On the Convergence of European Lookback Options with Floating Strike in the Binomial Model

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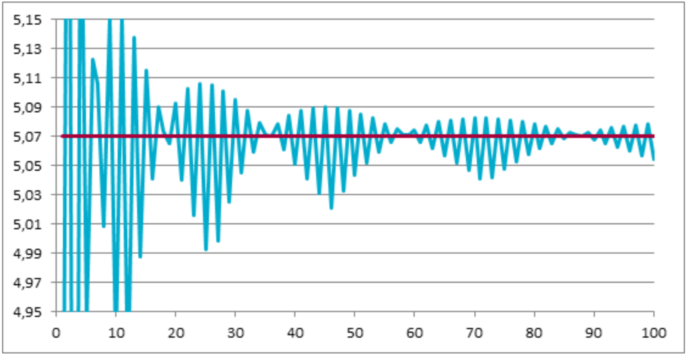
An intriguing function



Asymptotic expansion

Study of the behavior of the option price as a function of the number of steps n in the Cox-Ross-Rubinstein model. In particular, to write this price as

$$\Pi_n^{fl} = \Pi_{BS}^{fl} + \frac{\Pi_1}{\sqrt{n}} + \frac{\Pi_2}{n} + O\left(\frac{1}{n^{3/2}}\right),$$

where the coefficients Π_i are bounded functions of n.

Approximation for vanilla options

Diener-Diener (2004)

The evaluation of the European vanilla call in the Cox-Ross-Rubinstein model satisfies the relation

$$C_n^V = C_{BS}^V + \frac{S_0 e^{-\frac{d_1^2}{2}}}{24\sigma\sqrt{2\pi T}} \frac{A - 12\sigma^2 T(\Delta_n^2 - 1)}{n} + O\left(\frac{1}{n^{3/2}}\right),$$

with

- $\Delta_n = 1 - 2\left\{\frac{\ln(S_0/K) - n\sigma\sqrt{T/n}}{2\sigma\sqrt{T/n}}\right\},$
- $A = -\sigma^2 T(6 + d_1^2 + d_2^2) + 4rT(d_1^2 - d_2^2) - 12r^2T^2.$

Other results

Results for some other options are also known:

- binary options by Chang and Palmer (2007);
- barrier options by Lin and Palmer (2013).

Lookback option with floating strike

In the case of the European option:

- Payoff for the call: $f(T) = S_T - \min_{0 \le t \le T} S_t$,
- Payoff for the put: $f(T) = \max_{0 \le t \le T} S_t - S_T$,

where S_t is the price of the underlying at time t and T the time to maturity.

Other common notations

- r the risk free interest rate,
- σ the volatility of the underlying,
- $u_n = e^{\sigma\sqrt{T/n}}$ the proportional upward jump,
- $d_n = u_n^{-1}$ the proportional downward jump.

Cheuk-Vorst lattice (1997)

- Modified tree V.
- Backward induction.
- Value associated with a specific node depends only on the time and on the difference (in powers of u_n) between the present and the lowest value of the underlying from time $t = 0$ to the present time.
- For the call this difference is the value j such that

$$S_m = \left(\min_{0 \leq i \leq m} S_i \right) u_n^j.$$

Option evaluation

- Value inside the tree is not the option price.
- This value is the number by which we have to multiply the underlying price to obtain the corresponding option price:

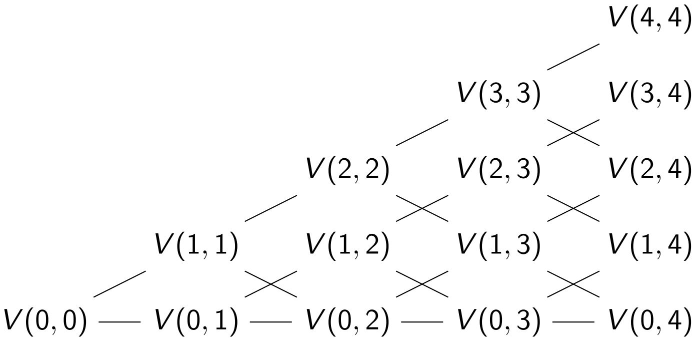
$$C_n^{fl}(m) = S_m\, V(j,m).$$

- A previous node is the expectation of the following two nodes with respect to the probability

