating the use of the two assumptions. For $i \geq 1$, we take $\gamma_{i}^{\prime} \in \Lambda^{\prime}$ such that $\gamma_{i} \gamma_{i}^{\prime} \in \Gamma_{m}^{\text {lose }}$ (using the second assumption), and we then take $\gamma_{i+1} \in \Lambda$ such that $\gamma_{i+1} \gamma_{i}^{\prime} \in \Gamma_{m}^{\text {win }}$ (using the first assumption).

We consider the infinite sequence $\gamma_{1} \gamma_{1}^{\prime} \gamma_{2} \gamma_{2}^{\prime} \gamma_{3} \ldots \in(M \times C)^{\omega}$ such that for all $i \geq 1$, $\gamma_{i} \gamma_{i}^{\prime} \in \Gamma_{m}^{\text {lose }}$ and $\gamma_{i}^{\prime} \gamma_{i+1} \in \Gamma_{m}^{\text {win }}$ (we use that the order of cycles on $m$ does not matter, shown in Lemma 24). We show that the infinite word $\operatorname{col}^{\omega}\left(\gamma_{1} \gamma_{1}^{\prime} \gamma_{2} \gamma_{2}^{\prime} \ldots\right)$ is both in $m^{-1} W$ and in $m^{-1} \bar{W}$ by pairing cycles in two different ways:

- the infinite sequence $\left(\gamma_{1} \gamma_{1}^{\prime}\right)\left(\gamma_{2} \gamma_{2}^{\prime}\right) \ldots$ is a sequence of losing cycles on $m$ and its projection to colors is therefore in $m^{-1} \bar{W}$ by using $\mathcal{M}$-cycle-consistency.
- the infinite word $\operatorname{col}^{\omega}\left(\gamma_{1}\left(\gamma_{1}^{\prime} \gamma_{2}\right)\left(\gamma_{2}^{\prime} \gamma_{3}\right) \ldots\right)$ is in $m^{-1} W$ if and only if $\operatorname{col}^{\omega}\left(\left(\gamma_{1}^{\prime} \gamma_{2}\right)\left(\gamma_{2}^{\prime} \gamma_{3}\right) \ldots\right)$ is in $m^{-1} W$ by using that $\gamma_{1} \in \Gamma_{m}$ and $\mathcal{M}$-prefix-independence of $W$. The sequence $\left(\gamma_{1}^{\prime} \gamma_{2}\right)\left(\gamma_{2}^{\prime} \gamma_{3}\right) \ldots$ is a sequence of winning cycles on $m$ and its projection to colors is in $m^{-1} W$ by using $\mathcal{M}$-cycle-consistency.
As $m^{-1} W \cap m^{-1} \bar{W}=\emptyset$, we have our contradiction.
- Lemma 26 (Repetition independence). Let $m \in M$. Let $\gamma, \gamma^{\prime} \in \Gamma_{m}$ such that $\gamma \gamma^{\prime} \in \Gamma_{m}^{\text {win }}$. We have $\gamma\left(\gamma^{\prime}\right)^{+} \subseteq \Gamma_{m}^{\text {win }}$.

Proof. We have that $\gamma$ or $\gamma^{\prime}$ is in $\Gamma_{m}^{\text {win }}$ - otherwise, $\gamma \gamma^{\prime}$ would be in $\Gamma_{m}^{\text {lose }}$ by $\mathcal{M}$-cycleconsistency. If $\gamma^{\prime}$ is in $\Gamma_{m}^{\text {win }}$, we notice that any element of $\gamma\left(\gamma^{\prime}\right)^{+}$can be written as $\left(\gamma \gamma^{\prime}\right)\left(\gamma^{\prime}\right)^{n}$ for some $n \geq 0$, which is a combination of winning cycles on $m$. Using $\mathcal{M}$-cycle-consistency, we thus get that $\gamma\left(\gamma^{\prime}\right)^{+} \subseteq \Gamma_{m}^{\text {win }}$.

It is left to deal with the case $\gamma \in \Gamma_{m}^{\text {win }}$ and $\gamma^{\prime} \in \Gamma_{m}^{\text {lose }}$. We first show by induction that for $n \geq 1, \gamma^{n}\left(\gamma^{\prime}\right)^{n} \in \Gamma_{m}^{\text {win }}$. This is true by hypothesis for $n=1$. We now assume it is true for some $n \geq 1$, and we show it is true for $n+1$. By Lemma 24 we have that $\gamma^{n+1}\left(\gamma^{\prime}\right)^{n+1} \in \Gamma_{m}^{\text {win }}$ if and only if $\gamma^{n}\left(\gamma^{\prime}\right)^{n+1} \gamma=\left(\gamma^{n}\left(\gamma^{\prime}\right)^{n}\right)\left(\gamma^{\prime} \gamma\right) \in \Gamma_{m}^{\text {win }}$, by swapping the order of $\gamma$ and $\gamma^{n}\left(\gamma^{\prime}\right)^{n+1}$. By induction hypothesis, $\gamma^{n}\left(\gamma^{\prime}\right)^{n} \in \Gamma_{m}^{\text {win }}$; by hypothesis and by Lemma $24 \gamma^{\prime} \gamma \in \Gamma_{m}^{\text {win }}$. Therefore, by $\mathcal{M}$-cycle-consistency, $\left(\gamma^{n}\left(\gamma^{\prime}\right)^{n}\right)\left(\gamma^{\prime} \gamma\right)$ is also in $\Gamma_{m}^{\text {win }}$.

We now define $\Lambda=\gamma^{+}$and $\Lambda^{\prime}=\left(\gamma^{\prime}\right)^{+}$. We have that for all elements $\left(\gamma^{\prime}\right)^{n}$ of $\Lambda^{\prime}$ (with $n \geq 1$ ), we have that $\gamma^{n}$ (an element of $\Lambda$ ) is such that $\gamma^{n}\left(\gamma^{\prime}\right)^{n} \in \Gamma_{m}^{\text {win }}$. Therefore the
hypothesis of Lemma 25 is verified for $\Lambda$ and $\Lambda^{\prime}$, which implies that there exists $n \geq 1$ such that $\gamma^{n}\left(\gamma^{\prime}\right)^{+} \subseteq \Gamma_{m}^{\text {win }}$.

We assume w.l.o.g. that $n=\min \left\{n \in \mathbb{N} \mid \gamma^{n}\left(\gamma^{\prime}\right)^{+} \subseteq \Gamma_{m}^{\text {win }}\right\}$. For all $k, k \geq n$, we also have that $\gamma^{k}\left(\gamma^{\prime}\right)^{+}=\gamma^{k-n}\left(\gamma^{n}\left(\gamma^{\prime}\right)^{+}\right) \subseteq \Gamma_{m}^{\text {win }}$ by $\mathcal{M}$-cycle-consistency. We intend to show that $n=1$, which would end the proof of the lemma as this would show that $\gamma^{1}\left(\gamma^{\prime}\right)^{+}=\gamma\left(\gamma^{\prime}\right)^{+} \subseteq \Gamma_{m}^{\text {win }}$.

We assume by contradiction that $n>1$. Then there must exist $k \in \mathbb{N}$ such that $\gamma^{n-1}\left(\gamma^{\prime}\right)^{k} \in \Gamma_{m}^{\text {lose }}$. We also have that $\left(\gamma^{\prime}\right)^{k} \gamma^{n-1}$ is in $\Gamma_{m}^{\text {lose }}$ by Lemma 24 which implies that $\gamma^{n-1}\left(\gamma^{\prime}\right)^{k}\left(\gamma^{\prime}\right)^{k} \gamma^{n-1}$ is also in $\Gamma_{m}^{\text {lose }}$ by $\mathcal{M}$-cycle-consistency. But then by Lemma 24 this cycle has the same value as $\gamma^{2 n-2}\left(\gamma^{\prime}\right)^{2 k}$, which must therefore be in $\Gamma_{m}^{\text {lose }}$. This is a contradiction since $n>1$ implies that $2 n-2 \geq n$.

We conclude that $\gamma\left(\gamma^{\prime}\right)^{+} \subseteq \Gamma_{m}^{\text {win }}$.

Thanks to this result, we can now show that any two consecutive cycles can always be swapped without changing the value of a longer cycle.

- Corollary 27 (Cycle-order independence). Let $m \in M$. Let $\gamma_{1}, \gamma_{2}, \gamma_{3} \in \Gamma_{m}$. Then, $\operatorname{val}\left(\gamma_{1} \gamma_{2} \gamma_{3}\right)=\operatorname{val}\left(\gamma_{1} \gamma_{3} \gamma_{2}\right)$.

Proof. We assume by contradiction that cycles $\gamma_{1} \gamma_{2} \gamma_{3}$ and $\gamma_{1} \gamma_{3} \gamma_{2}$ have a different value; w.l.o.g., that $\gamma_{1} \gamma_{2} \gamma_{3} \in \Gamma_{m}^{w i n}$ and that $\gamma_{1} \gamma_{3} \gamma_{2} \in \Gamma_{m}^{\text {lose }}$. By $\mathcal{M}$-cycle-consistency, at least one cycle among $\gamma_{1}, \gamma_{2}$ and $\gamma_{3}$ is winning and one is losing. We assume w.l.o.g. that $\gamma_{1} \in \Gamma_{m}^{\text {win }}$ and $\gamma_{2} \in \Gamma_{m}^{\text {lose }}$. We also assume that $\gamma_{3} \in \Gamma_{m}^{\text {win }}$; the other case can be dealt with by symmetry. Notice that we necessarily have that $\gamma_{3} \gamma_{2}$ is in $\Gamma_{m}^{\text {lose }}$; otherwise, $\gamma_{1} \gamma_{3} \gamma_{2}=\gamma_{1}\left(\gamma_{3} \gamma_{2}\right)$ would be in $\Gamma_{m}^{w i n}$ by $\mathcal{M}$-cycle-consistency. For the same reason, $\gamma_{2} \gamma_{1}$ is in $\Gamma_{m}^{\text {lose }}$. We have

$$
\begin{aligned}
\operatorname{win} & =\operatorname{val}\left(\gamma_{1} \gamma_{2} \gamma_{3}\right) & & \text { by hypothesis } \\
& =\operatorname{val}\left(\left(\gamma_{3} \gamma_{1}\right) \gamma_{2}\right) & & \text { by Lemma } 24 \\
& =\operatorname{val}\left(\left(\gamma_{3} \gamma_{1}\right)\left(\gamma_{2}\right)^{2}\right) & & \text { by Lemma } 26 \\
& =\operatorname{val}\left(\gamma_{2} \gamma_{3} \gamma_{1} \gamma_{2}\right) & & \text { by Lemma } 24
\end{aligned}
$$

However, this last cycle can be written as a combination of two losing cycles ( $\gamma_{2} \gamma_{3}$ ) and $\left(\gamma_{1} \gamma_{2}\right)$, and should therefore be losing by $\mathcal{M}$-cycle-consistency. This is a contradiction.

Combining cycles on different skeleton states. We can now also strengthen Lemma 26 ("repetition independence") to show that even "non-consecutive subcycles" in a longer cycle can be repeated without affecting the value of the long cycle.

- Corollary 28 (Repetition independence, strong version). Let $m_{1}, m_{2} \in M$. Let $\gamma_{1} \in \Gamma_{m_{1}}$, $\gamma_{2} \in \Gamma_{m_{2}}, \bar{\gamma}_{1} \in \Pi_{m_{1}, m_{2}}$, and $\bar{\gamma}_{2} \in \Pi_{m_{2}, m_{1}}$. Then, $\operatorname{val}\left(\gamma_{1} \bar{\gamma}_{1} \gamma_{2} \bar{\gamma}_{2}\right)=\operatorname{val}\left(\gamma_{1}\left(\bar{\gamma}_{1} \bar{\gamma}_{2}\right)^{n} \bar{\gamma}_{1} \gamma_{2} \bar{\gamma}_{2}\right)$ for all $n \geq 0$.

The situation is depicted in Figure5. Notice first that we can see $\gamma_{1} \bar{\gamma}_{1} \gamma_{2} \bar{\gamma}_{2}$ as a combination of two cycles $\gamma_{1}$ and $\bar{\gamma}_{1} \gamma_{2} \bar{\gamma}_{2}$ on $m_{1}$, we therefore already know that the value of $\gamma_{1} \bar{\gamma}_{1} \gamma_{2} \bar{\gamma}_{2}$ on $m_{1}$ is the same as the one of $\left(\gamma_{1}\right)^{k}\left(\bar{\gamma}_{1} \gamma_{2} \bar{\gamma}_{2}\right)^{l}$ for all $k, l \geq 1$. This second cycle can be seen as two cycles $\bar{\gamma}_{2} \bar{\gamma}_{1}$ and $\gamma_{2}$ on $m_{2}$, we therefore know that the value of $\bar{\gamma}_{1} \gamma_{2} \bar{\gamma}_{2}$ on $m_{2}$ is the same as the one of $\left(\gamma_{2}\right)^{k}\left(\bar{\gamma}_{2} \bar{\gamma}_{1}\right)^{l}$ for all $k, l \geq 1$. However, these two facts do not directly give the result as cycle $\bar{\gamma}_{1} \bar{\gamma}_{2}$ does not appear "consecutively" in $\gamma_{1} \bar{\gamma}_{1} \gamma_{2} \bar{\gamma}_{2}$.


Figure 5 Depiction of the statement of Corollary 28 (left) and Lemma 29 (right).

Proof. We have that

$$
\begin{aligned}
\operatorname{val}\left(\gamma_{1} \bar{\gamma}_{1} \gamma_{2} \bar{\gamma}_{2}\right) & =\operatorname{val}\left(\gamma_{1}\left(\bar{\gamma}_{1} \gamma_{2} \bar{\gamma}_{2}\right)\left(\bar{\gamma}_{1} \gamma_{2} \bar{\gamma}_{2}\right)\right) & & \text { by Lemma } 26 \text { on } m_{1} \\
& =\operatorname{val}\left(\gamma_{1} \bar{\gamma}_{1}\left(\gamma_{2}\right)\left(\bar{\gamma}_{2} \bar{\gamma}_{1}\right) \gamma_{2} \bar{\gamma}_{2}\right) & & \\
& =\operatorname{val}\left(\gamma_{1} \bar{\gamma}_{1}\left(\bar{\gamma}_{2} \bar{\gamma}_{1}\right)\left(\gamma_{2}\right) \gamma_{2} \bar{\gamma}_{2}\right) & & \text { by Lemma } 27 \text { on } m_{2} \\
& =\operatorname{val}\left(\gamma_{1}\left(\bar{\gamma}_{1} \bar{\gamma}_{2}\right) \bar{\gamma}_{1}\left(\gamma_{2}\right)^{2} \bar{\gamma}_{2}\right) & & \\
& =\operatorname{val}\left(\gamma_{1}\left(\bar{\gamma}_{1} \bar{\gamma}_{2}\right) \bar{\gamma}_{1} \gamma_{2} \bar{\gamma}_{2}\right) & & \text { by Lemma } 26 \text { on } m_{2} .
\end{aligned}
$$

This shows the result for $n=1$; applying Lemma 26 gives the result for all $n \geq 1$.
Another important property that will help define an interesting preorder on cycles is that the value of a combination of two cycles is independent from the skeleton state chosen to compare pairs of cycles: if two cycles both go through two different states $m_{1}$ and $m_{2}$ of $\mathcal{M}$, then combining them around $m_{1}$ or around $m_{2}$ yields the same value.

- Lemma 29 (Crossing-point independence). Let $m_{1}, m_{2} \in M$ be two states of $\mathcal{M}$. Let $\pi_{1}, \pi_{1}^{\prime} \in \Pi_{m_{1}, m_{2}}$ and $\pi_{2}, \pi_{2}^{\prime} \in \Pi_{m_{2}, m_{1}}$. We have that $\operatorname{val}\left(\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}^{\prime}\right)=\operatorname{val}\left(\pi_{2} \pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime}\right)$.

The intuition of this lemma is that if we take two cycles (in the statement, $\pi_{1} \pi_{2}$ and $\pi_{1}^{\prime} \pi_{2}^{\prime}$ ) that have two common states $\left(m_{1}\right.$ and $\left.m_{2}\right)$, the chosen starting state to combine the two cycles (the combination is $\left(\pi_{1} \pi_{2}\right)\left(\pi_{1}^{\prime} \pi_{2}^{\prime}\right)$ if $m_{1}$ is chosen, and $\left(\pi_{2} \pi_{1}\right)\left(\pi_{2}^{\prime} \pi_{1}^{\prime}\right)$ if $m_{2}$ is chosen) has no impact on the value of the combination. This situation is depicted in Figure 5

Proof. If $\pi_{1} \pi_{2}$ and $\pi_{1}^{\prime} \pi_{2}^{\prime}$ are both in $\Gamma_{m_{1}}^{\text {win }}$ or both in $\Gamma_{m_{1}}^{\text {lose }}$, then $\pi_{2} \pi_{1}$ and $\pi_{2}^{\prime} \pi_{1}^{\prime}$ are also respectively both in $\Gamma_{m_{2}}^{\text {win }}$ or both in $\Gamma_{m_{2}}^{\text {lose }}$ by Lemma 24 Therefore, we have our result using $\mathcal{M}$-cycle-consistency.

For the remaining cases, we assume w.l.o.g. that $\pi_{1} \pi_{2} \in \Gamma_{m_{1}}^{\text {win }}$ and $\pi_{1}^{\prime} \pi_{2}^{\prime} \in \Gamma_{m_{2}}^{\text {lose }}$. We will assume (again w.l.o.g.) that combining them is winning, i.e., that $\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}^{\prime} \in \Gamma_{m_{1}}^{\text {win }}$. Our goal is to show that $\pi_{2} \pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime}$ is also in $\Gamma_{m_{2}}^{w i n}$. We assume by contradiction that it is not, i.e., that $\pi_{2} \pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime} \in \Gamma_{m_{2}}^{\text {lose }}$.

Observe that as $\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}^{\prime} \in \Gamma_{m_{1}}^{\text {win }}$, we also have $\left(\pi_{2} \pi_{1}^{\prime}\right)\left(\pi_{2}^{\prime} \pi_{1}\right) \in \Gamma_{m_{2}}^{\text {win }}$ by Lemma 24 Hence, at least one of $\pi_{2} \pi_{1}^{\prime}$ or $\pi_{2}^{\prime} \pi_{1}$ must be a winning cycle on $m_{2}$, otherwise their combination would be losing on $m_{2}$ by $\mathcal{M}$-cycle-consistency. Equivalently, by Lemma 24 at least one of $\pi_{1}^{\prime} \pi_{2}$ or $\pi_{1} \pi_{2}^{\prime}$ must be a winning cycle on $m_{1}$.

Similarly, as $\pi_{2} \pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime}$ is in $\Gamma_{m_{2}}^{\text {lose }}$, we have that $\left(\pi_{1} \pi_{2}^{\prime}\right)\left(\pi_{1}^{\prime} \pi_{2}\right)$ is in $\Gamma_{m_{1}}^{\text {lose }}$. Hence, at least one of $\pi_{1} \pi_{2}^{\prime}$ and $\pi_{1}^{\prime} \pi_{2}$ is a losing cycle on $m_{1}$ by $\mathcal{M}$-cycle-consistency.

