# *On conformal anomalies and invariants in arbitrary dimensions*

*General solution of the Wess-Zumino consistency condition*

Nicolas Boulanger

Service de *Physique de l'Universe, Champs et Gravitation*, Université de Mons (UMONS), Belgium

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# Plan

1. Conformal Anomalies
   - Introduction
   - Wess-Zumino consistency conditions
   - Solution of the WZ conditions for the anomaly
   - The results
2. Conformal Invariants
   - Another day another cohomology
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In 1973, Derek Capper and Michael J. Duff discovered that the invariance under Weyl rescaling of the metric tensor

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x)$$

displayed by classical massless field systems in interaction with gravity no longer survives in the quantum theory.

$\hookrightarrow$ Weyl (or conformal) anomaly

# Conformal massless fields coupled to gravity

Examples of spin-1, spin-1/2 and spin-0 field theories :

- $$S[A_\mu, g_{\mu\nu}] = \frac{1}{4}\int d^4x\sqrt{-g}\,F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}$$
  where $\quad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$

- $$S[\Psi, e^a_\mu] = \frac{1}{2}\int d^nx\, e(\bar{\Psi}\gamma^a\nabla_a\Psi - \nabla_a\bar{\Psi}\gamma^a\Psi)$$

- $$S[\phi, g_{\mu\nu}] = \frac{1}{2}\int \sqrt{-g}\,\left[g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \xi(n)\,\mathscr{R}\,\Phi^2\right]\,d^nx$$
  with $\quad \xi(n) = \dfrac{1}{4}\,[(n-2)/(n-1)]\,.$

# A parenthesis : Indices, tensors and all that...

- Spacetime indices $\to$ Greek letters, e.g. Riemann tensor $R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} + \ldots$, Christoffel symbols $\Gamma^{\mu}{}_{\nu\rho}$, Ricci tensor $\mathscr{R}_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$ and scalar curvature $\mathscr{R} = g^{\alpha\beta}\mathscr{R}_{\alpha\beta}$ ; Curvature two-form $R^{\mu}{}_{\nu} = \frac{1}{2}\, R^{\mu}{}_{\nu\rho\sigma} dx^{\rho} dx^{\sigma}$ .
- Frame (tangent bundle) indices $\to$ Latin letters. The frame fields are $e_a = e_a^{\mu}\partial_{\mu}$ in coordinates $x^{\mu}$. Determinant $e = \det e_{\mu}^{a}$ where $e_{\mu}^{a} e_{a}^{\nu} = \delta_{\mu}^{\nu}$.
- For Dirac spinors : Clifford algebra $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$ where $\gamma_a$ denote Dirac's matrices and $\eta =$diag$(+,-,-,-)$ ; $\nabla_a \Psi = e_a^{\mu}(\partial_{\mu} - \frac{i}{2}\, \omega_{\mu}{}^{bc}\Sigma_{bc})\Psi$ , where $\Sigma_{bc} = \frac{i}{4}\,[\gamma_b, \gamma_c]$ and $\omega_{\mu}{}^{bc} = \omega_{\mu}{}^{bc}(e)$ is the Levi-Civita spin-connection.

# Trace of stress-tensor

- These matter systems coupled to gravity are invariant under the local Weyl rescalings

$$\left.\begin{array}{ll} g_{\mu\nu} & \rightarrow \Omega^2(x)\, g_{\mu\nu} \\ e_\mu^a & \rightarrow \Omega\, e_\mu^a \\ \Psi & \rightarrow \Omega^{(1-n)/2}\, \Psi \\ \phi & \rightarrow \Omega^{(2-n)/2}\, \phi \end{array}\right\} \tag{1}$$

- This is reflected in the (on-shell) tracelessness of the corresponding (matter) symmetric stress-tensors : (1) $\Rightarrow g^{\mu\nu} T_{\mu\nu} = 0\,.$

# Local symmetries

Clearly, by construction these actions are also invariant under diffeomorphisms.

