# Frobenius-Chern-Simons gauge theory 

Roberto Bonezzi ${ }^{1}$, Nicolas Boulanger ${ }^{1}$, Ergin Sezgin ${ }^{2}$ and Per Sundell ${ }^{3}$<br>${ }^{1}$ Groupe de Mécanique et Gravitation, Physique Théorique et Mathématique, Université de Mons - UMONS, 20 Place du Parc, B-7000 Mons, Belgium nicolas.boulanger@umons.ac.be, roberto.bonezzi@umons.ac.be<br>${ }^{2}$ George and Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy Texas A\& M University, College Station, TX 77843, USA<br>sezgin@tamu.edu<br>${ }^{3}$ Departamento de Ciencias Físicas, Universidad Andres Bello, Republica 220, Santiago de Chile per.anders.sundell@gmail.com


#### Abstract

Given a set of differential forms on an odd-dimensional noncommutative manifold valued in an internal associative algebra $\mathcal{H}$, we show that the most general cubic covariant Hamiltonian action, without mass terms, is controlled by an $\mathrm{Z}_{2}$-graded associative algebra $\mathcal{F}$ with a graded symmetric nondegenerate bilinear form. The resulting class of models provide a natural generalization of the Frobenius-Chern-Simons model (FCS) that was proposed in arXiv:1505.04957 as an off-shell formulation of the minimal bosonic four-dimensional higher spin gravity theory. If $\mathcal{F}$ is unital and the $Z_{2}$-grading is induced from a Klein operator that is outer to a proper Frobenius subalgebra, then the action can be written on a form akin to topological open string field theory in terms of a superconnection valued in $\mathcal{H} \otimes \mathcal{F}$. We give a new model of this type based on a twisting of $\mathrm{C}\left[\mathrm{Z}_{2} \times \mathrm{Z}_{4}\right]$, which leads to selfdual complexified gauge fields on $A d S_{4}$. If $\mathcal{F}$ is 3 -graded, the FCS model can be truncated consistently as to zero-form constraints on-shell. Two examples thereof are a twisting of $\mathrm{C}\left[\left(\mathrm{Z}_{2}\right)^{3}\right]$ that yields the original model, and the Clifford algebra $\mathcal{C} \ell_{2 n}$ which provides an FCS formulation of the bosonic Konstein-Vasiliev model with gauge algebra $h u\left(4^{n-1}, 0\right)$.


## Contents

1 Introduction ..... 2
2 Review of the minimal FCS gauge theory ..... 5
3 Generalized FCS gauge theory ..... 11
3.1 Cubic action ..... 12
$3.2 \quad Z_{2}$-graded quasi-Frobenius algebra ..... 14
3.3 Polarization in target space ..... 17
3.4 Global formulation ..... 20
4 Unital algebras with Klein operators ..... 22
4.1 Trace operation and outer Klein operator ..... 22
4.2 Superconnection ..... 23
4.3 Component formulation ..... 24
5 3-grading ..... 26
5.1 On-shell free differential algebra ..... 26
5.2 3-grading from inner Klein operator of $\mathcal{F}_{0}$ ..... 27
6 Examples ..... 28
6.1 3-graded matrix algebra ..... 29
6.2 3-graded Clifford algebra ..... 31
6.3 Twisted group algebra of $Z_{2} \times Z_{2 n}$ ..... 33
7 Conclusions ..... 36
A Twisted group algebras ..... 39

## 1 Introduction

In 1], a modified version of Vasiliev's four-dimensional higher spin gravity [2] has been introduced, with enlarged gauge symmetry and a dynamical two-form master field but with the same master zero-form as the original model. As a result, the two models propagate the same local degrees of freedom but the new one has (much) fewer higher spin gauge invariant observables, which could be an advantage in finding an effective action along the topological field theory inspired approach proposed and studied in [3, 4, 5, 5, 6, 7].

More specifically, the enlarged master field content consists of the original Weyl zeroform; two one-forms gauging one-sided actions of a complexified higher spin algebra; a twoform that contains topological degrees of freedom (including moduli for the star product algebra on the internal twistor space and fluxes in spacetime); corresponding bulk Lagrange multipliers; and, finally, the master fields required for a gapless duality extension of the model [4]. The key to the extension is the fact that the Cartan integrability of the resulting enlarged system of unfolded equations of motion is controlled by an an internal eight-dimensional $Z_{2^{-}}$ graded Frobenius algebra, such that the full field content can be assembled into a single flat superconnection valued in a direct product of this algebra and the original associative higher spin star product algebra.

An action principle can then be constructed following the Alexandrov-Kontsevitch-SchwarzZaboronsky (AKSZ) procedure [14], by introducing an auxiliary fifth commuting dimension and writing a covariant Hamiltonian action for the superconnection on the resulting ninedimensional base manifold; further references and a review can be found in [8. Formally, the superconnection is an odd element of an underlying associative superbundle, whose superdifferential can be used to write down a Chern-Simons-like cubic action 1 , leading to what shall henceforth refer to as the Frobenius-Chern-Simons (FCS) formulation of four-dimensional higher spin gravity, or the minimal FCS gauge theory.

The mathematical structure of the FCS model suggests a number of generalizations worthy of investigation, in particular in relation to the proposed relationship between (massless) higher spin gravity and topological open strings [10], later verified directly at the level of

[^0]amplitudes in [7, 11.
In this paper we shall examine the structure of the most general FCS model consisting of a set of even and odd forms on an odd-dimensional noncommutative base manifold valued in an internal associative algebra $\mathcal{H}$ and with canonical kinetic terms and general cubic Hamiltonian without mass terms. As we shall see, the gauge symmetry of the action, or equivalently, the Cartan integrability of the equations of motion, results in an action for a superconnection valued in $\mathcal{H} \otimes \mathcal{F}$ where $\mathcal{F}$ is a $Z_{2}$-graded associative algebra with a graded symmetric nondegenerate bilinear form, that we refer to as a $Z_{2}$-graded quasi-Frobenius algebra, as it does not have to contain a unity. In general, the resulting field equations may contain integrable zero-form constraints, which can be treated within the AKSZ scheme.

We then focus on unital algebras in which the $Z_{2_{2}}$-grading is generated by a Klein operator that is outer to a proper Frobenius subalgebra $\mathcal{F}_{0} \subset \mathcal{F}$. Simple examples of these generalized FCS models, that we shall present below, are based on matrix algebras and twisted group algebras [12, 13]. In this category, we shall present a simple model based on an eightdimensional Frobenius algebra that leads to a variant of the original FCS model with a zero-form constraint, containing a branch consisting of self-dual complexified gauge fields along the lines of [20].

A subset of the $Z_{2}$-graded models exhibit a refined 3-grading that can used to truncate the top-forms consistently together with some of the zero-forms and next-to-top forms, as to obtain a subclass of FCS models without zero-form constraints. We shall describe a specific truncation scheme, that employs an inner Klein operator in defining the 3 -grading, and provide examples thereof based on matrix algebras, Clifford algebras and twisted group algebras. In particular, the original FCS model arises within this subclass from a twisting the $\left(Z_{2}\right)^{3}$ algebra. Another set of examples based on matrix and Clifford algebras furnish novel off-shell formulations of a class of bosonic Konstein-Vasiliev [15, 16] models, that differ from their direct FCS extensions (based on the direct product of the Frobenius algebra of minimal FCS model and the Konstein-Vasiliev matrix algebra).

We would like to stress that the generalized FCS gauge theories to be constructed in what follows may have applications beyond higher spin gravity. With this in mind, we shall not make any definite choice for the internal associative algebra $\mathcal{H}$, that we shall hence
treat formally, sidestepping temporarily the important issues of choices of bases for the star product algebras, related function classes and the finiteness of the Lagrangian. Our focus is instead on how the nature of the Frobenius algebra $\mathcal{F}$ is affected by the gauge invariance and the existence of a polarization in target space $2^{2}$, such that the theory can be defined globally on a manifold $\mathcal{M}$ given by the direct product of a commutative manifold with boundaries (that may contains spacetimes), and a closed noncommutative manifold. Eventually, we hope to be able to address the global formulation on more general noncommutative manifolds (obtained from differential Poisson structures and their homotopy associative extensions), though the simplified geometries to be considered here nonetheless exhibits enough structure in order to lead to nontrivial constraints on the underlying Frobenius algebra.