Our conclusions imply that exactly one of $\pi_{1} \pi_{2}^{\prime}$ and $\pi_{1}^{\prime} \pi_{2}$ is winning on $m_{1}$, and one is losing on $m_{1}$. Without loss of generality, we assume that $\pi_{1} \pi_{2}^{\prime} \in \Gamma_{m_{1}}^{\text {win }}$ and $\pi_{1}^{\prime} \pi_{2} \in \Gamma_{m_{1}}^{\text {lose }}$.

We now have a value for all four two-word cycles on $m_{1}$ (and therefore for all four two-word cycles on $m_{2}$ by Lemma 24: $\pi_{1} \pi_{2}$ and $\pi_{1} \pi_{2}^{\prime}$ are in $\Gamma_{m_{1}}^{\text {win }}$, and $\pi_{1}^{\prime} \pi_{2}$ and $\pi_{1}^{\prime} \pi_{2}^{\prime}$ are in $\Gamma_{m_{1}}^{\text {lose }}$. If we look at four-word cycles, we have already assumed w.l.o.g. that $\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}^{\prime} \in \Gamma_{m_{1}}^{\text {win }}$ and
that $\pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime} \pi_{2} \in \Gamma_{m_{1}}^{\text {lose }}$. We still do not know whether $\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}$ and $\pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime} \pi_{2}^{\prime}$ are winning or losing - no matter how we express them as two two-word cycles, one two-word cycle is winning and the other one is losing. We study the value of these two four-word cycles.

Consider the cycle $\left(\pi_{2} \pi_{1}^{\prime} \pi_{2}^{\prime} \pi_{1}\right)\left(\pi_{2}^{\prime} \pi_{1} \pi_{2} \pi_{1}^{\prime}\right)$ on $m_{2}$. It is winning, since $\pi_{2} \pi_{1}^{\prime} \pi_{2}^{\prime} \pi_{1}$ and $\pi_{2}^{\prime} \pi_{1} \pi_{2} \pi_{1}^{\prime}$ are both in $\Gamma_{m_{2}}^{\mathrm{win}}$ : this can be shown using Lemma 24 and the fact that $\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}^{\prime}$ is in $\Gamma_{m_{1}}^{\text {win }}$. Therefore, by shifting the start of the cycle, $\left(\pi_{1}^{\prime} \pi_{2}^{\prime} \pi_{1} \pi_{2}^{\prime}\right)\left(\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}\right)$ is in $\Gamma_{m_{1}}^{\text {win }}$. By $\mathcal{M}$-cycle-consistency, this means that at least one of $\pi_{1}^{\prime} \pi_{2}^{\prime} \pi_{1} \pi_{2}^{\prime}$ (equivalently, $\pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime} \pi_{2}^{\prime}$ ) and $\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}$ is winning on $m_{1}$.

Similarly, we have that the cycle $\left(\pi_{2} \pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime}\right)\left(\pi_{2}^{\prime} \pi_{1}^{\prime} \pi_{2} \pi_{1}\right)$ is in $\Gamma_{m_{2}}^{\text {lose }}$. Therefore, by shifting the start of the cycle, $\left(\pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime} \pi_{2}^{\prime}\right)\left(\pi_{1}^{\prime} \pi_{2} \pi_{1} \pi_{2}\right)$ is in $\Gamma_{m_{1}}^{\text {lose }}$. This means that at least one of $\pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime} \pi_{2}^{\prime}$ or $\pi_{1}^{\prime} \pi_{2} \pi_{1} \pi_{2}$ (equivalently, $\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}$ ) is in $\Gamma_{m_{1}}^{\text {lose }}$.

Our conclusions imply that exactly one of $\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}$ and $\pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime} \pi_{2}^{\prime}$ is winning on $m_{1}$, and one is losing on $m_{1}$. We consider both cases and draw a contradiction in each case.

Assume that $\operatorname{val}\left(\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}\right)=$ lose. Now consider the cycle $\nu=\left(\pi_{2}^{\prime} \pi_{1}\right)\left(\pi_{2} \pi_{1}^{\prime}\right)^{2}$ on $m_{2}$. We have

$$
\begin{aligned}
\operatorname{val}(\nu) & =\operatorname{val}\left(\pi_{2}^{\prime} \pi_{1} \pi_{2} \pi_{1}^{\prime}\right) \\
& =\operatorname{val}\left(\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}^{\prime}\right) \\
& =\operatorname{win}
\end{aligned}
$$

by Lemma 26
by Lemma 24
by hypothesis.

However, we also have

$$
\operatorname{val}(\nu)=\operatorname{val}\left(\left(\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}\right)\left(\pi_{1}^{\prime} \pi_{2}^{\prime}\right)\right) \quad \text { by Lemma } 24
$$

and since $\operatorname{val}\left(\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}\right)=\operatorname{val}\left(\pi_{1}^{\prime} \pi_{2}^{\prime}\right)=$ lose, we also have $\operatorname{val}(\nu)=$ lose by $\mathcal{M}$-cycle-consistency. This is a contradiction.

Assume now that $\operatorname{val}\left(\pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime} \pi_{2}^{\prime}\right)=$ lose. Now consider the cycle $\nu=\left(\pi_{1} \pi_{2}\right)\left(\pi_{1}^{\prime} \pi_{2}^{\prime}\right)^{2}$ on $m_{1}$. We have

$$
\operatorname{val}(\nu)=\operatorname{val}\left(\pi_{1} \pi_{2} \pi_{1}^{\prime} \pi_{2}^{\prime}\right)
$$

$$
=\text { win } \quad \text { by hypothesis }
$$

However, we also have

$$
\operatorname{val}(\nu)=\operatorname{val}\left(\left(\pi_{2} \pi_{1}^{\prime}\right)\left(\pi_{2}^{\prime} \pi_{1}^{\prime} \pi_{2}^{\prime} \pi_{1}\right)\right) \quad \text { by Lemma } 24
$$

and since $\operatorname{val}\left(\pi_{1}^{\prime} \pi_{2}\right)=\operatorname{val}\left(\pi_{1} \pi_{2}^{\prime} \pi_{1}^{\prime} \pi_{2}^{\prime}\right)=$ lose, we also have $\operatorname{val}(\nu)=$ lose by $\mathcal{M}$-cycle-consistency. This is a contradiction.

- Remark 30. A consequence of the previous lemma is that when two cycles $\gamma, \gamma^{\prime} \in \Gamma_{\mathcal{M}}$ share at least one common state (i.e., $\operatorname{st}(\gamma) \cap \operatorname{st}\left(\gamma^{\prime}\right) \neq \emptyset$ ), we can write $\gamma \gamma^{\prime}$ for any cycle that, starting from a common state, sees first $\gamma$ and then $\gamma^{\prime}$, without necessarily specifying on which common state the cycle starts; we allow such a shortcut as the value of $\gamma \gamma^{\prime}$ is not impacted by the choice of the common skeleton state. This convention is used in the following definition.


## Competing cycles.

Definition 31. Let $\gamma, \gamma^{\prime} \in \Gamma_{\mathcal{M}}$ with $\operatorname{val}(\gamma) \neq \operatorname{val}\left(\gamma^{\prime}\right)$. We say that $\gamma$ and $\gamma^{\prime}$ are competing if there exists $\bar{\gamma} \in \Gamma_{\mathcal{M}}$ such that $\operatorname{st}(\gamma) \cap \operatorname{st}(\bar{\gamma}) \neq \emptyset$, $\operatorname{st}\left(\gamma^{\prime}\right) \cap \operatorname{st}(\bar{\gamma}) \neq \emptyset$, $\operatorname{val}(\gamma \bar{\gamma})=\operatorname{val}(\gamma)$, and $\operatorname{val}\left(\gamma^{\prime} \bar{\gamma}\right)=\operatorname{val}\left(\gamma^{\prime}\right)$. In this case, we say that $\bar{\gamma}$ is a witness that $\gamma$ and $\gamma^{\prime}$ are competing, or that the competition of $\gamma$ and $\gamma^{\prime}$ is witnessed by $\bar{\gamma}$.


Figure 6 Situation in the proof of Lemma 32

Our requirement for cycle $\bar{\gamma}$ means that it intersects the states of both $\gamma$ and $\gamma^{\prime}$, but is not influential enough to "alter" the values of $\gamma$ and $\gamma^{\prime}$ when it is combined with them. If $\operatorname{val}(\gamma) \neq \operatorname{val}\left(\gamma^{\prime}\right)$ and $\operatorname{st}(\gamma) \cap \operatorname{st}\left(\gamma^{\prime}\right) \neq \emptyset$, then if $\operatorname{val}\left(\gamma \gamma^{\prime}\right)=\operatorname{val}(\gamma)\left(\right.$ resp. $\left.\operatorname{val}\left(\gamma \gamma^{\prime}\right)=\operatorname{val}\left(\gamma^{\prime}\right)\right)$, we have that $\gamma^{\prime}$ (resp. $\gamma$ ) witnesses that $\gamma$ and $\gamma^{\prime}$ are competing (the argument uses Lemma 26). In short, any two cycles of opposite values that share a common state are competing, and two cycles of opposite values that do not share a common state may or may not be competing.

If two cycles are competing, we want to determine which one dominates the other. For $\bar{\gamma}$ a witness, for some $m \in \operatorname{st}(\gamma)$ and $m^{\prime} \in \operatorname{st}\left(\gamma^{\prime}\right)$, it is possible to decompose $\bar{\gamma}$ as two (possibly empty) paths $\bar{\gamma}_{1}$ and $\bar{\gamma}_{2}$ such that $\bar{\gamma}=\bar{\gamma}_{1} \bar{\gamma}_{2}, \bar{\gamma}_{1} \in \Pi_{m, m^{\prime}}$, and $\bar{\gamma}_{2} \in \Pi_{m^{\prime}, m}$. We define that $\gamma$ dominates $\gamma^{\prime}$ if $\operatorname{val}\left(\gamma \bar{\gamma}_{1} \gamma^{\prime} \bar{\gamma}_{2}\right)=\operatorname{val}(\gamma)$, and $\gamma^{\prime}$ dominates $\gamma$ if $\operatorname{val}\left(\gamma \bar{\gamma}_{1} \gamma^{\prime} \bar{\gamma}_{2}\right)=\operatorname{val}\left(\gamma^{\prime}\right)$. However, to be well-defined, the domination notion needs to be independent from the choice of witness.

- Lemma 32 (Witness independence). Let $\gamma, \gamma^{\prime} \in \Gamma_{\mathcal{M}}$ with $\operatorname{val}(\gamma) \neq \operatorname{val}\left(\gamma^{\prime}\right)$. Let $\bar{\gamma}_{1}, \bar{\gamma}_{2} \in \Gamma_{\mathcal{M}}$ be two witnesses that $\gamma$ and $\gamma^{\prime}$ are competing. Then, $\gamma$ dominates $\gamma^{\prime}$ taking $\bar{\gamma}_{1}$ as a witness if and only if $\gamma$ dominates $\gamma^{\prime}$ taking $\bar{\gamma}_{2}$ as a witness.

Proof. We assume w.l.o.g. $\operatorname{val}(\gamma)=\mathrm{win}$ and $\operatorname{val}\left(\gamma^{\prime}\right)=$ lose. As $\bar{\gamma}_{1}$ witnesses that $\gamma$ and $\gamma^{\prime}$ are competing, there exists $m_{1} \in \operatorname{st}(\gamma)$, $m_{1}^{\prime} \in \operatorname{st}\left(\gamma^{\prime}\right)$ such that $\bar{\gamma}_{1}=\bar{\gamma}_{1,1} \bar{\gamma}_{1,2}$ with $\bar{\gamma}_{1,1} \in \Pi_{m_{1}, m_{1}^{\prime}}$, $\bar{\gamma}_{1,2} \in \Pi_{m_{1}^{\prime}, m_{1}}$. Similarly, as $\bar{\gamma}_{2}$ witnesses that $\gamma$ and $\gamma^{\prime}$ are competing, there exists $m_{2} \in \operatorname{st}(\gamma)$, $m_{2}^{\prime} \in \operatorname{st}\left(\gamma^{\prime}\right)$ such that $\bar{\gamma}_{2}=\bar{\gamma}_{2,1} \bar{\gamma}_{2,2}$ with $\bar{\gamma}_{2,1} \in \Pi_{m_{2}, m_{2}^{\prime}}, \bar{\gamma}_{2,2} \in \Pi_{m_{2}^{\prime}, m_{2}}$. We can also write $\gamma=\gamma_{1} \gamma_{2}$ with $\gamma_{1} \in \Pi_{m_{1}, m_{2}}$ and $\gamma_{2} \in \Pi_{m_{2}, m_{1}}$, and $\gamma^{\prime}=\gamma_{1}^{\prime} \gamma_{2}^{\prime}$ with $\gamma_{1} \in \Pi_{m_{1}^{\prime}, m_{2}^{\prime}}$ and $\gamma_{2}^{\prime} \in \Pi_{m_{2}^{\prime}, m_{1}^{\prime}}$.

The situation is depicted in Figure 6 Note that it is possible that $m_{1}=m_{2}$ (in which case we can assume $\left.\gamma=\gamma_{1}, \gamma_{2}=\left(m_{1}, \perp\right)\right)$ or similarly that $m_{1}=m_{1}^{\prime}, m_{1}^{\prime}=m_{2}^{\prime}$, and/or $m_{2}=m_{2}^{\prime}$.

We assume by contradiction that $\gamma$ dominates $\gamma^{\prime}$ taking $\bar{\gamma}_{1}$ as a witness, but that $\gamma^{\prime}$ dominates $\gamma$ taking $\bar{\gamma}_{2}$ as a witness. In other words, $\operatorname{val}\left(\gamma_{1} \gamma_{2} \bar{\gamma}_{1,1} \gamma_{1}^{\prime} \gamma_{2}^{\prime} \bar{\gamma}_{1,2}\right)=$ win and $\operatorname{val}\left(\gamma_{2} \gamma_{1} \bar{\gamma}_{2,1} \gamma_{2}^{\prime} \gamma_{1}^{\prime} \bar{\gamma}_{2,2}\right)=$ lose. We consider the concatenation of both these cycles, shifting the second one to make it a cycle on $m_{1}$,

$$
\nu=\left(\gamma_{1} \gamma_{2} \bar{\gamma}_{1,1} \gamma_{1}^{\prime} \gamma_{2}^{\prime} \bar{\gamma}_{1,2}\right)\left(\gamma_{1} \bar{\gamma}_{2,1} \gamma_{2}^{\prime} \gamma_{1}^{\prime} \bar{\gamma}_{2,2} \gamma_{2}\right) .
$$

It is possible to express $\nu$ directly as a combination of two losing cycles: $\bar{\gamma}_{1,1} \gamma_{1}^{\prime} \gamma_{2}^{\prime} \bar{\gamma}_{1,2}$ is losing (by definition of witness), and

$$
\begin{array}{rlrl}
\operatorname{val}\left(\gamma_{1} \bar{\gamma}_{2,1} \gamma_{2}^{\prime} \gamma_{1}^{\prime}\right. & \left.\bar{\gamma}_{2,2} \gamma_{2} \gamma_{1} \gamma_{2}\right) & \\
& =\operatorname{val}\left(\bar{\gamma}_{2,1} \gamma_{2}^{\prime} \gamma_{1}^{\prime} \bar{\gamma}_{2,2}\left(\gamma_{2} \gamma_{1}\right)^{2}\right) & & \text { by Lemma } 24 \\
& =\operatorname{val}\left(\bar{\gamma}_{2,1} \gamma_{2}^{\prime} \gamma_{1}^{\prime} \bar{\gamma}_{2,2} \gamma_{2} \gamma_{1}\right) & & \text { by Lemma } 26 \\
& =\text { lose } & & \text { as } \gamma^{\prime} \text { dominates } \gamma \text { taking } \bar{\gamma}_{2} \text { as a witness. }
\end{array}
$$




Figure 7 A parity automaton ( $\mathcal{M}, p$ ) (left) with $C=\{a, b, c\}$ used in Example 33 Notation $c \mid k$ on a transition from a state $m$ means that $p(m, c)=k$. A diagram (right) showing the relations between the elements of $\Gamma_{\mathcal{M}} / \simeq$ ordered by partial preorder $\triangleleft$, discussed in Example 36

Cycle $\nu$ is therefore losing by $\mathcal{M}$-cycle-consistency.
Now, notice that $\nu$ can be written as three cycles on $m_{2}^{\prime}$ after being shifted in the following way:

$$
\operatorname{val}(\nu)=\operatorname{val}\left(\left(\gamma_{2}^{\prime} \bar{\gamma}_{1,2} \gamma_{1} \bar{\gamma}_{2,1}\right)\left(\gamma_{2}^{\prime} \gamma_{1}^{\prime}\right)\left(\bar{\gamma}_{2,2} \gamma_{2} \gamma_{1} \gamma_{2} \bar{\gamma}_{1,1} \gamma_{1}^{\prime}\right)\right)
$$

By Lemma 27("cycle-order independence"), it has the same value as

$$
\operatorname{val}(\nu)=\operatorname{val}\left(\left(\gamma_{2}^{\prime} \bar{\gamma}_{1,2} \gamma_{1} \bar{\gamma}_{2,1}\right)\left(\bar{\gamma}_{2,2} \gamma_{2} \gamma_{1} \gamma_{2} \bar{\gamma}_{1,1} \gamma_{1}^{\prime}\right)\left(\gamma_{2}^{\prime} \gamma_{1}^{\prime}\right)\right)
$$

As before, this cycle can be shifted and written as two winning cycles $\gamma_{1} \bar{\gamma}_{2,1} \bar{\gamma}_{2,2} \gamma_{2}$ and $\gamma_{1} \gamma_{2} \bar{\gamma}_{1,1}\left(\gamma_{1}^{\prime} \gamma_{2}^{\prime}\right)^{2} \bar{\gamma}_{1,2}$ and is therefore winning by $\mathcal{M}$-cycle-consistency.

Cycle $\nu$ is both winning and losing, a contradiction.

- Example 33. We illustrate competition and domination of cycles on a parity automaton (even though at this point in the proof, we have not yet shown that $W$ is $\omega$-regular). We consider the parity automaton $(\mathcal{M}, p)$ from Figure 7. with winning condition $W=L_{(\mathcal{M}, p)}$. Condition $W$ is $\mathcal{M}$-prefix-independent and $\mathcal{M}$-cycle-consistent (Lemma 9). We give a few examples of competition and domination between cycles. The winning cycle $\left(m_{1}, a\right)\left(m_{2}, a\right)$ dominates losing cycle $\left(m_{1}, b\right)$ but is dominated by losing cycle $\left(m_{1}, c\right)$. Cycle $\left(m_{2}, b\right)$ is winning but is not competing with $\left(m_{1}, b\right)$. In particular, their competition is not witnessed by $\left(m_{1}, a\right)\left(m_{2}, a\right)$ since combining it with $\left(m_{1}, b\right)$ alters its value $\left(\left(m_{1}, b\right)\right.$ is losing but $\left(m_{1}, b\right)\left(m_{1}, a\right)\left(m_{2}, a\right)$ is winning $)$ - another potential witness is $\left(m_{1}, c\right)\left(m_{1}, a\right)\left(m_{2}, a\right)$, but combining it with $\left(m_{2}, b\right)$ alters its value. However, cycle $\left(m_{2}, b\right)$ is competing with and dominated by $\left(m_{1}, c\right)$ : their competition is witnessed, e.g., by $\left(m_{1}, a\right)\left(m_{2}, a\right)$ and by $\left(m_{1}, b\right)\left(m_{1}, a\right)\left(m_{2}, a\right)$.