To summarize, the local symmetries of these conformally invariant massless systems coupled to gravity are

## Local symmetries :

- Diffeomorphism invariance
- Weyl rescaling invariance

# Both symmetries cannot survive

It turns out that, after regularization, both symmetries cannot survive at the same time. One always chooses to maintain diffeomorphism invariance (conservation of energy-momentum). This is done at the price of a

Weyl anomaly

$$\hookrightarrow A = g^{\mu\nu} \langle T_{\mu\nu} \rangle_{reg} \neq 0$$

Note : Weyl anomalies are also called "Trace anomalies" or "Conformal anomalies" for obvious reasons.

# Generating functionals

- Generating functional of Green's functions :

$$Z[J] \;=\; \int \mathscr{D}\Phi \, e^{\frac{i}{\hbar} \int d^n x \, [\mathscr{L}(\Phi,\partial\Phi) + J(x)\Phi(x)]}$$

- Generating functional of *connected* Green's functions :

$$W[J] \;=\; -i \ln Z[J]$$

- The generating functional of 1PI Green's functions

$$\Gamma[\Phi_c] \;=\; W[J] - \int d^n x \, \Phi_c(x) J(x) \,, \quad \Phi_c(x) := \frac{\delta W[J]}{\delta J(x)} \,.$$

The functional $\Gamma$ is also called *quantum action* or *effective action*. In following, write $\Phi$ in place of $\Phi_c$.

# The anomalies cannot be anything

## 1. An anomaly in QFT ...

Anomalies occur when quantization spoils symmetries of the classical action, i.e. if $\Gamma[\Phi]$ cannot be made invariant under infinitesimal transformations $s$ by a suitable choice of local counterterms.

## 2. ... is an infinitesimal variation

To lowest order in $\hbar$ the variation $A = s\,\Gamma[\Phi]$ is local. It is an anomaly if it cannot be written as $A = s\,C$ for any local functional $C$ .

# Consistency conditions

Because an anomaly is a variation

$$\boxed{A = s\,\Gamma[\Phi]}$$

it is not arbitrary but constrained to obey consistency conditions. Similar to integrability conditions $\nabla \times F = 0$ which a gradient $F = \nabla\varphi$ has to satisfy.
$\Rightarrow$ An anomaly must satisfy the

Wess-Zumino consistency conditions [1971]

# BRST-cohomological rephrasing

The analysis of WZ consistency conditions simplifies in the

Becchi-Rouet-Stora-Tyutin (BRST) formulation.

$\hookrightarrow$ one introduces a ghost for each gauge parameter ;

$\hookrightarrow$ one suitably defines the transformations of the ghosts so that

$$\boxed{s^2 = 0}$$

## Local cohomology of $s$

The WZ consistency conditions take the simple form

$$s\,A = 0\,, \quad A \neq s\,C$$

where $A$ and $C$ are local functionals $A = \int a_1^n([\Phi], x)$ , $C = \int b_1^n([\Phi], x)$ and $s$ is the BRST differential.

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# Wess-Zumino Consistency condition

Central equations for candidate anomalies in QFT : Wess-Zumino (WZ) consistency conditions. By using these conditions, the general structure of all the know anomalies (except the conformal one) had been determined by purely algebraic methods featuring descent equations à la Stora-Zumino.

Determining the general solution of the WZ consistency conditions is tantamount to computing the cohomology of the corresponding Becchi-Rouet-Stora-Tyutin (BRST) differential $s$ in the space of local functionals with ghost number one.

# Stora-Zumino descent of equations ?

With $A = \int a_1^n$ , the WZ conditions get translated to

$$\boxed{s\, a_1^n + d\, a_2^{n-1} = 0\,, \quad a_1^n \sim a_1^n + s\, c_0^n + d\, c_1^{n-1}} \tag{2}$$

with the total exterior derivative $d = dx^\mu \frac{\partial}{\partial x^\mu}$ . One has

$$s^2 = 0\,, \qquad d^2 = 0\,,$$
$$\{s, d\} := s\, d + d\, s = 0\,.$$

Acting on (2) with $s$ and using the above relations :

$$d\,(s\, a_2^{n-1}) = 0 \quad \overset{\text{algebraic Poincaré lemma}}{\Longrightarrow} \quad \boxed{s\, a_2^{n-1} + d\, a_3^{n-2} = 0}\,.$$

Apply $s$ again on this equations, ...