The plan of the paper is as follows: In Section 2 we review the original FCS model [1]. In Section 3, we tackle the problem of how this model can be generalized by studying, under some assumptions, the most general cubic action for a set of odd and even form master fields on an odd-dimensional noncommutative manifold, which leads to the emergence of $Z_{2^{-}}$ graded quasi-Frobenius algebras. We then show how the global formulation on direct product manifolds with boundaries can be achieved using a polarization in target space, yielding the generalized FCS gauge theory action (3.49); notably, the attendant inner product need not be a trace operation as the algebra need not be unital. In Section 4 we introduce a unit element and an outer Kleinian operator such that the action can be written as an integral of a Chern-Simons-like Lagrangian density expressed using a trace and a single odd master field, referred to as the superconnection [17]. In Section [5, we provide a general scheme for the elimination of all zero-form constraints by employing a 3-grading of the Frobenius algebra. Section 6 contains a set of examples based on matrix algebras and twisted group algebras, containing two new models of interest to higher spin gravity, namely a $Z_{2}$-graded model with zero-form constraints containing a self-dual branch, and an FCS generalization of a set of bosonic Konstein-Vasiliev models (with internal symmetries). We conclude in Section 7, pointing to future directions involving homotopy associative algebras and tightening correspondency to underlying first-quantized model. The appendix contains a summary of basic properties of twisted group algebras, and a demonstration of the fact that Frobenius algebra of the original FCS model is a twisting of the $\left(Z_{2}\right)^{3}$ group algebra.

[^1]
## 2 Review of the minimal FCS gauge theory

The FCS model of [1] is formulated on a direct product manifold

$$
\begin{equation*}
\mathcal{M}_{9}=\mathcal{X}_{5} \times \mathcal{Z}_{4} \tag{2.1}
\end{equation*}
$$

where $\mathcal{Z}_{4}$ is a four-dimensional closed noncommutative manifold and $\mathcal{X}_{5}$ is a five-dimensional open commutative manifold whose boundary contains spacetime (possibly as an open subset). The model consists of locally defined differential forms 3 on $\mathcal{M}_{9}$, referred to as master fields, valued in an associative higher spin algebra $\mathcal{W} \otimes \mathcal{K}$, where $\mathcal{W}$ is a Weyl algebra extended by inner Klein operators, and $\mathcal{K}=C\left(Z_{2} \times Z_{2}\right)$ is the (untwisted) group algebra generated by two outer Klein operators of $\Omega\left(\mathcal{Z}_{4}\right) \otimes \mathcal{W}$; for further details, see [1].

The spectrum of master fields make up a superconnection $Z$ in a generalized bundle space $\sqrt[4]{ }$ $\mathcal{E}$ with fiber given by $\mathcal{A}=\mathcal{W} \otimes \mathcal{K} \otimes \mathcal{F}$, where

$$
\begin{equation*}
\mathcal{F}=\mathcal{F}_{0} \oplus h \mathcal{F}_{0}=\mathcal{F}^{(-1)} \oplus \mathcal{F}^{(0)} \oplus \mathcal{F}^{(+1)} \tag{2.2}
\end{equation*}
$$

is the associative algebra built from

$$
\begin{equation*}
\mathcal{F}_{0}=\operatorname{mat}_{2}(\mathrm{C})=\bigoplus_{i, j=1,2} \mathrm{C} \otimes e_{i j} \tag{2.3}
\end{equation*}
$$

and an outer Klein operator $h$, subject to the product rules

$$
\begin{equation*}
e_{i j} e_{k l}=\delta_{j k} e_{i l}, \quad h e_{i j}=(-1)^{i-j} e_{i j} h, \quad h^{2}=1 \tag{2.4}
\end{equation*}
$$

[^2]which provides $\mathcal{F}_{0}$ with a $Z_{2}$-grading that can be further refined into a 3 -grading by declaring
\[

$$
\begin{equation*}
e_{i j}, h e_{i j} \in \mathcal{F}^{(j-i)} \tag{2.5}
\end{equation*}
$$

\]

The algebra can be realized as

$$
\begin{gather*}
e:=e_{11}=\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \otimes 1, \quad \tilde{e}:=e_{22}=\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right) \otimes 1,  \tag{2.6}\\
f:=e_{12}=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) \otimes \sigma_{1}, \quad \tilde{f}:=e_{21}=\left(\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right) \otimes \sigma_{1},  \tag{2.7}\\
h:=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \otimes \sigma_{3} . \tag{2.8}
\end{gather*}
$$

Alternatively, in order to make manifest the 3-grading, one may display the algebra as

$$
\mathcal{F}=\left[\begin{array}{ll}
e \oplus h e & f \oplus h f  \tag{2.9}\\
\tilde{f} \oplus h \tilde{f} & \tilde{e} \oplus h \tilde{e}
\end{array}\right],
$$

where

$$
\begin{equation*}
\mathcal{F}^{(-1)}=\tilde{f} \oplus h \tilde{f}, \quad \mathcal{F}^{(0)}=e \oplus h e \oplus \tilde{e} \oplus h \tilde{e}, \quad \mathcal{F}^{(+1)}=f \oplus h f . \tag{2.10}
\end{equation*}
$$

The superconnection can thus be expanded as

$$
\begin{equation*}
Z=h X+P \tag{2.11}
\end{equation*}
$$

where

$$
X=\sum_{i, j} X^{i j} e_{i j}=\left(\begin{array}{cc}
A & B  \tag{2.12}\\
\widetilde{B} & \widetilde{A}
\end{array}\right), \quad P=\sum_{i, j} P^{i j} e_{i j}=\left(\begin{array}{cc}
V & U \\
\widetilde{U} & \widetilde{V}
\end{array}\right)
$$

whose entries are $\mathcal{W} \otimes \mathcal{K}$-valued master fields decomposing under the form degree on $\mathcal{M}_{9}$ as follow: $5^{5}$

$$
\begin{gather*}
\operatorname{deg}(B, A, \widetilde{A}, \widetilde{B}) \in\{(2 n, 1+2 n, 1+2 n, 2+2 n)\}_{n=0,1,2,3},  \tag{2.13}\\
\operatorname{deg}(\widetilde{U}, V, \widetilde{V}, U)=\{(8-2 n, 7-2 n, 7-2 n, 6-2 n)\}_{n=0,1,2,3} . \tag{2.14}
\end{gather*}
$$

[^3]Introducing the Z-valued superdegree $\operatorname{map} \operatorname{deg}_{\mathcal{E}}$ given by the sum of form degree and Frobenius degree, viz.

$$
\begin{equation*}
\operatorname{deg}_{\mathcal{E}}:=\operatorname{deg}_{\mathcal{M}_{9}}+\operatorname{deg}_{\mathcal{F}} \tag{2.15}
\end{equation*}
$$

and the superdifferential

$$
\begin{equation*}
q:=h d \tag{2.16}
\end{equation*}
$$

it follows that $Z$, and hence $X$ and $P$, are odd elements in strictly positive superdegrees, that is

$$
\begin{equation*}
\operatorname{deg}_{\mathcal{E}}(Z), \operatorname{deg}_{\mathcal{E}}(X), \operatorname{deg}_{\mathcal{E}}(P) \in\{1,3, \ldots\} \tag{2.17}
\end{equation*}
$$

and that $q$ is a nilpotent differential of superdegree $\operatorname{deg}_{\mathcal{E}}(q)=1$, viz.