Preorder on cycles. For a winning (resp. losing) cycle $\gamma \in \Gamma_{\mathcal{M}}$, we define $\operatorname{comp}(\gamma)$ as the set of losing (resp. winning) cycles that $\gamma$ is competing with, and domBy $(\gamma)$ as the set of losing (resp. winning) cycles dominated by $\gamma$.

We define an ordering $\triangleleft$ of cycles based on these notions. We write $\gamma^{\prime} \triangleleft \gamma$ if $\gamma^{\prime} \in \operatorname{domBy}(\gamma)$. We extend this definition to cycles with the same value: if $\operatorname{val}\left(\gamma_{1}\right)=\operatorname{val}\left(\gamma_{2}\right)$, we write $\gamma_{2} \triangleleft \gamma_{1}$ if there exists a cycle $\gamma^{\prime}$ such that $\operatorname{val}\left(\gamma^{\prime}\right) \neq \operatorname{val}\left(\gamma_{1}\right)$ with $\gamma_{2} \in \operatorname{domBy}\left(\gamma^{\prime}\right)$ and $\gamma^{\prime} \in \operatorname{domBy}\left(\gamma_{1}\right)$ - intuitively, this condition implies that $\gamma_{2}$ is less powerful than $\gamma_{1}$ as we can find a cycle dominating $\gamma_{2}$ that is itself dominated by $\gamma_{1}$. We show that relation $\triangleleft$ is a strict preorder (which is not total in general).

- Lemma 34. Relation $\triangleleft$ is a strict preorder.

Proof. We first prove that $\triangleleft$ is irreflexive, i.e., that for all $\gamma \in \Gamma_{\mathcal{M}}, \gamma \nexists \gamma$. If $\gamma \triangleleft \gamma$, since $\operatorname{val}(\gamma)=\operatorname{val}(\gamma)$, there exists $\gamma^{\prime} \in \Gamma_{\mathcal{M}}$ such that $\operatorname{val}\left(\gamma^{\prime}\right) \neq \operatorname{val}(\gamma), \gamma \in \operatorname{domBy}\left(\gamma^{\prime}\right)$, and


Figure 8 Situation to show transitivity of $\triangleleft$ in Lemma 34
$\gamma^{\prime} \in \operatorname{domBy}(\gamma)$. But that is not possible since when $\gamma$ and $\gamma^{\prime}$ are competing, they cannot both dominate the other (no matter the choice of witness, as shown in Lemma 32).

We now prove that $\triangleleft$ is transitive. We distinguish four cases (we rename cycles in each case to ease the reading by making it so that cycles with a prime symbol have a different value from cycles without a prime symbol).

If $\gamma_{2} \triangleleft \gamma^{\prime}$ and $\gamma^{\prime} \triangleleft \gamma_{1}$ with $\operatorname{val}\left(\gamma_{2}\right) \neq \operatorname{val}\left(\gamma^{\prime}\right)$ and $\operatorname{val}\left(\gamma^{\prime}\right) \neq \operatorname{val}\left(\gamma_{1}\right)$, then $\operatorname{val}\left(\gamma_{2}\right)=\operatorname{val}\left(\gamma_{1}\right)$, and $\gamma_{2} \triangleleft \gamma_{1}$ by definition.

Let $\gamma_{2}^{\prime} \triangleleft \gamma_{2}$ and $\gamma_{2} \triangleleft \gamma_{1}$ with $\operatorname{val}\left(\gamma_{2}^{\prime}\right) \neq \operatorname{val}\left(\gamma_{2}\right)$ and $\operatorname{val}\left(\gamma_{2}\right)=\operatorname{val}\left(\gamma_{1}\right)$. We assume w.l.o.g. that $\operatorname{val}\left(\gamma_{2}\right)=\operatorname{val}\left(\gamma_{1}\right)=$ win, so there exists $\gamma_{1}^{\prime} \in \Gamma_{\mathcal{M}}$ such that $\operatorname{val}\left(\gamma_{1}^{\prime}\right)=$ lose, $\gamma_{2} \triangleleft \gamma_{1}^{\prime}$ and $\gamma_{1}^{\prime} \triangleleft \gamma_{1}$. We assume that $\gamma_{1}^{\prime} \triangleleft \gamma_{1}$ is witnessed by $\bar{\gamma}$, that $\gamma_{2} \triangleleft \gamma_{1}^{\prime}$ is witnessed by $\bar{\gamma}^{\prime}$ and that $\gamma_{2}^{\prime} \triangleleft \gamma_{2}$ is witnessed by $\bar{\gamma}^{\prime \prime}$. We assume that $\bar{\gamma}=\bar{\gamma}_{1} \bar{\gamma}_{2}, \gamma_{1}^{\prime}=\gamma_{1,1}^{\prime} \gamma_{1,2}^{\prime}, \bar{\gamma}^{\prime}=\bar{\gamma}_{1}^{\prime} \bar{\gamma}_{2}^{\prime}, \gamma_{2}=\gamma_{2,1} \gamma_{2,2}$, and $\bar{\gamma}^{\prime \prime}=\bar{\gamma}_{1}^{\prime \prime} \bar{\gamma}_{2}^{\prime \prime}$. We use Figure 8 to illustrate the situation and explain where the common states of these cycles are.

We want to show that $\gamma_{2}^{\prime} \triangleleft \gamma_{1}$. To do so, we show that for $\nu_{1}=\bar{\gamma}_{1} \gamma_{1,1}^{\prime} \bar{\gamma}_{1}^{\prime} \gamma_{2,1} \bar{\gamma}_{1}^{\prime \prime}$ and $\nu_{2}=\bar{\gamma}_{2}^{\prime \prime} \gamma_{2,2} \bar{\gamma}_{2}^{\prime} \gamma_{1,2}^{\prime} \bar{\gamma}_{2}, \nu=\nu_{1} \nu_{2}$ is a witness that $\gamma_{1}$ and $\gamma_{2}^{\prime}$ are competing, and that $\gamma_{1} \nu_{1} \gamma_{2}^{\prime} \nu_{2}$ is winning:

- Cycle $\gamma_{1} \nu$ is winning. We can split this cycle into $\gamma_{1,2}^{\prime} \bar{\gamma}_{2} \gamma_{1} \bar{\gamma}_{1} \gamma_{1,1}^{\prime}$ and $\bar{\gamma}_{1}^{\prime} \gamma_{2,1} \bar{\gamma}_{1}^{\prime \prime} \bar{\gamma}_{2}^{\prime \prime} \gamma_{2,2} \bar{\gamma}_{2}^{\prime}$. The former cycle can be shifted (Lemma 24) to $\gamma_{1} \bar{\gamma}_{1} \gamma_{1,1}^{\prime} \gamma_{1,2}^{\prime} \bar{\gamma}_{2}$, which is winning since $\gamma_{1}^{\prime} \triangleleft \gamma_{1}$. The latter cycle has the same value as $\bar{\gamma}_{1}^{\prime}\left(\gamma_{2,2} \gamma_{2,1}\right) \gamma_{2,1} \bar{\gamma}_{1}^{\prime \prime} \bar{\gamma}_{2}^{\prime \prime} \gamma_{2,2} \bar{\gamma}_{2}^{\prime}$ (Lemma 28), which can be shifted to $\left(\bar{\gamma}_{2}^{\prime} \bar{\gamma}_{1}^{\prime} \gamma_{2,2} \gamma_{2,1}\right)\left(\gamma_{2,1} \bar{\gamma}_{1}^{\prime \prime} \bar{\gamma}_{2}^{\prime \prime} \gamma_{2,2}\right)$. Both these cycles are winning since $\bar{\gamma}^{\prime}$ and $\bar{\gamma}^{\prime \prime}$ are witnesses for competitions involving $\gamma_{2}$.
- Cycle $\gamma_{2}^{\prime} \nu$ is losing. We can split this cycle into $\bar{\gamma}_{1}^{\prime \prime} \gamma_{2}^{\prime} \bar{\gamma}_{2}^{\prime \prime}$ (losing because $\bar{\gamma}^{\prime \prime}$ witnesses a competition involving $\gamma_{2}^{\prime}$ ) and $\gamma_{2,2} \bar{\gamma}_{2}^{\prime} \gamma_{1,2}^{\prime} \bar{\gamma}_{2} \bar{\gamma}_{1} \gamma_{1,1}^{\prime} \bar{\gamma}_{1}^{\prime} \gamma_{2,1}$. This latter cycle has the same value as the cycle $\gamma_{2,2} \bar{\gamma}_{2}^{\prime}\left(\gamma_{1,2}^{\prime} \gamma_{1,1}^{\prime}\right) \gamma_{1,2}^{\prime} \bar{\gamma}_{2} \bar{\gamma}_{1} \gamma_{1,1}^{\prime} \bar{\gamma}_{1}^{\prime} \gamma_{2,1}$ (Lemma 28, which can itself be split into $\gamma_{1,2}^{\prime} \bar{\gamma}_{2} \bar{\gamma}_{1} \gamma_{1,1}^{\prime}$ (losing because $\bar{\gamma}$ witnesses a competition involving $\gamma^{\prime}$ ) and $\bar{\gamma}_{2}^{\prime} \gamma_{1,2}^{\prime} \gamma_{1,1}^{\prime} \bar{\gamma}_{1}^{\prime} \gamma_{2,1} \gamma_{2,2}$ (losing because $\gamma_{2} \triangleleft \gamma_{1}^{\prime}$ ).
- Cycle $\gamma_{1} \nu_{1} \gamma_{2}^{\prime} \nu_{2}$ is winning. Using Lemma 28 we can show that $\gamma_{1} \nu_{1} \gamma_{2}^{\prime} \nu_{2}$ has the same value as $\gamma_{1} \nu_{1}\left(\bar{\gamma}_{2}^{\prime \prime} \gamma_{2,2} \gamma_{2,1} \bar{\gamma}_{1}^{\prime \prime}\right) \gamma_{2}^{\prime} \nu_{2}$. We can split this cycle into $\gamma_{1} \nu_{1} \nu_{2}=\gamma_{1} \nu$ (which is winning, as shown above) and $\bar{\gamma}_{2}^{\prime \prime} \gamma_{2,2} \gamma_{2,1} \bar{\gamma}_{1}^{\prime \prime} \gamma_{2}^{\prime}$ (winning since $\gamma_{2}^{\prime} \triangleleft \gamma_{2}$ ).
This shows that $\gamma_{2}^{\prime} \triangleleft \gamma_{1}$.
There are still two cases left to consider. The case $\gamma_{2}^{\prime} \triangleleft \gamma_{1}^{\prime}$ and $\gamma_{1}^{\prime} \triangleleft \gamma_{1}$ with $\operatorname{val}\left(\gamma_{2}^{\prime}\right)=\operatorname{val}\left(\gamma_{1}^{\prime}\right)$ and $\operatorname{val}\left(\gamma_{1}^{\prime}\right) \neq \operatorname{val}\left(\gamma_{1}\right)$ can be dealt with in the same way as the previous case (after noticing that there exists $\gamma_{2} \in \Gamma_{\mathcal{M}}$ such that $\gamma_{2}^{\prime} \triangleleft \gamma_{2}, \gamma_{2} \triangleleft \gamma_{1}^{\prime}$ and $\left.\operatorname{val}\left(\gamma_{2}\right)=\operatorname{val}\left(\gamma_{1}\right)\right)$.

If $\gamma_{3} \triangleleft \gamma_{2}$ and $\gamma_{2} \triangleleft \gamma_{1}$ with $\operatorname{val}\left(\gamma_{3}\right)=\operatorname{val}\left(\gamma_{2}\right)=\operatorname{val}\left(\gamma_{1}\right)$, then there exists in particular $\gamma^{\prime} \in \Gamma_{\mathcal{M}}$ such that $\gamma_{3} \triangleleft \gamma^{\prime}, \gamma^{\prime} \triangleleft \gamma_{2}$ and $\operatorname{val}\left(\gamma_{3}\right) \neq \operatorname{val}\left(\gamma^{\prime}\right)$. By a previous case, we conclude from $\gamma^{\prime} \triangleleft \gamma_{2}$ and $\gamma_{2} \triangleleft \gamma_{1}$ that $\gamma^{\prime} \triangleleft \gamma_{1}$. As $\gamma_{3} \triangleleft \gamma^{\prime}$ and $\gamma^{\prime} \triangleleft \gamma_{1}$, we have $\gamma_{3} \triangleleft \gamma_{1}$ as desired.

We define an equivalence relation on the cycles: we write $\gamma_{1} \simeq \gamma_{2}$ if $\operatorname{val}\left(\gamma_{1}\right)=\operatorname{val}\left(\gamma_{2}\right)$, $\operatorname{comp}\left(\gamma_{1}\right)=\operatorname{comp}\left(\gamma_{2}\right)$, and $\operatorname{domBy}\left(\gamma_{1}\right)=\operatorname{domBy}\left(\gamma_{2}\right)$. We show that cycles that are equivalent for $\simeq$ are in relation with the same elements for $\triangleleft$.

- Lemma 35. Let $\gamma_{1}, \gamma_{2}, \gamma^{\prime} \in \Gamma_{\mathcal{M}}$. If $\gamma_{1} \simeq \gamma_{2}$ and $\gamma^{\prime} \triangleleft \gamma_{1}$, then $\gamma^{\prime} \triangleleft \gamma_{2}$. If $\gamma_{1} \simeq \gamma_{2}$ and $\gamma_{1} \triangleleft \gamma^{\prime}$, then $\gamma_{2} \triangleleft \gamma^{\prime}$. In other words, preorder $\triangleleft$ is compatible with $\simeq$.

Proof. The first item is trivial, as the elements smaller than $\gamma_{1}$ for $\triangleleft$ are determined by $\operatorname{domBy}\left(\gamma_{1}\right)$, and $\operatorname{dom} \operatorname{By}\left(\gamma_{1}\right)=\operatorname{dom} \operatorname{By}\left(\gamma_{2}\right)$. For the second item, we distinguish two cases:

- if $\operatorname{val}\left(\gamma_{1}\right) \neq \operatorname{val}\left(\gamma^{\prime}\right)$, then $\gamma_{1} \triangleleft \gamma^{\prime}$ means that $\gamma^{\prime} \in \operatorname{comp}\left(\gamma_{1}\right)$ and $\gamma^{\prime} \notin \operatorname{domBy}\left(\gamma_{1}\right)$. If $\gamma_{1} \simeq \gamma_{2}$, the same properties also hold for $\gamma_{2}$, so $\gamma_{2} \triangleleft \gamma^{\prime}$.
- if $\operatorname{val}\left(\gamma_{1}\right)=\operatorname{val}\left(\gamma^{\prime}\right)$, then $\gamma_{1} \triangleleft \gamma^{\prime}$ means that there exists $\gamma^{\prime \prime}$ with $\operatorname{val}\left(\gamma_{1}\right) \neq \operatorname{val}\left(\gamma^{\prime \prime}\right)$ such that $\gamma_{1} \in \operatorname{domBy}\left(\gamma^{\prime \prime}\right)$ and $\gamma^{\prime \prime} \in \operatorname{domBy}\left(\gamma^{\prime}\right)$. By the previous case, if $\gamma_{1} \simeq \gamma_{2}$, then $\gamma_{2} \triangleleft \gamma^{\prime \prime}$, so $\gamma_{2} \triangleleft \gamma^{\prime}$ as well.
Partial preorder $\triangleleft$ therefore also induces a partial preorder on $\Gamma_{\mathcal{M}} / \simeq$.
Example 36. We represent the relations between all elements of $\Gamma_{\mathcal{M}} / \simeq$ for the parity automaton considered in Example 33 in their "Hasse diagram", depicted in Figure 7 Elements that are linked by a line segment are comparable for $\triangleleft$, and elements that are above are greater for $\triangleleft$. There are four equivalence classes of cycles, two of them winning and two of them losing. Notice for instance that any cycle going through transition $\left(m_{1}, c\right)$ is equivalent (for $\simeq$ ) to cycle ( $m_{1}, c$ ): indeed, it is necessarily a losing cycle competing with and dominating all the winning cycles in this skeleton. Other examples are given by $\left(m_{1}, a\right)\left(m_{2}, a\right) \simeq\left(m_{1}, a\right)\left(m_{2}, b\right)\left(m_{2}, a\right)$ and $\left(m_{2}, b\right) \simeq\left(m_{2}, c\right)$.

We now prove finiteness of the index of $\simeq$, by showing that

- the height of partial preorder $\triangleleft$ is finite, i.e., there is no infinite increasing nor decreasing sequence for $\triangleleft$ (Lemma 39);
- the width of partial preorder $\triangleleft$ on $\Gamma_{\mathcal{M}} / \simeq$ is finite, i.e., there is no infinite set of elements in $\Gamma_{\mathcal{M}} / \simeq$ that are all pairwise incomparable for $\triangleleft$ (Lemma 40 ).
We start with two technical lemmas about competition between cycles.
- Lemma 37. Let $\gamma_{1}, \gamma_{2}, \gamma^{\prime} \in \Gamma_{\mathcal{M}}$ be such that $\operatorname{val}\left(\gamma_{1}\right)=\operatorname{val}\left(\gamma_{2}\right) \neq \operatorname{val}\left(\gamma^{\prime}\right), \gamma_{2} \triangleleft \gamma^{\prime}$, and $\gamma^{\prime} \triangleleft \gamma_{1}$. Let $\bar{\gamma}$ be a witness that $\gamma_{2}$ and $\gamma^{\prime}$ are competing such that $\operatorname{st}(\bar{\gamma}) \cap \operatorname{st}\left(\gamma_{1}\right) \cap \operatorname{st}\left(\gamma_{2}\right) \neq \emptyset$. Then, $\bar{\gamma}$ also witnesses that $\gamma_{1}$ and $\gamma^{\prime}$ are competing.