# A ladder of equations

... one obtains the following descent equations

$$
\begin{aligned}
s\,a_1^n + d\,a_2^{n-1} &= 0 \quad , \\
s\,a_2^{n-1} + d\,a_3^{n-2} &= 0 \quad , \\
&\vdots \\
s\,a_q^{n-q+1} + d\,a_{q+1}^{n-q} &= 0 \quad , \\
s\,a_{q+1}^{n-q} &= 0 \quad (0 \leqslant q \leqslant n)\,.
\end{aligned}
$$

If $q = 0$, the descent is *trivial* : $s\,a_1^n = 0\,.$

## Dubois-Violette, Talon, Viallet (1985)

- In order to find $a_1^n \in H^{1,n}(s|d)$, find the $a_{q+1}^{n-q} \in H(s)$ that can be lifted up to a top form.

# Bonora et al.

Cohomological consideration, although without any descent equation analysis $\hookrightarrow$ pioneering works by Bonora, Cotta-Ramusino, Reina, Pasti and Bregola [1983–1985]. Results up to even dimension $n = 6$.

They found :

(I) Euler term

$$e_1^n \quad = \quad \sqrt{-g}\,\omega\,(R^{\mu_1\nu_1}\ldots R^{\mu_m\nu_m})\,\varepsilon_{\mu_1\nu_1\ldots\mu_m\nu_m}\ ,$$

plus

(II) strictly Weyl-invariant scalar densities. In $n = 4$ e.g.

$$a_1^4 = \omega\sqrt{-g}g^{\sigma\tau}g^{\lambda\kappa}W^{\mu}{}_{\rho\sigma\lambda}W^{\rho}{}_{\mu\tau\kappa}d^4x$$

where $W^{\mu}{}_{\rho\sigma\lambda}$ : conformally invariant Weyl tensor, traceless part of Riemann curvature tensor $R^{\mu}{}_{\rho\sigma\lambda}$.

- Using dimensional regularization, Deser and Schwimmer confirmed the structure obtained by Bonora et al.

  The Euler term from class (i) was called type-A Weyl anomaly, while the terms of (ii) were called type-B anomalies;

- From the structure of the poles in the variation of the effective action, they observed that the type-A anomaly appears in a similar way to the non-Abelian chiral anomaly in Yang-Mills gauge theory. That the type-A anomaly should arise via some *descent equations* was therefore conjectured.

# BRST TRANSFORMATIONS

- Apart from $g_{\mu\nu}$, the other fields of the problem are the Weyl ghost $\omega$ and the diffeomorphisms ghosts $\xi^\mu$ , $gh(\xi^\mu) = gh(\omega) = 1$ .

- The BRST transformations on the fields $\Phi^A = \{g_{\mu\nu}, \omega, \xi^\mu\}$ read

$$s_{_D} g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho} \, , \; s_{_W} g_{\mu\nu} = 2\omega g_{\mu\nu}$$
$$s_{_D} \xi^\mu = \xi^\rho \partial_\rho \xi^\mu \, , \quad s_{_D} \omega = \xi^\rho \partial_\rho \omega \, , \quad s_{_W} \xi^\mu = 0 = s_{_W} \omega \, .$$

where the BRST differentials $s_{_W}$ and $s_{_D}$ implement the Weyl transformations and the diffeomorphisms, respectively.

# WZ conditions for WA

Upon quantization one always chooses to preserve diffeomorphism invariance. With $s = s_{_W} + s_{_D}$, decomposing $s\,a + d\,b = 0$, $a \sim a + s\,c + d\,f$ w.r.t. the Weyl ghost degree gives the WZ consistency conditions for the Weyl anomalies in terms of local forms :

$$
(*) \left\{ \begin{array}{rcl} s_{_W} a_1^n + d\, b_2^{n-1} = & 0\,, & a_1^n \neq s_{_W} p_0^n + d\, f_1^{n-1}\,, \\ \\ s_{_D} a_1^n + d\, c_2^{n-1} = & 0\,, & \forall\, p_0^n \quad s.t. \quad s_{_D} p_0^n + d\, h_1^{n-1} = 0\,. \end{array} \right.
$$

