On the expressiveness and decidability of o-minimal hybrid systems *

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Abstract. This paper is driven by a general motto: bisimulate a hybrid system by a finite symbolic dynamical system. In the case of o-minimal hybrid systems, the continuous and discrete components can be decoupled, and hence, the problem reduces in building a finite symbolic dynamical system for the continuous dynamics of each location. We show that this can be done for a quite general class of hybrid systems defined on o-minimal structures. In particular, we recover the main result of a paper by Lafferriere G., Pappas G.J. and Sastry S. on o-minimal hybrid systems. We also study related decidability questions.
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1 Introduction

Hybrid systems consist of finite state machines equipped with a continuous dynamics. This notion has been intensively studied [ACH+,HKPV,Hen95] (see [Hen96] for a survey), and is a generalization of timed automata [AD]. Hybrid systems encompass many interesting applications such as air traffic management [TPS] and highway systems [LGS].

As an example, consider the following situation. The entrance of an highly secure building is controlled by an electronic system. Anyone who wants to enter the building have to possess an *access card* and to know a *password*. Before entering the building he will have to insert his access card into the system and to type the password on a keyboard. If the person commits a single mistake when typing the password, an alarm will immediately warn. If the password is ABC, the finite state system of Figure 1 accurately describes the process. Since variables of the system takes only finitely many values, we will say that the dynamics of the system is discrete.

Suppose now, we want to make the previous system even more secure, by adding some *timed constraints*. Once the access card have been introduced into the device, each letter of the password have to be typed within one unit of time,

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if typing is too slow, the alarm will fire. This more complex process can be described by Figure 2, where X is the time variable and so can take uncountably many values. Since states of the system are characterized by values of the variables and locations, such a system has uncountably many states due to the presence of *time*. This is a simple example of a hybrid system, i.e. a system where both discrete and continuous transitions coexist, the last ones being in practice governed by differential equations (in this example $\dot{X} = 1$)

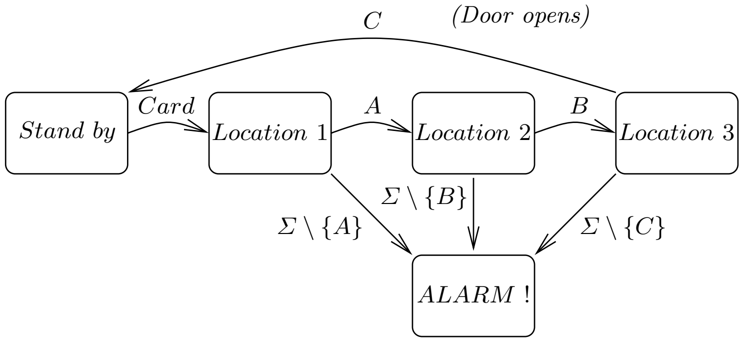

Fig. 1. A digital code $\Sigma = \{A, B, \cdots, Z, Card\}$

Given a hybrid system, a natural question is to know whether the system can reach some (prohibited) state (the Alarm state for example). This question is known as *the reachability problem*. Since the state space is usually uncountable it is necessary to have an algorithmic approach to this problem. The main difficulty is the richness of continuous dynamics and its interaction with a discrete dynamics. Several results on decidability and undecidability of the reachability problem have been developed in [ACH+,HKPV].

One approach to solve the reachability problem is to study equivalence relations preserving reachability and to find finite state systems equivalent to the original one. Building *bisimulations* is a way to achieve this goal. This is the point of view adopted in this paper. Bisimulations have many other interesting properties (e.g. they preserve the temporal logic CTL (Computation Tree Logic), [AHLP]).

In [LPS], the notion of *o-minimal hybrid system* is defined. This class of hybrid systems have a particularly rich continuous dynamics, in particular it may be non-linear. Through this paper, we adopt the conventions introduced in [LPS, p. 6] for the discrete transitions. This allows to decouple the discrete and continuous components of the hybrid system. Hence the problem to find a finite bisimulation of such a hybrid system is equivalent to find a finite bisimulation, on each location, which respects some initial partition induced by resets, guards, initial and final regions. In [LPS, p. 12], the continuous dynamics of an o-minimal hybrid system is given by a smooth complete vector field F from $\mathbb{R}^n$ to $\mathbb{R}^n$ and the flow is assumed to be definable in an o-minimal extension of $\langle \mathbb{R}, <, +, - \rangle$. In particular, the system is time-invariant, the flow is injective w.r.t. the time and thus the trajectories are non self-intersecting. We relax these assumptions by permitting the system to be time-varying and to have self-intersecting trajectories, which are natural features of many real systems. The continuous transition relation of such systems is therefore much richer (see Section 2.3). Moreover the generalization allows for general dynamics instead of flow, for an output space M^{k_2} distinct from the input space M^{k_1} where M is the domain of any linearly ordered structure $\mathcal{M}$. In particular this level of generality allows fuzzy dynamics, by encoding dynamics given by a definable relation into a function with value in an output space of larger dimension (see subsection 2.3).

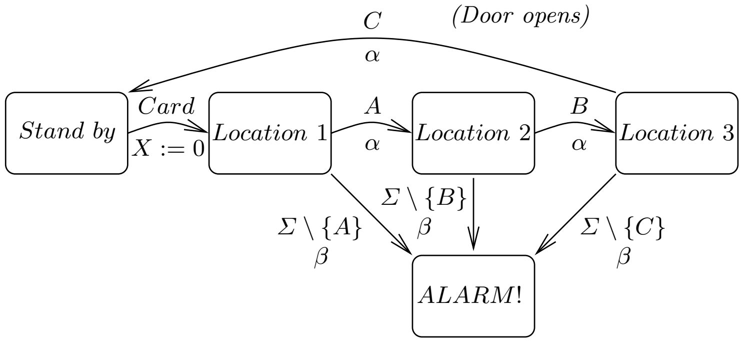

Fig. 2. A timed digital code where $\alpha \equiv (X \leqslant 1)\ ;\ (X := 0)$ means the discrete transition is allowed if the value of X is less or equal to one and the value is reset to zero after the transition, the same holds for the discrete transition $\beta \equiv (X > 1)\ ;\ (X := 0)$. In each location, the continuous transitions are ruled by $\dot{X} = 1$

In Section 3 of this paper, we present a general construction to associate words with trajectories of a continuous dynamics w.r.t. an initial partition of the space. Let us mention that this kind of idea already appears in the literature (see for example [ASY]). The concept of *dynamical type of a point* plays a central role in our construction. By using this general tool, a finite symbolic dynamical system is associated with any *o-minimal dynamical system*, the states of which are represented by words (see Section 4).

Under the extra assumption that there is a unique word encoding the dynamics of any point of the output space, we show that this finite symbolic dynamical system bisimulates the original one (4.17 and 4.20). As a byproduct of this result, we obtain a simple proof of the main result of [LPS] which asserts that every o-minimal hybrid system admits a finite bisimulation (Corollary 4.18).

Our construction of the bisimulation are not clearly effective, and so our results do not show that the reachability problem is decidable. In Section 5 we address this problem on differents aspects. We first make precise the problem by pointing how it depends on the model of computability chosen and that it naturally expresses in the framework of the BSS model of computation. We show that, under natural extra assumptions, the decidability of the reachability problem for o-minimal dynamical system is equivalent to the decidability of the existential theory of the underlying first-order structure $\mathcal{M}$. In particular, when the structure $\mathcal{M}$ is the field of real numbers with the exponentials, it reduces to *Schanuel Conjecture,* a famous unsolved problem in transcendental number theory. We show that the same kind of results hold for the effectiveness of the so-called *bisimulation algorithm.* Finally we brieffly discuss complexity issues about the finite state systems build in Section 4.

In the last section, we try to delimit the border between o-minimal systems which admits finite bisimulations and the others. To settle this goal, we closely look at examples which are in some sense generic in their class. Firstly we examine the effect to weaken the assumption on the uniqueness of the word encoding the dynamics of a point. Secondly we consider classes of o-minimal hybrid systems where stronger deterministic reset are allowed. In all cases we exhibit an o-minimal hybrid system which does not admit a finite bisimulation w.r.t. some initial partition, setting in this way some limits to our results of Section 4.

Some results were already proved in [BMRT] under slightly stronger hypothesis. When it is the case we mention it in regard of the results. Let us finally mention that the main result of [LPS] is, amongst a lot of other interesting results, also proved in the paper [Da], of which we were not aware at the time we wrote [BMRT]. Our method of proof is completely different from the one used in [Da,LPS].

2 Preliminaries

In this section, we recall some basic definitions and results. However we do not recall classical definitions about hybrid systems, they can be found for example in [Hen96]. For o-minimal hybrid systems and their extensions treated in the paper, we refer to [LPS] and give the bases in Section 4.

2.1 Transition systems and bisimulation

Definition 2.1. A *transition system* $T = (Q, \Sigma, \rightarrow)$ consists of a set of states Q (which may be uncountable), Σ an alphabet of events, and $\rightarrow \subseteq Q \times \Sigma \times Q$ a transition relation.

A transition $(q_1, a, q_2) \in \rightarrow$ is denoted by $q_1 \xrightarrow{a} q_2$. A transition system is said finite if Q is finite. If the alphabet of events is reduced to a singleton, $\Sigma = \{a\}$, we will denote the transition system $(Q, \rightarrow)$ and omit the event a.

Definition 2.2. Given two transition systems on the same alphabet of events, $T_1 = (Q_1, \Sigma, \rightarrow_1)$ and $T_2 = (Q_2, \Sigma, \rightarrow_2)$, a *partial simulation of* T_1 *by* T_2 is a binary relation $\sim \subseteq Q_1 \times Q_2$ which satisfies the following condition:

$$\forall q_1, q_1' \in Q_1,\ \forall q_2 \in Q_2,\ \forall a \in \Sigma,$$
$$\big(q_1 \sim q_2 \text{ and } q_1 \xrightarrow{a}_1 q_1'\big) \Rightarrow \big(\exists q_2',\ q_1' \sim q_2' \text{ and } q_2 \xrightarrow{a}_2 q_2'\big)$$

This condition is read T_2 *simulates* T_1.

Definition 2.3. Given $\sim$ a partial simulation of T_1 by T_2, we say that $\sim$ is a *simulation of* T_1 *by* T_2 if, for each $q_1 \in Q_1$, there exists $q_2 \in Q_2$ such that $q_1 \sim q_2$.

Definition 2.4. Given two transition systems on the same alphabet of events, $T_1 = (Q_1, \Sigma, \rightarrow_1)$ and $T_2 = (Q_2, \Sigma, \rightarrow_2)$, a *bisimulation between* T_1 *and* T_2 is a relation $\sim \subseteq Q_1 \times Q_2$ such that $\sim$ is a simulation of T_1 by T_2 and the *inverse relation*[1] $\sim^{-1}$ is a simulation of T_2 by T_1.