$$
\begin{equation*}
q(f \star g)=q(f) \star g+(-1)^{\operatorname{deg}_{\mathcal{E}}(f)} f \star q g, \quad f, g \in \mathcal{E} \tag{2.18}
\end{equation*}
$$

Turning to the action, it requires a (cyclic) trace operation

$$
\begin{equation*}
\operatorname{Tr}_{\mathcal{A}}=\operatorname{Tr}_{\mathcal{W} \otimes \mathcal{K}} \operatorname{Tr}_{\mathcal{F}} \tag{2.19}
\end{equation*}
$$

where $\operatorname{Tr}_{\mathcal{W} \otimes \mathcal{K}}=\operatorname{Tr}_{\mathcal{W}} \operatorname{Tr}_{\mathcal{K}}$ is composed out of the standard trace on $\mathcal{K}$ and a modified supertrace on $\mathcal{W}$ (making use of the inner Klein operators to achieve cyclicity), and

$$
\begin{equation*}
\operatorname{Tr}_{\mathcal{F}}\left(e_{i j}\right)=\delta_{i j}, \quad \operatorname{Tr}_{\mathcal{F}}\left(h e_{i j}\right)=0 \tag{2.20}
\end{equation*}
$$

Letting $\pi_{h}$ be the automorphism that sends $h$ to $-h$, the action proposed in [1] reads

$$
\begin{equation*}
S=\int_{\mathcal{M}_{9}} \operatorname{Tr}_{\mathcal{A}}\left(\frac{1}{2} Z \star q Z+\frac{1}{3} Z \star Z \star Z\right)-\frac{1}{4} \oint_{\partial \mathcal{M}_{9}} \operatorname{Tr}_{\mathcal{A}}\left[h \pi_{h}(Z) \star Z\right] \tag{2.21}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
S=\int_{\mathcal{M}_{9}} \operatorname{Tr}_{\mathcal{A}}\left(P \star F^{X}+\frac{1}{3} P \star P \star P\right) \tag{2.22}
\end{equation*}
$$

where

$$
\begin{equation*}
F^{X}:=d X+h X h \star X, \tag{2.23}
\end{equation*}
$$

and it is assumed that the locally defined configurations are glued together (see footnote 3)
such that the Lagrangian is globally defined and that $\int_{\mathcal{M}_{9}} \operatorname{Tr}_{\mathcal{A}}$ and $\oint_{\partial_{\mathcal{M}_{9}}} \operatorname{Tr}_{\mathcal{A}}$ are cyclic and graded cyclic operations, respectively, that are non-degenerate and obey Stokes' theorem.

Taking $X$ to fluctuate freely at $\partial \mathcal{M}_{9}$, the variational principle implies ${ }^{6}$

$$
\begin{equation*}
R:=q Z+Z \star Z \approx 0,\left.\quad P\right|_{\partial \mathcal{M}_{9}} \approx 0 \tag{2.24}
\end{equation*}
$$

which do not contain any zero-form constraints $\operatorname{since}^{\operatorname{deg}_{\mathcal{E}}}(Z \star Z) \geqslant 1$. Equivalently, by decomposing $R=R^{X}+R^{P}$, where $R^{X}$ and $h R^{P}$ are $h$-independent, the equations of motion can be written as

$$
\begin{equation*}
R^{X}:=F^{X}+P \star P \approx 0, \quad R^{P}:=Q P \approx 0 \tag{2.25}
\end{equation*}
$$

with

$$
\begin{equation*}
Q f:=q f+h X \star f-(-1)^{\operatorname{deg}_{\mathcal{E}}(f)} f \star h X \tag{2.26}
\end{equation*}
$$

obeying the graded Leibniz rule

$$
\begin{equation*}
Q(f \star g)=Q(f) \star g+(-1)^{\operatorname{deg}_{\mathcal{E}}(f)} f \star Q g \tag{2.27}
\end{equation*}
$$

The equations of motion form a Cartan integrable system with Bianchi identities

$$
\begin{equation*}
q R+[Z, R]_{\star}=0 \tag{2.28}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
Q R^{X}+\left[P, R^{P}\right]_{\star} \equiv 0, \quad Q R^{P}-\left[R^{X}, P\right]_{\star} \equiv 0 \tag{2.29}
\end{equation*}
$$

as can be seen using the ordinary Bianchi identities

$$
\begin{equation*}
Q^{2} f=\left[F^{X}, f\right]_{\star}, \quad Q F^{X}=0 \tag{2.30}
\end{equation*}
$$

The generalized Bianchi identities ensure invariance of the action under the gauge transformations

$$
\begin{equation*}
\delta Z=q \theta+[Z, \theta]_{\star} \tag{2.31}
\end{equation*}
$$

[^4]up to total derivatives. Decomposing
\[

$$
\begin{equation*}
\theta=\epsilon^{X}+h \epsilon^{P} \tag{2.32}
\end{equation*}
$$

\]

one finds that an $\epsilon^{X}$-transformation leaves the action invariant, while an $\epsilon^{P}$-transformation yields a total derivative that vanishes provided that $\epsilon^{P}$ belongs to the same section as $P$ and

$$
\begin{equation*}
\left.\epsilon^{P}\right|_{\partial \mathcal{M}_{9}}=0 \tag{2.33}
\end{equation*}
$$

Using the basis (2.10) to decompose

$$
\begin{equation*}
Z=\mathbf{A}+\mathbf{B} \tag{2.34}
\end{equation*}
$$

where

$$
\mathbf{A}=\left(\begin{array}{cc}
h A+V & 0  \tag{2.35}\\
0 & h \widetilde{A}+\widetilde{V}
\end{array}\right), \quad \mathbf{B}=\left(\begin{array}{cc}
0 & h B+U \\
h \widetilde{B}+\widetilde{U} & 0
\end{array}\right)
$$

are odd and even forms, respectively, the action (2.21) takes the form

$$
\begin{align*}
S= & \int_{\mathcal{M}_{9}} \operatorname{Tr}_{\mathcal{A}}\left[\frac{1}{2} \mathbf{A} \star q \mathbf{A}+\frac{1}{3} \mathbf{A} \star \mathbf{A} \star \mathbf{A}+\frac{1}{2} \mathbf{B} \star(q \mathbf{B}+\mathbf{A} \star \mathbf{B}+\mathbf{B} \star \mathbf{A})\right] \\
& -\frac{1}{4} \int_{\partial \mathcal{M}_{9}} \operatorname{Tr}_{\mathcal{A}}\left[h \pi_{h}(\mathbf{A}) \star \mathbf{A}+h \pi_{h}(\mathbf{B}) \star \mathbf{B}\right]  \tag{2.36}\\
= & \int_{\mathcal{M}_{9}} \operatorname{Tr}_{\mathcal{W} \otimes \mathcal{K}}\left[V \star\left(F-B \star \widetilde{B}+\frac{1}{3} V \star V+U \star \widetilde{U}\right)+\widetilde{U} \star D B\right. \\
& \left.\quad+\widetilde{V} \star\left(\widetilde{F}-\widetilde{B} \star B+\frac{1}{3} \widetilde{V} \star \widetilde{V}+\widetilde{U} \star U\right)+U \star \widetilde{D} \widetilde{B}\right] \tag{2.37}
\end{align*}
$$

where we have defined

$$
\begin{array}{cc}
F=d A+A \star A, & \widetilde{F}=d \widetilde{A}+\widetilde{A} \star \widetilde{A} \\
D B=d B+A \star B-B \star \widetilde{A}, & \widetilde{D} \widetilde{B}=d \widetilde{B}+\widetilde{A} \star \widetilde{B}-\widetilde{B} \star A \tag{2.39}
\end{array}
$$