# Stora's trick

Denoting $\boxed{\tilde{s}_{W} = s_{W} + d}$ and similarly for $s_{D}$, the problem (*) amounts to determining the $\tilde{s}_{D}$-invariant $(n+1)$-local total forms $\alpha(\mathscr{W})$ satisfying

$$\tilde{s}_{W}\,\alpha(\mathscr{W}) = 0\,, \quad \alpha(\mathscr{W}) \neq \tilde{s}_{W}\,\zeta(\mathscr{W}) + constant\,, \tag{3}$$

$$\boxed{TotalDeg = formdeg + gh}$$

where $\zeta(\mathscr{W})$ must be $\tilde{s}_{D}$-invariant.

Using very general results obtained in [Friedemann Brandt, CMP 1996], we know that the solution of (3) will take the form

$$\alpha(\mathscr{W}) = 2\omega\,\tilde{C}^{N_1}\ldots\tilde{C}^{N_n}\,a_{N_1\ldots N_n}(\mathscr{T})\,. \tag{4}$$

# $\mathscr{T}$ : Weyl-covariant tensors

$\hookrightarrow$ The space $\mathscr{T}$ is generated by the (invertible) metric $g_{\mu\nu}$ together with the $W$-tensors $\{W_{\Omega_i}\}$, $i \in \mathbb{N}$ that contain $W^{\mu}{}_{\nu\rho\sigma}$, $\nabla_\tau W^{\mu}{}_{\nu\rho\sigma}$ and tower $\{\mathscr{D}_{\alpha_1} \dots \mathscr{D}_{\alpha_n} W^{\mu}{}_{\nu\rho\sigma}\}$, $n \in \mathbb{N}$, where $\boxed{[\mathscr{D}, \mathscr{D}] \sim \text{Weyl} + \text{Cotton}}$.

$\hookrightarrow$ One can write the Weyl tensor as

$$W^{\mu}{}_{\nu\rho\sigma} \quad = \quad R^{\mu}{}_{\nu\rho\sigma} - 2\Big(\delta^{\mu}_{[\rho} P_{\sigma]\nu} - g_{\nu[\rho} P_{\sigma]}{}^{\mu}\Big)\,,$$

where the Schouten tensor $P_{\mu\nu}$ is

$$P_{\mu\nu} \quad = \quad \frac{1}{n-2}\left(\mathscr{R}_{\mu\nu} - \frac{1}{2(n-1)}\, g_{\mu\nu}\mathscr{R}\right).$$

# Generalized connections

The generalized connections $\tilde{C}^N$ in (4) :

$$\{\tilde{C}^N\} = \{2\omega\,, dx^\nu\,, \tilde{C}^\mu{}_\nu\,, \tilde{\omega}_\alpha\}\,,$$

$$\tilde{C}^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho}\, dx^\rho\,, \quad \tilde{\omega}_\alpha = \omega_\alpha - P_{\alpha\rho}\, dx^\rho\,, \quad \omega_\alpha = \partial_\alpha \omega\,.$$

Note that $\tilde{\omega}_\alpha$ is a local total form of degree 1, the sum of piece $\omega_\alpha$ with ghost number 1 but form degree 0 plus $P_{\alpha\rho}\, dx^\rho$ of ghost degree zero but form degree 1 :

$$\boxed{TotalDeg = formdeg + gh}$$

***Theorem*** : Let $\psi_{\mu_1\dots\mu_{2p}}$ be the local total form

$$\begin{aligned}\psi_{\mu_1\dots\mu_{2p}} &= \frac{\omega}{\sqrt{-g}}\,\varepsilon^{\alpha_1\dots\alpha_r}{}_{\nu_1\dots\nu_r\mu_1\dots\mu_{2p}}\tilde{\omega}_{\alpha_1}\dots\tilde{\omega}_{\alpha_r}\,dx^{\nu_1}\dots dx^{\nu_r}\,,\\ p &= m-r\,,\quad m=n/2\,,\quad 0\leqslant r\leqslant m\end{aligned}$$

and let $W^{\mu\nu}$ denote the tensor-valued two-form $W^{\mu\nu}=W^{\mu}{}_{\rho}\,g^{\rho\nu}$ , then the local total forms $\Phi_r^{[n-r]}$ $(0\leqslant r\leqslant m)$