Definition 2.5. Given $\sim$ a bisimulation between T_1 and T_2 if $\sim$ is a function from Q_1 to Q_2, we call it a *functional bisimulation.*

Remarks 2.6. Given a transition system $T = (Q, \Sigma, \rightarrow)$, we can look at bisimulations on $Q \times Q$; they are called *bisimulations on* T.
Given T_1, T_2 two transition systems and $\sim \subseteq Q_1 \times Q_2$ a bisimulation between T_1 and T_2, the kernel[2] $\mathrm{Ker}(\sim)$ is a bisimulation on T_1.
Given $\sim$ a functional bisimulation between T_1 and T_2, we have that $\mathrm{Ker}(\sim)$ is an equivalence relation on Q_1; moreover there is a bisimulation between $T_1/\mathrm{Ker}(\sim)$ and T_2 (these statements and their proofs can be found in [Cau]).

Definition 2.7. Given T a transition system, $\mathcal{P}$ a partition of Q and $\sim \subseteq Q \times Q$ a bisimulation which is an equivalence relation on Q, we say that the bisimulation $\sim$ *respects the partition* $\mathcal{P}$ if any $P \in \mathcal{P}$ is an union of equivalence classes for $\sim$. We will speak of *bisimulations w.r.t.* $\mathcal{P}$.

Since the reachability problem for a finite state system (effectively described) is trivially decidable, it is an important question to know whether a given infinite system admits a finite bisimulation. In the case where such a finite bisimulation exists, the next question is to know how effectively we can compute it. The first question is partially settled by the following so-called *bisimulation algorithm* which appears in [BFH,Hen96]. The second question is discussed later in Section 5, in the restricted framework of *o-minimal hybrid systems.*

Given a transition system $T = (Q, \Sigma, \rightarrow)$ and $\mathcal{P}$ a finite partition of Q, the bisimulation algorithm iterates the computation of predecessors[3] of the pieces of the partition, let us recall it:

Algorithm 2.8.
Initialization: $Q/\!\sim \;:= \mathcal{P}$
While $\exists P, P' \in Q/\!\sim$ such that $\varnothing \neq P \cap \mathrm{Pre}(P') \neq P$
  Set $P_1 = P \cap \mathrm{Pre}(P')$ and $P_2 = P \setminus \mathrm{Pre}(P')$
  Refine $Q/\!\sim \;:= (Q/\!\sim \setminus \{P\}) \cup \{P_1, P_2\}$
End while

[1] If $\sim = \{(q_1, q_2) \in Q_1 \times Q_2 | q_1 \sim q_2\}$, then $\sim^{-1} = \{(q_2, q_1) \in Q_2 \times Q_1 | q_1 \sim q_2\}$.
[2] $\mathrm{Ker}(\sim) = \sim \circ \sim^{-1} = \{(p, q) \in Q_1 \times Q_1 \mid \exists r \in Q_2,\ p \sim r \text{ and } q \sim r\}$.
[3] Given T a transition system and $q \in Q$, the set of predecessors of q, denoted $\mathrm{Pre}(q)$, is defined by $\mathrm{Pre}(q) = \{q' \in Q | \exists a \in \Sigma,\ q' \xrightarrow{a} q\}$, and if $P \subseteq Q$, $\mathrm{Pre}(P) = \bigcup_{q \in P} \mathrm{Pre}(q)$.

Remark 2.9. The bisimulation algorithm is a priori not a Turing algorithm. But let us assume that the partition and the predecessor relation are first-order definable in a structure $\mathcal{M}$. Then the bisimulation algorithm can be seen as a BSS algorithm over $\mathcal{M}$ with an oracle for the emptiness problem for first-order definable sets of $\mathcal{M}$ (for information about BSS algorithms the interested reader can look at [BCSS,Poi]).

Let us recall the main result on the bisimulation algorithm.

Lemma 2.10. *Given T a transition system and $\mathcal{P}$ a finite partition of Q, the bisimulation algorithm terminates if and only if there exists a finite bisimulation on T w.r.t. $\mathcal{P}$.*

When this pseudo algorithm for the class of o-minimal hybrid systems becomes an algorithm in some (well-chosen) model of computation will be discussed in Section 5.

2.2 O-minimality, definability and decidability

Let $\mathcal{M}$ be a first-order structure. In this paper when we say that some relation, subset, function is definable, we mean it is first-order definable (possibly with parameters) in the sense of the structure $\mathcal{M}$. A general reference for first-order logic is [CK]. All the notions related to o-minimality and an extensive bibliography can be found in [vdD98]. Let us recall the definition of an o-minimal structure:

Definition 2.11. [PS] A totally ordered structure $\mathcal{M} = \langle M, <, ...\rangle$ is *o-minimal* if every definable subset of M is a finite union of points and open intervals (possibly unbounded).

In other words the definable subsets of M are the simplest possible: the ones which are definable with parameters in $\langle M, <\rangle$. This assumption implies that definable subsets of M^n (in the sense of M) admit very nice structure theorems (like *Cell decomposition*) or Theorem 2.13. In the early 1980s van den Dries noticed that many properties of semialgebraic sets and maps could derive from a few simple axioms [vdD84], essentially the axioms defining "o-minimal structures", as their models came to be called in an influential article of Pillay and Steinhorn [PS]. After Wilkie established in 1991 that the exponential field of real numbers is o-minimal [Wi96] the subject has grown rapidly. The following are examples of o-minimal structures.

Example 2.12. The field of reals $\langle \mathbb{R}, <, +, \cdot, 0, 1\rangle$, the group of rationals $\langle \mathbb{Q}, <, +, \cdot, 0, 1\rangle$, the field of reals with exponential function $\langle \mathbb{R}, <, +, \cdot, 0, 1, e^x\rangle$, the field of reals expanded by restricted pfaffian functions and the exponential function (see [vdD98,Wi96]). There exists many more interesting o-minimal structures.

The main result we use on o-minimal structures is (see [vdD98, Corollary 3.6, p. 60]):

Theorem 2.13 (Uniform Finiteness). *Let $S \subseteq M^m \times M^n$ be definable, we denote by S_a the fiber $\{y \in M^n | (a, y) \in S\}$. Then there is a number $N_S \in \mathbb{N}$ such that for each $a \in M^m$ the set $S_a \subseteq M^n$ has at most N_S definably connected components.*

Let us remark that Theorem 2.13 holds in structure which are not o-minimal, e.g. in algebraically closed fields, differentially closed fields (see [MMP]).

Finally let us recall that if o-minimal structures share a fine analysis of their definable sets, there is no general results about quantifier elimination in these structures or about the decidability of their theories. An old and celebrated result in this direction is Tarski Theorem which asserts there exists a (Turing) effective quantifier elimination procedure for real closed fields. Another particularly spectacular and recent result is the model-completeness of the theory of the field of real numbers expanded by the exponential function proved by Wilkie ([Wi96]) but it is not known whether the theory of $\langle \mathbb{R}, <, +, \cdot, 0, 1, e^x \rangle$ is decidable ([MW,Wi97]).

2.3 Dynamics

Definition 2.14. A *dynamical system* is a pair $(\mathcal{M}, \gamma)$ where:

- $\mathcal{M} = \langle M, <, ... \rangle$ is a totally ordered structure,
- $\gamma : M^{k_1} \times M \to M^{k_2}$ is a definable function of $\mathcal{M}$.

The function γ is called the *dynamics* of the dynamical system. More generally, we can consider the case where γ is defined on definable subsets of $\mathcal{M}$ that is $\gamma : V_1 \times V \to V_2$ with $V_1 \subseteq M^{k_1}$, $V \subseteq M$ and $V_2 \subseteq M^{k_2}$.

Classically, when M is the field of the reals, we see M as the time, $M^{k_1} \times M$ as the space-time, M^{k_2} as the (output) space and M^{k_1} as the input space. We keep this terminology in the more general context of a structure $\mathcal{M}$.

Remark 2.15. The fact we allow in our definition of a dynamical system to have different dimensions for the input space and output space has some interesting features. For example, it allows to use relations instead of functions in order to define the dynamics γ. More precisely this means that γ can be a relation included in $M^{k_1} \times M \times M^{k_2}$. Given such a relation γ it is always possible to associate with it a function γ_f from $M^{k_1} \times M$ to the power set of M^{k_2}, $\gamma_f(x, t) = \{y \mid \gamma(x, t, y)\}$. Since the description of these sets $\gamma_f(x, t)$ are uniformly given by the formula $\gamma(x, t, y)$, it is in general natural to attach with it a n-tuple of terms in (x, t) which completely characterizes $\gamma_f(x, t)$. Let us illustrate this process in the following two examples.
The first situation is given in Figure 3. In this case the dynamics is given by a cone which expresses a *differential inclusion* and is related to *rectangular hybrid automata* (see [HKPV] for example). The relation $\gamma \subseteq \mathbb{R}^5$ is given by $\{(x_1, x_2, t, y_1, y_2) \mid y_1 = x_1 + t \text{ and } x_2 + \frac{1}{2}t \leqslant y_2 \leqslant x_2 + t\}$. Let us notice that given $(x_1, x_2, t) \in M^2 \times M$, $\{(y_1, y_2) \in M^2 \mid \gamma(x_1, x_2, t, y_1, y_2)\}$ does not reduce to a point anymore but is an interval in the output space. In some sense,

this interval represents the set of potential positions at time t with input condition (x_1, x_2). A possible function to characterize γ is given by $\gamma_f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3$ with $\gamma_f(x_1, x_2, t) = (x_1 + t, x_2 + \frac{1}{2}t, x_2 + t)$.
The second situation is given in Figure 4 and is in some sense related to plane trajectories as treated in [TPS]. In this situation, the dynamics is given by the relation $\gamma = \Big\{(x_1, x_2, t, y_1, y_2) \mid (y_1 - (x_1 + t))^2 + \big(y_2 - (x_2 + \frac{1}{2}t)\big)^2 \leqslant r^2\Big\} \subseteq \mathbb{R}^5$. Given $(x_1, x_2, t) \in M^2 \times M$ the set of potential positions at time t with input condition (x_1, x_2) is a disc in the output space.

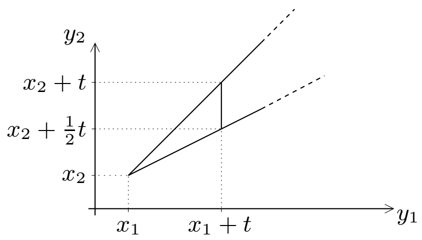

Fig. 3. Intervals dynamics

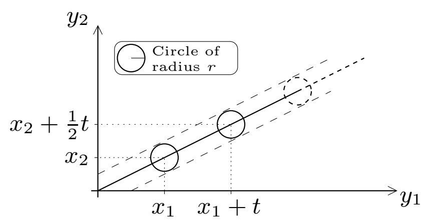

Fig. 4. Discs dynamics

Definition 2.16. If we fix a point $x \in M^{k_1}$, the set $\Gamma_x = \{\gamma(x, t) \mid t \in M\} \subseteq M^{k_2}$ is called the trajectory determined by x.