Finally, on $\partial \mathcal{M}_{9}$, where $(U, \widetilde{U} ; V, \widetilde{V})$ vanish, the equations of motion read

$$
\begin{align*}
& F-B \star \widetilde{B} \approx 0, \quad D B \approx 0  \tag{2.40}\\
& \widetilde{F}-\widetilde{B} \star B \approx 0, \quad \widetilde{D} \widetilde{B} \approx 0 \tag{2.41}
\end{align*}
$$

which can be shown to contain Vasiliev's equations upon expanding around a vacuum expectation value the dynamical two-form in $\widetilde{B}$ and fixing a gauge for $\widetilde{A}-A$; in the simplest setting

$$
\begin{equation*}
\widetilde{A}=A=W, \quad \widetilde{B}=J, \tag{2.42}
\end{equation*}
$$

where $J$ is a closed a central two-form and the reduced system takes the form

$$
\begin{equation*}
d W+W \star W+B \star J=0, \quad d B+W \star B-B \star W=0 \tag{2.43}
\end{equation*}
$$

Extending the model by $\widetilde{B}_{[0]}, \widetilde{U}_{[0]}, B_{[8]}, U_{[8]}$ and the top-forms $A_{[9]}, \widetilde{A}_{[9]}, V_{[9]}$ and $\widetilde{V}_{[9]}$, yields a gauge invariant action for a superconnection with

$$
\begin{equation*}
\operatorname{deg}_{\mathcal{E}}(Z), \operatorname{deg}_{\mathcal{E}}(X), \operatorname{deg}_{\mathcal{E}}(P) \in\{-1,1,3, \ldots\} \tag{2.44}
\end{equation*}
$$

leading to quadratic zero-form constraints on-shell that are compatible with the differential constraints on the zero-form: 7 ; on the boundary

$$
\begin{equation*}
B_{[0]} \star \widetilde{B}_{[0]} \approx 0, \quad \widetilde{B}_{[0]} \star B_{[0]} \approx 0 \tag{2.45}
\end{equation*}
$$

The equations of motion can furthermore be extended by ten-form curvature constraints for the top-forms, which yields a universal quasi-free differential algebra (from which the top-form gauge transforms can be read off).

The algebra $\mathcal{F}$ reflected in the master field content of the model is an example of a Frobenius algebra, that is, a unital associative algebra with a nondegenerate invariant bilinear form 8. In what follows, we shall generalize the above model by arranging the master fields

[^5]using $Z_{2}$-graded but not necessarily unital associative algebras with nondegenerate bilinear forms, which we shall refer to as $Z_{2}$-graded quasi-Frobenius algebras. Just as in the model above, the $Z_{2}$-grading will be crucial for on-shell integrability and gauge invariance, while the stronger 3-grading, which facilitates the removal of top-forms off-shell and hence zeroform constraints on-shell, is an optional requirement $t^{9}$. Although unitality is optional as well, the unital case, which contains (twisted) group algebras and more general Hopf algebras, is nonetheless interesting as it permits the usage of inner Klein operators to generate the polarization.

## 3 Generalized FCS gauge theory

In this section we shall generalize the FCS model of Section 2 to models consisting of a finite numbers of master differential forms on odd-dimensional noncommutative manifolds with boundaries, valued in an associative algebra with a trace operation. Under natural assumptions on the resulting differential form algebra, i.e. Leibniz' rule, Stokes' law and cyclicity of the integration operation, we shall demonstrate that the most general cubic covariant Hamiltonian action with canonical kinetic term (without terms containing more than one exterior derivative nor mass terms) is governed by a $Z_{2}$-graded quasi Frobenius algebra.

We remark that on a commutative manifold $\mathcal{M}$, the off-shell gauge invariance of a general covariant Hamiltonian action (with general symplectic potential and including mass terms) is equivalent to on-shell Cartan integrability. In the noncommutative case, this equivalence continues to hold in the case of a canonical kinetic term. Below, in the cubic case, we shall also keep track of boundary terms and deal with global formulations by means of specifications of polarizations.

[^6]
### 3.1 Cubic action

We consider a Lagrangian of the form 10

$$
\begin{align*}
S= & \int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left(\frac{1}{2} A^{I} \star d A^{J} \Sigma_{I J}+\frac{1}{2} B^{P} \star d B^{Q} \Omega_{P Q}+\frac{1}{3} t_{I J K} A^{I} \star A^{J} \star A^{K}+s_{I P Q} A^{I} \star B^{P} \star B^{Q}\right) \\
& +\frac{1}{4} \oint_{\partial \mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[A^{I} \star A^{J} \Theta_{I J}-B^{P} \star B^{Q} \Xi_{P Q}\right] \tag{3.1}
\end{align*}
$$

where $\mathcal{M}$ is a noncommutative manifold of dimension $2 n+1$ with boundary $\partial \mathcal{M}$, and

$$
\begin{align*}
A^{I} & =A_{[1]}^{I}+A_{[3]}^{I}+\cdots+A_{[2 n+1]}^{I}  \tag{3.2}\\
B^{P} & =B_{[0]}^{P}+B_{[2]}^{P}+\cdots+B_{[2 n]}^{P}, \tag{3.3}
\end{align*}
$$

where $I=1, \ldots, N_{+}$and $P=1, \ldots, N_{-}$, which we shall refer to as the master fields, are differential forms on $\mathcal{M}$ valued in an associative algebra $\mathcal{H}$, that is, elements of $\Omega(\mathcal{M}) \otimes \mathcal{H}$. Initially, we shall assume that the master fields are defined globally on $\mathcal{M}$; in Section 3.4 we shall relax this condition and provide a global formulation in terms of locally defined fields of a special type on direct product manifolds.

In (3.1), the $\star$ denotes the combined associative product on $\Omega(\mathcal{M}) \otimes \mathcal{H}$ and $\Omega(\partial \mathcal{M}) \otimes \mathcal{H}$. It is assumed that the operation of restricting to the boundary commutes with the star product operation, i.e. $\left.(f \star g)\right|_{\partial \mathcal{M}}=\left.\left.(f)\right|_{\partial \mathcal{M}} \star(g)\right|_{\partial \mathcal{M}}$ for $f, g \in \Omega(\mathcal{M}) \otimes \mathcal{H}$. The combined operations $\int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H}}$ and $\oint_{\partial \mathcal{M}} \operatorname{Tr}_{\mathcal{H}}$, where $\operatorname{Tr}_{\mathcal{H}}$ denotes a trace operation on $\mathcal{H}$, are assumed to be cyclic and graded cyclic linear maps on $\Omega(\mathcal{M}) \otimes \mathcal{H}$ and $\Omega(\partial \mathcal{M}) \otimes \mathcal{H}$, respectively. We shall assume that Leibniz' rule holds together with Stokes' theorem, viz. $\int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H}} d(\cdot)=\oint_{\partial \mathcal{M}} \operatorname{Tr}_{\mathcal{H}}(\cdot)$. It follows that

$$
\begin{align*}
& \Sigma_{I J}=\Sigma_{J I}, \quad \Omega_{P Q}=-\Omega_{Q P}, \quad t_{I J K}=t_{J K I} \\
& \Theta_{I J}=-\Theta_{J I}, \quad \Xi_{P Q}=\Xi_{Q P} \tag{3.4}
\end{align*}
$$

[^7]We assume that $\Sigma_{I J}$ and $\Omega_{P Q}$ are nondegenerate with inverses defined by

$$
\begin{equation*}
\Omega^{R P} \Omega_{Q P}=\delta_{Q}^{R}, \quad \Sigma^{I K} \Sigma_{J K}=\delta_{J}^{I} \tag{3.5}
\end{equation*}
$$

We assume that all master fields are real, i.e.

$$
\begin{equation*}
\left(A^{I}\right)^{\dagger}=A^{I}, \quad\left(B^{P}\right)^{\dagger}=B^{P} \tag{3.6}
\end{equation*}
$$

where the dagger denotes the hermitian conjugation map of $\Omega(\mathcal{M}) \otimes \mathcal{H}$, which is assumed to obey

$$
\begin{equation*}
\left(\int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H}}(f \star g)\right)^{\dagger}=\int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left(g^{\dagger} \star f^{\dagger}\right), \quad f, g \in \Omega(\partial \mathcal{M}) \otimes \mathcal{H} \tag{3.7}
\end{equation*}
$$