$$\Phi_r^{[n-r]}=\frac{(-1)^p}{2^p}\,\frac{m!}{r!\,p!}\,\psi_{\mu_1\dots\mu_{2p}}\,W^{\mu_1\mu_2}\dots W^{\mu_{2p-1}\mu_{2p}}$$

obey a descent equations so that the following relations hold :

$$\tilde{s}_W\,\alpha=0=\tilde{s}_W\,\beta$$

with

$$\alpha:=\sum_{r=1}^{m}\Phi_r^{[n-r]}\,,\quad \beta:=\Phi_0^{[n]}\,.$$

## Theorem (A)

*The top form-degree component $a_1^n$ of $\alpha$ (cf. Theorem 1) satisfies the WZ consistency conditions for the Weyl anomalies. The WZ conditions for $a_1^n$ give rise to a non-trivial descent and $a_1^n$ is the unique anomaly with such a property, up to the addition of trivial terms and anomalies satisfying a trivial descent.*

## Theorem (B)

*The top form-degree component $e_1^n$ of $(\alpha + \beta)$ is proportional to the Euler density of the manifold $\mathscr{M}_n$ :*

$$e_1^n \quad = \quad \frac{(-1)^m}{2^m} \sqrt{-g}\, \omega\, (R^{\mu_1\nu_1} \dots R^{\mu_m\nu_m})\, \varepsilon_{\mu_1\nu_1 \dots \mu_m\nu_m} \; .$$

*The anomaly $\beta = \Phi_0^{[n]}$ — a contraction of a product of Weyl tensors — satisfies a trivial descent. It is a type-B anomaly.*

# A regularization-free understanding

- Universal structure of Weyl anomalies established in a purely algebraic manner, independently of any regularization scheme and in *arbitrary* dimensions $n$. The type-A Weyl anomaly : The *unique* Weyl anomaly satisfying a non-trivial descent of equations.

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# From anomalies to invariants

- Conformal anomalies are related to *global conformal invariant*. The Deser-Schwimmer paper triggered the interest of mathematicians working in the field of conformal geometry.
- Global conformal invariants are given by the integral over a $n$-dimensional (pseudo) Riemannian manifold $\mathscr{M}_n(g)$ of linear combinations of strictly Weyl-invariant scalar densities with scalar densities that are invariant under Weyl rescalings only up to a total derivative.
- What is the general structure of the latter ?
  $\hookrightarrow$ relevant for deformations of Weyl-invariant Lagrangians densities.

- By the assumption of *locality*, a global invariant is a ghost-zero scalar density whose Hodge dual $a^{0,n}$ obeys the cocycle equation

$$sa^{0,n} + db^{1,n-1} = 0 \ .$$

- The local conformal invariants are (the integral of) scalar densities that are strictly Weyl invariant. They can be built using various techniques, be them algebraic or geometric [tractor calculus].

- The global invariants are scalar densities that are Weyl invariant only up to a total derivative $\Rightarrow$ Produce a non-trivial descent equations.

- Non-trivial *descent equations* :

$$\left.\begin{aligned} s\,a^{0,n} + d\,a^{1,n-1} &= 0 \\ s\,a^{1,n-1} + d\,a^{2,n-2} &= 0 \\ &\vdots \\ s\,a^{p-1,n-p+1} + d\,a^{p,n-p} &= 0 \\ s\,a^{p,n-p} &= 0 \end{aligned}\right\}$$

It stops either because $p = n$ or because one encounters an $s$-cocycle $a^{p,n-p}$.

- Decomposing the first equation wrt Weyl-ghost degree :

$$\left\{\begin{array}{rl} s_{_D}\,a^{0,n} + d\,f^{1,n-1} = & 0\,, \\ s_{_W}\,a^{0,n} + d\,g^{1,n-1} = & 0\,, \end{array}\right. \quad a^{0,n} \neq d\,b^{0,n-1}\;.$$

- The classification of global conformal invariants is also given by the cohomology of the associated BRST differential in top form degree $n$, but this time, at ghost number *zero*, i.e., $H^{0,n}(s|d)$. The two cohomological groups $H^{1,n}(s|d)$ (anomalies) and $H^{0,n}(s|d)$ present some similarities but also important differences. The latter group is the larger!
- The conjecture of Deser and Schwimmer on the structure of Weyl anomalies led the mathematician Spyros Alexakis to study the problem of the *classification of global conformal invariants*.
  $\hookrightarrow$ Gave rise to several long publications culminating with a monograph "The Decomposition of Global Conformal Invariants" in the Annals of Mathematics Studies series at Princeton U. Press, 2012.