Proof. We already know that $\operatorname{val}\left(\gamma^{\prime} \bar{\gamma}\right)=\operatorname{val}\left(\gamma^{\prime}\right)$ and that $\operatorname{st}\left(\gamma^{\prime}\right) \cap \operatorname{st}(\bar{\gamma}) \neq \emptyset$ (as $\bar{\gamma}$ witnesses a competition involving $\gamma^{\prime}$ ) and that $\operatorname{st}\left(\gamma_{1}\right) \cap \operatorname{st}(\bar{\gamma}) \neq \emptyset$ (by hypothesis). It is left to show that $\operatorname{val}\left(\gamma_{1} \bar{\gamma}\right)=\operatorname{val}\left(\gamma_{1}\right)$. Let $m \in \operatorname{st}(\bar{\gamma}) \cap \operatorname{st}\left(\gamma_{1}\right) \cap \operatorname{st}\left(\gamma_{2}\right)$; we represent the situation in Figure 9 with $\bar{\gamma}=\bar{\gamma}_{1} \bar{\gamma}_{2}$. Consider first cycle $\bar{\gamma}_{1} \bar{\gamma}_{2} \gamma_{2}$ : this cycle is a witness that $\gamma_{1}$ and $\gamma^{\prime}$ are competing, since it has common states with those cycles, $\operatorname{val}\left(\gamma_{1}\left(\bar{\gamma}_{1} \bar{\gamma}_{2} \gamma_{2}\right)\right)=\operatorname{val}\left(\gamma_{1}\right)\left(\right.$ both $\gamma_{1}$ and $\bar{\gamma}_{1} \bar{\gamma}_{2} \gamma_{2}$ have the same value), and $\operatorname{val}\left(\gamma^{\prime}\left(\bar{\gamma}_{2} \gamma_{2} \bar{\gamma}_{1}\right)\right)=\operatorname{val}\left(\gamma^{\prime}\right)$ (since $\gamma_{2} \triangleleft \gamma^{\prime}$ and $\bar{\gamma}$ is a witness of the competition). Therefore, as $\gamma^{\prime} \triangleleft \gamma_{1}$, the cycle $\nu=\gamma_{1} \bar{\gamma}_{1} \gamma^{\prime} \bar{\gamma}_{2} \gamma_{2}$ has the same value as $\gamma_{1}$. By Lemma 28, cycle $\nu$ has the same value as $\gamma_{1}\left(\bar{\gamma}_{1} \bar{\gamma}_{2}\right) \bar{\gamma}_{1} \gamma^{\prime} \bar{\gamma}_{2} \gamma_{2}$, which can be split into $\bar{\gamma}_{1} \gamma^{\prime} \bar{\gamma}_{2} \gamma_{2}$ (which has the same value as $\gamma^{\prime}$ since $\gamma_{2} \triangleleft \gamma^{\prime}$ and $\bar{\gamma}$ is a witness of the competition) and $\gamma_{1} \bar{\gamma}_{1} \bar{\gamma}_{2}$. Therefore, $\gamma_{1} \bar{\gamma}_{1} \bar{\gamma}_{2}=\gamma_{1} \bar{\gamma}$ cannot have the same value as $\gamma^{\prime}$, otherwise $\nu$ would also have the same value as $\gamma^{\prime}$ by $\mathcal{M}$-cycle-consistency.

- Lemma 38. Let $\gamma, \gamma_{1}^{\prime} \in \Gamma_{\mathcal{M}}$ be such that $\operatorname{val}(\gamma) \neq \operatorname{val}\left(\gamma_{1}^{\prime}\right)$ and $\gamma_{1}^{\prime} \triangleleft \gamma$. Let $\gamma_{2}^{\prime}$ be a cycle such that $\operatorname{val}\left(\gamma_{2}^{\prime}\right)=\operatorname{val}\left(\gamma_{1}^{\prime}\right)$ and $\operatorname{st}\left(\gamma_{2}^{\prime}\right) \cap \operatorname{st}\left(\gamma_{1}^{\prime}\right) \neq \emptyset$. Then, $\gamma$ and $\gamma_{2}^{\prime}$ are competing.
Proof. Let $\bar{\gamma}$ be a witness that $\gamma$ and $\gamma_{1}^{\prime}$ are competing; we represent the situation in Figure 9 . We show that $\bar{\gamma} \gamma_{1}^{\prime}$ is a witness that $\gamma$ and $\gamma_{2}^{\prime}$ are competing. As st $(\bar{\gamma}) \cap \operatorname{st}(\gamma) \neq \emptyset$, we have $\operatorname{st}\left(\bar{\gamma} \gamma_{1}^{\prime}\right) \cap \operatorname{st}(\gamma) \neq \emptyset$. Similarly, as st $\left(\gamma_{1}^{\prime}\right) \cap \operatorname{st}\left(\gamma_{2}^{\prime}\right) \neq \emptyset$, we have $\operatorname{st}\left(\bar{\gamma} \gamma_{1}^{\prime}\right) \cap \operatorname{st}\left(\gamma_{2}^{\prime}\right) \neq \emptyset$. As $\gamma_{1}^{\prime} \triangleleft \gamma$ with witness $\bar{\gamma}$, we have that $\operatorname{val}\left(\gamma\left(\bar{\gamma} \gamma_{1}^{\prime}\right)\right)=\operatorname{val}(\gamma)$. Moreover, since $\bar{\gamma}$ is a witness for $\gamma_{1}^{\prime}$ (and $\left.\gamma^{\prime}\right)$, $\operatorname{val}\left(\bar{\gamma} \gamma_{1}^{\prime}\right)=\operatorname{val}\left(\gamma_{1}^{\prime}\right)$. Therefore $\operatorname{val}\left(\bar{\gamma} \gamma_{1}^{\prime}\right)=\operatorname{val}\left(\gamma_{2}^{\prime}\right)$, which implies by $\mathcal{M}$-cycle-consistency that $\operatorname{val}\left(\gamma_{2}^{\prime}\left(\bar{\gamma} \gamma_{1}^{\prime}\right)\right)=\operatorname{val}\left(\gamma_{2}^{\prime}\right)$.



Figure 9 Situation in the proof of Lemma 37 (left) and in the proof of Lemma 38 (right).


Figure 10 Situation in the proof of Lemma 39 Competition of $\gamma_{i}$ and $\gamma_{i}^{\prime}$ is witnessed by $\bar{\gamma}_{i}$, and competition of $\gamma_{i}^{\prime}$ and $\gamma_{i+1}$ is witnessed by $\bar{\gamma}_{i}^{\prime}$. State $m^{\prime}$ appears somewhere along $\gamma_{i}^{\prime}$ and is not represented.

- Lemma 39. The height of partial preorder $\triangleleft$ is finite.

Proof. By contradiction, let $\gamma_{0} \triangleright \gamma_{0}^{\prime} \triangleright \gamma_{1} \triangleright \gamma_{1}^{\prime} \triangleright \gamma_{2} \triangleright \ldots$ be an infinitely decreasing sequence for $\triangleleft$. We assume w.l.o.g. that for all $i \geq 0, \operatorname{val}\left(\gamma_{i}\right)=\operatorname{win}$ and $\operatorname{val}\left(\gamma_{i}^{\prime}\right)=$ lose (if two consecutive cycles are, for example, both winning, we can always insert an intermediate losing cycle between them, by definition).

For $i \geq 0$, let $m$ (resp. $m^{\prime}$ ) be a state of $\mathcal{M}$ that is part of infinitely many sets st $\left(\gamma_{i}\right)$ (resp. $\left.\operatorname{st}\left(\gamma_{i}^{\prime}\right)\right)$ - such states necessarily exist as the state space of $\mathcal{M}$ is finite. Thanks to transitivity of $\triangleleft$ (Lemma 34), by keeping only winning cycles $\gamma_{i}$ such that $m \in \operatorname{st}\left(\gamma_{i}\right)$ alternating with losing cycles $\gamma_{i}^{\prime}$ such that $m^{\prime} \in \operatorname{st}\left(\gamma_{i}^{\prime}\right)$, we keep an infinitely decreasing sequence for $\triangleleft$. We can therefore assume w.l.o.g., up to renaming cycles, that for all $i \in \mathbb{N}, m \in \operatorname{st}\left(\gamma_{i}\right)$ and $m^{\prime} \in \operatorname{st}\left(\gamma_{i}^{\prime}\right)$.

We show that the competition of each contiguous pair in sequence $\gamma_{0}, \gamma_{0}^{\prime}, \gamma_{1}, \gamma_{1}^{\prime}, \gamma_{2}, \ldots$ has a witness that intersects the winning cycle at $m$, and the losing cycle at $m^{\prime}$. For all $i \geq 0$, let $\bar{\gamma}_{i}=\bar{\gamma}_{i, 1} \bar{\gamma}_{i, 2}$ be a witness that $\gamma_{i}$ and $\gamma_{i}^{\prime}$ are competing, and $\bar{\gamma}_{i}^{\prime}=\bar{\gamma}_{i, 1}^{\prime} \bar{\gamma}_{i, 2}^{\prime}$ be a witness that $\gamma_{i}^{\prime}$ and $\gamma_{i+1}$ are competing. Let $i \geq 0$; we depict part of the situation in Figure 10 with $\gamma_{i}^{\prime}=\gamma_{i, 1}^{\prime} \gamma_{i, 2}^{\prime}$. Based on the cycles that we already know, we consider the cycle $\bar{\nu}_{i}=\bar{\gamma}_{i, 1} \gamma_{i, 1}^{\prime} \bar{\gamma}_{i, 1}^{\prime} \gamma_{i+1} \bar{\gamma}_{i, 2}^{\prime} \gamma_{i, 2}^{\prime} \bar{\gamma}_{i, 2}$. We have that $m, m^{\prime} \in \operatorname{st}\left(\bar{\nu}_{i}\right)$ since $\gamma_{i+1}$ and $\gamma_{i}^{\prime}$ are part of $\bar{\nu}$. We show that $\bar{\nu}_{i}$ witnesses that $\gamma_{i}$ and $\gamma_{i}^{\prime}$ are competing:

- $\operatorname{val}\left(\gamma_{i} \bar{\nu}_{i}\right)=$ win since $\gamma_{i} \bar{\nu}_{i}$ can be split into $\gamma_{i, 2}^{\prime} \bar{\gamma}_{i, 2} \gamma_{i} \bar{\gamma}_{i, 1} \gamma_{i, 1}^{\prime}$ (winning since $\gamma_{i} \triangleright \gamma_{i}^{\prime}$ ) and $\bar{\gamma}_{i, 1}^{\prime} \gamma_{i+1} \bar{\gamma}_{i, 2}^{\prime}$ (winning since $\bar{\gamma}_{i}^{\prime}$ witnesses a competition involving $\gamma_{i+1}$ );
- $\operatorname{val}\left(\gamma_{i}^{\prime} \bar{\nu}_{i}\right)=$ lose since $\gamma_{i}^{\prime} \bar{\nu}_{i}$ can be split into $\gamma_{i}^{\prime} \bar{\gamma}_{i, 2} \bar{\gamma}_{i, 1}$ (losing since $\bar{\gamma}_{i}$ witnesses a competition involving $\gamma_{i}^{\prime}$ ) and $\gamma_{i, 1}^{\prime} \bar{\gamma}_{i, 1}^{\prime} \gamma_{i+1} \bar{\gamma}_{i, 2}^{\prime} \gamma_{i, 2}^{\prime}\left(\right.$ losing since $\left.\gamma_{i}^{\prime} \triangleright \gamma_{i+1}\right)$. We use Remark 30 in order to write " $\gamma_{i}^{\prime} \bar{\nu}_{i}$ ".

We can perform a symmetric reasoning to show that the competition of any pair $\gamma_{i}^{\prime}, \gamma_{i+1}$, $i \geq 0$, is witnessed by a cycle $\bar{\nu}_{i}^{\prime} \in \Gamma_{\mathcal{M}}$ such that $m^{\prime}, m \in \operatorname{st}\left(\bar{\nu}_{i}^{\prime}\right)$.

By Lemma $37 \bar{\nu}_{i}^{\prime}$ is not only a witness that $\gamma_{i}^{\prime}$ and $\gamma_{i+1}$ are competing, but also that $\gamma_{i}$ and $\gamma_{i}^{\prime}$ are competing (indeed, $\operatorname{val}\left(\gamma_{i}\right)=\operatorname{val}\left(\gamma_{i+1}\right) \neq \operatorname{val}\left(\gamma_{i}^{\prime}\right), \gamma_{i+1} \triangleleft \gamma_{i}^{\prime}, \gamma_{i}^{\prime} \triangleleft \gamma_{i}, \bar{\nu}_{i}^{\prime}$ witnesses that $\gamma_{i+1}$ and $\gamma_{i}^{\prime}$ are competing, and $\left.m \in \operatorname{st}\left(\bar{\nu}_{i}^{\prime}\right) \cap \operatorname{st}\left(\gamma_{i}\right) \cap \operatorname{st}\left(\gamma_{i+1}\right)\right)$.

For $i \geq 0$, we can write $\bar{\nu}_{i}^{\prime}=\bar{\nu}_{i, 1}^{\prime} \bar{\nu}_{i, 2}^{\prime}$ with $\bar{\nu}_{i, 1}^{\prime} \in \Pi_{m^{\prime}, m}$ and $\bar{\nu}_{i, 2}^{\prime} \in \Pi_{m, m^{\prime}}$. We now consider the infinite sequence

$$
\xi=\gamma_{0} \bar{\nu}_{0,2}^{\prime} \gamma_{0}^{\prime} \bar{\nu}_{0,1}^{\prime} \gamma_{1} \bar{\nu}_{1,2}^{\prime} \gamma_{1}^{\prime} \bar{\nu}_{1,1}^{\prime} \gamma_{2} \ldots
$$

Notice that for all $i \geq 0, \gamma_{i} \bar{\nu}_{i, 2}^{\prime} \gamma_{i}^{\prime} \bar{\nu}_{i, 1}^{\prime}$ is a winning cycle on $m$ since $\gamma_{i} \triangleright \gamma_{i}^{\prime}$; hence col ${ }^{\omega}(\xi) \in$ $m^{-1} W$ by $\mathcal{M}$-cycle-consistency. Also, for all $i \geq 0, \bar{\nu}_{i, 2}^{\prime} \gamma_{i}^{\prime} \bar{\nu}_{i, 1}^{\prime} \gamma_{i+1}$ is a losing cycle on $m$ since $\gamma_{i}^{\prime} \triangleright \gamma_{i+1}$; hence col ${ }^{\omega}(\xi) \in m^{-1} \bar{W}$ by $\mathcal{M}$-prefix-independence and $\mathcal{M}$-cycle-consistency. This is a contradiction.

A proof for infinitely increasing sequences can be done in a symmetric way.

- Lemma 40. The width of partial preorder $\triangleleft$ on $\Gamma_{\mathcal{M}} / \simeq$ is finite.

Proof. We recall that $\mathcal{M}=\left(M, m_{\text {init }}, \alpha_{\text {upd }}\right)$. We will show that any two cycles $\gamma_{1}$ and $\gamma_{2}$ such that $\operatorname{st}\left(\gamma_{1}\right)=\operatorname{st}\left(\gamma_{2}\right)$ are necessarily comparable for $\simeq$ or $\triangleleft$. This will show that the cardinality of a maximal set of pairwise incomparable (for $\triangleleft$ ) elements in $\Gamma_{\mathcal{M}} / \simeq$ is necessarily bounded by $2^{|M|}$, which implies that the width of partial preorder $\triangleleft$ is finite. Let $\gamma_{1}$ and $\gamma_{2}$ be two cycles such that $\operatorname{st}\left(\gamma_{1}\right)=\operatorname{st}\left(\gamma_{2}\right)$ (we recall that there are infinitely many transitions in $\mathcal{M}$ if $C$ is infinite, and that two cycles going through the same states may use different transitions and have a different value).

If $\operatorname{val}\left(\gamma_{1}\right) \neq \operatorname{val}\left(\gamma_{2}\right)$, then as $\gamma_{1}$ and $\gamma_{2}$ share a common state, they are competing - we have either $\gamma_{1} \triangleleft \gamma_{2}$ or $\gamma_{2} \triangleleft \gamma_{1}$ (depending on the value of $\gamma_{1} \gamma_{2}$ ).

We now assume that $\operatorname{val}\left(\gamma_{1}\right)=\operatorname{val}\left(\gamma_{2}\right)$; we assume w.l.o.g. that $\gamma_{1}$ and $\gamma_{2}$ are winning. If $\gamma_{1}$ and $\gamma_{2}$ are such that $\operatorname{comp}\left(\gamma_{1}\right)=\operatorname{comp}\left(\gamma_{2}\right)$, then they can necessarily be compared with $\simeq$ or $\triangleleft$; indeed,

- if $\operatorname{dom} \operatorname{By}\left(\gamma_{1}\right)=\operatorname{dom} \operatorname{By}\left(\gamma_{2}\right)$, then $\gamma_{1} \simeq \gamma_{2}$;
- if $\operatorname{dom} \operatorname{By}\left(\gamma_{1}\right) \neq \operatorname{dom} \operatorname{By}\left(\gamma_{2}\right)$, then there is $i \in\{1,2\}$ and a losing cycle $\gamma^{\prime}$ in $\operatorname{domBy}\left(\gamma_{i}\right)$ that is competing with $\gamma_{3-i}$ but that is not an element of $\operatorname{dom} \operatorname{By}\left(\gamma_{3-i}\right)$. Therefore, $\gamma_{3-i} \triangleleft \gamma^{\prime} \triangleleft \gamma_{i}$, which means that $\gamma_{3-i} \triangleleft \gamma_{i}$.
It is left to deal with the case $\operatorname{comp}\left(\gamma_{1}\right) \neq \operatorname{comp}\left(\gamma_{2}\right)$. W.l.o.g., let $\gamma^{\prime}$ be in $\operatorname{comp}\left(\gamma_{1}\right) \backslash \operatorname{comp}\left(\gamma_{2}\right)$. There are two cases to discuss: whether $\gamma_{1} \triangleleft \gamma^{\prime}$ or $\gamma^{\prime} \triangleleft \gamma_{1}$.
- Assume $\gamma_{1} \triangleleft \gamma^{\prime}$. By Lemma 38, as $\operatorname{val}\left(\gamma_{1}\right)=\operatorname{val}\left(\gamma_{2}\right)$ and $\operatorname{st}\left(\gamma_{1}\right) \cap \operatorname{st}\left(\gamma_{2}\right) \neq \emptyset, \gamma^{\prime}$ is also competing with $\gamma_{2}$, which is a contradiction.
- Assume $\gamma^{\prime} \triangleleft \gamma_{1}$. Let $\bar{\gamma}$ be a witness that $\gamma_{1}$ and $\gamma^{\prime}$ are competing. We therefore have that $\gamma_{1} \bar{\gamma}$ is winning and $\gamma^{\prime} \bar{\gamma}$ is losing. As $\gamma_{2}$ is not competing with $\gamma^{\prime}, \bar{\gamma}$ cannot be a witness that $\gamma_{2}$ and $\gamma^{\prime}$ are competing. Since $\operatorname{st}\left(\gamma_{1}\right)=\operatorname{st}\left(\gamma_{2}\right)$ has a non-empty intersection with $\operatorname{st}(\bar{\gamma})$, the only possibility for that to happen is that $\gamma_{2} \bar{\gamma}$ is losing (all other conditions are satisfied). This means that $\bar{\gamma}$ must itself be a losing cycle. But then, observe that $\bar{\gamma}$ is competing both with $\gamma_{1}$ and $\gamma_{2}$ (as $\bar{\gamma}$ has a common state with and a different value than $\gamma_{1}$ and $\gamma_{2}$ ) and $\gamma_{2} \triangleleft \bar{\gamma} \triangleleft \gamma_{1}$ (as $\gamma_{2} \bar{\gamma}$ is losing and $\gamma_{1} \bar{\gamma}$ is winning). This implies that $\gamma_{2} \triangleleft \gamma_{1}$.