Definition 2.17. Given $(\mathcal{M}, \gamma)$ a dynamical system, we define a *transition system* $T_\gamma = (Q, \to_\gamma)$ *associated with the dynamical system* by:

- the set Q of states is M^{k_2};
- the transition relation $y_1 \to_\gamma y_2$ is defined by:

$$\exists x \in M^{k_1},\ \exists t_1, t_2 \in M,\ \big(t_1 \leqslant t_2 \text{ and } \gamma(x, t_1) = y_1 \text{ and } \gamma(x, t_2) = y_2\big)$$

Let us make an *important* observation. Given a transition $y_1 \to_\gamma y_2$, we denote the couple of instants of time corresponding to the positions y_1, y_2 by (t_1, t_2). If there exists a position y and different times $t < t'$ such that $\gamma(x, t) = \gamma(x, t') = y$ (see Figures 5 and 8 for example), then the transition relation $\to_\gamma$ allows the following sequence of transitions: $y_1 \to_\gamma y \to_\gamma y_2$ with couples of time (t_1, t') and (t, t_2). Let us look at a simple example of this behavior, in Figure 5, there clearly exists $t < t'$ such that $\gamma(x, t) = \gamma(x, t') = y$. The composition of transitions as explained above allows an arbitrary large number of passages in the loop. Later in this paper, we will show that such a dynamics can encompass reset of variables (see Remarks 6.6).

3 Encoding trajectories by words

In this section, we describe the general tools that we use further on.

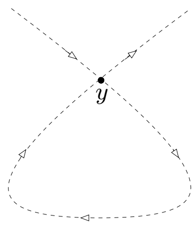

Fig. 5. A simple loop

Given a dynamical system $(\mathcal{M}, \gamma)$ and $\mathcal{P}$ a finite definable partition of the space M^{k_2}, $\mathcal{P} = \{P_1, \ldots, P_s\}$, we want to encode the trajectories on M^{k_2} as words[4] on the finite alphabet $\mathcal{P}$.

Let us first remark that the partition $\mathcal{P}$ of the space M^{k_2} induces a partition $\tilde{\mathcal{P}}$ on the space-time $M^{k_1} \times M$ defined by the preimages of the P_i's under γ. The preimage of trajectory Γ_x is the line $\{x\} \times M$ in the space-time $M^{k_1} \times M$. This line crosses the regions $\tilde{P}_i$'s and looking to this crossing, when time is increasing, naturally gives a word on the alphabet $\tilde{\mathcal{P}}$. Replacing each letter $\tilde{P}_i$ by its corresponding letter P_i gives the word ω_x on the alphabet $\mathcal{P}$ we want to associate with Γ_x. An example of this construction is given in Figure 6 where we assumed that any point of the closed trajectory is periodically reached. For the sake of completeness, we mathematically formalize this idea. Given $x \in M^{k_1}$, we consider the sets $\{t \mid \gamma(x,t) \in P_i\}$ for $i = 1, \ldots, s$. This gives a partition of the time M. We associate a word on $\mathcal{P}$ with the trajectory determined by x such that two consecutive letters are different. Let $\mathcal{F}_x$ be the set of intervals[5] defined

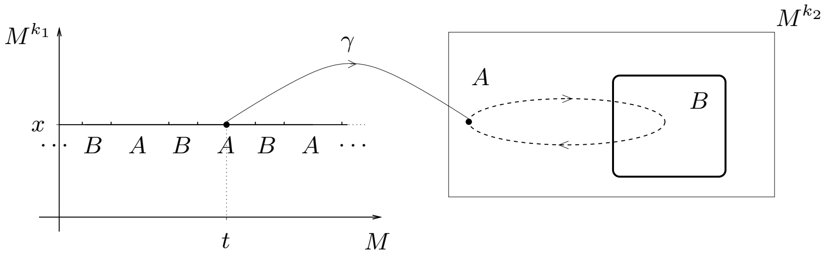

Fig. 6. Construction of $\omega_x = \ldots BABABA \ldots$

4 In this general (possibly uncountable) context, a word is a function from M (or from a quotient of M induced by a partition on M) to $\mathcal{P}$.

5 For each $x \in M^{k_1}$, $\mathcal{F}_x$ can be viewed as a definable set, each interval $I \in \mathcal{F}_x$ being represented by its end points. Formally we need a couple to represent a point in order to recover $-\infty$ and $+\infty$ (as in the projective line case). The formula defining $\mathcal{F}_x$ is tedious to write, any reader could convince himself that it works.

by:

$$\mathcal{F}_x = \big\{ I \mid I \text{ is a time interval and is maximal for the property} \\ \exists i \in \{1, \ldots, s\},\ \forall t \in I,\ \gamma(x,t) \in P_i \big\}.$$

For each x, the set $\mathcal{F}_x$ is totally ordered by the order induced from M. By analogy with the work of [Tr], we introduce a family of functions of *coloration* $\mathcal{C}_x : \mathcal{F}_x \to \mathcal{P}$ defined by:

$$\mathcal{C}_x(I) = P_i \quad \Leftrightarrow \quad \exists t \in I,\ \gamma(x,t) \in P_i\ .$$

The word ω_x is defined by:

$$\omega_x \text{ is the sequence } \left(\mathcal{C}_x(I)\right)_{I \in \mathcal{F}_x} .$$

We denote by Ω the set of words associated with $(\mathcal{M}, \gamma)$ w.r.t. $\mathcal{P}$. In the sequel we will have to consider this construction w.r.t. different partitions.

Example 3.1. Consider the dynamical system and the partition $\mathcal{P} = \{A, B\}$ described in Figure 7. In this situation, we have $\Omega = \{A, ABA, ABABA\}$.

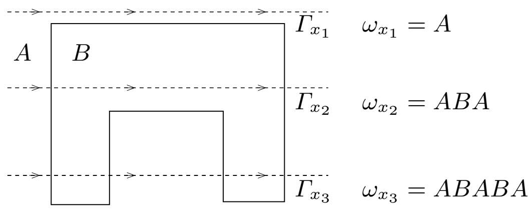

Fig. 7. Encoding trajectories by words

By encoding trajectories by words, we give a description of the "support" of the dynamical system. But, in order to recover the dynamics of a point in the trajectory, we need to encode more information: given a point (x,t) of the space-time, we want to know what the "position of $\gamma(x,t)$" in ω_x is. Given $(x,t) \in M^{k_1} \times M$, we associate a unique *dotted word* $\dot{\omega}_{(x,t)}$ in the following way: let $I \in \mathcal{F}_x$ be the unique interval such that $t \in I$, we add a dot on $\mathcal{C}_x(I)$ in ω_x. The set of dotted words associated with $(\mathcal{M}, \gamma)$ w.r.t. $\mathcal{P}$ is denoted by $\dot{\Omega}$.

Example 3.2. If we now consider the dotted words associated with Figure 7, we have $\dot{\Omega} = \{\dot{A}, \dot{A}BA, A\dot{B}A, AB\dot{A}, \dot{A}BABA, \ldots, ABAB\dot{A}\}$.

Remark 3.3. In general, γ is not injective and so a point y of the space M^{k_2} has more than one preimage (x,t). So several words ω_x and dotted words $\dot{\omega}_{(x,t)}$ are associated with y.

In view to describe this general situation, we introduce the notion of *dynamical type* W_y, for $y \in M^{k_2}$:

$$W_y = \{\dot{\omega}_{(x,t)} \mid \exists (x,t) \in M^{k_1} \times M,\ \gamma(x,t) = y\}.$$

Definition 3.4. A dotted word is said *associated with a point* y if and only if it belongs to the dynamical type of y.

We denote by Δ the set of dynamical types associated with $(\mathcal{M}, \gamma)$ with respect to $\mathcal{P}$. Let us consider the partition given by the equivalence relation on the space M^{k_2} "*to have same dynamical type*". We can now repeat the previous construction w.r.t. this new partition (we will also denote it by Δ). So, we naturally obtain a set of words on Δ, denoted Ω_Δ. Let us notice that Δ is a refinement of $\mathcal{P}$. Given $x \in M^{k_1}$, we denote u_x the word on Δ associated with Γ_x, $\mathcal{F}_x^\Delta$ the ordered set of intervals induced on M and $\mathcal{C}_x^\Delta : \mathcal{F}_x^\Delta \to \Delta$ the coloration function.

Example 3.5. Figure 8 represents the trajectory Γ_x of some dynamical system through the partition $\mathcal{P} = \{A, B, C\}$, the word ω_x associated with the trajectory is $ABCBA$. For $y \in \Gamma_x$, there exists seven different dynamical types: $W_1 = \{\dot{A}BCBA\}, \ldots, W_5 = \{ABCB\dot{A}\}$, $W_6 = \{\dot{A}BCBA, ABCB\dot{A}\}$ and $W_7 = \{A\dot{B}CBA, ABC\dot{B}A\}$. The word u_x associated with the trajectory is $W_1 W_6 W_1 W_2 W_7 W_2 W_3 W_4 W_7 W_4 W_5 W_6 W_5$.

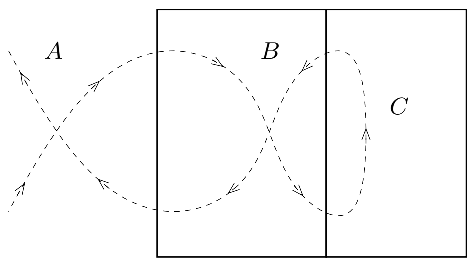

Fig. 8. Double loop

Given a trajectory Γ_x for some $x \in M^{k_1}$ and $y \in \Gamma_x$, we want to know "the position of y" in u_x. But by Remark 3.3 this position is not necessarily unique. We introduce a unique *multidotted word* $\ddot{u}_{(x,y)}$ in the following way: we add dots on $\mathcal{C}_x^\Delta(I)$ for all interval $I \in \mathcal{F}_x^\Delta$ such that there exists $t \in I$ with $\gamma(x,t) = y$.

We denote by $\ddot{\Omega}_\Delta$ the set of multidotted words associated with $(\mathcal{M}, \gamma)$ w.r.t. Δ. The definition of dynamical type naturally extends to the context of the multidotted words.

4 O-minimal hybrid system

We have just described the general framework. In the sequel we will be interested by *o-minimal hybrid systems* whose the building blocks are finitely many *o-minimal dynamical systems*. A run of such a system is roughly a composition of the action of its dynamical systems on a point of the output space which has to respect the conditions induced by the *guards* and *resets*.

In particular, we discuss two special and interesting cases in Sections 4.2 and 4.3. We freely use the notations introduced in the previous sections.

Now we give precise definitions.

Definition 4.1. An *o-minimal dynamical system* $(\mathcal{M}, \gamma)$ is a dynamical system where $\mathcal{M}$ is an o-minimal structure.