Thus, the action is then real provided that

$$
\begin{gather*}
\left(\Sigma_{I J}\right)^{\dagger}=\Sigma_{I J}, \quad\left(\Omega_{P Q}\right)^{\dagger}=\Omega_{P Q}  \tag{3.8}\\
\left(t_{I J K}\right)^{\dagger}=t_{I K J}, \quad\left(s_{I P Q}\right)^{\dagger}=s_{I Q P}, \quad\left(\Theta_{I J}\right)^{\dagger}=-\Theta_{I J}, \quad\left(\Xi_{P Q}\right)^{\dagger}=\Xi_{P Q} \tag{3.9}
\end{gather*}
$$

The total variation

$$
\begin{align*}
\delta S= & \int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[\delta A^{I} \star R^{J} \Sigma_{I J}+\delta B^{P} \star R^{Q} \Omega_{P Q}\right] \\
& +\frac{1}{2} \oint_{\partial \mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[\delta A^{I} \star A^{J}\left(\Theta_{I J}+\Sigma_{I J}\right)-\delta B^{P} \star B^{Q}\left(\Xi_{P Q}+\Omega_{P Q}\right)\right] \tag{3.10}
\end{align*}
$$

where the generalized curvatures

$$
\begin{align*}
& R^{I}:=d A^{I}+t^{I}{ }_{J K} A^{J} \star A^{K}+s^{I}{ }_{P Q} B^{P} \star B^{Q}  \tag{3.11}\\
& R^{P}:=d B^{P}-s_{I Q}{ }^{P} A^{I} \star B^{Q}-s_{I}{ }_{Q}{ }_{Q} B^{Q} \star A^{I} \tag{3.12}
\end{align*}
$$

and the indices are raised and lowered using the conventions

$$
\begin{equation*}
A^{I}=\Sigma^{I J} A_{J}, \quad B^{P}=\Omega^{P Q} B_{Q} \tag{3.13}
\end{equation*}
$$

Provided that the coefficients $\left(t_{I J K}, s_{I P Q}\right)$ obey the quadratic constraints

$$
\begin{gather*}
t^{J}{ }_{M K} t^{I}{ }_{L J}=t^{J}{ }_{L M} t^{I}{ }_{J K}, \quad s^{I}{ }_{P Q} s_{I R T}=-s^{I}{ }_{T P} s_{I Q R},  \tag{3.14}\\
s_{J P}{ }^{R} s_{I R Q}=-t^{K}{ }_{I J} s_{K P Q}, \quad s_{I}{ }^{P}{ }_{Q} s_{J P R}=s_{J Q}{ }^{P} s_{I R P}, \tag{3.15}
\end{gather*}
$$

the curvatures obey the generalized Bianchi identities

$$
\begin{align*}
d R^{I}-t^{I}{ }_{J K}\left(R^{J} \star A^{K}-A^{J} \star R^{K}\right)-s^{I}{ }_{P Q}\left(R^{P} \star B^{Q}+B^{P} \star R^{Q}\right) \equiv 0,  \tag{3.16}\\
d R^{P}+s_{I}{ }^{P}{ }_{Q}\left(R^{Q} \star A^{I}+B^{Q} \star R^{I}\right)+s_{I Q}{ }^{P}\left(R^{I} \star B^{Q}-A^{I} \star R^{Q}\right) \equiv 0 . \tag{3.17}
\end{align*}
$$

In particular, the variation of the action with respect to $A_{[2 n+1]}^{I}$ yield the zero-form constraint

$$
\begin{equation*}
R_{[0]}^{I} \equiv s_{P Q}^{I} B_{[0]}^{P} \star B_{[0]}^{Q} \approx 0 \tag{3.18}
\end{equation*}
$$

Its exterior derivative is proportional to $R_{[1]}^{P}$, that is, the bulk equations of motion define a quasi-free differential algebra with a zero-form constraint. Moreover, the curvature $R_{[2 n+2]}^{I}$ of $A_{[2 n+1]}^{I}$ does not appear in the variation of the action but can nonetheless be introduced within the context of a universal quasi-free differential algebra.

The Cartan gauge transformations

$$
\begin{align*}
\delta A^{I} & =d \epsilon^{I}+t^{I}{ }_{J K}\left(A^{J} \star \epsilon^{K}-\epsilon^{J} \star A^{K}\right)-s^{I}{ }_{P Q}\left(\eta^{P} \star B^{Q}+B^{P} \star \eta^{Q}\right)  \tag{3.19}\\
\delta B^{P} & =d \eta^{P}+s_{I Q}{ }^{P}\left(\epsilon^{I} \star B^{Q}-A^{I} \star \eta^{Q}\right)+s_{I}{ }^{P}{ }_{Q}\left(\eta^{Q} \star A^{I}+B^{Q} \star \epsilon^{I}\right), \tag{3.20}
\end{align*}
$$

transform the curvatures covariantly and leave the action invariant up to boundary terms, which will be studied below in a more streamlined notation.

## $3.2 \quad \mathrm{Z}_{2}$-graded quasi-Frobenius algebra

The constraints (3.4), (3.5) and (3.15) are equivalent the existence of a $Z_{2}$-graded associative algebra

$$
\begin{equation*}
\mathcal{F}=\mathcal{F}^{+} \oplus \mathcal{F}^{-} \tag{3.21}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{F}^{+}=\bigoplus_{I=1}^{N^{+}} \mathrm{C} \otimes e_{I}, \quad \mathcal{F}^{-}=\bigoplus_{P=1}^{N^{-}} \mathrm{C} \otimes f_{P}, \tag{3.22}
\end{equation*}
$$

in terms of generators obeying the product laws

$$
\begin{align*}
& e_{I} e_{J}=e_{K} t^{K}{ }_{I J},  \tag{3.23}\\
& e_{I} f_{R}=-f_{P} s_{I R}{ }^{P}, \quad f_{P} f_{Q}=-e_{I} s^{I}{ }_{P Q}=f_{P} s_{I}{ }^{P}{ }_{R}, \tag{3.24}
\end{align*}
$$

with a non-degenerate bilinear form

$$
\begin{equation*}
\left(e_{I}, e_{J}\right)_{\mathcal{F}}=\Sigma_{I J}, \quad\left(f_{P}, f_{Q}\right)_{\mathcal{F}}=\Omega_{P Q} \tag{3.25}
\end{equation*}
$$

obeying the invariance condition

$$
\begin{equation*}
(a, b c)_{\mathcal{F}}=(a b, c)_{\mathcal{F}}, \quad a, b, c \in \mathcal{F} \tag{3.26}
\end{equation*}
$$

and the graded symmetry property

$$
\begin{equation*}
\left(a, b^{\sigma}\right)_{\mathcal{F}}=\sigma\left(b^{\sigma}, a\right)_{\mathcal{F}}, \quad a \in \mathcal{F} . \quad b^{\sigma} \in \mathcal{F}^{\sigma}, \quad \sigma= \pm . \tag{3.27}
\end{equation*}
$$

The associativity conditions $e(e e)=(e e) e, f(f f)=(f f) f, e(e f)=(e e) f$ and $e(f e)=(e f) e$ imply the constraints in (3.15). The invariance conditions $\left(e_{I} e_{J}, e_{K}\right)_{\mathcal{F}}=\left(e_{I}, e_{J} e_{K}\right)_{\mathcal{F}}$ and $\left(e_{I} f_{P}, f_{Q}\right)_{\mathcal{F}}=\left(e_{I}, f_{P} f_{Q}\right)_{\mathcal{F}}$, respectively, hold by virtue of the cyclicity of $t_{K I J}$ and the fact that both $e_{I} f_{P}$ and $f_{P} f_{Q}$ are given in terms of $s_{I P Q}$.