Lemmas 39 and 40 imply together that $\simeq$ has finite index, and thus that $\triangleleft$ (partially) orders only finitely many classes of cycles in $\Gamma_{\mathcal{M}} / \simeq$. Therefore, for some $n \in \mathbb{N}$, there exists a function $p_{\Gamma}: \Gamma_{\mathcal{M}} / \simeq \rightarrow\{0, \ldots, n\}$ that is a monotonic function (assuming $\Gamma_{\mathcal{M}} / \simeq$ is preordered with $\triangleleft$ and $\{0, \ldots, n\}$ is ordered with the usual order on $\mathbb{N}$ ); such a function is sometimes called a linear extension of the partial order. We extend it to a function
$p_{\Gamma}: \Gamma_{\mathcal{M}} \rightarrow\{0, \ldots, n\}$ such that $p_{\Gamma}(\gamma)=p_{\Gamma}([\gamma] \simeq)$. Moreover, we assume w.l.o.g. that $\operatorname{val}(\gamma)=$ win if and only if $p_{\Gamma}(\gamma)$ is even (this might require to increase $n$, but it is always possible).

We fix $n$ and any such function $p_{\Gamma}$ for the rest of the proof.

Parity automaton on top of $\mathcal{M}$. At this point, it would already be possible to describe words of $W$ in terms of the cycles of $\mathcal{M}$ that they visit through function skel (there may be multiple such decompositions) and their values by $p_{\Gamma}$, but that does not directly correspond to a classical acceptance condition for automata on infinite words. We can actually obtain something more satisfying: we show that we can assign priorities to transitions of $\mathcal{M}$ to recognize $W$, in a way that corresponds to a parity acceptance condition on transitions. We transfer function $p_{\Gamma}$ to transitions of $\mathcal{M}$ : for $(m, c) \in M \times C$, we define

$$
\begin{equation*}
p(m, c)=\min \left\{p_{\Gamma}(\gamma) \mid \gamma \in \Gamma_{\mathcal{M}},(m, c) \in \gamma\right\} . \tag{1}
\end{equation*}
$$

We now have a well-defined function assigning priorities to every transition of $\mathcal{M}$.

- Example 41. We illustrate our definitions for $p_{\Gamma}$ and $p$. We again consider the example from Figure 7 (for which, unlike $W$, we already know that it describes an $\omega$-regular language). For the sake of the example, let us ignore the already-defined priority function $p$ of this parity automaton. We show that we can recover priorities defining the same language starting from our preorder $\triangleleft$ and our definitions for $p_{\Gamma}$ and $p$. Function $p_{\Gamma}$ must be any function that respects the order given by the diagram and that assigns even priorities to winning classes of cycles, and odd priorities to losing classes. One such possible choice is $p_{\Gamma}\left(\left[\left(m_{1}, c\right)\right]_{\simeq}\right)=5, p_{\Gamma}\left(\left[\left(m_{1}, a\right)\left(m_{2}, a\right)\right]_{\simeq}\right)=2, p_{\Gamma}\left(\left[\left(m_{2}, b\right)\right]_{\simeq}\right)=4$, and $p_{\Gamma}\left(\left[\left(m_{1}, b\right)\right] \simeq\right)=1$. From this choice of function $p_{\Gamma}$, our definition (1) of function $p$ entails $p\left(\left(m_{1}, c\right)\right)=5$, $p\left(\left(m_{1}, a\right)\right)=p\left(\left(m_{2}, a\right)\right)=2, p\left(\left(m_{2}, b\right)\right)=p\left(\left(m_{2}, c\right)\right)=4$, and $p\left(\left(m_{1}, b\right)\right)=1$. This choice of priorities defines the same language as the original parity automaton.

We will prove a property of function $p:(\mathcal{M}, p)$ recognizes the language $W$. In our proof, we will need to relate the cycles dominated by a cycle $\gamma$ and the ones dominated by cycles in a "decomposition" of $\gamma$, i.e., cycles that can be obtained from iteratively removing cycles from $\gamma$. We formally define this notion and prove two related results.

- Definition 42 (Cycle decomposition). Let $\gamma=\left(m_{1}, c_{1}\right) \ldots\left(m_{k}, c_{k}\right) \in \Gamma_{\mathcal{M}}$, and $\gamma_{1}, \ldots, \gamma_{l} \in$ $\Gamma_{\mathcal{M}}$. We say that $\left(\gamma_{1}, \ldots, \gamma_{l}\right)$ is a cycle decomposition of $\gamma$ if
- either $l=1$ and $\gamma=\gamma_{1}$,
- or $l>1$ and there exist $i, i^{\prime} \in\{1, \ldots, k\}, i \leq i^{\prime}$, such that $\gamma_{1}=\left(m_{i}, c_{i}\right) \ldots\left(m_{i^{\prime}}, c_{i^{\prime}}\right)$, and $\left(\gamma_{2}, \ldots, \gamma_{l}\right)$ is a cycle decomposition of $\left(m_{1}, c_{1}\right) \ldots\left(m_{i-1}, c_{i-1}\right)\left(m_{i^{\prime}+1}, c_{i^{\prime}+1}\right) \ldots\left(m_{k}, c_{k}\right)$.
- Lemma 43. Let $\gamma, \gamma_{1}, \gamma_{2}, \gamma^{\prime} \in \Gamma_{\mathcal{M}}$ be cycles such that $\gamma=\gamma_{1} \gamma_{2}$. If $\gamma^{\prime} \triangleleft \gamma_{1}$, then $\gamma^{\prime} \triangleleft \gamma$.

Proof. We assume $\gamma^{\prime} \triangleleft \gamma_{1}$.
If $\operatorname{val}\left(\gamma_{1}\right) \neq \operatorname{val}(\gamma)$, then $\gamma_{1} \triangleleft \gamma$ - indeed, they share at least one state and $\gamma_{1} \gamma=\left(\gamma_{1}\right)^{2} \gamma_{2}$ has the same value as $\gamma_{1} \gamma_{2}=\gamma$ by Lemma 26 . Therefore, by transitivity of $\triangleleft$ (Lemma 34), $\gamma^{\prime} \triangleleft \gamma$.

We now assume $\operatorname{val}\left(\gamma_{1}\right)=\operatorname{val}(\gamma)$ and $\operatorname{val}\left(\gamma^{\prime}\right) \neq \operatorname{val}\left(\gamma_{1}\right)$. Let $\bar{\gamma}$ be a witness that $\gamma^{\prime}$ and $\gamma_{1}$ are competing. We prove that $\bar{\gamma}$ also witnesses that $\gamma^{\prime}$ and $\gamma$ are competing: to do so, it is left to show that $\nu=\bar{\gamma} \gamma$ has the same value as $\gamma$. We have that $\nu$ can be written as $\bar{\gamma} \gamma_{1,1} \gamma_{2} \gamma_{1,2}$ for some paths $\gamma_{1,1}$ and $\gamma_{1,2}$ such that $\gamma_{1}=\gamma_{1,1} \gamma_{1,2}$. Cycle $\nu$ has the same value as $\bar{\gamma}\left(\gamma_{1,1} \gamma_{1,2}\right) \gamma_{1,1} \gamma_{2} \gamma_{1,2}$ by Lemma 28 This last cycle can be split into $\bar{\gamma} \gamma_{1}$ and $\gamma$, which both


Figure 11 Situation in the proof of Lemma 45, with $\gamma=e_{1} \ldots e_{k}$.
have the same value as $\gamma$. Therefore $\bar{\gamma}$ is also a witness that $\gamma^{\prime}$ and $\gamma$ are competing. We can show with a very similar argument that $\gamma^{\prime} \bar{\gamma}_{1} \gamma \bar{\gamma}_{2}$ also has the same value as $\gamma$, hence $\gamma^{\prime} \triangleleft \gamma$.

If $\operatorname{val}\left(\gamma_{1}\right)=\operatorname{val}(\gamma)$ and $\operatorname{val}\left(\gamma^{\prime}\right)=\operatorname{val}\left(\gamma_{1}\right)$, then there exists $\gamma^{\prime \prime}$ with $\operatorname{val}\left(\gamma^{\prime \prime}\right) \neq \operatorname{val}\left(\gamma_{1}\right)$ such that $\gamma^{\prime} \in \operatorname{domBy}\left(\gamma^{\prime \prime}\right)$ and $\gamma^{\prime \prime} \in \operatorname{domBy}\left(\gamma_{1}\right)$, so $\gamma^{\prime} \triangleleft \gamma^{\prime \prime} \triangleleft \gamma_{1}$. By the previous case, $\gamma^{\prime \prime} \triangleleft \gamma$, and by transitivity, $\gamma^{\prime} \triangleleft \gamma$.

- Lemma 44. Let $\gamma$ be a cycle of $\mathcal{M}$ and $\left(\gamma_{1}, \ldots, \gamma_{l}\right)$ be a cycle decomposition of $\gamma$. For all $i \in\{1, \ldots, l\}$, for all $\gamma^{\prime} \in \Gamma_{\mathcal{M}}$, if $\gamma^{\prime} \triangleleft \gamma_{i}$, then $\gamma^{\prime} \triangleleft \gamma$.

Proof. We proceed by induction on $l$. If $l=1$, then the statement is trivial as $\gamma=\gamma_{1}$. For $l>1$, we now assume that the property holds for $l-1$, and we show that it also holds for $l$. Up to a shift of $\gamma$ and of the cycle decomposition, we assume that $\gamma$ is equal to $\gamma_{1} \nu$, where $\left(\gamma_{2}, \ldots, \gamma_{l}\right)$ is a cycle decomposition of $\nu$.

Let $\gamma^{\prime} \in \Gamma_{\mathcal{M}}$ be such that $\gamma^{\prime} \triangleleft \gamma_{i}$ for some $i \in\{1, \ldots, l\}$. This implies that $\gamma^{\prime} \triangleleft \gamma_{1}$ if $i=1$ or, using the induction hypothesis, that $\gamma^{\prime} \triangleleft \nu$. In any case, by Lemma 43, we immediately have that $\gamma^{\prime} \triangleleft \gamma$.

We can now prove that $W$ is recognized by the parity automaton $(\mathcal{M}, p)$. We do this in the next two results. First, we show that winning cycles of $\mathcal{M}$ are exactly the ones that have an even maximal priority given by $p$. It is then straightforward to conclude that infinite words in $W$ are exactly the ones whose maximal infinitely visited priority is even.

- Lemma 45. Let $\gamma=\left(m_{1}, c_{1}\right) \ldots\left(m_{k}, c_{k}\right) \in \Gamma_{\mathcal{M}}$. Then, $\gamma$ is winning if and only if $\max _{1 \leq i \leq k} p\left(m_{i}, c_{i}\right)$ is even.
Proof. For conciseness, let $p^{*}=\max _{1 \leq i \leq k} p\left(m_{i}, c_{i}\right)$ and $e_{i}=\left(m_{i}, c_{i}\right)$. By definition of function $p$, for all $i \in\{1, \ldots, k\}, p\left(e_{i}\right) \leq p_{\Gamma}(\gamma)$. Hence, $p^{*} \leq p_{\Gamma}(\gamma)$. We want to show that $\gamma$ is winning if and only if $p^{*}$ is even. By contradiction, we assume that we do not have this equivalence. We assume w.l.o.g. that $\gamma$ is losing and that $p^{*}$ is even; we could obtain in a symmetric way a contradiction for $\gamma$ winning and $p^{*}$ odd.

As $\gamma$ is losing, we have that $p_{\Gamma}(\gamma)$ is odd - as $p^{*}$ is even, $p^{*}<p_{\Gamma}(\gamma)$. We assume (up to a shift of the transitions) that $p^{*}=p\left(e_{1}\right)$. Since $p^{*}<p_{\Gamma}(\gamma)$, there exists, for all $i \in\{1, \ldots, k\}$, a cycle $\gamma_{i} \neq \gamma$ such that $e_{i} \in \gamma_{i}$ and $p\left(e_{i}\right)=p_{\Gamma}\left(\gamma_{i}\right)$. We assume $\gamma_{i}=e_{i} \pi_{i}$. The situation is represented in Figure 11.

The rest of the proof consists in exhibiting two cycles, building on the ones we know, showing that one of them is winning and one of them is losing, and finally showing that they must have the same value, which provides a contradiction.

We will first consider cycle $e_{1} \ldots e_{k} \pi_{k} \ldots \pi_{1}$ (on $m_{1}$ ). We will prove by induction that it is winning. First, $e_{1} \pi_{1}$ is winning since $p_{\Gamma}\left(e_{1} \pi_{1}\right)=p^{*}$ is even. Assume now that for $1<l \leq k$, $e_{1} \ldots e_{l-1} \pi_{l-1} \ldots \pi_{1}$ is winning. We show that $e_{1} \ldots e_{l-1}\left(e_{l} \pi_{l}\right) \pi_{l-1} \ldots \pi_{1}$ is winning.

- If $e_{l} \pi_{l}$ is a winning cycle, it follows from $\mathcal{M}$-cycle-consistency.
- If $e_{l} \pi_{l}$ is a losing cycle, we distinguish two cases. If $e_{2} \ldots e_{l-1}\left(e_{l} \pi_{l}\right) \pi_{l-1} \ldots \pi_{2}$ is winning, then so is $e_{1} \ldots e_{l-1}\left(e_{l} \pi_{l}\right) \pi_{l-1} \ldots \pi_{1}$ because we just concatenate the winning cycle $\pi_{1} e_{1}$ to a winning cycle ( $\mathcal{M}$-cycle-consistency). If $e_{2} \ldots e_{l-1}\left(e_{l} \pi_{l}\right) \pi_{l-1} \ldots \pi_{2}$ is losing, then $e_{2} \ldots e_{l-1} \pi_{l-1} \ldots \pi_{2}$ witnesses that $e_{1} \pi_{1}$ and $e_{l} \pi_{l}$ are competing. Since $p_{\Gamma}\left(e_{l} \pi_{l}\right)$ is odd, and $p_{\Gamma}\left(e_{1} \pi_{1}\right)$ is even and is equal to the maximum of $i \mapsto p_{\Gamma}\left(e_{i} \pi_{i}\right)$, we have that $p_{\Gamma}\left(e_{l} \pi_{l}\right)<p_{\Gamma}\left(e_{1} \pi_{1}\right)$. Since $p_{\Gamma}$ is monotonic and $e_{1} \pi_{1}$ and $e_{l} \pi_{l}$ are competing, this implies $e_{l} \pi_{l} \triangleleft e_{1} \pi_{1}$. Thus $e_{1} \ldots e_{l-1}\left(e_{l} \pi_{l}\right) \pi_{l-1} \ldots \pi_{1}$ is winning.

We now consider the cycle $e_{1}\left(\pi_{1} e_{1}\right) \ldots e_{k}\left(\pi_{k} e_{k}\right)$ (on $\left.m_{1}\right)$. We show by induction that it is losing. We start from $\gamma$, which is losing by hypothesis, and we add cycles $\pi_{i} e_{i}$ one by one. We denote $\gamma^{(l)}=e_{1}\left(\pi_{1} e_{1}\right) \ldots e_{l}\left(\pi_{l} e_{l}\right) e_{l+1} \ldots e_{k}$. Assume that $\gamma^{(l-1)}$ is losing for $1<l \leq k$. We want to show that $\gamma^{(l)}$ is also losing.

- If $\pi_{l} e_{l}$ is a losing cycle, it follows from $\mathcal{M}$-cycle-consistency.
- If $\pi_{l} e_{l}$ is a winning cycle, then as $p_{\Gamma}\left(\pi_{l} e_{l}\right) \leq p^{*}<p_{\Gamma}(\gamma)$ and $\pi_{l} e_{l}$ is competing with $\gamma$ (they share common states), we have $\pi_{l} e_{l} \triangleleft \gamma$. Notice that $\left(\pi_{1} e_{1}, \ldots, \pi_{l-1} e_{l-1}, \gamma\right)$ is a cycle decomposition of $\gamma^{(l-1)}$ as in Definition 42 Thus by Lemma 44 as $\pi_{l} e_{l} \triangleleft \gamma$, we also have $\pi_{l} e_{l} \triangleleft \gamma^{(l-1)}$. We conclude that $\gamma^{(l)}$ is also losing.

We have now considered two cycles on $m_{1}$ : the winning $e_{1} \ldots e_{k} \pi_{k} \ldots \pi_{1}$ and the losing $e_{1}\left(\pi_{1} e_{1}\right) \ldots e_{k}\left(\pi_{k} e_{k}\right)$. We show that it is possible to transform the latter into the former using only value-preserving transformations (given by Lemmas 26 and 27), which provides the desired contradiction.

We show inductively that for all $l \in\{1, \ldots, k\}$, cycle $e_{1}\left(\pi_{1} e_{1}\right) \ldots e_{k}\left(\pi_{k} e_{k}\right)$ can be transformed into

$$
\nu^{(l)}=\left(e_{1} \ldots e_{l} \pi_{l} \ldots \pi_{1} e_{1} \ldots e_{l}\right) e_{l+1}\left(\pi_{l+1} e_{l+1}\right) \ldots e_{k}\left(\pi_{k} e_{k}\right)
$$

using value-preserving transformations. Notice that $e_{1}\left(\pi_{1} e_{1}\right) \ldots e_{k}\left(\pi_{k} e_{k}\right)$ is equal to $\nu^{(1)}$, which deals with the base case of the induction. Now assume that $e_{1}\left(\pi_{1} e_{1}\right) \ldots e_{k}\left(\pi_{k} e_{k}\right)$ has the same value as $\nu^{(l-1)}$ for $1<l \leq k$. In the expression of $\nu^{(l-1)}$, notice that $\pi_{l-1} \ldots \pi_{1} e_{1} \ldots e_{l-1}$ and $e_{l} \pi_{l}$ are two consecutive cycles on $m_{l}$. By Lemma 27, they can thus be swapped while keeping a cycle with the same value. Notice that this gives exactly the cycle $\nu^{(l)}$.