Definition 4.2. Given $\mathcal{M}$ an o-minimal structure, an *o-minimal hybrid system on* $\mathcal{M}$, $\mathcal{H} = (Loc, \Sigma, Edg, Dyn, Inv, \mathcal{G}, \mathcal{R})$, consists of:

- Loc is a finite set of locations (discrete states),
- Σ is a finite alphabet of events,
- $Edg \subseteq Loc \times \Sigma \times Loc$ is a finite set of edges,
- Dyn assigns to each location a continuous dynamics (for each $l \in Loc$, $Dyn(l) = \gamma_l$ where $\gamma_l : M^{k_1} \times M \to M^{k_2}$ is a function definable in $\mathcal{M}$), i.e. $(\mathcal{M}, \gamma_l)$ *is an o-minimal dynamical system*,
- Inv assigns to each location a definable subset of M^{k_2} called *invariant* (for each $l \in Loc$, $Inv(l) = Inv_l$ where Inv_l is a definable set in M^{k_2}),
- $\mathcal{G}$ assigns to each edge a definable subset of M^{k_2} called *guard* (for each $e \in Edg$, $\mathcal{G}(e) = \mathcal{G}_e$ where $\mathcal{G}_e$ is a definable set in M^{k_2}),
- $\mathcal{R}$ assigns to each edge a definable subset of M^{k_2} called *reset* (for each $e \in Edg$, $\mathcal{R}(e) = \mathcal{R}_e$ where $\mathcal{R}_e$ is a definable set in M^{k_2}).

Example 4.3. The introduction example, the timed digital code (see Figure 2) is an hybrid o-minimal system. Let us make it precise in view of the definition above:

- $Loc = \{Sb, L1, L2, L3, Alarm\,!\}$
  where Sb stands for *Stand by* and Li for *Location i* $(i = 1, 2, 3)$,
- $\Sigma = \{A, B, \cdots, Z, Card\}$
- $e_1 = (Sb, Card, L1) \in Edg$, $e_2 = (L1, A, L2) \in Edg$, ...
- For each $l \in Loc$, $Dyn(l) = \gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and is defined by $\gamma(x, t) = x + t$,
- For each $l \in Loc$, $Inv(l) = \mathbb{R}$,
- $\mathcal{G}(e_1) = \mathbb{R}$, $\mathcal{G}(e_2) = \{x \in \mathbb{R} \mid x \leqslant 1\}$, ...
- $\mathcal{R}(e_1) = \{0\}$, $\mathcal{R}(e_2) = \{0\}$, ...

Definition 4.4. Given $\mathcal{H}$ an o-minimal hybrid system, we define a *transition system* $T_{\mathcal{H}} = \big(Q_{\mathcal{H}}, \to_{\mathcal{H}}, \Sigma \cup \{\gamma_l \mid l \in Loc\}\big)$ *associated with the hybrid system* by:

- the set of states $Q_{\mathcal{H}}$ is $Loc \times M^{k_2}$,

– the transition relation $\rightarrow_{\mathcal{H}}$ can be of two types:
  - discrete transition:

$$(l_1, y_1) \xrightarrow{a}_{\mathcal{H}} (l_2, y_2) \Leftrightarrow e = (l_1, a, l_2) \in Edg \text{ and } y_1 \in \mathcal{G}(e) \text{ and } y_2 \in \mathcal{R}(e)$$

  - continuous transition:

$$(l_1, y_1) \xrightarrow{\gamma}_{\mathcal{H}} (l_2, y_2) \Leftrightarrow l_1 = l_2 \text{ and } y_1 \rightarrow_{\gamma_{l_1}} y_2 \text{ without leaving } Inv(l_1)^6$$

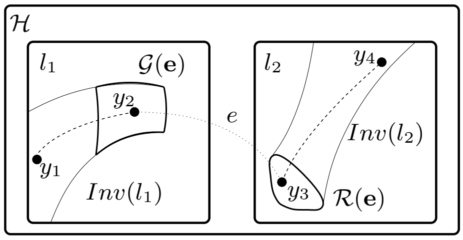

Fig. 9. $(l_1, y_1) \xrightarrow{\gamma}_{\mathcal{H}} (l_1, y_2) \xrightarrow{e}_{\mathcal{H}} (l_2, y_3) \xrightarrow{\gamma}_{\mathcal{H}} (l_2, y_4)$

Example 4.5. We continue the illustration of the definitions by giving a finite sequence of transitions for the timed digital code (see Figure 2):

$$(Sb, 0) \xrightarrow{\gamma}_{\mathcal{H}} (Sb, 7.35) \xrightarrow{e_1}_{\mathcal{H}} (L1, 0) \xrightarrow{\gamma}_{\mathcal{H}} (L1, 0.58) \xrightarrow{e_2}_{\mathcal{H}} (L2, 0) \xrightarrow{\gamma}_{\mathcal{H}} (L2, 0.99)$$

Figure 9 is a representation of Definition 4.4.

Remark 4.6. Our definition allows only resets as defined in [LPS]. We will justify later this choice (see Section 6).

Notations 4.7. In the figures in order to distinguish guard and reset conditions, we use the following notation. Given $e \in Edg$, we denote the associated discrete transition by $y \in \mathcal{G}(e)$; $y := \mathcal{R}(e)$. We already use this notation in Figure 2.

Remark 4.8. By the argument that decouples the continuous and discrete components of the hybrid system given in [LPS, p. 6], we only need to prove that there exists a finite bisimulation on each location which respects the initial finite partition given by the resets, guards and invariants which are definable in the o-minimal structure we are working in, by assumption. That is why, in the sequel, we focus our study on o-minimal dynamical systems.

[6] i.e. $\exists x \in M^{k_1}$, $\exists t_1, t_2 \in M$, $\big(t_1 \leqslant t_2$ and $\gamma(x, t_1) = y_1$ and $\gamma(x, t_2) = y_2\big)$ and $\forall t \in M\ ((t_1 \leqslant t \leqslant t_2) \Rightarrow (\gamma(x, t) \in Inv(l_1))$

4.1 Symbolic o-minimal dynamical system

In Section 3 we gave a description of the trajectories of any dynamical system in term of words. In the case of an o-minimal dynamical system, finitely many finite words are enough to describe the trajectories. This will allow us to define *finite* transition systems on the words. The results of this section already appear in [BMRT].

Lemma 4.9. *Given* $(\mathcal{M}, \gamma)$ *an o-minimal dynamical system and a finite definable partition* $\mathcal{P}$*, the set of words* Ω *is a finite set of finite words.*

Proof. Let us recall from Section 3 that the partition $\mathcal{P}$ of the space induces a definable partition of the space-time whose regions are the $\tilde{P}_i$'s. Given $x \in M^{k_1}$, we have that $\mathcal{F}_x$ exactly consists in the connected components of the fibers of the $\tilde{P}_i$'s: $(\tilde{P}_i)_x = \{t \in M \mid \gamma(x,t) \in P_i\}$. By the Uniform Finiteness Theorem 2.13, we have that the number of connected components of the $(\tilde{P}_i)_x$'s is uniformly finite w.r.t. x, this implies that the length of the ω_x's is uniformly bounded. So since the number of P_i's is finite, we have that Ω is finite. □

The next result is a trivial consequence of Lemma 4.9 and the definition of $\dot{\Omega}$.

Corollary 4.10. $\dot{\Omega}$ *is finite.*

Remark 4.11. Let us remark that in the proof of Lemma 4.9, we just used the Uniform Finiteness Theorem 2.13. So this result holds in all the structures admitting the Uniform Finiteness Theorem 2.13.

The following technical lemma will be useful in the next section.

Lemma 4.12. *In an o-minimal structure, given* W *a dynamical type,* $y \in M^{k_2}$*, "*y *is of dynamical type* W*" is definable.*

Proof. Suppose $W = \{\dot{\omega}_1, ..., \dot{\omega}_l\}$, y is of type W if and only if $\dot{\omega}_i$ is associated to y for $1 \leqslant i \leqslant l$ and y is associated to none of the others $\dot{\omega} \in \dot{\Omega}$ (which is a finite set by Corollary 4.10). It remains to show that *y is associated to $\dot{\omega}$* is definable for some $\dot{\omega} \in \dot{\Omega}$. Suppose $\dot{\omega} = P_{i_1}...\dot{P}_{i_k}...P_{i_n}$. We have that $\dot{\omega}$ is associated to y if and only the following formula holds:

$$\exists x \in M^{k_1} \ \Big(\mathcal{F}_x = \{I_1, ..., I_n\} \text{ and } \bigwedge_{j=1}^{n} (\mathcal{C}_x(I_j) = P_{i_j}) \\ \text{and } \exists t \in I_k \ (\gamma(x,t) = y)\Big)$$

This concludes the proof since $\mathcal{F}_x$ and $\mathcal{C}_x$ are definable (see Section 3). □

Now we introduce a symbolic transition system which is finite under the assumption of o-minimality.

We define $T_{\dot{\Omega}}$, a finite transition system on the dotted words. In order to mathematically formalize $T_{\dot{\Omega}}$, we need to introduce two functions: UNDOT : $\dot{\Omega} \to$

Ω which gives the word ω corresponding to $\dot{\omega}$ without dot; DOT : $\dot{\Omega} \to \mathbb{N}$ which gives the position of the dot on $\dot{\omega}$. Given $x \in M^{k_1}$, the set $\mathcal{F}_x$ can be described as a finite ordered sequence of intervals $I_0 < I_1 < \cdots < I_k$ with $k < N_S$. If we consider $\dot{\omega}_x$ a dotted word constructed from ω_x, we have the following relation: the dot of $\dot{\omega}_x$ is on $\mathcal{C}_x(I_i)$ with $I_i \in \mathcal{F}_x$ if and only if DOT$(\dot{\omega}_x) = i$. We can now define $T_{\dot{\Omega}} = (\dot{\Omega}, \to_{\dot{\Omega}})$:

- the set of states Q is $\dot{\Omega}$
- the transition relation $\dot{\omega}_1 \to_{\dot{\Omega}} \dot{\omega}_2$ is defined by: the undotted words $\dot{\omega}_1$ and $\dot{\omega}_2$ are equal and the dot on $\dot{\omega}_2$ is on a righter (or on the same) position as the dot on $\dot{\omega}_1$. This can be formalized by:

$$\begin{gathered}\dot{\omega}_1 \to_{\dot{\Omega}} \dot{\omega}_2 \\ \Updownarrow \\ \textsc{undot}(\dot{\omega}_1) = \textsc{undot}(\dot{\omega}_2) \text{ and } \textsc{dot}(\dot{\omega}_1) \leqslant \textsc{dot}(\dot{\omega}_2)\end{gathered}$$

Example 4.13. Here is an example of transition on the dotted words w.r.t. Figure 7: $A\dot{B}ABAB \to_{\dot{\Omega}} ABAB\dot{A}B$

Lemma 4.14. *Given $(\mathcal{M}, \gamma)$ an o-minimal dynamical system and a finite definable partition $\mathcal{P}$, the set of words Ω_Δ is a finite set of finite words.*

Proof. We first notice that the number of dynamical types is finite since $|\Delta| \leqslant 2^{|\dot{\Omega}|}$ and $\dot{\Omega}$ is finite by Corollary 4.10. Since *being of dynamical type W_y* for some $W_y \in \Delta$ is definable in an o-minimal structure (see Lemma 4.12), this induces a finite definable partition Δ of the space M^{k_2} and so we can use the same argument as in the proof of Lemma 4.9. □

The next result is a trivial consequence of Lemma 4.14 and the definition of $\ddot{\Omega}_\Delta$.