# Pursuing the cohomological analysis

- From
$$\begin{cases} s_{_D} a^{0,n} + d\, f^{1,n-1} = & 0\,, \\ s_{_W} a^{0,n} + d\, g^{1,n-1} = & 0\,, \end{cases} \quad a^{0,n} \neq d\, b^{0,n-1}\;,$$

  $\hookrightarrow$ Find the cocycles of the differential $s_{_W}$ modulo $d$, in the cohomology of the diffeomorphism-invariant local $n$-forms.

- The latter cohomology class already been worked out in [Brandt-Dragon-Kreuzer89] and [Barnich-Brandt-Henneaux95].

- Denote by $f_K := \mathrm{Tr}(R^{m(K)})$, $K \in \{1, \ldots, r = [n/2]\}$, the invariant polynomials of the Lorentz algebra $so(1, n-1)$ and $q^0_K$ the corresponding Chern-Simons $(2m(K)-1)$-forms obeying $dq^0_K = f_K$. The general solution of the first equation above decomposes into two main classes :

- Two main classes :

$$a^{0,n} = \underbrace{\sqrt{-g}\, L(\nabla, R, g) d^n x}_{class\ I} + \underbrace{\sum_m \sum_{K:m(K)=m} q^0_K \frac{\partial}{\partial f_K} P_m(f_1, \ldots, f_r)}_{class\ II} \;.$$

- The second class only contributes for spacetimes of dimensions $n = 4p - 1$, $p \in \mathbb{N}^*$. Taking $n = 7$ as a definite example, the second class gives two structures

$$\mathrm{Tr}(\Gamma d\Gamma + \tfrac{2}{3}\,\Gamma^3)\mathrm{Tr}(R^2) \equiv L^3_{CS}\, \mathrm{Tr}(R^2) \text{ and } L^7_{CS} = \mathrm{Tr}(I_7)\;,$$
$$I_7 = \Gamma(d\Gamma)^3 + \tfrac{8}{5}(d\Gamma)^2\Gamma^3 + \tfrac{4}{5}\Gamma(\Gamma d\Gamma)^2 + 2\,\Gamma^5 d\Gamma + \tfrac{4}{7}\Gamma^7,$$

where $\Gamma$ denotes the matrix-valued 1-form $dx^\mu\, \Gamma^\alpha{}_{\beta\mu}$ whose components $\Gamma^\alpha{}_{\beta\mu}$ are the Christoffel symbols and $\mathrm{Tr}(\cdot)$ denotes the matrix trace. $\mathrm{Tr}R^2 \equiv R^\alpha{}_\beta R^\beta{}_\alpha$ for $R^\alpha{}_\beta = \frac{1}{2}\, dx^\mu dx^\nu\, R^\alpha{}_{\beta\mu\nu}$ the curvature 2-form.

## Lemma 1 :

Let $\psi_{\mu_1\dots\mu_{2p}}$ be the local total form

$$\begin{aligned} \psi_{\mu_1\dots\mu_{2p}} &= \frac{1}{\sqrt{-g}}\,\varepsilon^{\alpha_1\dots\alpha_r}{}_{\nu_1\dots\nu_r\mu_1\dots\mu_{2p}}\tilde{\omega}_{\alpha_1}\dots\tilde{\omega}_{\alpha_r}\,dx^{\nu_1}\dots dx^{\nu_r}\,, \\ p &= m-r\,, \quad m=n/2\,, \quad r\in\{0,\dots,m\}\,. \end{aligned}$$