We obtain that $e_{1}\left(\pi_{1} e_{1}\right) \ldots e_{k}\left(\pi_{k} e_{k}\right)$ has the same value as $\nu^{(k)}=e_{1} \ldots e_{k} \pi_{k} \ldots \pi_{1} e_{1} \ldots e_{k}$, which has the same value as $e_{1} \ldots e_{k} \pi_{k} \ldots \pi_{1}$ by Lemma 26

- Proposition 46. Let $w=c_{1} c_{2} \ldots \in C^{\omega}$ with $\operatorname{skel}(w)=\left(m_{1}, c_{1}\right)\left(m_{2}, c_{2}\right) \ldots \in(M \times C)^{\omega}$. Then,

$$
w \in W \text { if and only if } \limsup _{i>1} p\left(m_{i}, c_{i}\right) \text { is even. }
$$

Proof. Let $p^{*}=\lim \sup _{i \geq 1} p\left(m_{i}, c_{i}\right)$. Let $j \geq 1$ be an index such that for all $i \geq j$, $p\left(m_{i}, c_{i}\right) \leq p^{*}$. Let $I^{*}=\left\{i \geq j \mid p\left(m_{i}, c_{i}\right)=p^{*}\right\}$ be the infinite set of indices of transitions with priority $p^{*}$ occurring after index $j$. We write $i_{1}, i_{2}, \ldots$ for the elements of $I^{*}$ in order. Let $m^{*}$ be a state appearing infinitely often in $\left\{m_{i} \mid i \in I^{*}\right\}$ (such a state exists necessarily as the state space of $\mathcal{M}$ is finite). This implies that $\operatorname{skel}(w)$ can be written as the concatenation of a finite prefix $\left(m_{1}, c_{1}\right) \ldots\left(m_{i_{1}-1}, c_{i_{1}-1}\right)$ and infinitely many cycles $\gamma_{k}=\left(m_{i_{k}}, c_{i_{k}}\right) \ldots\left(m_{i_{k+1}-1}, c_{i_{k+1}-1}\right)$ with $m_{i_{k}}=m^{*}$ and $p\left(m_{i_{k}}, c_{i_{k}}\right)=p^{*}$, for $k \geq 1$.

For all $k \geq 1$, we have that $\max _{i_{k} \leq i<i_{k+1}} p\left(m_{i}, c_{i}\right)=p^{*}$ (it is $\leq p^{*}$ as $i_{k} \geq j$, and it is $\geq p^{*}$ as $p\left(m_{i_{k}}, c_{i_{k}}\right)=p^{*}$ ). By Lemma 45, we conclude that cycles $\gamma_{k}$ are all cycles on
$m^{*}$ that have the same value: they are winning if $p^{*}$ is even, and losing if $p^{*}$ is odd. By $\mathcal{M}$-prefix-independence and $\mathcal{M}$-cycle-consistency, $w$ is in $W$ if $p^{*}$ is even, and $w$ is in $\bar{W}$ if $p^{*}$ is odd.

We have therefore reached our goal for this section.
Corollary 47 (Second item of Theorem 10). If there exists a skeleton $\mathcal{M}$ such that $W$ is $\mathcal{M}$-prefix-independent and $\mathcal{M}$-cycle-consistent, then $W$ is $\omega$-regular and can be recognized by a deterministic parity automaton defined on top of $\mathcal{M}$.

- Remark 48. As discussed in Remark 4. our proof shows as a by-product that even if $C$ is infinite, many colors can be assumed to be equal (w.r.t. $W$ ) - there are only finitely many classes of truly different colors.


## 6 Applications

We provide a thorough application of our concepts to a discounted-sum winning condition. We then discuss more briefly mean-payoff and total-payoff winning conditions.

### 6.1 Discounted sum

We apply our results to a discounted-sum condition in order to illustrate our notions. A specificity of this example is that its $\omega$-regularity depends on some chosen parameters we use our results to characterize the parameters for which it is $\omega$-regular or, equivalently (Theorem 11), chromatic-finite-memory determined. The $\omega$-regularity of discounted-sum conditions has also been studied in [14, 2] with different techniques and goals.

Let $C \subseteq \mathbb{Q}$ be non-empty and bounded. For $\lambda \in(0,1) \cap \mathbb{Q}$, we define the discounted-sum function $\mathrm{DS}_{\lambda}: C^{\omega} \rightarrow \mathbb{R}$ such that for $w=c_{1} c_{2} \ldots \in C^{\omega}$,

$$
\mathrm{DS}_{\lambda}(w)=\sum_{i=1}^{\infty} \lambda^{i-1} \cdot c_{i}
$$

This function is always well-defined for a bounded $C$, and takes values in $\left[\frac{\inf C}{1-\lambda}, \frac{\sup C}{1-\lambda}\right]$.
We define the winning condition $\mathrm{DS}_{\lambda}^{\geq 0}=\left\{w \in C^{\omega} \mid \mathrm{DS}_{\lambda}(w) \geq 0\right\}$ as the set of infinite words whose discounted sum is non-negative, and let $\sim$ be its right congruence. We will analyze cycle-consistency and prefix-independence of $\mathrm{DS}_{\lambda}^{\geq 0}$ to conclude under which conditions (on $C$ and $\lambda$ ) it is chromatic-finite-memory determined. First, we discuss a few properties of the discounted-sum function.

Basic properties. We extend function $\mathrm{DS}_{\lambda}$ to finite words in a natural way: for $w \in C^{*}$, we define $\mathrm{DS}_{\lambda}(w)=\mathrm{DS}_{\lambda}\left(w 0^{\omega}\right)$. For $w \in C^{*}$, we define $|w|$ as the length of $w$ (so $w \in C^{|w|}$ ). First, we notice that for $w \in C^{*}$ and $w^{\prime} \in C^{\omega}$, we have

$$
\mathrm{DS}_{\lambda}\left(w w^{\prime}\right)=\mathrm{DS}_{\lambda}(w)+\lambda^{|w|} \mathrm{DS}_{\lambda}\left(w^{\prime}\right)
$$

Therefore,

$$
w w^{\prime} \in \mathrm{DS}_{\lambda}^{\geq 0} \Longleftrightarrow \frac{\mathrm{DS}_{\lambda}(w)}{\lambda^{|w|}} \geq-\mathrm{DS}_{\lambda}\left(w^{\prime}\right)
$$

This provides a characterization of the winning continuations of a finite word $w \in C^{*}$ by comparing their discounted sum to the value $\frac{\mathrm{DS}_{\lambda}(w)}{\lambda^{|w|}}$.


Figure 12 Arena with infinitely many edges in which $\mathcal{P}_{1}$ needs infinite memory to win for condition $\mathrm{DS}_{\lambda}^{\geq 0}$ from $s_{1}$ for any $\lambda \in(0,1) \cap \mathbb{Q}$, with $C=[-k, k] \cap \mathbb{Q}$ for $k$ sufficiently large.

This leads us to define the gap of a finite word $w \in C^{*}$, following ideas in [5] as

$$
\operatorname{gap}(w)= \begin{cases}\top & \text { if } \frac{\mathrm{DS}_{\lambda}(w)}{\lambda^{w \mid}} \geq-\frac{\inf C}{1-\lambda} \\ \perp & \text { if } \frac{\mathrm{DS}_{\lambda}(w)}{\lambda^{|w|}}<-\frac{\sup C}{1-\lambda} \\ \frac{\mathrm{DS}_{\lambda}(w)}{\lambda^{|w|}} & \text { otherwise }\end{cases}
$$

Intuitively, the gap of a finite word $w \in C^{*}$ represents how far it is from going back to 0 : if $w^{\prime} \in C^{\omega}$ is such that $\mathrm{DS}_{\lambda}\left(w^{\prime}\right)=-\operatorname{gap}(w)$, then $\mathrm{DS}_{\lambda}\left(w w^{\prime}\right)=0$. We can see that for all words $w \in C^{*}$, if $\operatorname{gap}(w)=\top$, then all continuations are winning (i.e., $w^{-1} W=C^{\omega}$ ) as it is not possible to find an infinite word with a discounted sum less than $\frac{\inf C}{1-\lambda}$. Similarly, if $\operatorname{gap}(w)=\perp$, then all continuations are losing (i.e., $w^{-1} W=\emptyset$ ).

Cycle-consistency. We can show that $\mathrm{DS}_{\lambda}^{\geq 0}$ is always $\mathcal{M}_{\text {triv-cycle-consistent. }}$

- Proposition 49. For all bounded $C \subseteq \mathbb{Q}, \lambda \in(0,1) \cap \mathbb{Q}$, winning condition $\mathrm{DS}_{\lambda}^{\geq 0}$ is $\mathcal{M}_{\text {triv-cycle-consistent. }}$

The proof of this result is elementary and is provided in Appendix B.
Prefix-independence. If $C=[-k, k] \cap \mathbb{Q}$ for some $k \in \mathbb{N} \backslash\{0\}$, winning condition $\mathrm{DS}_{\lambda}^{\geq 0}$ is not $\mathcal{M}$-prefix-independent for any $\mathcal{M}$, as $\sim$ has infinite index. Indeed, for $i \geq 1$ and $w_{i}=\frac{1}{i} \in C^{*}$, we have $w_{1} \succ w_{2} \succ \ldots$ - we can see how to use this to exhibit an arena in which $\mathcal{P}_{1}$ can win but needs infinite memory to do so in Figure 12 ,

For finite $C \subseteq \mathbb{Z}$, the picture is more complicated; for $C=[-k, k] \cap \mathbb{Z}$ for some $k \in \mathbb{N}$, we characterize when $D S_{\lambda}^{\geq 0}$ is $\mathcal{M}$-prefix-independent for some finite skeleton $\mathcal{M}$. We give an intuition of the two situations in which that happens: $(i)$ if $C$ is too small, then the first non-zero color seen determines the outcome of the game, as it is not possible to compensate this color to change the sign of the discounted sum; (ii) if $\lambda=\frac{1}{n}$ for some integer $n \geq 1$, then the gap function actually takes only finitely many values, which is not the case for a different $\lambda$.

- Proposition 50. Let $\lambda \in(0,1) \cap \mathbb{Q}, k \in \mathbb{N}$, and $C=[-k, k] \cap \mathbb{Z}$. Then, the right congruence $\sim$ of $\mathrm{DS}_{\lambda}^{\geq 0}$ has finite index if and only if $k<\frac{1}{\lambda}-1$ or $\lambda$ is equal to $\frac{1}{n}$ for some integer $n \geq 1$.
Proof. We define $\operatorname{maxDS}=\frac{\sup C}{1-\lambda}=\frac{k}{1-\lambda}$ and $\operatorname{minDS}=-\frac{k}{1-\lambda}$, as respectively the maximal and minimal discounted-sum value achievable with colors in $C$.

The key property that we will show is that gaps characterize equivalence classes of prefixes: for $w_{1}, w_{2} \in C^{*}$,

$$
\begin{equation*}
w_{1} \sim w_{2} \Longleftrightarrow \operatorname{gap}\left(w_{1}\right)=\operatorname{gap}\left(w_{2}\right) \tag{2}
\end{equation*}
$$

Once this is proven, it is left to determine the number of different gap values, which will correspond to the index of $\sim$. The right-to-left implication of $(2)$ is clear: if the gaps are $T$,


Figure 13 Minimal-state automaton of $\sim_{D_{S} \geq 0}$ for $\lambda=\frac{1}{2}$ and $C=\{-2,-1,0,1,2\}$. The value in a state is the gap characterizing the equivalence class of $\sim$ corresponding to that state. Here, $\frac{\sup C}{1-\lambda}=4$ and $\frac{\inf C}{1-\lambda}=-4$. The asymmetry around 0 comes from the $\geq 0$ inequality in the definition of the condition: when state -4 is reached, there is exactly one winning continuation $\left(2^{\omega}\right)$, but a state with gap value 4 would only have winning continuations (hence, it is part of state T). Notice that we can define a parity condition on top of this automaton that recognizes $\mathrm{DS}_{\lambda}^{\geq 0}$ : an infinite word is winning as long as it does not reach $\perp$.
all the continuations are winning; if the gaps are $\perp$, all the continuations are losing; else, for any continuation, the final discounted-sum values will have the same sign. We prove the left-to-right implication for each case of the disjunction from the statement and discuss the number of gap values.

We first assume $k<\frac{1}{\lambda}-1$. The case $k=0$ is trivial (as all words are winning) - we assume $k \geq 1$. The inequality $k<\frac{1}{\lambda}-1$ is equivalent to $\frac{1}{\lambda}>\frac{k}{1-\lambda}$. In this case, there are only three possible gaps:

- for $w \in 0^{*}, \operatorname{gap}(w)=0$.
- for $w \in 0^{*} c$ with $c \geq 1$, then $\frac{\mathrm{DS}_{\lambda}(w)}{\lambda^{|w|}}=\frac{c}{\lambda} \geq \frac{1}{\lambda}>\frac{k}{1-\lambda}=-\operatorname{minDS}-$ so for any word $w \in 0^{*} c C^{*}, \operatorname{gap}(w)=T$.
- for $w \in 0^{*} c C^{*}$ with $c \leq-1$, symmetrically, $\operatorname{gap}(w)=\perp$.

These three possible gaps clearly correspond to different equivalence classes of the right congruence $\sim$, so there are three such equivalence classes. Hence the minimal-state automaton $\mathcal{M}_{\sim}$ has three states $[\varepsilon]_{\sim},[1]_{\sim}$, and $[-1]_{\sim}$.

We now assume that $k \geq\left\lceil\frac{1}{\lambda}-1\right\rceil$. The left-to-right implication of (2) is clear in the cases in which all, or none, of the continuations are winning. The difficult case is when both $w_{1}$ and $w_{2}$ have a rational gap. We show that if their gaps are different rational numbers, then they have different winning continuations. We assume w.l.o.g. that $\operatorname{gap}\left(w_{2}\right)<\operatorname{gap}\left(w_{1}\right)$. We show that there is an infinite continuation that has a discounted sum exactly equal to $-\operatorname{gap}\left(w_{1}\right)$ : this infinite continuation is winning after $w_{1}$ but losing after $w_{2}$.

Showing that there exists $w \in C^{\omega}$ such that $\mathrm{DS}_{\lambda}(w)=-\operatorname{gap}\left(w_{1}\right)$ amounts to showing that there is a representation of $-\operatorname{gap}\left(w_{1}\right)$ in the (rational but not necessarily integral) base $\frac{1}{\lambda}$ with digits in $C$, with one digit before the decimal point. We can adapt the well-known greedy expansion [40] to our context to show this (details in Appendix B. Proposition 54).

It is left to show that there are finitely many gap values if and only if $\lambda$ equals $\frac{1}{n}$ for some integer $n \geq 1$. One implication is clear: if $\lambda=\frac{1}{n}$ for some integer $n \geq 1$, then there are finitely many possible gaps as gaps are then always integers between minDS and maxDS, $\top$, or $\perp$. We illustrate this implication by depicting the minimal-state automaton of $\sim$ for $\lambda=\frac{1}{2}$ and $k=2$ in Figure 13 The proof of the other implication is provided in Appendix B Proposition 55

- Corollary 51. Let $\lambda \in(0,1) \cap \mathbb{Q}, k \in \mathbb{N}$, and $C=[-k, k] \cap \mathbb{Z}$.
- If $k<\frac{1}{\lambda}-1$, then $\mathrm{DS}_{\lambda}^{\geq 0}$ is memoryless-determined.
- If $k \geq\left\lceil\frac{1}{\lambda}-1\right\rceil$, then $\mathrm{DS}_{\lambda}^{\geq 0}$ is chromatic-finite-memory determined if and only if $\lambda$ is equal to $\frac{1}{n}$ for some integer $n \geq 1$.

Proof. This follows from Propositions 49 and 50, thanks to Theorem 10. The only thing to clarify is that memoryless strategies suffice in case $k<\frac{1}{\lambda}-1$. The proof of Proposition 50 tells us that in this case, $\mathrm{DS}_{\lambda}^{\geq 0}$ is $\omega$-regular and can be recognized by a parity automaton that can be defined on top of $\mathcal{M}_{\sim} \otimes \mathcal{M}_{\text {triv }}$, which has three states. To use this skeleton as a memory, we can notice that the game is already over in states $[1]_{\sim}$ and $[-1]_{\sim}$ (as every continuation wins or every continuation loses). Thus, it is not necessary to use these states to play, and we can consider that we always stay in state $[\varepsilon]_{\sim}$.

### 6.2 Other winning conditions

Mean payoff. Let $C \subseteq \mathbb{Q}$ be non-empty. We define the mean-payoff function MP: $C^{\omega} \rightarrow$ $\mathbb{R} \cup\{-\infty, \infty\}$ such that for $w=c_{1} c_{2} \ldots \in C^{\omega}$,

$$
\mathrm{MP}(w)=\limsup _{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} c_{i} .
$$

We define the winning condition $\mathrm{MP}{ }^{\geq 0}=\left\{w \in C^{\omega} \mid \mathrm{MP}(w) \geq 0\right\}$ as the set of infinite words whose mean payoff is non-negative. This condition is $\mathcal{M}_{\text {triv }}$-prefix-independent for any set of colors. However, it is known that infinite-memory strategies may be required to play optimally in some infinite arenas [39, Section 8.10]; the example provided uses infinitely many colors. Here, we show that chromatic-finite-memory strategies do not suffice to play optimally even when $C=\{-1,1\}$. Let us analyze cycle-consistency of $\mathrm{MP}^{\geq 0}$. If we consider, for $n \in \mathbb{N}$,

$$
w_{n}=\underbrace{1 \ldots 1}_{n \text { times }} \underbrace{-1 \ldots-1}_{n+1 \text { times }},
$$

we have that $\left(w_{n}\right)^{\omega}$ is losing for all $n \in \mathbb{N}$, but the infinite word $w_{0} w_{1} w_{2} \ldots$ has a mean payoff of 0 and is thus winning. This shows directly that $M P \geq 0$ is not $\mathcal{M}_{\text {triviv }}$-cycle-consistent. The argument can be adapted to show that $M P{ }^{\geq 0}$ is not $\mathcal{M}$-cycle-consistent for any skeleton $\mathcal{M}$ (see Appendix B Lemma 57).

- Remark 52. As memoryless strategies suffice to play optimally for both players in finite arenas for mean-payoff games [19], winning condition $\mathrm{MP}^{\geq 0}$ distinguishes finite-memory determinacy in finite and in infinite arenas. For a skeleton $\mathcal{M}, \mathcal{M}$-determinacy in finite arenas has also been shown [23, 6] to be equivalent to the conjunction of a property about prefixes ( $\mathcal{M}$-monotony) and a property about cycles ( $\mathcal{M}$-selectivity). The concepts of $\mathcal{M}$-selectivity and $\mathcal{M}$-cycle-consistency share the similar idea that combining losing cycles cannot produce a winning word, but they are distinct notions with different quantification on the families of cycles considered. Here, $\mathrm{MP}^{\geq 0}$ is $\mathcal{M}_{\text {triv- }}$-selective but not $\mathcal{M}_{\text {triv-cycle-consistent }}$.

Total payoff. Let $C \subseteq \mathbb{Q}$ be non-empty. We define the total-payoff function TP: $C^{\omega} \rightarrow$ $\mathbb{R} \cup\{-\infty, \infty\}$ such that for $w=c_{1} c_{2} \ldots \in C^{\omega}$,

$$
\mathrm{TP}(w)=\limsup _{n \rightarrow \infty} \sum_{i=1}^{n} c_{i} .
$$

We define the winning condition $\mathrm{TP}^{\geq 0}=\left\{w \in C^{\omega} \mid \mathrm{TP}(w) \geq 0\right\}$ as the set of infinite words whose total payoff is non-negative.