Corollary 4.15. *$\ddot{\Omega}_\Delta$ is finite.*

We define also $T_{\ddot{\Omega}_\Delta}$, a finite transition system on the multidotted words. To mathematically formalize $T_{\ddot{\Omega}_\Delta}$, we need to introduce three functions: UNDOT : $\ddot{\Omega}_\Delta \to \Omega_\Delta$ gives the word u corresponding to $\ddot{u}$ without dot; MINDOT : $\ddot{\Omega}_\Delta \to \mathbb{N}$ gives the position of the left most dot on $\ddot{u}$ and MAXDOT : $\ddot{\Omega}_\Delta \to \mathbb{N}$ gives the position of the right most dot on $\ddot{u}$.

Given $x \in M^{k_1}$, the set $\mathcal{F}_x^\Delta$ can be described as a finite ordered sequence of intervals $I_0 < I_1 < \cdots < I_k$ with $k < N_S^\Delta$. If we consider a multidotted word $\ddot{u}_{(x,y)}$, constructed from u_x and y on the trajectory Γ_x, let W be the element of Δ such that $y \in W$. Those letters W correspond to some intervals $I_i \in \mathcal{F}_x^\Delta$ such that MINDOT$(\ddot{u}_{(x,y)}) \leqslant i \leqslant$ MAXDOT$(\ddot{u}_{(x,y)})$. We can now define $T_{\ddot{\Omega}_\Delta} = (\ddot{\Omega}_\Delta, \to_{\ddot{\Omega}_\Delta})$:

- the set of states is $\ddot{\Omega}_\Delta$

– the transition relation $\ddot{u}_1 \to_{\ddot{\Omega}_\Delta} \ddot{u}_2$ is defined by: the undotted words $\dot{u}_1$ and $\dot{u}_2$ are equal and the right most dot on $\ddot{u}_2$ is on a righter (or the same) position than the left most dot on $\dot{\omega}_1$. This can be formalized by:

$$\begin{array}{c} \ddot{u}_1 \to_{\ddot{\Omega}_\Delta} \ddot{u}_2 \\ \Updownarrow \\ \textsc{undot}(\ddot{u}_1) = \textsc{undot}(\ddot{u}_2) \quad \text{and} \quad \textsc{mindot}(\ddot{u}_1) \leqslant \textsc{maxdot}(\ddot{u}_2) \end{array}$$

Example 4.16. Here is an example of transition on multidotted words w.r.t. Figure 8:

$$\begin{aligned} &W_1\dot{W}_6W_1W_2W_7W_2W_3W_4W_7W_4W_5\dot{W}_6W_5 \\ &\qquad \to_{\ddot{\Omega}_\Delta} W_1W_6W_1W_2\dot{W}_7W_2W_3W_4\dot{W}_7W_4W_5W_6W_5 \end{aligned}$$

4.2 Word determinism case

The first situation that we will be interested in is the following: we suppose that there is a unique dotted word associated with each point y of the output space M^{k_2}. This hypothesis encompasses the relevant situation where a unique non self-intersecting trajectory goes through each point y (which is the case treated in [LPS], see also [Da,BMRT]).

Theorem 4.17. *Let $(\mathcal{M}, \gamma)$ be an o-minimal dynamical system, let T_γ be the associated transition system on M^{k_2}, and let $\mathcal{P}$ be a finite definable partition of M^{k_2}. If there exists a unique dotted word associated with each $y \in M^{k_2}$, then there exists a finite bisimulation of T_γ that respects $\mathcal{P}$.*

Proof. To prove this theorem, we will show that there exists a bisimulation between the transition systems T_γ and $T_{\dot{\Omega}}$. Let us first recall that $T_{\dot{\Omega}}$ is a finite transition system by Corollary 4.10. We define a binary relation $\sim \subseteq M^{k_2} \times \dot{\Omega}$ as follows:

$$y \sim \dot{\omega} \quad \Leftrightarrow \quad \exists (x,t) \in M^{k_1} \times M,\ \big(\dot{\omega}_{(x,t)} = \dot{\omega} \text{ and } \gamma(x,t) = y\big).$$

We begin by showing that $T_{\dot{\Omega}}$ simulates T_γ. Given y_1, $y_2 \in M^{k_2}$ and $\dot{\omega}_1 \in \dot{\Omega}$ such that $y_1 \to_\gamma y_2$ and $y_1 \sim \dot{\omega}_1$, we have to find $\dot{\omega}_2 \in \dot{\Omega}$ such that $\dot{\omega}_1 \to_{\dot{\Omega}} \dot{\omega}_2$ and $y_2 \sim \dot{\omega}_2$. By definition of $\to_\gamma$, there exists $x \in M^{k_1}$ and $t_1 \leqslant t_2 \in M$ such that $\gamma(x, t_1) = y_1$ and $\gamma(x, t_2) = y_2$. By assumptions, there exists a unique $\dot{\omega}_1$ such that $y_1 \sim \dot{\omega}_1$. So we have that $\dot{\omega}_1 = \dot{\omega}_{(x,t_1)}$. We set that $\dot{\omega}_2 = \dot{\omega}_{(x,t_2)}$. We have clearly that $y_2 \sim \dot{\omega}_2$. To prove that $\dot{\omega}_1 \to_{\dot{\Omega}} \dot{\omega}_2$, we first remark that $\textsc{undot}(\dot{\omega}_1) = \textsc{undot}(\dot{\omega}_2) = \omega_x$. Since $t_1 \leqslant t_2$, we have that $t_1 \in I_i$ an $t_2 \in I_j$, for some I_i, $I_j \in \mathcal{F}_x$, with $i \leqslant j$, so $\textsc{dot}(\dot{\omega}_1) \leqslant \textsc{dot}(\dot{\omega}_2)$.

Conversely[7] let us prove that T_γ simulates $T_{\dot{\Omega}}$. Given $y_1 \in M^{k_2}$ and $\dot{\omega}_1$, $\dot{\omega}_2 \in \dot{\Omega}$ such that $\dot{\omega}_1 \to_{\dot{\Omega}} \dot{\omega}_2$ and $\dot{\omega}_1 \sim^{-1} y_1$, we have to find $y_2 \in M^{k_2}$ such that

[7] Let us notice that the uniqueness of the dotted word does not play any role in this second part of the proof.

$y_1 \rightarrow_\gamma y_2$ and $\dot{\omega}_2 \sim^{-1} y_2$. Since $\dot{\omega}_1 \in \dot{\Omega}$, there exists $(x, t_1) \in M^{k_1} \times M$ such that $\dot{\omega}_1 = \dot{\omega}_{(x,t_1)}$ and $t_1 \in I_i$ for some $I_i \in \mathcal{F}_x$. We can find $I_j \in \mathcal{F}_x$ with $I_i \leqslant I_j$ such that if we add the dot corresponding to I_j on ω_x we obtain $\dot{\omega}_2$. We take $t_2 \in I_j$, and set $y_2 = \gamma(x, t_2)$, we clearly have that $y_1 \rightarrow_\gamma y_2$ and $\dot{\omega}_2 \sim^{-1} y_2$.

We have proved that $\sim \subseteq M^{k_2} \times \dot{\Omega}$ is a bisimulation. By assumtions, $\sim$ is a functional bisimulation. By Remark 2.6, $\sim$ induces a finite bisimulation on $M^{k_2} \times M^{k_2}$ given by Ker($\sim$); moreover, by definition of $\sim$ and Ker($\sim$), this bisimulation is an equivalence relation which respects $\mathcal{P}$. □

Corollary 4.18. *[LPS, Theorem (4.3), p.11] Every o-minimal hybrid system (as defined in [LPS]) admits a finite bisimulation.*

Proof. By assumptions, $\gamma(.,.)$ is the definable flow of a vector field $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which does not depend of the time [LPS, p. 12], so in particular $\gamma(x, .)$ is injective [LPS, p. 13], therefore from every $y \in \mathbb{R}^n$ $(= M^{k_2})$, there exists a unique trajectory which does not self-intersect. In such a situation, a unique dotted word is associated with any point $y \in \mathbb{R}^n$, so we can apply Theorem 4.17. And this concludes the proof by Remark 4.8. □

Remarks 4.19. We can remark that in the proof of Theorem 4.17, we only use the Uniform Finiteness Theorem 2.13. In the proof of [LPS] *Cell decomposition* and the fact that *connectedness and arc-connectedness are equivalent* are used, so their proof fully uses the power of o-minimality assumption contrary to ours (see Remark 4.11).
If we were interested in bisimulations on the space-time, the proof of Theorem 4.17 shows that there always exists a finite bisimulation of $(\mathcal{M}, \gamma)$ that respects $\mathcal{P}$.

4.3 Loop case

In this section, we consider the more general situation where a unique multidotted word is associated with each point y of the output space M^{k_2}. In particular, this hypothesis allows us to consider self-intersecting trajectories (i.e. loops, Figure 8 is an example of this situation). Let us remark that the self intersection set can be an arbitrary definable set.

Theorem 4.20. *Let $(\mathcal{M}, \gamma)$ be an o-minimal dynamical system, let T_γ be the associated transition system on M^{k_2}, and let $\mathcal{P}$ be a finite definable partition of M^{k_2}. If there exists a unique multidotted word associated with each $y \in M^{k_2}$, then there exists a finite bisimulation of T_γ that respects $\mathcal{P}$.*

Proof. As in the proof of Theorem 4.17, we show that there exists a bisimulation between T_γ and $T_{\ddot{\Omega}_\Delta}$, which is a finite transition system by Corollary 4.15. We define a binary relation $\sim \subseteq M^{k_2} \times \ddot{\Omega}_\Delta$ in the following way:

$$y \sim \ddot{u} \quad \Leftrightarrow \quad \exists (x,t) \in M^{k_1} \times M, \ \big(\ddot{u}_{(x,y)} = \ddot{u} \text{ and } \gamma(x,t) = y\big).$$

First, we prove that $T_{\ddot{\Omega}_\Delta}$ simulates T_γ. Given $y_1,\ y_2 \in M^{k_2}$ and $\ddot{u}_1 \in \ddot{\Omega}_\Delta$ such that $y_1 \rightarrow_\gamma y_2$ and $y_1 \sim \ddot{u}_1$, we have to find $\ddot{u}_2 \in \ddot{\Omega}_\Delta$ such that $\ddot{u}_1 \rightarrow_{\ddot{\Omega}_\Delta} \ddot{u}_2$ and $y_2 \sim \ddot{u}_2$. By definition of $\rightarrow_\gamma$, there exists $x \in M^{k_1}$ and $t_1 \leqslant t_2 \in M$ such that $\gamma(x, t_1) = y_1$ and $\gamma(x, t_2) = y_2$. Since there is a unique multidotted word associated with y_1, we have that $\ddot{u}_1 = \ddot{u}_{(x,y_1)}$. By choosing $\ddot{u}_2 = \ddot{u}_{(x,y_2)}$, we have clearly that $y_2 \sim \ddot{u}_2$. Moreover we have that $\textsc{undot}(\ddot{u}_1) = \textsc{undot}(\ddot{u}_2)$. Since $t_1 \leqslant t_2$, $t_1 \in I_i$ and $t_2 \in I_j$ for some $I_i, I_j \in \mathcal{F}_x^\Delta$ with $i \leqslant j$ and so $\textsc{mindot}(\ddot{u}_1) \leqslant i \leqslant j \leqslant \textsc{maxdot}(\ddot{u}_2)$.