$$q_n = p_n u_n e^{-\frac{rT}{n}},$$

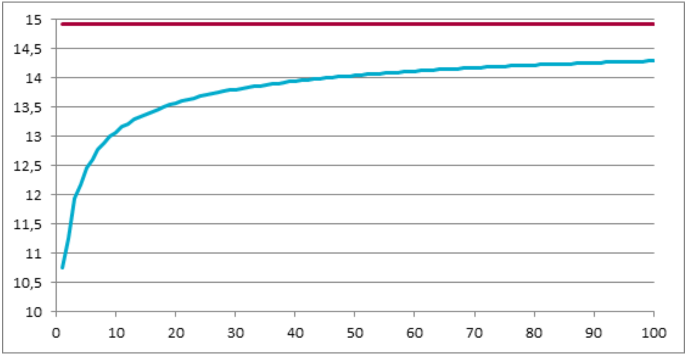
with p_n the traditional risk-neutral probability.

Example for $n = 4$



Value function of the lookback option

Example with $S_0 = 80$, $\sigma = 0.2$, $r = 0.08$ and $T = 1$.



Difference with the traditional tree

Each final level can be reached by several different numbers of upward jumps. Because of this, $V_n(0,0)$ can be written as a double sum

$$V_n(0,0) = \sum_{j=0}^{n} (1 - u_n^{-j}) \sum_{k=j}^{l} \Lambda_{j,k,n}\, q_n^k (1 - q_n)^{n-k}$$

with

$$l = \left\lfloor \frac{n+j}{2} \right\rfloor, \qquad \Lambda_{j,k,n} = \binom{n}{k-j} - \binom{n}{k-j-1},$$

if $k > j$ and $\Lambda_{j,k,n} = 1$ if $k = j$.

Price for a fixed number n

The call price (with n steps) can be deduced from this construction and is $S_0\, V_n(0,0)$, with

$$V_n(0,0) = \frac{Q_n(1-d_n)}{(1-Q_n)(1-Q_n d_n)}\,\phi_1 - \frac{1}{1-Q_n}\,\phi_2 + \frac{e^{-rT}}{1-Q_n d_n}\,\phi_3,$$

where

- $Q_n = \frac{q_n}{1-q_n}$,
- $\phi_1 = \mathcal{B}_{n,q_n}(\lfloor n/2 \rfloor) - Q_n\, \mathcal{B}_{n,q_n}(\lfloor n/2 \rfloor - 1)$,
- $\phi_2 = Q_n\, \mathcal{B}_{n,1-q_n}(\lfloor n/2 \rfloor) - \mathcal{B}_{n,1-q_n}(\lfloor n/2 \rfloor - 1)$,
- $\phi_3 = Q_n d_n\, \mathcal{B}_{n,1-p_n}(\lfloor n/2 \rfloor) - u_n\, \mathcal{B}_{n,1-p_n}(\lfloor n/2 \rfloor - 1)$,

and $\mathcal{B}_{n,p}$ the cumulative distribution function of the binomial distribution with parameters n and p.

The results (1/2)

Approximation of the complementary cumulative function for some binomial distributions (H. 2013)

Suppose that $p_n = \frac{1}{2} + \frac{\alpha}{\sqrt{n}} + \frac{\beta}{n} + \frac{\gamma}{n^{3/2}} + \frac{\delta}{n^2} + O\left(\frac{1}{n^{5/2}}\right)$ and $j_n = \frac{n}{2} + a\sqrt{n} + \frac{1}{2} + b_n + \frac{c}{\sqrt{n}} + \frac{d}{n} + O\left(\frac{1}{n^{3/2}}\right)$, where the sequence $(b_n)_n$ is bounded. Then

$$\sum_{k=j_n}^{n} \binom{n}{k} p_n^k (1-p_n)^{n-k}$$

$$= \Phi(A) + \frac{e^{-A^2/2}}{\sqrt{2\pi}} \left(\frac{B_n}{\sqrt{n}} + \frac{C_0 - C_2 B_n^2}{n} + \frac{D_0 - D_1 B_n - D_3 B_n^3}{n^{3/2}} \right) + O\left(\frac{1}{n^2}\right),$$

where $A = 2(\alpha - a)$, $B_n = 2(\beta - b_n)$, $C_0 = 2(\alpha^2 A + \gamma - c) + (2\alpha/3 - A/12)(1 - A^2)$, $C_2 = A/2$, $D_0 = 2(2\alpha\beta A + \delta - d) + 2(1 - A^2)\beta/3$, $D_1 = (1 - 4A^2 + A^4)/12 - 2\alpha(\alpha - A - \alpha A^2 + A^3/3) + 2A(\gamma - c)$, $D_3 = (1 - A^2)/6$, and Φ is the cumulative distribution function of the standard normal distribution.