The right congruence $\sim$ of $\mathrm{TP}^{\geq 0}$ does not have finite index, even for $C=\{-1,1\}$ : we indeed have that $(-1) \succ(-1)(-1) \succ \ldots$. Condition $\mathrm{TP}^{\geq 0}$ is therefore not $\mathcal{M}$-prefixindependent for any skeleton $\mathcal{M}$. We can also show that $\mathrm{TP}^{\geq 0}$ is not $\mathcal{M}$-cycle-consistent for
any $\mathcal{M}$, using the exact same argument as for $\mathrm{MP}^{\geq 0}$. Chromatic-finite-memory strategies are therefore insufficient to play optimally for $\mathrm{TP}^{\geq 0}$ in infinite arenas. Once again, this situation contrasts with the case of finite arenas, in which memoryless strategies suffice to play optimally [22].

## 7 Conclusion

We proved an equivalence between chromatic-finite-memory determinacy of a winning condition in games on infinite graphs and $\omega$-regularity of the corresponding language of infinite words, generalizing a result by Colcombet and Niwiński [17]. A "strategic" consequence of our result is that chromatic-finite-memory determinacy in one-player games of both players implies the seemingly stronger chromatic-finite-memory determinacy in zero-sum games. A "language-theoretic" consequence is a relation between the representation of $\omega$-regular languages by parity automata and the memory structures used to play optimally in zero-sum games, using as a tool the minimal-state automata classifying the equivalence classes of the right congruence.

For future work, one possible improvement over our result is to deduce tighter chromatic memory requirements in two-player games compared to one-player games. Our proof technique gives as an upper bound on the two-player memory requirements a product between the minimal-state automaton and a sufficient skeleton for one-player arenas, but smaller skeletons often suffice. We do not know whether the product with the minimal-state automaton is necessary in general in order to play optimally in two-player arenas (although it is necessary in Theorem 10 to describe $W$ using a parity automaton). This behavior contrasts with the case of finite arenas, in which it is known that a skeleton sufficient for both players in finite one-player arenas also suffices in finite two-player arenas [6, 8]. More generally, it would be interesting to characterize precisely the (chromatic) memory requirements of $\omega$-regular winning conditions, extending work on the subclass of Muller conditions [18, 11].
_ References
1 Dana Angluin and Dana Fisman. Regular $\omega$-languages with an informative right congruence. Inf. Comput., 278:104598, 2021. doi:10.1016/j.ic.2020.104598
2 Suguman Bansal, Swarat Chaudhuri, and Moshe Y. Vardi. Comparator automata in quantitative verification. CoRR, abs/1812.06569, 2018. URL: http://arxiv.org/abs/1812.06569 arXiv:1812.06569,
3 Alessandro Bianco, Marco Faella, Fabio Mogavero, and Aniello Murano. Exploring the boundary of half-positionality. Ann. Math. Artif. Intell., 62(1-2):55-77, 2011. doi:10.1007/ s10472-011-9250-1
4 Roderick Bloem, Krishnendu Chatterjee, and Barbara Jobstmann. Graph games and reactive synthesis. In Edmund M. Clarke, Thomas A. Henzinger, Helmut Veith, and Roderick Bloem, editors, Handbook of Model Checking, pages 921-962. Springer, 2018. doi:10.1007/978-3-319-10575-8_27
5 Udi Boker, Thomas A. Henzinger, and Jan Otop. The target discounted-sum problem. In 30th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2015, Kyoto, Japan, July 6-10, 2015, pages 750-761. IEEE Computer Society, 2015. doi:10.1109/LICS.2015.74
6 Patricia Bouyer, Stéphane Le Roux, Youssouf Oualhadj, Mickael Randour, and Pierre Vandenhove. Games where you can play optimally with arena-independent finite memory. In Igor Konnov and Laura Kovács, editors, 31st International Conference on Concurrency Theory, CONCUR 2020, September 1-4, 2020, Vienna, Austria (Virtual Conference), volume

171 of LIPIcs, pages 24:1-24:22. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.CONCUR.2020.24
7 Patricia Bouyer, Stéphane Le Roux, and Nathan Thomasset. Finite-memory strategies in two-player infinite games. In Florin Manea and Alex Simpson, editors, 30th EACSL Annual Conference on Computer Science Logic, CSL 2022, February 14-19, 2022, Göttingen, Germany (Virtual Conference), volume 216 of LIPIcs, pages 8:1-8:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.CSL.2022.8
8 Patricia Bouyer, Youssouf Oualhadj, Mickael Randour, and Pierre Vandenhove. Arenaindependent finite-memory determinacy in stochastic games. In Serge Haddad and Daniele Varacca, editors, 32nd International Conference on Concurrency Theory, CONCUR 2021, August 24-27, 2021, Virtual Conference, volume 203 of LIPIcs, pages 26:1-26:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.CONCUR.2021.26
9 Patricia Bouyer, Mickael Randour, and Pierre Vandenhove. Characterizing omega-regularity through finite-memory determinacy of games on infinite graphs. In Petra Berenbrink and Benjamin Monmege, editors, 39th International Symposium on Theoretical Aspects of Computer Science, STACS 2022, March 15-18, 2022, Marseille, France (Virtual Conference), volume 219 of LIPIcs, pages 16:1-16:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.STACS. 2022.16
10 Tomás Brázdil, Václav Brozek, and Kousha Etessami. One-counter stochastic games. In Kamal Lodaya and Meena Mahajan, editors, IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2010, December 15-18, 2010, Chennai, India, volume 8 of LIPIcs, pages 108-119. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2010. doi:10.4230/LIPIcs.FSTTCS.2010.108
11 Antonio Casares. On the minimisation of transition-based Rabin automata and the chromatic memory requirements of Muller conditions. In Florin Manea and Alex Simpson, editors, 30th EACSL Annual Conference on Computer Science Logic, CSL 2022, February 14-19, 2022, Göttingen, Germany (Virtual Conference), volume 216 of LIPIcs, pages 12:1-12:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.CSL. 2022.12.
12 Antonio Casares, Thomas Colcombet, and Nathanaël Fijalkow. Optimal transformations of games and automata using Muller conditions. In Nikhil Bansal, Emanuela Merelli, and James Worrell, editors, 48 th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference), volume 198 of LIPIcs, pages 123:1-123:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. doi: 10.4230/LIPIcs.ICALP. 2021.123

13 Krishnendu Chatterjee and Laurent Doyen. Perfect-information stochastic games with generalized mean-payoff objectives. In Martin Grohe, Eric Koskinen, and Natarajan Shankar, editors, Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '16, New York, NY, USA, July 5-8, 2016, pages 247-256. ACM, 2016. doi:10.1145/2933575.2934513.
14 Krishnendu Chatterjee, Laurent Doyen, and Thomas A. Henzinger. Expressiveness and closure properties for quantitative languages. In Proceedings of the 24 th Annual IEEE Symposium on Logic in Computer Science, LICS 2009, 11-14 August 2009, Los Angeles, CA, USA, pages 199-208. IEEE Computer Society, 2009. doi:10.1109/LICS.2009.16.
15 Krishnendu Chatterjee and Nathanaël Fijalkow. Infinite-state games with finitary conditions. In Simona Ronchi Della Rocca, editor, Computer Science Logic 2013 (CSL 2013), CSL 2013, September 2-5, 2013, Torino, Italy, volume 23 of LIPIcs, pages 181-196. Schloss Dagstuhl -Leibniz-Zentrum für Informatik, 2013. doi:10.4230/LIPIcs.CSL.2013.181
16 Thomas Colcombet, Nathanaël Fijalkow, and Florian Horn. Playing safe. In Venkatesh Raman and S. P. Suresh, editors, 34th International Conference on Foundation of Software Technology and Theoretical Computer Science, FSTTCS 2014, December 15-17, 2014, New Delhi, India, volume 29 of LIPIcs, pages 379-390. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2014. doi:10.4230/LIPIcs.FSTTCS.2014.379

17 Thomas Colcombet and Damian Niwiński. On the positional determinacy of edge-labeled games. Theor. Comput. Sci., 352(1-3):190-196, 2006. doi:10.1016/j.tcs.2005.10.046
18 Stefan Dziembowski, Marcin Jurdzinski, and Igor Walukiewicz. How much memory is needed to win infinite games? In Proceedings, 12th Annual IEEE Symposium on Logic in Computer Science, Warsaw, Poland, June 29 - July 2, 1997, pages 99-110. IEEE Computer Society, 1997. doi:10.1109/LICS.1997.614939.

19 Andrzej Ehrenfeucht and Jan Mycielski. Positional strategies for mean payoff games. Int. Journal of Game Theory, 8(2):109-113, 1979. doi:10.1007/BF01768705
20 Hugo Gimbert. Parity and exploration games on infinite graphs. In Jerzy Marcinkowski and Andrzej Tarlecki, editors, Computer Science Logic, 18th International Workshop, CSL 2004, 13th Annual Conference of the EACSL, Karpacz, Poland, September 20-24, 2004, Proceedings, volume 3210 of Lecture Notes in Computer Science, pages 56-70. Springer, 2004. doi:10.1007/978-3-540-30124-0_8.
21 Hugo Gimbert and Edon Kelmendi. Submixing and shift-invariant stochastic games. CoRR, abs/1401.6575, 2014. arXiv:1401.6575
22 Hugo Gimbert and Wieslaw Zielonka. When can you play positionally? In Jirí Fiala, Václav Koubek, and Jan Kratochvíl, editors, Mathematical Foundations of Computer Science 2004, 29th International Symposium, MFCS 2004, Prague, Czech Republic, August 22-27, 2004, Proceedings, volume 3153 of Lecture Notes in Computer Science, pages 686-697. Springer, 2004. doi:10.1007/978-3-540-28629-5_53.

23 Hugo Gimbert and Wieslaw Zielonka. Games where you can play optimally without any memory. In Martín Abadi and Luca de Alfaro, editors, CONCUR 2005 - Concurrency Theory, 16th International Conference, CONCUR 2005, San Francisco, CA, USA, August 23-26, 2005, Proceedings, volume 3653 of Lecture Notes in Computer Science, pages 428-442. Springer, 2005. doi:10.1007/11539452_33

24 Hugo Gimbert and Wieslaw Zielonka. Pure and stationary optimal strategies in perfectinformation stochastic games with global preferences. Unpublished, 2009. URL: https: //hal.archives-ouvertes.fr/hal-00438359.
25 Erich Grädel and Igor Walukiewicz. Positional determinacy of games with infinitely many priorities. Log. Methods Comput. Sci., 2(4), 2006. doi:10.2168/LMCS-2(4:6)2006
26 Stefan Kiefer, Richard Mayr, Mahsa Shirmohammadi, Patrick Totzke, and Dominik Wojtczak. How to play in infinite MDPs (invited talk). In Artur Czumaj, Anuj Dawar, and Emanuela Merelli, editors, 47 th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany (Virtual Conference), volume 168 of LIPIcs, pages 3:1-3:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi: 10.4230/LIPIcs.ICALP. 2020.3

27 Nils Klarlund. Progress measures, immediate determinacy, and a subset construction for tree automata. Ann. Pure Appl. Log., 69(2-3):243-268, 1994. doi:10.1016/0168-0072(94) 90086-8
28 Eryk Kopczyński. Half-positional determinacy of infinite games. In Michele Bugliesi, Bart Preneel, Vladimiro Sassone, and Ingo Wegener, editors, Automata, Languages and Programming, 33rd International Colloquium, ICALP 2006, Venice, Italy, July 10-14, 2006, Proceedings, Part II, volume 4052 of Lecture Notes in Computer Science, pages 336-347. Springer, 2006. doi:10.1007/11787006_29
29 Eryk Kopczyński. Omega-regular half-positional winning conditions. In Jacques Duparc and Thomas A. Henzinger, editors, Computer Science Logic, 21st International Workshop, CSL 2007, 16th Annual Conference of the EACSL, Lausanne, Switzerland, September 11-15, 2007, Proceedings, volume 4646 of Lecture Notes in Computer Science, pages 41-53. Springer, 2007. doi:10.1007/978-3-540-74915-8_7.
30 Eryk Kopczyński. Half-positional Determinacy of Infinite Games. PhD thesis, Warsaw University, 2008.

31 Alexander Kozachinskiy. One-to-two-player lifting for mildly growing memory. In Petra Berenbrink and Benjamin Monmege, editors, 39th International Symposium on Theoretical Aspects of Computer Science, STACS 2022, March 15-18, 2022, Marseille, France (Virtual Conference), volume 219 of LIPIcs, pages 43:1-43:21. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.STACS.2022.43
32 Stéphane Le Roux. Time-aware uniformization of winning strategies. In Marcella Anselmo, Gianluca Della Vedova, Florin Manea, and Arno Pauly, editors, Beyond the Horizon of Computability - 16th Conference on Computability in Europe, CiE 2020, Fisciano, Italy, June 29 - July 3, 2020, Proceedings, volume 12098 of Lecture Notes in Computer Science, pages 193-204. Springer, 2020. doi:10.1007/978-3-030-51466-2_17
33 Stéphane Le Roux and Arno Pauly. Extending finite-memory determinacy to multi-player games. Inf. Comput., 261(Part):676-694, 2018. doi:10.1016/j.ic.2018.02.024
34 Stéphane Le Roux, Arno Pauly, and Mickael Randour. Extending finite-memory determinacy by Boolean combination of winning conditions. In Sumit Ganguly and Paritosh K. Pandya, editors, 38th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2018, December 11-13, 2018, Ahmedabad, India, volume 122 of LIPIcs, pages 38:1-38:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018. doi: 10.4230/LIPIcs.FSTTCS.2018.38

35 Oded Maler and Ludwig Staiger. On syntactic congruences for omega-languages. Theor. Comput. Sci., 183(1):93-112, 1997. doi:10.1016/S0304-3975(96)00312-X
36 Andrzej Wlodzimierz Mostowski. Regular expressions for infinite trees and a standard form of automata. In Andrzej Skowron, editor, Computation Theory - Fifth Symposium, Zaborów, Poland, December 3-8, 1984, Proceedings, volume 208 of Lecture Notes in Computer Science, pages 157-168. Springer, 1984. doi:10.1007/3-540-16066-3_15
37 Anil Nerode. Linear automaton transformations. Proceedings of the American Mathematical Society, 9(4):541-544, 1958. URL: http://www.jstor.org/stable/2033204 doi:10.2307/ 2033204.

38 Amir Pnueli. The temporal logic of programs. In 18th Annual Symposium on Foundations of Computer Science, Providence, Rhode Island, USA, 31 October - 1 November 1977, pages 46-57. IEEE Computer Society, 1977. doi:10.1109/SFCS.1977.32
39 Martin L. Puterman. Markov Decision Processes: Discrete Stochastic Dynamic Programming. Wiley Series in Probability and Statistics. Wiley, 1994. doi:10.1002/9780470316887.
40 Alfréd Rényi. Representations for real numbers and their ergodic properties. Acta Mathematica Academiae Scientiarum Hungarica, 8:477-493, 1957. doi:10.1007/BF02020331
41 Lloyd S. Shapley. Stochastic games. Proceedings of the National Academy of Sciences, 39(10):1095-1100, 1953. doi:10.1073/pnas.39.10.1095
42 Ludwig Staiger. Finite-state omega-languages. J. Comput. Syst. Sci., 27(3):434-448, 1983. doi:10.1016/0022-0000(83)90051-X
43 Wieslaw Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. Theor. Comput. Sci., 200(1-2):135-183, 1998. doi:10.1016/S0304-3975(98) 00009-7.

## A Proof of claim from Section 1

We argue that the result on the strategy complexity of finitary games in [15, Theorem 2], mentioned in Section 1 (paragraph Related works), also apply to our setting. The main difference is that our setting considers arenas with edges labeled with colors rather than states. As we have discussed, this difference may in general have an impact on the strategy complexity [25, 17]. However, we argue that this difference has no impact for the strategy complexity of finitary games as defined in [15]. A similar argument was stated in [17] to transfer the memoryless determinacy of parity conditions from state-labeled to edge-labeled arenas. We refer to [15] for formal definitions of finitary conditions. Informally, finitary Büchi is a winning condition defined over the alphabet of colors $C=\{0,1\}$. It contains the infinite words for which there exists a uniform bound $N \in \mathbb{N}$ such that each 1 is followed by a 0 after at most $N$ steps.

Lemma 53 ([15 Theorem 2] for edge-labeled arenas). For finitary Büchi games, $\mathcal{P}_{1}$ has a memoryless optimal strategy in all infinite (edge-labeled) arenas.

Proof (sketch). Let $C=\{0,1\}$. Let $\mathcal{A}=\left(S, S_{1}, S_{2}, E\right)$ be an (edge-labeled) arena, as defined in Section 2, and $W$ be a finitary Büchi condition. We transform $\mathcal{A}$ into a state-labeled arena $\mathcal{A}^{\prime}=\left(S^{\prime}, S_{1}^{\prime}, S_{2}^{\prime}, E^{\prime}, c\right)$ such that $E^{\prime} \subseteq S \times S$ and $c: S \rightarrow C$ labels each state with a color. For each edge $e \in E$, we define a new state $s_{e}$ to which we assign the color col $(e)$, and we insert this state between in $(e)$ and out $(e)$. To any other state, we assign the color 1 . We define

```
\(S_{1}^{\prime}=S_{1} \uplus\left\{s_{e} \mid e \in E, \operatorname{in}(e) \in S_{1}\right\}, S_{2}^{\prime}=S_{2} \uplus\left\{s_{e} \mid e \in E, \operatorname{in}(e) \in S_{2}\right\}, S^{\prime}=S_{1}^{\prime} \uplus S_{2}^{\prime}\),
\(E^{\prime}=\left\{\left(\operatorname{in}(e), s_{e}\right) \mid e \in E\right\} \uplus\left\{\left(s_{e}, \operatorname{out}(e)\right) \mid e \in E\right\}\),
for \(s \in S, c(s)=1\), and for \(e \in E, c\left(s_{e}\right)=\operatorname{col}(e)\).
```

There is a natural bijection between strategies on $\mathcal{A}$ and strategies on $\mathcal{A}^{\prime}$ which preserves the memoryless feature of strategies. Moreover, the winning feature of strategies is also preserved: the infinite words of colors seen in the state-labeled arena will be the same as the ones in the edge-labeled arenas, with extra 1's in every other position. This alteration does not change the winning status of infinite words for the finitary Büchi condition. We can therefore reuse [15, Theorem 2] on $\mathcal{A}^{\prime}$, and recover a memoryless optimal strategy of $\mathcal{P}_{1}$ on $\mathcal{A}$.

An example showing that $\mathcal{P}_{2}$ needs infinite memory to play optimally in some statelabeled arena for finitary Büchi conditions [15] Example 1] can also be easily transformed to work with edge-labeled arenas. Similar arguments can be used to show that $\mathcal{P}_{1}$ has optimal strategies based on a skeleton for finitary parity conditions in edge-labeled arenas.

## B Missing proofs of Section 6

We prove statements that were given without proof in Section 6.1 about the discountedsum winning condition: Proposition 49, as well as the two properties used in the proof of Proposition 50 (Propositions 54 and 55 .