Conversely[8] let us prove that T_γ simulates $T_{\ddot{\Omega}_\Delta}$. Given $y_1 \in M^{k_2}$ and $\ddot{u}_1$, $\ddot{u}_2 \in \ddot{\Omega}_\Delta$ such that $\ddot{u}_1 \rightarrow_{\ddot{\Omega}_\Delta} \ddot{u}_2$ and $\ddot{u}_1 \sim^{-1} y_1$, we have to find $y_2 \in M^{k_2}$ such that $y_1 \rightarrow_\gamma y_2$ and $\ddot{u}_2 \sim^{-1} y_2$. Since $\ddot{u}_1 \sim^{-1} y_1$, we have that $\ddot{u}_1 = \ddot{u}_{(x,y_1)}$ for some $x \in M^{k_1}$ and $y_1 = \gamma(x, t_1)$ for some $t_1 \in M$. We take $t_0 \in I_{\textsc{mindot}(\ddot{u}_1)} \in \mathcal{F}_x^\Delta$ such that $\gamma(x, t_0) = y_1$. Since $\textsc{mindot}(\ddot{u}_1) \leqslant \textsc{maxdot}(\ddot{u}_2)$, it is always possible to choose $t_2 \in I_{\textsc{maxdot}(\ddot{u}_2)} \in \mathcal{F}_x^\Delta$ such that $t_0 \leqslant t_2$. We now set $y_2 = \gamma(x, t_2)$. All this construction respects the rules given for the composition of transitions (see the observation mentioned after Definition 2.17).

We have proved that $\sim\ \subseteq M^{k_2} \times \ddot{\Omega}_\Delta$ is a bisimulation. By assumptions, it is a functional bisimulation. By Remark 2.6, $\sim$ induces a finite bisimulation on $M^{k_2} \times M^{k_2}$ given by Ker($\sim$). Moreover this bisimulation is an equivalence and clearly respects Δ, and so $\mathcal{P}$ since Δ is finer than $\mathcal{P}$. □

The assumptions of Theorem 4.20 encompass the the following corollary.

Corollary 4.21. *Let $(\mathcal{M}, \gamma)$ be an o-minimal dynamical system, let T_γ be the associated transition system on M^{k_2}, and let $\mathcal{P}$ be a finite definable partition of M^{k_2}. If there exists a unique trajectory which could self-intersect associated with each $y \in M^{k_2}$, then there exists a finite bisimulation of T_γ that respects $\mathcal{P}$.*

Remark 4.22. If we look at a different transition system on $(\mathcal{M}, \gamma)$ where the set of states Q is given by $M^{k_1} \times M^{k_2}$ and the transition relation $(x_1, y_1) \rightarrow_{\tilde{\gamma}} (x_2, y_2)$ is defined by: $(x_1 = x_2) \wedge \exists t_1 \leqslant t_2 \in M \ \big((\gamma(x_1, t_1) = y_1) \wedge (\gamma(x_2, t_2) = y_2)\big)$, the proof of Theorem 4.20 shows that any such o-minimal dynamical system admits a finite bisimulation which respects a given finite definable partition $\mathcal{P}$.

Remark 4.23. The main assumption of Theorems 4.17 and 4.20 is the uniqueness of the dotted or multidotted word associated to any point of the output space M^{k_2}. This restricts the behavior of the dynamics through the partition. However, this does not restrict at all the behavior of the dynamics into the pieces of the partition as illustrated in Figure 10: pieces are black boxes w.r.t. this analysis.

5 Decidability

In this section we discuss the decidability of the *reachability problem*, particularly in the context of the o-minimal hybrid systems. In the previous sections we

[8] Let us notice that the uniqueness of the multidotted word does not play any role in this second part of the proof.

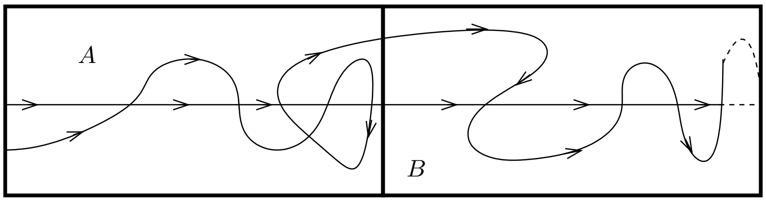

Fig. 10. $\dot{\Omega} = \left\{\dot{A}B, A\dot{B}\right\}$

show that under some assumptions an o-minimal hybrid system is bisimilar to a finite state system for which the reachability problem is obviously decidable. But this construction is not clearly effective; the same is true for the so-called bisimulation algorithm 2.8. In the sequel we explore the relationship between the effectiveness of this construction and the decidability of the (existential) theory of the underlying first-order structure $\mathcal{M}$. This last problem was and is still widely explored in the literature, see for example Matiyasevich [Ma], Shlapentokh [Shl],. . .
We freely use the notations, abbreviations and conventions previously introduced for an o-minimal dynamical system $(\mathcal{M}, \gamma)$.

5.1 Reachability

The *reachability problem*, as explained in our introduction, is a fundamental problem in verification theory, but also in practice (see [ACH+]). So the algorithmic and complexity aspects of this problem are important.

Given an o-minimal dynamical system $(\mathcal{M}, \gamma)$, an initial region $Init \subseteq M^{k_2}$ and a final region $Fin \subseteq M^{k_2}$, both definable in $\mathcal{M}$, the reachability problem can be translated by the following first-order sentence :

$$\exists y_1 \in Init \;\; \exists y_2 \in Fin \text{ such that } y_1 \rightarrow_\gamma y_2$$

or more explicitely by :

$$\begin{aligned} &\exists y_1 \in Init \;\; \exists y_2 \in Fin \;\; \exists x \in M^{k_1} \;\; \exists t_1, t_2 \in M \\ &\qquad \big(t_1 \leqslant t_2 \text{ and } \gamma(x, t_1) = y_1 \text{ and } \gamma(x, t_2) = y_2\big) \end{aligned}$$

The complexity of this sentence clearly depends on the complexity of the formulas defining the subsets $Init$ and Fin. If the structure $\mathcal{M}$ admits elimination of quantifiers, the sentence is equivalent to a quantifier free one. Hence in the case where there exists a (Turing) algorithm which performs this elimination, decidability of the reachability problem on $\mathcal{M}$ is equivalent to establish the truth of a boolean combination of atomic formulas. This is for example the case when $\mathcal{M}$ is the field of the real numbers (by Tarski Theorem). But this still does not mean that the decidability problem is Turing decidable! Indeed, the formulas

defining $Init$, Fin and γ may have real parameters like π; testing the truth of an atomic formula with parameters needs to be able to effectively decide the equality of two real numbers. Without extra assumptions on the involved parameters we do not have decidablity in the Turing model and so it seems us that the natural model of computability in order to discuss these theoretical issues is the Blum-Shub and Smale model of computability introduced in [BSS] (see also for example [BCSS] and [MM]). And in any case if we consider the decidability of the question to know whether there is a continuous transition from a given point $y_1 \in M^{k_2}$ to an other point $y_2 \in M^{k_2}$ ($y_1 \rightarrow_\gamma y_2$), we cannot escape to decide formulas with elements of M. The BSS model has been widely studied from the end of 80's and has been developped in the general framework of a first-order structure $\mathcal{M}$ by Poizat (see [Poi]).
For the remaining of the discussion we assume that the subsets $Init$ and Fin (or more generally the guards, the resets and invariants) and the dynamics γ are given by quantifier free formulas without parameters. Those are common assumptions in the literature (see [AD,HKPV]). In this case the sentence above is clearly an existential one and the Turing decidability of $Th_\exists(\mathcal{M})$ implies the Turing decidability of the reachability problem.

Conversely decidability of the reachability problem implies decidability of $Th_\exists(\mathcal{M})$. Indeed consider the following (maybe unrealistic) dynamical system $(\mathcal{M}, \gamma)$ where $\mathcal{M}$ is any first-order structure with at least two elements (we will denote these two elements 0 and 1) and $\gamma : M \times M \rightarrow M^2$ is defined as follows.

$$\gamma(x,t) = \begin{cases} (t,1) & \text{iff } \varphi(x) \\ (t,0) & \text{iff } \neg\varphi(x) \end{cases}$$

where $\varphi(x)$ is a quantifier free formula of $\mathcal{M}$. Under the assumption that the reachability problem is decidable, we can in particular decide whether $(0,1) \rightarrow_\gamma (0,1)$. Clearly this is equivalent to decide whether $\exists x\ \varphi(x)$.

The whole above discussion shows that in the particular case of an o-minimal dynamical system defined in $\langle \mathbb{R}, +, \cdot, 0, 1, e^x \rangle$ the reachability problem reduces to *Schanuel conjecture*, a famous unsolved problem in transcendental number theory (see [MW,Wi97]).

5.2 Effectiveness of the bisimulation construction

Theorems 4.17 and 4.20 state that under some conditions an o-minimal dynamical system $(\mathcal{M}, \gamma)$ admits a finite bisimulation. The following theorem gives a condition under which the construction of a finite bisimulation is effective; it is a generalisation which encompasses previous results of [LPY]. In the sequel we make the extra assumption that the first-order theory of the structure $\mathcal{M}$ is model-complete; this hypothesis assures that all formulas are equivalent to existential ones, in particular we can assume tthat the formulas defining the guards, the resets and invariants, and the dynamics γ are existential.

Theorem 5.1. *Let $(\mathcal{M}, \gamma)$ be an o-minimal dynamical system, let T_γ be the associated transition system on M^{k_2}, and let $\mathcal{P}$ be a finite definable partition* *of M^{k_2}. If $Th(\mathcal{M})$ is model complete and decidable and if there exists a unique multidotted word associated with each $y \in M^{k_2}$, then there exists an algorithm which computes a finite bisimulation of T_γ that respects $\mathcal{P}$.*

Proof. By Theorem 4.20 we know there exists a finite bisimulation of T_γ that respects $\mathcal{P}$. On the other hand, we know that under the assumption of the existence of a finite bisimulation, the (pseudo) bisimulation algorithm 2.8 terminates. Since we assume the decidability of the theory of $\mathcal{M}$ each step of the algorithm is computable[9].

In the remaining of this subsection we exhibit an o-minimal dynamical system on each o-minimal expansion of a field which shows that the decidability of $Th_\exists\mathcal{M}$ is an essential assumption w.r.t. the problem of termination of the bisimulation algorithm.