Then, the local total forms

$$\Phi_r^{[n-r]} = \frac{(-1)^p}{2^p}\,\frac{m!}{r!\,p!}\,\psi_{\mu_1\dots\mu_{2p}}\,W^{\mu_1\mu_2}\dots W^{\mu_{2p-1}\mu_{2p}}$$

satisfy non-trivial descent equations and give solutions

$$\begin{aligned} \tilde{s}_{W}\,\alpha = \;& 0 \;= \tilde{s}_{W}\,\beta \quad \text{for} \\ \alpha \;&=\; \sum_{r=1}^{m}\Phi_r^{[n-r]} \quad \text{and} \quad \beta=\Phi_0^{[n]}\,. \end{aligned}$$

## [Lemma 2 Invariants of class I]

The top form-degree component $a^{0,n}$ of $\alpha$ in Lemma 1 satisfies the cocycle condition for the conformal invariants. It gives rise to a non-trivial descent in $H(s_W|d)$ . The invariant $\beta = \Phi_0^{[n]}$ satisfies a trivial descent and is obtained by taking contractions of products of Weyl tensors ($m$ of them in dimension $n = 2m$). The top form-degree component $e^{0,n}$ of $\alpha + \beta$ is proportional to the Euler density of the manifold $\mathscr{M}_n$ :

$$e^{0,n} = \frac{(-1)^m}{2^m}\sqrt{-g}\,\varepsilon_{\alpha_1\beta_1\ldots\alpha_m\beta_m}\,(R^{\alpha_1\beta_1}\wedge\ldots\wedge R^{\alpha_m\beta_m})$$

It is the *only* conformal invariant of the class I that satisfies a non-trivial descent in $H(s_W|d)$ .

## Lemma 3 [Invariants of class II]

Let $\alpha^{4p-1}_{[2m-1]}$ be the total $(4p-1)$-form of degree $2m-1$ in the connection 1-form $\Gamma$, defined by

$$\alpha^{4p-1}_{[2m-1]} := -\tfrac{1}{2m-1}\,\mathrm{Tr}\left([\omega dx - R]^{2p-m}\Gamma^{2m-1}\right)\ , \quad m = 1, 2, \dots 2p\ ,$$
$$\alpha^{4p-1}_{[0]} := 2\omega(d\omega)^{2p-1}\ ,$$

where $[\omega dx - R]$ stands for the matrix-valued total 2-form with components $\omega^\alpha dx_\beta - R^\alpha{}_\beta$ and $\Gamma$ denotes the matrix-valued 1-form with $\Gamma^\alpha{}_\beta$ for components. Then, the total form

$$\tilde{\alpha}^{4p-1} := \alpha^{4p-1}_{[0]} + \sum_{m=1}^{2p} \alpha^{4p-1}_{[2m-1]}$$

obeys the equation

$$\tilde{s}_W \tilde{\alpha}^{4p-1} = \mathrm{Tr} R^{2p}\ .$$

By decomposing the equation $\tilde{s}_W \tilde{\alpha}^{4p-1} = \mathrm{Tr} R^{2p}$ with respect to the form degree, we obtain, in dimension $n = 4p-1$, the descent equations

$$\begin{aligned} \mathrm{Tr} R^{2p} &= dL^n_{CS} \;, \\ s_W L^n_{CS} + da^{1,n-1} &= 0 \;, \\ s_W a^{1,n-1} + da^{2,n-2} &= 0 \;, \\ &\vdots \\ s_W a^{2p-1,2p} + da^{2p,2p-1} &= 0 \;, \\ s_W a^{2p,2p-1} &= 0 \;, \quad a^{2p,2p-1} \equiv \alpha^{4p-1}_{[0]} \;. \end{aligned} \tag{5}$$

Equation (5) is the WZ consistency condition for a conformal anomaly in a submanifold of co-dimension 1 wrt $\mathscr{M}_{4p-1}$ . The consistent Weyl anomaly is the integral, over this co-dimension one submanifold $\mathscr{M}_{4p-2}$ , of

$$a^{1,n-1} = \sum_{m=1}^{2p} \tfrac{(-1)^m}{2m-1}\, dx^\mu g_{\mu\alpha} [\Gamma^{2m-1} R^{2p-m-1}]^\alpha{}_\beta \, g^{\beta\sigma} \omega_\sigma \;.$$