Proposition 49. For all bounded $C \subseteq \mathbb{Q}, \lambda \in(0,1) \cap \mathbb{Q}$, winning condition $\mathrm{DS}_{\lambda}^{\geq 0}$ is $\mathcal{M}_{\text {triv-cycle-consistent }}$.

Proof. Let $w \in C^{*}$. We show that $\left(\mathrm{c}_{\mathcal{M}_{\text {triv }}}^{\text {lose } w}\right)^{\omega} \subseteq w^{-1} \overline{\mathrm{DS}_{\lambda}^{\geq 0}}$ - we discuss how to adapt the proof to show that $\left(\mathrm{c} \Gamma_{\mathcal{M}_{\text {triv }}}^{\text {win } w}\right)^{\omega} \subseteq w^{-1} \mathrm{DS}_{\lambda}^{\geq 0}$ at the end. Let $w_{1}, w_{2}, \ldots \in \mathrm{c} \Gamma_{\mathcal{M}_{\text {triv }}}^{\text {lose, }}$. We want to show that $w w_{1} w_{2} \ldots \in \mathrm{DS}_{\lambda}^{\geq 0}$, i.e., that $\mathrm{DS}_{\lambda}\left(w w_{1} w_{2} \ldots\right)<0$.

For $k \geq 1$, as $w_{k} \in \Gamma_{\mathcal{M}_{\text {triv }}}^{\text {lose }, w}$, we have $\mathrm{DS}_{\lambda}\left(w w_{k}^{\omega}\right)<0$. Since, moreover,

$$
\begin{aligned}
\mathrm{DS}_{\lambda}\left(w w_{k}^{\omega}\right) & =\mathrm{DS}_{\lambda}(w)+\lambda^{|w|} \mathrm{DS}_{\lambda}\left(w_{k}^{\omega}\right) \\
& =\mathrm{DS}_{\lambda}(w)+\lambda^{|w|} \sum_{i=0}^{\infty} \lambda^{i\left|w_{k}\right|} \mathrm{DS}_{\lambda}\left(w_{k}\right) \\
& =\mathrm{DS}_{\lambda}(w)+\lambda^{|w|} \mathrm{DS}_{\lambda}\left(w_{k}\right) \frac{1}{1-\lambda^{\left|w_{k}\right|}}
\end{aligned}
$$

we obtain

$$
\begin{equation*}
\mathrm{DS}_{\lambda}\left(w_{k}\right)<-\mathrm{DS}_{\lambda}(w) \frac{1-\lambda^{\left|w_{k}\right|}}{\lambda^{|w|}} \tag{3}
\end{equation*}
$$

In particular, for $k=1$, there exists $\varepsilon>0$ such that

$$
\begin{equation*}
\mathrm{DS}_{\lambda}\left(w_{1}\right)=-\varepsilon-\mathrm{DS}_{\lambda}(w) \frac{1-\lambda^{\left|w_{1}\right|}}{\lambda^{|w|}} \tag{4}
\end{equation*}
$$

We have that

$$
\begin{aligned}
\mathrm{DS}_{\lambda}\left(w w_{1} w_{2} \ldots\right) & =\mathrm{DS}_{\lambda}(w)+\lambda^{|w|} \sum_{k=1}^{\infty} \lambda^{\sum_{i=1}^{k-1}\left|w_{i}\right|} \mathrm{DS}_{\lambda}\left(w_{k}\right) \\
& \leq \mathrm{DS}_{\lambda}(w)-\lambda^{|w|} \varepsilon-\lambda^{|w|} \sum_{k=1}^{\infty} \lambda^{\sum_{i=1}^{k-1}\left|w_{i}\right|} \mathrm{DS}_{\lambda}(w) \frac{1-\lambda^{\left|w_{k}\right|}}{\lambda^{|w|}} \\
& \leq \mathrm{DS}_{\lambda}(w)-\lambda^{|w|} \varepsilon-\mathrm{DS}_{\lambda}(w) \sum_{k=1}^{\infty} \lambda^{\sum_{i=1}^{k-1}\left|w_{i}\right|}\left(1-\lambda^{\left|w_{k}\right|}\right)
\end{aligned}
$$

where the second line uses (4) for $k=1$, and (3) for $k \geq 2$. The series

$$
\sum_{k=1}^{\infty} \lambda^{\sum_{i=1}^{k-1}\left|w_{i}\right|}\left(1-\lambda^{\left|w_{k}\right|}\right)=\sum_{k=1}^{\infty} \lambda^{\sum_{i=1}^{k-1}\left|w_{i}\right|}-\lambda^{\sum_{i=1}^{k}\left|w_{i}\right|}
$$

is telescoping (we can expand it as $1-\lambda^{\left|w_{1}\right|}+\lambda^{\left|w_{1}\right|}-\lambda^{\left|w_{1}\right|+\left|w_{2}\right|}+\lambda^{\left|w_{1}\right|+\left|w_{2}\right|}-\ldots$ ). As $\lim _{k \rightarrow \infty} \lambda^{\sum_{i=1}^{k}\left|w_{i}\right|}=0$, this series converges to 1 . We conclude that

$$
\mathrm{DS}_{\lambda}\left(w w_{1} w_{2} \ldots\right) \leq-\lambda^{|w|} \varepsilon<0,
$$

as required. A proof that $\left(c \Gamma_{\mathcal{M}_{\text {triv }}}^{w i n}\right)^{\omega} \subseteq w^{-1} \mathrm{DS}_{\lambda}^{\geq 0}$ can be done in a similar way, with no need to extract an $\varepsilon$ as we are then only looking for a non-strict inequality.

We now prove the two properties used in Proposition 50 whose proofs were omitted. We use notations from the proof of Proposition 50 itself.

Proposition 54. Let $\lambda \in(0,1) \cap \mathbb{Q}, k \in \mathbb{Z}$ such that $k \geq\left\lceil\frac{1}{\lambda}-1\right\rceil$, and $C=[-k, k] \cap \mathbb{Z}$. For any real number $x, \frac{-k}{1-\lambda} \leq x \leq \frac{k}{1-\lambda}$, there exists $w \in C^{\omega}$ such that $x=\mathrm{DS}_{\lambda}(w)$.
Proof. This problem can be rephrased as a number representation problem: we are looking for a sequence of "digits" $\left(x_{i}\right)_{i \geq 0}$ in $C$ such that $x=x_{0} \cdot x_{1} x_{2} \ldots$ in base $\frac{1}{\lambda}$, i.e., such that

$$
x=\sum_{i=0}^{\infty} x_{i} \lambda^{i}
$$

Notice that $\sum_{i=0}^{\infty} x_{i} \lambda^{i}=\mathrm{DS}_{\lambda}\left(x_{0} x_{1} \ldots\right)$. It is known that every number $x \in[0,1)$ has (at least) one representation $0 . x_{1} x_{2} \ldots$ in base $\frac{1}{\lambda}$ with digits in $\left\{0,1, \ldots,\left\lceil\frac{1}{\lambda}-1\right\rceil\right\}$, and one such representation can be found using the greedy expansion [40].

We adapt this greedy expansion to our setting (for a potentially greater $x$ and larger set $C$ of digits). Let $x \in \mathbb{R}$ such that $\frac{-k}{1-\lambda} \leq x \leq \frac{k}{1-\lambda}$. We deal with the case $x \geq 0-$ the negative case is symmetric. We set $x_{0}=\min (k,\lfloor x\rfloor)$; clearly, $x_{0} \leq x$. Then inductively, if $x_{0}, \ldots, x_{l-1}$ have been defined, we define $x_{l}$ as the greatest integer in $\{0, \ldots, k\}$ such that

$$
\sum_{i=0}^{l} x_{i} \lambda^{i} \leq x
$$

The series $\sum_{i=0}^{\infty} x_{i} \lambda^{i}$ is converging, as every term is non-negative and it is bounded from above by $x$. We show that it converges to $x$, which ends the proof. Let $\varepsilon \geq 0$. Assume by contradiction that $\sum_{i=0}^{\infty} x_{i} \lambda^{i} \leq x-\varepsilon$. Let $j$ be the least index such that $\lambda^{j} \leq \varepsilon$. Clearly, for any $j^{\prime} \geq j, x_{j^{\prime}}=k$ - otherwise, a greater digit could have been picked during the inductive greedy selection. Still, not every digit $x_{0}, x_{1}, \ldots$ can be $k$, as $\sum_{i=0}^{\infty} k \lambda^{i}=\frac{k}{1-\lambda}>x-\varepsilon$. Let $l$ be the greatest index such that $x_{l} \neq k$. We show that a digit $\geq x_{l}+1$ should have been picked instead of $x_{l}$ for the digit at index $l$, leading to a contradiction. To do so, it is sufficient to show that

$$
\left(x_{l}+1\right) \lambda^{l}+\sum_{i=0}^{l-1} x_{i} \lambda^{i} \leq x .
$$

We have

$$
\begin{aligned}
\left(x_{l}+1\right) \lambda^{l}+\sum_{i=0}^{l-1} x_{i} \lambda^{i} & =\sum_{i=0}^{\infty} x_{i} \lambda^{i}+\lambda^{l}-\sum_{i=l+1}^{\infty} x_{i} \lambda^{i} \\
& =\sum_{i=0}^{\infty} x_{i} \lambda^{i}+\lambda^{l}-\sum_{i=l+1}^{\infty} k \lambda^{i} \quad \text { as } x_{i}=k \text { for } i \geq l+1 \\
& =\sum_{i=0}^{\infty} x_{i} \lambda^{i}+\lambda^{l}-\frac{k \lambda^{l+1}}{1-\lambda} \\
& =\sum_{i=0}^{\infty} x_{i} \lambda^{i}+\lambda^{l}\left(1-\frac{k \lambda}{1-\lambda}\right) .
\end{aligned}
$$

Since $\frac{k \lambda}{1-\lambda} \geq \frac{\left\lceil\frac{1}{\lambda}-1\right\rceil \lambda}{1-\lambda}=\left\lceil\frac{1-\lambda}{\lambda}\right\rceil \frac{\lambda}{1-\lambda} \geq 1$, we have that $\lambda^{l}\left(1-\frac{k \lambda}{1-\lambda}\right) \leq 0$, which implies that

$$
\left(x_{l}+1\right) \lambda^{l}+\sum_{i=0}^{l-1} x_{i} \lambda^{i} \leq \sum_{i=0}^{\infty} x_{i} \lambda^{i}<x
$$

a contradiction. We conclude that $x=\mathrm{DS}_{\lambda}\left(x_{0} x_{1} \ldots\right)$.

- Proposition 55. Let $\lambda \in(0,1) \cap \mathbb{Q}, k \in \mathbb{Z}$ such that $k \geq\left\lceil\frac{1}{\lambda}-1\right\rceil$, and $C=[-k, k] \cap \mathbb{Z}$. If $\lambda \neq \frac{1}{n}$ for all integers $n \geq 1$, the gap function takes infinitely many values.

Proof. We assume that $\lambda=\frac{p}{q}$ with $p, q \in \mathbb{N}$ co-prime, $p \geq 2$ and $q>p$, and we show that the gap function takes infinitely many values. To do so, we exhibit an infinite word $w=c_{1} c_{2} \ldots \in C^{\omega}$ such that the sequence of rationals $\left(\operatorname{gap}\left(c_{1} \ldots c_{i}\right)\right)_{i \geq 1}$ never takes the same value twice.

We will use the following inductive property of gaps: for $w \in C^{*}$ and $c \in C$,

$$
\begin{equation*}
\operatorname{gap}(w c)=\frac{\operatorname{gap}(w)}{\lambda}+\frac{\lambda^{|w|-1} c}{\lambda^{|w|}}=\frac{1}{\lambda}(\operatorname{gap}(w)+c), \tag{5}
\end{equation*}
$$

unless some gap in this equation equals $\rceil$ or $\perp$. Notice that under our hypotheses, $\left\lceil\frac{1}{\lambda}-1\right\rceil=$ $\left\lfloor\frac{1}{\lambda}\right\rfloor$ (this equality is not verified when $\lambda=\frac{1}{n}$ for some integer $n \geq 1$ ).

We set $c_{1}=1$. Then, $\operatorname{gap}\left(c_{1}\right)=\frac{1}{\lambda}=\frac{q}{p}$. Inductively, if $c_{1}, \ldots, c_{i-1}$ are defined, we set $c_{i}=-\left\lfloor\operatorname{gap}\left(c_{1} \ldots c_{i-1}\right)\right\rfloor$ (we remove the largest possible integer from the current gap, while keeping a positive gap value). We set $g_{i}=\operatorname{gap}\left(c_{1} \ldots c_{i}\right)$ for conciseness.

We first show that if all $g_{i}$ 's are rational, then no two $g_{i}$ 's can be equal. To do so, we show inductively that the reduced denominator of fraction $g_{i}$ is $p^{i}$ for all $i \geq 1$. This is true for $i=1$. For $i>1$, assume it is true for $i-1$. Then, $g_{i-1}=\frac{m}{p^{i-1}}$ for some $m$ co-prime with $p$. Using (5),

$$
g_{i}=\frac{1}{\lambda} \cdot\left(g_{i-1}+c_{i}\right)=\frac{1}{\lambda} \cdot\left(\frac{m}{p^{i-1}}+c_{i}\right)=\frac{q\left(m+c_{i} p^{i-1}\right)}{p^{i}} .
$$

This last fraction is irreducible: $q$ and $p$ are co-prime, and the fact that $m$ and $p$ are co-prime implies that $m+c_{i} p^{i-1}$ and $p$ are co-prime.

We now prove by induction that our scheme is well-defined by showing that for all $i \geq 1$, $c_{i} \in C$ and $0<g_{i} \leq \frac{1}{\lambda}$. This is true for $i=1$ (as $k \geq 1$ for any possible value of $\lambda$ ). For $i>1$, if this is true for $i-1$, then $-c_{i}=\left\lfloor g_{i-1}\right\rfloor \leq\left\lfloor\frac{1}{\lambda}\right\rfloor$, so $c_{i} \in C$. Moreover, $g_{i}=\frac{1}{\lambda}\left(g_{i-1}+c_{i}\right)$. Since $g_{i}$ 's cannot be integers (as their reduced denominator is not 1 by an earlier property), we have that $g_{i-1}+c_{i}$ is not an integer either. Therefore, $0<g_{i-1}+c_{i}<1$, so $0<g_{i}<\frac{1}{\lambda}$. As $\frac{1}{\lambda}<\frac{k}{1-\lambda}=\operatorname{maxDS}$, the values of the considered gaps are never $T$ or $\perp$.

- Remark 56. For $\lambda \in(0,1) \cap \mathbb{Q}$ with $\lambda \neq \frac{1}{n}$ for all integers $n \geq 1, k \geq\left\lceil\frac{1}{\lambda}-1\right\rceil$, and $C=[-k, k] \cap \mathbb{Z}$, Proposition 50 along with Theorem 10 implies that any chromatic skeleton is insufficient to play optimally (for at least one player). However, this does not directly give an explicit arena in which some player requires infinite memory to play optimally. Here, we show how to construct such an arena given the extra results from Appendix B

The proof of Proposition 55 gives us $c_{1} c_{2} \ldots \in C^{\omega}$ such that $\left(\operatorname{gap}\left(c_{1} \ldots c_{i}\right)\right)_{i \geq 1}$ is a sequence of distinct values in [0, maxDS]. Hence, by compacity of [ $0, \operatorname{maxDS}$ ], there exists a subsequence $\left(i_{j}\right)_{j \geq 1}$ and $x \in[0, \operatorname{maxDS}]$ such that $\lim _{j \rightarrow \infty} \operatorname{gap}\left(c_{1} \ldots c_{i_{j}}\right)=x$. We can further extract a subsequence such that either all elements are greater than $x$, or all elements are less than $x$. We assume w.l.o.g. that for all $j \geq 1$, gap $\left(c_{1} \ldots c_{i_{j}}\right)<x$ (this implies that $x \neq 0$ ). The proof is symmetric if all the gaps are greater than $x$ (which would imply that $x \neq \operatorname{maxDS}$ ).

By Proposition 54 for all $\varepsilon>0$ sufficiently small, there exists $w_{\varepsilon} \in C^{\omega}$ such that $\mathrm{DS}_{\lambda}\left(w_{\varepsilon}\right)=-x+\varepsilon$. We can define an infinite arena in which $\mathcal{P}_{2}$ needs infinite memory to win, depicted in Figure 14 In this arena, $\mathcal{P}_{1}$ may choose to reach a gap arbitrarily close (but not equal) to $x$ in $s_{2}$, and then $\mathcal{P}_{2}$ is always able to bring the discounted sum below 0 by choosing a word reaching a discounted sum sufficiently close to $-x$.

We now prove the claim about the mean-payoff condition (Section 6.2) stating that it is not $\mathcal{M}$-cycle-consistent for any $\mathcal{M}$.

- Lemma 57. For all skeletons $\mathcal{M}, \mathrm{MP}^{\geq 0}$ is not $\mathcal{M}$-cycle-consistent.

Proof. Let $\mathcal{M}=\left(M, m_{\text {init }}, \alpha_{\text {upd }}\right)$ be a skeleton. For $n \in \mathbb{N}$, let

$$
w_{n}=\underbrace{1 \ldots 1}_{n \text { times }} \underbrace{-1 \ldots-1}_{n+1 \text { times }} .
$$



Figure 14 Infinite arena in which $\mathcal{P}_{2}$ needs infinite memory to win from $s_{1}$ for condition $\mathrm{DS}_{\lambda}^{\geq 0}$ for $\lambda \in(0,1) \cap \mathbb{Q}$ with $\lambda \neq \frac{1}{n}$ for all integers $n \geq 1, k=\left\lceil\frac{1}{\lambda}-1\right\rceil$, and $C=[-k, k] \cap \mathbb{Z}$.

Let $M_{n}$ be the set of states $m \in M$ such that there exists $k \geq 1$ with $m=\alpha_{\text {upd }}^{*}\left(m, w_{n}^{k}\right)$. Each $M_{n}$ is non-empty as, $M$ being finite, iterating the function $m \mapsto \alpha_{\text {upd }}^{*}\left(m, w_{n}\right)$ necessarily goes multiple times through at least one state. Let $m \in M$ be a state in set $M_{n}$ for infinitely many $n$ 's, and $w$ be any finite word in $c \Pi_{m_{\text {init }}, m}$. Let $n_{1}, n_{2}, \ldots$ be the indices such that $m \in M_{n_{i}}$, and let $k_{1}, k_{2}, \ldots$ be such that $m=\alpha_{\text {upd }}^{*}\left(m, w_{n_{i}}^{k_{i}}\right)$. Every word $w_{n_{i}}^{k_{i}}$ is a losing cycle after any finite word (in particular after $w$ ). However, it is possible to find a subsequence of $\left(w_{n_{i}}^{k_{i}}\right)_{i \geq 1}$ with a non-negative mean payoff by always taking a word $w_{n_{i}}$ that bring the sum of the colors above 0 during the first $n_{i} 1$ 's.


[^0]:    1 We argue in Appendix A that their result also applies to our slightly different setting.