Let $\mathcal{M}$ be an o-minimal structure expansion of a field, x_0 is a fixed point of M^{k_1} and $\gamma : M^{k_1} \times M \to M^{k_1}$ a dynamics defined as follows,

$$\gamma(x,t) = x_0.t + x.(1-t)$$

Consider the initial partition of M^{k_1} given by $\mathcal{P} = \big\{\{x_0\}, D, M^{k_1} \setminus (D \cup \{x_0\})\big\}$ with $D = \{x \mid \varphi(x)\}$ is a definable set and $x_0 \notin D$.

If we apply the bisimulation algorithm (2.8), the first step of the algorithm have to test the following equality:

$$\mathrm{Pre}(x_0) \cap D = \varnothing \tag{1}$$

By definition of the dynamics $\mathrm{Pre}(x_0) = M^{k_2}$, so the equality (1) is false if and only if $\exists x\ \varphi(x)$. This observation allows us to state that if the first step of the bisimulation algorithm is computable for this kind systems then $Th_\exists(\mathcal{M})$ is decidable.

Let us end this subsection by a remark about the effectiveness of the construction of the symbolic transition systems $T_{\dot{\Omega}}$ and $T_{\ddot{\Omega}_\Delta}$. A tedious analysis shows that if ω is a word in $\mathcal{P}^*$ we can build a first-order formula which expresses that a trajectory Γ_x is encoded by the word ω. This is similar to Lemma 4.12. Hence if we have an a priori bound on length of the possible words for encoding trajectories, and if $Th(\mathcal{M})$ is decidable we can effectively build the symbolic transition systems $T_{\dot{\Omega}}$ and $T_{\ddot{\Omega}_\Delta}$ (this requires to pursue tedious definability analysis of our construction). This consideration motivates the next subsection.

5.3 Complexity issues

In this subsection, we collect some well-known results on the number of connected components for definable sets in particular o-minimal structures. Indeed our Theorems 4.17 and 4.20 show that the length of the finite words (respectively on

[9] Again the distinction between Turing and BSS computability holds in this context. Hence if we want a Turing algorithm we need to assume that all the defining formulas are existential ones without parameters.

the initial partition $\mathcal{P}$ and on the partition Δ induced by the dynamical type) is bounded (see Lemmas 4.9 and 4.14). Let us denote these bounds respectively c and mc. In this way, independently of the effectiveness of our constructions, we can yield a rough bound on the size of the finite state system provided by our results ($T_{\dot{\Omega}}$ and $T_{\ddot{\Omega}_\Delta}$). If the cardinal of the initial partition $\mathcal{P}$ is s and if c is the number of connected components then the number of words in Ω is bounded by s^c. Consequently[10] , the cardinal of the dotted words $\dot{\Omega}$ is at most $c.s^c$ and the cardinal of multidotted words $\ddot{\Omega}_\Delta$ is at most $2^{mc}.\delta^{mc}$ where δ is a bound on the number of dynamical type. This last bound can be easily estimated, it is bounded by the power set of $\dot{\Omega}$, i.e. $\delta \leqslant 2^{c.s^c}$. We hope that finer computations on the number of connected components induced by a partition $\mathcal{P}$ on $\gamma(x,.)$ will provide bounds for c and mc. To illustrate this idea let us cite a classical result on definable subsets in the field of real numbers.

Theorem 5.2. *[BCSS, proposition 7, p. 314]*
Let $S \subseteq \mathbb{R}^n$ be defined by

$$\begin{cases} f_i(x) = 0, \, i = 1, ..., p \\ f_i(x) \geqslant 0, \, i = p+1, ..., p+l \\ f_i(x) > 0, \, i = p+l+1, ..., k \end{cases}$$

and let $d = max\{degree\ f_1, ..., degree\ f_k\}$. Then the number of connected components of S is bounded by $(kd+1).(2kd+1)^{n+1}.(4kd+1)^n$.

In our case, $S = \{t \mid \gamma(x,t) \in P\}$ for some $x \in M^{k_1}$ and $P \in \mathcal{P}$; since here $n = 1$, if s the cardinal of $\mathcal{P}$, a rough bound for c is given by $s.(kd+1).(2kd+1)^2.(4kd+1)$ where d os the maximum of the degree of the polynomials involve in the description of S, which can be computed from the degree of the polynomials defining γ and $\mathcal{P}$.
Today there is a lot of works trying to improve such bounds in the framework of o-minimal structures, for example let us cite [Kho,PV,Pe].

6 Limits of our results

In this section, we try to delimit the border between o-minimal systems which admits finite bisimulations and the others. To settle this aim, we closely look at examples which are in some sense generic in their class. Firstly we examine the effect of weaker assumptions in Theorem 4.17 and 4.20. Secondly we consider classes of o-minimal hybrid systems where stronger deterministic reset are allowed.

[10] This calculation seems to not take in account words of length less than c. But consecutive letters of words in Ω are different by definition (see Section 3) and so the number of such words of length at most c is bounded by the number of words of length c.

6.1 Relaxing the assumptions on the continuous dynamics

In Sections 4.2 and 4.3, we proved that if the continuous dynamics of a point $y \in M^{k_2}$ w.r.t. a given finite partition (in some sense the orbit of y under the dynamics) can be "uniformly encoded by a unique finite word" we obtain a finite bisimulation of the space. Our first example is an o-minimal dynamical system on the torus where at first view the dynamics of a point y requires several words for its encoding (see also [BMRT]). We show that this system does not admit a finite bisimulation w.r.t. a particular partition. As usual, to establish the lack of finite bisimulation w.r.t. the partition, it is sufficient to show the non-termination of the bisimulation algorithm (2.8).

We work in the structure $\mathcal{M} = \langle \mathbb{R}, <, +, \cdot, 0, 1, \sin\restriction_{[0,4\pi]} \rangle$ which is o-minimal, as it can be seen from [vdD96]. A torus is a definable set of $\mathcal{M}$ since it is given by the following equations :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (R + r\cos u)\cos v \\ (R + r\cos u)\sin v \\ r\sin u \end{pmatrix} =: \varphi(u,v)$$

with $u, v \in [0, 2\pi[$.

We define a dynamics $\gamma : [0, 2\pi[^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ on the torus : for all $t \in [0, 2\pi[$,

$$\gamma(u_0, v_0, a, t) = \begin{cases} \varphi(u_0 + t, v_0 + t) & \text{if } a = 1, \\ \varphi(u_0 + t, v_0 + 2t) & \text{if } a = 2, \\ \varphi(u_0, v_0) & \text{otherwise.} \end{cases}$$

The dynamics is definable in $\mathcal{M}$, so $(\mathcal{M}, \gamma)$ is an o-minimal dynamical system and the transition relation is the one given in Definition 2.17. The torus can be represented by a *square of length* 2π *where the opposite sides are identified*. We adopt this description in order to study the dynamics on the torus. Therefore the trajectories on the torus are given by pieces of lines on the square. We note that trajectories are closed curves. In this context, the equation of the dynamics $\gamma : [0, 2\pi[^2 \times \mathbb{R} \times \mathbb{R} \to [0, 2\pi[^2$ becomes :

$$\gamma(u_0, v_0, a, t) = \begin{cases} (u_0 + t, v_0 + t) \bmod 2\pi & \text{if } a = 1 \text{ and } t \in [0, 2\pi[, \\ (u_0 + t, v_0 + 2t) \bmod 2\pi & \text{if } a = 2 \text{ and } t \in [0, 2\pi[, \\ (u_0, v_0) & \text{otherwise.} \end{cases}$$

Given a point $(u_0, v_0) \in [0, 2\pi[^2$, three behaviors of the dynamics are possible: it can follow a line of slope 1 or 2, or it can remain stationary (see Figure 11).

We consider the following initial partition of the square $\mathcal{P} = \{P_0, P_1, P_2, P_3\}$ where:

$$P_0 = \big\{(0,0)\big\}, \qquad P_1 = \big\{(0,v) \bigm| v \in]0, 2\pi[\big\},$$
$$P_2 = \big\{(u,0) \bigm| u \in]0, 2\pi[\big\}, \qquad P_3 = [0, 2\pi[^2 \setminus (P_0 \cup P_1 \cup P_2).$$

This induces a definable (in the sense of the structure $\mathcal{M}$) partition of the torus.

We will now apply the bisimulation algorithm (2.8) and show that it does not terminate when we take this initial partition.

To formalize the non-termination of the algorithm we need to compute the set of predecessors of a given point (y_1, y_2) of the space. By the previous observation, we have that :

$$\mathrm{Pre}(y_1, y_2) = \big\{(y_1 + t, y_2 + t) \bmod 2\pi \bigm| t \in [0, 2\pi]\big\} \cup \big\{(y_1 + t, y_2 + 2t) \bmod 2\pi \bigm| t \in [0, 2\pi]\big\}$$

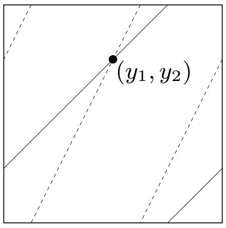

Fig. 11. $\mathrm{Pre}(y_1, y_2)$

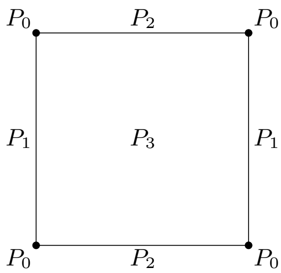

Fig. 12. The partition

We observe that the sets $\mathrm{Pre}(y_1, y_2) \cap P_1$ and $\mathrm{Pre}(y_1, y_2) \cap P_2$ are finite. The iterations of the **While** instruction of the bisimulation algorithm isolates[11] an infinite number of points. The next lemma formalizes this :

Lemma 6.1. *For each $n \geqslant 0$, there exists odd integers k, k' such that the algorithm isolates the points $(k\pi/2^n, 0)$ and $(0, k'\pi/2^n)$.*

Proof. We proceed by induction on n.

(1) In the case $n = 0$, we isolate $(\pi, 0)$ starting from $\{(0, 0)\}$ and then we isolate $(0, \pi)$ by using the new isolated point $\{(\pi, 0)\}$, as shown on Figure 13.

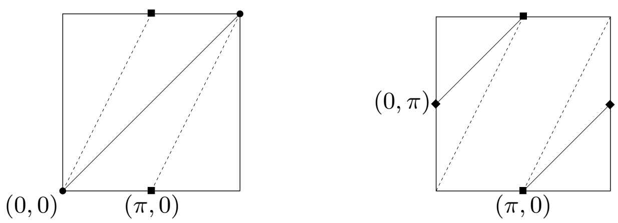

Fig. 13. Case $n = 0$

[11] By "isolating a point q" we mean that the algorithm has constructed $P \in Q/\!\sim$ such that $P = \{q\}$.