- Finally, descent equations associated with a product of the type $L_{CS}^{4p-1} f_{K_1} \dots f_{K_m}$ will be exactly the same as the descent associated with $L_{CS}^{4p-1}$, where each element $a^{q,n-q}$ is obtained from the corresponding one in the descent for $L_{CS}^{4p-1}$ upon taking the wedge product with $f_{K_1} \dots f_{K_m}$. In other words, the products of the type $f_{K_1} \dots f_{K_m}$ are completely spectators in a descent of $s_W$ modulo $d$. That the $f_K$'s are $s_W$-closed is trivial once one realizes the identity $\mathrm{Tr}(R^{m(K)}) \equiv \mathrm{Tr}(W^{m(K)})$ that is obtained from the relation $R^{ab} = W^{ab} + 2e^{[a}P^{b]}$ where $e^a$ are the vielbein 1-forms and $P^a$ is the Schouten 1-form.

# Action and field equations

- Given a pseudo-Riemannian spacetime $\mathscr{M}_{4p-1}$ of dimension $n = 4p-1$ with an orientation, consider the functional

$$I[g_{\mu\nu}] = \frac{1}{2p}\int_{\mathscr{M}_{4p-1}} L_{CS}^{4p-1}\ .$$

- The Euler-Lagrange derivative (wrt the metric) of the functional is

$$\mathscr{E}^{\mu\nu} := \frac{\delta I}{\delta g_{\mu\nu}} \equiv \frac{1}{2^{2p-1}}\nabla^{\lambda}\mathscr{A}^{(\mu|\nu)}{}_{\lambda}\ ,$$

where

$$\mathscr{A}^{\mu|\nu}{}_{\lambda} := \varepsilon^{\mu\nu_2\nu_3\dots\nu_{4p-1}}\ [R_{\nu_2\nu_3}\dots R_{\nu_{4p-2}\nu_{4p-1}}]^{\nu}{}_{\lambda}\ .$$

and $[R_{\nu_2\nu_3}\dots R_{\nu_{4p-2}\nu_{4p-1}}]^{\nu}{}_{\lambda}$ denotes the $(2p-1)$-fold product of the 2-form valued matrix $[R_{\nu_2\nu_3}]^{\alpha}{}_{\beta} \equiv R^{\alpha}{}_{\beta\nu_2\nu_3}$ .

- Weyl and diffeomorphism invariances of the action $I[g_{\mu\nu}]$ get translated into the Noether identites

$$g_{\mu\nu}\mathscr{E}^{\mu\nu} \equiv 0 \,, \qquad \text{and} \qquad \nabla_\mu \mathscr{E}^{\mu\nu} \equiv 0 \,.$$

- For the second identity, one must use

$$\varepsilon^{\nu_1 \dots \nu_{4p-1}} \,\mathrm{Tr}[R_{\nu_1\nu_2} \dots R_{\nu_{4p-3}\nu_{4p-2}} R_{\nu_{4p-1}\nu}] \equiv 0 \,,$$

(Schouten identity and cyclicity of the trace)

- Finally, one has the strict invariance under Weyl transformations :

$$s_W \mathscr{E}^{\mu\nu} = -2\,\omega\,\mathscr{E}^{\mu\nu} \;\Leftrightarrow\; s_W \mathscr{E}^{\mu}{}_{\nu} = 0 \,.$$

that can be seen by expressing

$$\mathscr{A}^{\mu|\nu}{}_{\lambda} = \varepsilon^{\mu\nu_2\nu_3\dots\nu_{4p-1}} \left[W_{\nu_2\nu_3} \dots W_{\nu_{4p-2}\nu_{4p-1}}\right]^{\nu}{}_{\lambda} \,.$$

# Conclusions

- As a consequence of our decomposition, global conformal invariants are *not* in one-to-one correspondence with the conformal anomalies. Indeed, multiplying the Lorentz Chern-Simons densities by the Weyl parameter $\sigma(x)$ does *not* produce any consistent conformal anomaly.
- Our work generalises the analyses devoted to the three-dimensional case $p = 1$ [see TMG and PvN] and completes the results obtained in the book by Alexakis, where the global conformal invariants related to the Lorentz Chern-Simons densities were overlooked.
- Prospect : Higher-derivative TMG ?