The results (2/2)

Asymptotics of the lookback call (H. 2013)

If $r \neq 0$ then the asymptotic formula for the price of the European lookback call option with floating strike is

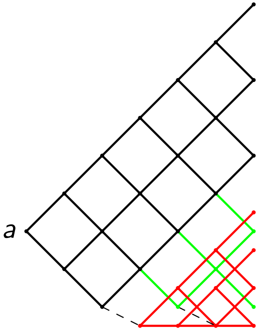
$$C_n^{fl} = C_{BS}^{fl} + \frac{\sigma\sqrt{T}}{2}(C_{BS}^{fl} - S_0)\frac{1}{\sqrt{n}} + \left[\frac{\sigma^2 T}{12}\left(C_{BS}^{fl} + 2S_0\left[\Phi(a_1) - e^{-rt}\Phi(a_2) - \frac{3}{2}\right]\right) + S_0\frac{\sigma\sqrt{T}}{2}\frac{e^{-a_1^2/2}}{\sqrt{2\pi}}\right]\frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right),$$

with $a_1 = (r/\sigma + \sigma/2)\sqrt{T}$ and $a_2 = (r/\sigma - \sigma/2)\sqrt{T}$.

Work in progress

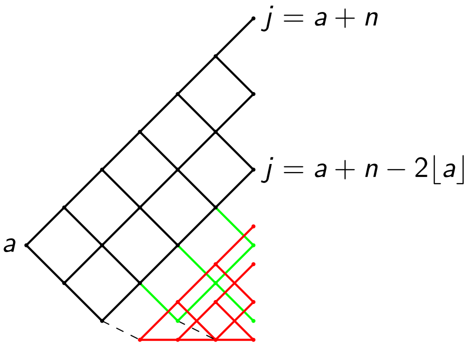
- Do not restrict the evaluation to time $t = 0$.
- At time t, we have the information for the minimal value of the underlying between times 0 and t.
- We divide the remaining time line in n steps.
- The form of the previous tree changes if this minimal value is less than the underlying price at time t.

Generalization of the tree



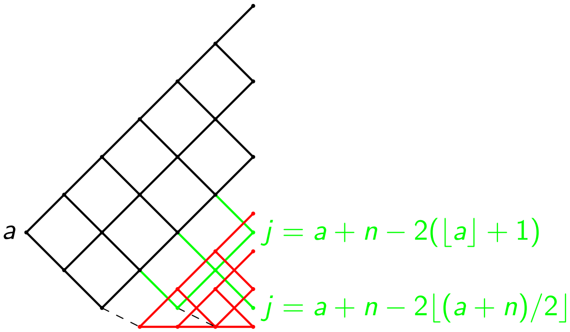
$$a(n) = \frac{\log(S_t/S_{\min})}{\sigma\sqrt{T/n}}$$

Generalization of the tree



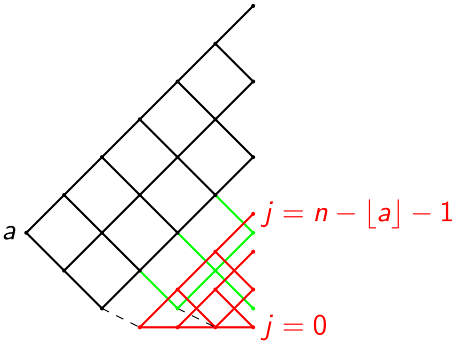
The number of paths to arrive at a black state can be found as in the binomial tree.

Generalization of the tree



The number of paths to arrive at a green state can be found as in the binomial tree without forgetting to subtract the paths that have been absorbed in the red tree.

Generalization of the tree



The number of paths to arrive at a red state can be found more or less as in the basic case where the option is evaluated at emission.

Thank you for your attention!

Bibliography

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