(2) Suppose now that we have isolated the points $(0, k\pi/2^n)$ and $(k'\pi/2^n, 0)$ with k, k' satisfying the required conditions, we show how to obtain the new isolated points:

- Consider first the intersection $A = \text{Pre}(0, k\pi/2^n) \cap (X \times \{0\})$ where $X \times \{0\}$ is an element of a sub-partition of P_2; by the characterization of the predecessors above, we have that

$$\begin{aligned}
&(x, 0) \in A \\
&\Leftrightarrow \exists t \in [0, 2\pi],\ \big(t = x \bmod 2\pi \ \text{ and } \ k\pi/2^n = -t \bmod 2\pi\big) \text{ or} \\
&\qquad\qquad\qquad\ \big(t = x \bmod 2\pi \ \text{ and } \ k\pi/2^n = -2t \bmod 2\pi\big) \\
&\Leftrightarrow x = 2\pi - k\pi/2^n \ \text{ or } \ x = 2\pi - k\pi/2^{n+1}
\end{aligned}$$

  The second part of this disjunction permits to isolate the new point $(2^{n+2} - k)\pi/2^{n+1}$ with $2^{n+2} - k = 1 \pmod 2$.
- Using the same argument when considering $B = \text{Pre}(k'n/2^n) \cap (\{0\} \times Y)$, we obtain the second isolated point of the lemma. □

Remark 6.2. Maybe the discussion above does not enlighten where the assumptions of Theorems 4.17 and 4.20 are not satisfied by the dynamics. In fact there are points y of the torus with several trajectories going through and even several dotted words associated with y. For example the multidotted words $\dot{P}_0$, $\dot{P}_0P_3$ and $\dot{P}_0P_3P_2$ are associated with $(0, 0)$.

6.2 Relaxing the assumptions on the discrete dynamics

Now we examine several examples of o-minimal hybrid systems where the continuous dynamics of the o-minimal dynamical systems build in are linear. First let us recall a precise definition of the discrete transition in an o-minimal hybrid system in the sense of [LPS] (which is the convention we adopted in this paper, see 4.4).

Definition 6.3. Memoryless discrete transition:

$$(l_1, y_1) \xrightarrow{a}_{\mathcal{H}} (l_2, y_2) \Leftrightarrow e = (l_1, a, l_2) \in Edg \text{ and } y_1 \in \mathcal{G}(e) \text{ and } y_2 \in \mathcal{R}(e)$$

If we consider Definition 6.3 componentwise, it says that $(l_1, y_1) \xrightarrow{a}_{\mathcal{H}} (l_2, y_2)$ if and only if *each component of* y is reset *non deterministically* into some definable set (see Figure 9). More precisely, the j^{th} component of y after resetting has its value in the projection of $\mathcal{R}(e)$ onto this component; $(y)_j := (\mathcal{R}(e))_j$: discrete transitions are *memoryless*. In the general model of hybrid systems ([Hen96]) or in timed automata ([AD]), all the components are not necessarily reset, they can keep their value.

The next series of examples shows why the memoryless condition is essential in order to obtain finite bisimulation or at least to have the decidability of the reachability problem. Our first hope was to obtain extension of Theorem 4.20

to o-minimal hybrid system with some deterministic discrete transition. This attempt was motivated by several recent results in the literature about hybrid automata, see the papers [AD,Hen96,HKPV].

First we give a general definition of discrete transition which encompasses the previous one and allows to naturally express the classical reset condition of the literature. Discrete transition of *general types* can be defined this way: we add the function $\mathcal{U} : Edg \to 2^{\{1,\cdots,k_2\}}$ to the definition of o-minimal hybrid system. The function $\mathcal{U}$ picks the components of y which will keep their value *unchanged*, the ones which are not reset[12]. We now define $(l_1, y_1) \xrightarrow{a}_{\mathcal{H}} (l_2, y_2)$ this way:

Definition 6.4. General type discrete transition:

$$(l_1, y_1) \xrightarrow{a}_{\mathcal{H}} (l_2, y_2) \Leftrightarrow e = (l_1, a, l_2) \in Edg \text{ and } y_1 \in \mathcal{G}(e) \text{ and } y_2 \in \mathcal{R}(e) \\ \text{and } \forall j \in \mathcal{U}(e) \text{ we have that } (y_1)_j = (y_2)_j$$

The model of *timed automata* can be defined as an o-minimal hybrid system with discrete transitions of general type. Timed automata admit finite bisimulations given by the *region graph* (see [AD]). But we can easily find folk examples of o-minimal hybrid systems with discrete transitions of general type who does not have finite bisimulation. Consider the hybrid system $\mathcal{H}_1$ of Figure 14, it is clearly an o-minimal hybrid system with discrete transition of general type. The guards and resets of $\mathcal{H}_1$ imply that the point $\{(1, 1)\}$ is isolated in the initial partition $\mathcal{P}$. By iterating the predecessors on $(1, 1)$, we show that bisimulation algorithm does not terminate. On Figure 15 we have represented the dynamics of the system by solid lines for the continuous transitions and dashed lines for the discrete transitions. By using the same argument than in Section 6.1 we show that the bisimulation algorithm isolates infinitely many points. So $\mathcal{H}_1$ does not admit finite bisimulation. But the result is even worst, since [ACH+,HKPV,KPSY]

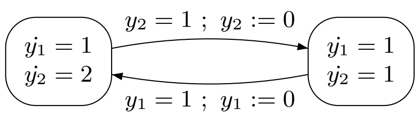

Fig. 14. Hybrid system $\mathcal{H}_1$

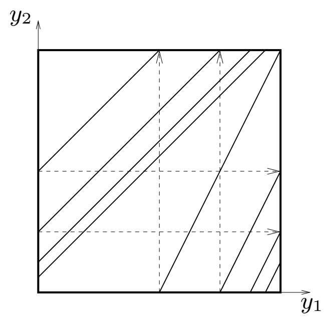

Fig. 15. Infinite bisimulation on $\mathcal{H}_1$

show that for different subclasses of o-minimal hybrid system with general type

[12] Let us remark that memoryless discrete transitions are the ones where the function $\mathcal{U}$ assigns the empty set to each edge.

discrete transitions the *reachability problem* is undecidable. Their proofs show that *the halting problem for 2-counter machines* is reducible to it.

On the other hand it is shown in [HKPV] that the so-called *initialized singular automata* admit finite bisimulations. In the sequel we discuss the case of *initialized o-minimal hybrid systems*. Let us first define the scope of the adjective *initialized*. *Initialized discrete transitions* are a special case of discrete transition of general type defined as follows. We consider the dynamics γ componentwise, and we force the j^{th} component of y to be reset on the discrete transition $e = (l_1, a, l_2)$ if the j^{th} component of the dynamics on l_1 is different from the j^{th} component of the dynamics on l_2. More precisely this condition is given by:

Definition 6.5. Initialized discrete transition:

$$\begin{aligned}(l_1, y_1) \xrightarrow{a}_{\mathcal{H}} (l_2, y_2) \;\Leftrightarrow\; & e = (l_1, a, l_2) \in Edg \text{ and } y_1 \in \mathcal{G}(e) \text{ and } y_2 \in \mathcal{R}(e)\\ & \text{and } \forall j \in \mathcal{U}(e) \text{ we have that } (y_1)_j = (y_2)_j\\ & \text{and } (\gamma_{l_1}(x,t))_j \neq (\gamma_{l_2}(x,t))_j \Rightarrow y_j \notin \mathcal{U}(e)\end{aligned}$$

Again, we can easily find examples of initialized o-minimal hybrid systems which do not have finite bisimulation. Consider the hybrid system $\mathcal{H}_2$ of Figure 16, it is clearly an initialized o-minimal hybrid system since the is only one continuous dynamics. The guards and resets of $\mathcal{H}_1$ imply that the point $\{(1,1)\}$ and the diagonal $D = \{(y_1, y_2) \mid 0 < y_1 = y_2 < 1\}$ are isolated in the initial partition $\mathcal{P}$. By iterating the predecessor computation of $\{(1,1)\}$ and intersecting the resulting sets with the diagonal D, we easily see that the bisimulation algorithm does not terminate (see Figure 17), by using the same argument than in Section 6.1. So $\mathcal{H}_2$ does not admit finite bisimulation. It is shown in [Mil]

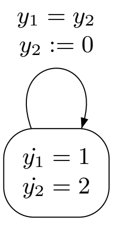

Fig. 16. Hybrid system $\mathcal{H}_2$

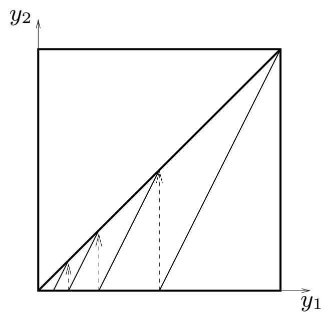

Fig. 17. Infinite bisimulation on $\mathcal{H}_2$

that the *reachability problem* is already undecidable for a subclass of initialized o-minimal hybrid systems[13]. The proof uses the undecidability of *the halting*

[13] The proof only uses timed automata with finitely many parameters in $\mathbb{Z}[\sqrt{2}]$. Usual timed automata only allows finitely many parameters in $\mathbb{Z}$.

problem for 2-counter machines and one can be easily convinced that the proof uses initialized o-minimal hybrid system.

Remarks 6.6. This section shows that the memoryless discrete transitions seem compulsory in order to keep finite bisimulation (and even decidability of the reachability problem). This a posteriori justifies the definition of o-minimal hybrid systems of [LPS] (see Remark 4.6). However, the richness of the continuous dynamics, allowed in the assumption of Theorem 4.20, encompasses some encoding of resets of variables.

Let us look at the following example (Figure 18) which is clearly in the class of o-minimal dynamical system with self-intersecting behavior which satisfies assumptions of Corollary 4.21. Consider the *dynamics* together with the partition $\mathcal{P}$ of Figure 18. Due to the transition system defined on dynamical system (see Definition 2.17), we have that the hybrid system given by Figure 19 is in a some broader sense "bisimilar" to the o-minimal dynamical system of Figure 18. This example of encoding discrete transitions in a single continuous dynamical system (through the definition of the transition system associated with it) opens some perspectives for future work.

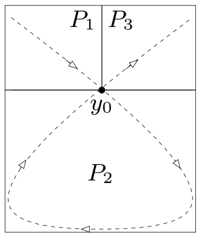

Fig. 18. $\mathcal{P} = \{y_0, P_1, P_2, P_3\}$

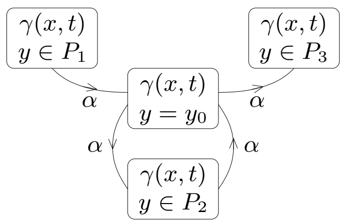

Fig. 19. $\alpha \equiv (y = y_0)$

Conclusion

In this paper, we introduce several notions of symbolic dynamics in the context of hybrid systems. A lot of them are quite general, we study their definability in a first-order structure, particularly in the case of o-minimal hybrid system. It allows us to show several results about the reachability problem and the existence of finite bisimulations. In particular, we are able to show that the result of [LPS] still holds when permitting more general continuous transitions for o-minimal hybrid systems. On the other hand, we show it is not possible to guarantee [LPS] result if we relax the assumptions about the discrete transitions.

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