Introducing the master fields 11

$$
\begin{equation*}
\boldsymbol{A}:=\sum_{I} A^{I} e_{I} \in \mathcal{H} \otimes \mathcal{F}^{+}, \quad \boldsymbol{B}:=\sum_{P} B^{P} f_{P} \in \mathcal{H} \otimes \mathcal{F}^{-} \tag{3.28}
\end{equation*}
$$

and corresponding curvatures

$$
\begin{equation*}
\boldsymbol{F}=d \boldsymbol{A}+\boldsymbol{A} \star \boldsymbol{A}, \quad \boldsymbol{D} \boldsymbol{B}=d \boldsymbol{B}+\boldsymbol{A} \star \boldsymbol{B}-\boldsymbol{B} \star \boldsymbol{A} \tag{3.29}
\end{equation*}
$$

[^8]the action (3.1) can be re-written as
\[

$$
\begin{align*}
S= & \int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[\frac{1}{2}(\boldsymbol{A}, d \boldsymbol{A})_{\mathcal{F}}+\frac{1}{3}(\boldsymbol{A}, \boldsymbol{A} \star \boldsymbol{A})_{\mathcal{F}}+\frac{1}{2}(\boldsymbol{B}, \boldsymbol{D} \boldsymbol{B})_{\mathcal{F}}\right] \\
& -\frac{1}{4} \oint_{\partial \mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[(\boldsymbol{A}, \Theta(\boldsymbol{A}))_{\mathcal{F}}-(\boldsymbol{B}, \Xi(\boldsymbol{B}))_{\mathcal{F}}\right] \tag{3.30}
\end{align*}
$$
\]

where we have defined the outer operators

$$
\begin{equation*}
\Theta\left(e_{I}\right):=\Theta_{I}^{J} e_{J}, \quad \Xi\left(f_{P}\right):=\Xi_{P}^{Q} f_{Q} \tag{3.31}
\end{equation*}
$$

Given the symmetry properties of $\Theta_{I J}$ and $\Xi_{P Q}$, these operators obey

$$
\begin{align*}
& \left(a^{+}, \Theta\left(b^{+}\right)\right)_{\mathcal{F}}=-\left(b^{+}, \Theta\left(a^{+}\right)\right)_{\mathcal{F}}=-\left(\Theta\left(a^{+}\right), b^{+}\right)_{\mathcal{F}},  \tag{3.32}\\
& \left(a^{-}, \Xi\left(b^{-}\right)\right)_{\mathcal{F}}=\left(b^{-}, \Xi\left(a^{-}\right)\right)_{\mathcal{F}}=-\left(\Xi\left(a^{-}\right), b^{-}\right)_{\mathcal{F}}, \tag{3.33}
\end{align*}
$$

for $a^{ \pm}, b^{ \pm} \in \mathcal{F}^{ \pm}$.
Using the above notation, the general variation reads

$$
\begin{align*}
\delta S= & \int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[\left(\delta \boldsymbol{A}, \boldsymbol{R}^{A}\right)_{\mathcal{F}}+\left(\delta \boldsymbol{B}, \boldsymbol{R}^{B}\right)_{\mathcal{F}}\right] \\
& +\frac{1}{2} \oint_{\partial \mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[(\delta \boldsymbol{A},(1-\Theta) \boldsymbol{A})_{\mathcal{F}}-(\delta \boldsymbol{B},(1-\Xi) \boldsymbol{B})_{\mathcal{F}}\right] \tag{3.34}
\end{align*}
$$

where the generalized curvatures

$$
\begin{equation*}
\boldsymbol{R}^{A}:=\boldsymbol{F}-\boldsymbol{B} \star \boldsymbol{B}, \quad \boldsymbol{R}^{B}:=\boldsymbol{D} \boldsymbol{B} \tag{3.35}
\end{equation*}
$$

obey the generalized Bianchi identities

$$
\begin{equation*}
\boldsymbol{D} \boldsymbol{R}^{A}+\left\{\boldsymbol{B}, \boldsymbol{R}^{B}\right\}_{\star} \equiv 0, \quad \boldsymbol{D} \boldsymbol{R}^{B}+\left[\boldsymbol{B}, \boldsymbol{R}^{A}\right]_{\star} \equiv 0 \tag{3.36}
\end{equation*}
$$

The Cartan gauge transformations are given by

$$
\begin{equation*}
\delta \boldsymbol{A}=\boldsymbol{D} \boldsymbol{\epsilon}+\{\boldsymbol{\eta}, \boldsymbol{B}\}_{\star}, \quad \delta \boldsymbol{B}=\boldsymbol{D} \boldsymbol{\eta}-[\boldsymbol{\epsilon}, \boldsymbol{B}]_{\star} . \tag{3.37}
\end{equation*}
$$

Under these transformations, the bulk term in (3.34) becomes a total derivative, viz.

$$
\begin{equation*}
\int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[\left(\delta \boldsymbol{A}, \boldsymbol{R}^{A}\right)_{\mathcal{F}}+\left(\delta \boldsymbol{B}, \boldsymbol{R}^{B}\right)_{\mathcal{F}}\right]=\oint_{\partial \mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[\left(\boldsymbol{\epsilon}, \boldsymbol{R}^{A}\right)_{\mathcal{F}}+\left(\boldsymbol{\eta}, \boldsymbol{R}^{B}\right)_{\mathcal{F}}\right] \tag{3.38}
\end{equation*}
$$

Taking into account the remaining boundary term in (3.34) and defining

$$
\begin{equation*}
\boldsymbol{A}_{ \pm}=\frac{1}{2}(1 \pm \Theta) \boldsymbol{A}, \quad \boldsymbol{B}_{ \pm}=\frac{1}{2}(1 \pm \Xi) \boldsymbol{B} \tag{3.39}
\end{equation*}
$$

the gauge variation of the action can be written as

$$
\begin{align*}
\delta_{\epsilon, \eta} S= & \oint_{\partial M} \operatorname{Tr}_{\mathcal{H}}\left[\left(\boldsymbol{\epsilon}, d \boldsymbol{A}_{+}+\boldsymbol{A}_{+} \star \boldsymbol{A}_{+}-\boldsymbol{A}_{-} \star \boldsymbol{A}_{-}-\boldsymbol{B}_{+} \star \boldsymbol{B}_{+}+\boldsymbol{B}_{-} \star \boldsymbol{B}_{-}\right)_{\mathcal{F}}\right. \\
& \left.+\left(\boldsymbol{\eta}, d \boldsymbol{B}_{+}+\left[\boldsymbol{A}_{+}, \boldsymbol{B}_{+}\right]_{\star}+\left[\boldsymbol{B}_{-}, \boldsymbol{A}_{-}\right]_{\star}\right)_{\mathcal{F}}\right] \tag{3.40}
\end{align*}
$$

As we shall see next, the expressions for the variations of the action given in (3.34) and (3.40), respectively, facilitates the global formulation of the model on topologically sufficiently simple base manifolds.

### 3.3 Polarization in target space

In what follows we shall give a set of conditions on $\Theta, \Xi$ and the structure coefficients of $\mathcal{F}$ such that the boundary terms in the variations (3.34) and (3.40) of the action can be expressed in terms of representations of a generalized structure group (whose transition elements are sums over forms of different degrees).

To this end, we begin by observing that since $\delta \boldsymbol{A}$ and $\delta \boldsymbol{B}$ are sections, it follows from the form of the total variation (3.34) that $\left(\boldsymbol{A}_{-}, \boldsymbol{B}_{-}\right)$and hence $\left(\boldsymbol{\epsilon}_{-}, \boldsymbol{\eta}_{-}\right)$must be sections as well. Thus, the maximal possible structure group is gauged by $\boldsymbol{A}_{+}$and $\boldsymbol{B}_{+}$.

Turning to the gauge variation (3.40), requiring it to be writable in terms of sections leads to constraints on the structure constants, the inner product, $\Theta_{I J}$ and $\Xi_{P Q}$, which we refer to as the polarization conditions. To exhibit these, we assume that $\frac{1}{2}(1 \pm \Theta)$ and $\frac{1}{2}(1 \pm \Xi)$ are projectors, that is

$$
\begin{equation*}
\Theta^{2}=\operatorname{Id}_{\mathcal{F}^{+}}, \quad \Xi^{2}=\operatorname{Id}_{\mathcal{F}^{-}} \tag{3.41}
\end{equation*}
$$

so that we can decompose

$$
\begin{equation*}
\mathcal{F}_{ \pm}^{+}:=\frac{1}{2}(1 \pm \Theta) \mathcal{F}^{+}, \quad \mathcal{F}_{ \pm}^{-}:=\frac{1}{2}(1 \pm \Xi) \mathcal{F}^{-}, \tag{3.42}
\end{equation*}
$$

where thus

$$
\begin{equation*}
\left(\mathcal{F}_{ \pm}^{\sigma}, \mathcal{F}_{ \pm}^{\sigma}\right)_{\mathcal{F}}=0, \quad \sigma= \pm \tag{3.43}
\end{equation*}
$$

in view of (3.33). Thus, requiring the gauge variation (3.40) to be expressible in terms of sections yields

$$
\begin{gather*}
\left(\mathcal{F}_{ \pm}^{+}\right)^{\star 2} \subseteq \mathcal{F}_{+}^{+}, \quad\left(\mathcal{F}_{ \pm}^{-}\right)^{\star 2} \subseteq \mathcal{F}_{+}^{+}  \tag{3.44}\\
\mathcal{F}_{ \pm}^{\sigma} \star \mathcal{F}_{ \pm}^{-\sigma} \subseteq \mathcal{F}_{+}^{-}, \quad \sigma= \pm \tag{3.45}
\end{gather*}
$$

which are linear constraints on the structure constants $\left(t_{I J K}, s_{I P Q}\right)$ that together with Eq. (3.41) form the aforementioned polarization conditions.

In order to exhibit the resulting structure, we define

$$
\begin{align*}
& (\mathcal{A}, \mathcal{B} ; \overline{\mathcal{U}}, \overline{\mathcal{V}}):=\left(\boldsymbol{A}_{+}, \boldsymbol{B}_{+} ; \boldsymbol{B}_{-}, \boldsymbol{A}_{-}\right)  \tag{3.46}\\
& \left(\epsilon^{\mathcal{A}}, \epsilon^{\mathcal{B}} ; \eta^{\overline{\mathcal{U}}}, \eta^{\overline{\mathcal{V}}}\right):=\left(\boldsymbol{\epsilon}_{+}, \boldsymbol{\eta}_{+} ; \boldsymbol{\eta}_{-}, \boldsymbol{\epsilon}_{-}\right) \tag{3.47}
\end{align*}
$$

where thus $\overline{\mathcal{U}}$ and $\overline{\mathcal{V}}$ and their gauge parameters belong to sections. Defining

$$
\begin{equation*}
\mathcal{F}=d \mathcal{A}+\mathcal{A} \star \mathcal{A}, \quad \mathcal{D B}=d \mathcal{B}+[\mathcal{A}, \mathcal{B}]_{\star} \tag{3.48}
\end{equation*}
$$

and combining (3.44) and (3.45) with the fact that the only nonvanishing inner products are $\left(a_{ \pm}^{\sigma}, b_{\mp}^{\sigma}\right)_{\mathcal{F}}$, from (3.30) we arrive at the following action:

$$
\begin{equation*}
S=\int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[(\overline{\mathcal{U}}, \mathcal{D B})_{\mathcal{F}}+\left(\overline{\mathcal{V}}, \mathcal{F}-\mathcal{B} \star \mathcal{B}+\frac{1}{3} \overline{\mathcal{V}} \star \overline{\mathcal{V}}-\overline{\mathcal{U}} \star \overline{\mathcal{U}}\right)_{\mathcal{F}}\right] \tag{3.49}
\end{equation*}
$$

which underlies the general Frobenius-Chern-Simons model based on a $Z_{2}$-graded quasiFrobenius algebra. In contrast to the case of minimal FCS model where a trace operation in the Frobenius algebra arises, here the inner product occurs. In summary, the route from the general Ansatz in (3.1) to the action (3.49) makes use of the equations (3.23), (3.25),
(3.28), (3.30), (3.39) and (3.46). If there are no even forms, the action is given by the difference of two generalized CS actions on $\mathcal{M}$ for odd forms $\mathcal{A}_{L}$ and $\mathcal{A}_{R}$ valued in $\mathcal{H} \otimes \mathcal{F}^{(+)}$ and with $\mathcal{A}=\mathcal{A}_{L}+\mathcal{A}_{R}$ and $\overline{\mathcal{V}}=\mathcal{A}_{L}-\mathcal{A}_{R}$.

The above action is of the covariant Hamiltonian form, that is, the Lagrange multipliers $(\overline{\mathcal{U}}, \overline{\mathcal{V}})$ and the fields $(\mathcal{B}, \mathcal{A})$ belong to dual spaces, since the nondegeneracy of the inner product together with Eq. (3.41) imply that

$$
\begin{equation*}
\operatorname{dim} \mathcal{F}_{+}^{\sigma}=\operatorname{dim} \mathcal{F}_{-}^{\sigma}=\frac{1}{2} N^{\sigma}, \quad \sigma= \pm \tag{3.50}
\end{equation*}
$$

The total variation of (3.49), which can also be obtained from (3.34), reads

$$
\begin{align*}
\delta S= & \int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[\left(\delta \overline{\mathcal{U}}, R^{\mathcal{B}}\right)_{\mathcal{F}}+\left(\delta \mathcal{B}, R^{\overline{\mathcal{U}}}\right)_{\mathcal{F}}+\left(\delta \overline{\mathcal{V}}, R^{\mathcal{A}}\right)_{\mathcal{F}}+\left(\delta \mathcal{A}, R^{\overline{\mathcal{V}}}\right)_{\mathcal{F}}\right] \\
& +\oint_{\partial \mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[(\delta \mathcal{A}, \overline{\mathcal{V}})_{\mathcal{F}}-(\delta \mathcal{B}, \overline{\mathcal{U}})_{\mathcal{F}}\right] \tag{3.51}
\end{align*}
$$

where the Cartan curvatures

$$
\begin{gather*}
R^{\mathcal{A}}:=\mathcal{F}-\mathcal{B} \star \mathcal{B}+\overline{\mathcal{V}} \star \overline{\mathcal{V}}-\overline{\mathcal{U}} \star \overline{\mathcal{U}}, \quad R^{\mathcal{B}}:=\mathcal{D B}-[\overline{\mathcal{U}}, \overline{\mathcal{V}}]_{\star}  \tag{3.52}\\
R^{\overline{\mathcal{V}}}:=\mathcal{D} \overline{\mathcal{V}}-\{\overline{\mathcal{U}}, \mathcal{B}\}_{\star}, \quad R^{\overline{\mathcal{U}}}:=\mathcal{D} \overline{\mathcal{U}}+[\overline{\mathcal{V}}, \mathcal{B}]_{\star} \tag{3.53}
\end{gather*}
$$

obey the generalized Bianchi identities

$$
\begin{align*}
& \mathcal{D} R^{\mathcal{A}}+\left\{\mathcal{B}, R^{\mathcal{B}}\right\}_{\star}+\left[\overline{\mathcal{V}}, R^{\overline{\mathcal{V}}}\right]_{\star}+\left\{\overline{\mathcal{U}}, R^{\overline{\mathcal{U}}}\right\}_{\star} \equiv 0  \tag{3.54}\\
& \mathcal{D} R^{\mathcal{B}}+\left[\mathcal{B}, R^{\mathcal{A}}\right]_{\star}+\left[\overline{\mathcal{U}}, R^{\overline{\mathcal{V}}}\right]_{\star}+\left\{\overline{\mathcal{V}}, R^{\overline{\mathcal{U}}}\right\}_{\star} \equiv 0  \tag{3.55}\\
& \mathcal{D} R^{\overline{\mathcal{V}}}+\left[\overline{\mathcal{V}}, R^{\mathcal{A}}\right]_{\star}+\left\{\overline{\mathcal{U}}, R^{\mathcal{B}}\right\}_{\star}+\left\{\mathcal{B}, R^{\overline{\mathcal{U}}}\right\}_{\star} \equiv 0  \tag{3.56}\\
& \mathcal{D} R^{\overline{\mathcal{U}}}+\left[\overline{\mathcal{U}}, R^{\mathcal{A}}\right]_{\star}+\left\{\overline{\mathcal{V}}, R^{\mathcal{B}}\right\}_{\star}+\left[\mathcal{B}, R^{\overline{\mathcal{V}}}\right]_{\star} \equiv 0 \tag{3.57}
\end{align*}
$$

The gauge transformations take the form

$$
\begin{align*}
& \delta \mathcal{A}=\mathcal{D} \epsilon^{\mathcal{A}}+\left\{\mathcal{B}, \epsilon^{\mathcal{B}}\right\}_{\star}+\left[\overline{\mathcal{V}}, \eta^{\overline{\mathcal{V}}}\right]_{\star}+\left\{\overline{\mathcal{U}}, \eta^{\overline{\mathcal{U}}}\right\}_{\star},  \tag{3.58}\\
& \delta \mathcal{B}=\mathcal{D} \epsilon^{\mathcal{B}}+\left[\mathcal{B}, \epsilon^{\mathcal{A}}\right]_{\star}+\left[\overline{\mathcal{U}}, \eta^{\overline{\mathcal{V}}}\right]_{\star}+\left\{\overline{\mathcal{V}}, \eta^{\overline{\mathcal{U}}}\right\}_{\star},  \tag{3.59}\\
& \delta \overline{\mathcal{V}}=\mathcal{D} \eta^{\overline{\mathcal{V}}}+\left[\overline{\mathcal{V}}, \epsilon^{\mathcal{A}}\right]_{\star}+\left\{\mathcal{B}, \eta^{\overline{\mathcal{U}}}\right\}_{\star}+\left\{\overline{\mathcal{U}}, \epsilon^{\mathcal{B}}\right\}_{\star},  \tag{3.60}\\
& \delta \overline{\mathcal{U}}=\mathcal{D} \eta^{\overline{\mathcal{U}}}+\left[\overline{\mathcal{U}}, \epsilon^{\mathcal{A}}\right]_{\star}+\left\{\overline{\mathcal{V}}, \epsilon^{\mathcal{B}}\right\}_{\star}+\left[\mathcal{B}, \eta^{\overline{\mathcal{V}}}\right]_{\star} . \tag{3.61}
\end{align*}
$$

The gauge variation of the action is given by

$$
\begin{equation*}
\delta_{\epsilon, \eta} S=\oint_{\partial \mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left[\left(\eta^{\overline{\mathcal{U}}}, \mathcal{D B}+[\overline{\mathcal{U}}, \overline{\mathcal{V}}]_{\star}\right)_{\mathcal{F}}+\left(\eta^{\overline{\mathcal{V}}}, \mathcal{F}-\mathcal{B} \star \mathcal{B}-\overline{\mathcal{V}} \star \overline{\mathcal{V}}+\overline{\mathcal{U}} \star \overline{\mathcal{U}}\right)_{\mathcal{F}}\right] . \tag{3.62}
\end{equation*}
$$

This result, as well as the result of general variation formula (3.51) will be used below in studying the global formulation.

### 3.4 Global formulation

In order to treat master fields that are defined locally we need to assume that the integration measure on $\mathcal{M}$ provides a cyclic trace operation on the algebra of locally defined forms.

The polarization introduced above suffices for nontrivial global formulations on direct product manifolds

$$
\begin{equation*}
\mathcal{M}=\mathcal{X} \times \mathcal{Z} \tag{3.63}
\end{equation*}
$$

where $\mathcal{X}$ is a commuting manifold consisting of charts $\mathcal{X}_{\xi}$ and $\mathcal{Z}$ is a closed noncommutative manifold for which $\int_{\mathcal{Z}}$ provides a trace operation on $\Omega(\mathcal{Z})$, that is cyclic and graded-cyclic, respectively, in case $\operatorname{dim}(\mathcal{Z})$ is odd and even. The master fields are taken to be locally defined forms on $\Omega\left(\mathcal{X}_{\xi} \times \mathcal{Z}\right)$. The locally defined configurations can be glued together into sections of a generalized bundle ${ }^{12}$, that we shall denote by $\mathcal{E}$, using transition functions $T_{\xi}^{\eta}$

[^9]generated by parameters
\[

$$
\begin{equation*}
\left.\left(\left(t^{\mathcal{A}}\right)_{\xi}^{\eta},\left(t^{\mathcal{B}}\right)_{\xi}^{\eta}\right) \in \Omega\left(\mathcal{X}_{\xi} \cap \mathcal{X}_{\eta}\right) \times \mathcal{Z}\right), \tag{3.64}
\end{equation*}
$$

\]

valued in subspaces of the spaces of $\mathcal{H} \otimes \mathcal{F}$ that contain the parameters $\left(\epsilon^{\mathcal{A}}, \epsilon^{\mathcal{B}}\right)$. Letting $\mathcal{X}_{\xi}^{\prime} \subseteq \mathcal{X}_{\xi}$ be patches such that ${ }^{13}$

$$
\begin{equation*}
\mathcal{X}=\cup_{\xi} \mathcal{X}_{\xi}^{\prime}, \tag{3.65}
\end{equation*}
$$

we may write

$$
\begin{equation*}
S=\sum_{\xi} \int_{\mathcal{X}_{\xi}^{\prime}} \check{\mathcal{L}}_{\xi}, \tag{3.66}
\end{equation*}
$$

where the locally defined Lagrangian

$$
\begin{equation*}
\check{\mathcal{L}}_{\xi}=\oint_{\mathcal{Z}} \operatorname{Tr}_{\mathcal{H}}\left[(\overline{\mathcal{U}}, \mathcal{D B})_{\mathcal{F}}+\left(\overline{\mathcal{V}}, \mathcal{F}-\mathcal{B} \star \mathcal{B}+\frac{1}{3} \overline{\mathcal{V}} \star \overline{\mathcal{V}}-\overline{\mathcal{U}} \star \overline{\mathcal{U}}\right)_{\mathcal{F}}\right]_{\xi} \tag{3.67}
\end{equation*}
$$

Since (3.62) does not contain the parameters $\left(\epsilon^{\mathcal{A}}, \epsilon^{\mathcal{B}}\right)$, it follows that $\check{\mathcal{L}}_{\xi}$ remains invariant (pointwise on $\mathcal{X}_{\xi}$ ) as the fields are transformed by transition functions. Thus

$$
\begin{equation*}
\check{\mathcal{L}}_{\xi}=\left.\check{\mathcal{L}}\right|_{\mathcal{X}_{\xi}}, \tag{3.68}
\end{equation*}
$$

where $\check{\mathcal{L}}$ is a globally defined top form on $\mathcal{X}$, which is to say that the action is globally defined modulo boundary terms. The total variation of the action on-shell as well as its gauge variation are thus given by terms evaluated at the boundary

$$
\begin{equation*}
\partial \mathcal{M}=\partial \mathcal{X} \times \mathcal{Z} \tag{3.69}
\end{equation*}
$$

that vanish provided that

$$
\begin{equation*}
\left.\left(\overline{\mathcal{U}}, \overline{\mathcal{V}} ; \eta^{\bar{u}}, \eta^{\bar{\nu}}\right)\right|_{\partial \mathcal{M}}=0, \tag{3.70}
\end{equation*}
$$

thus leading to a globally defined action including boundary terms.

[^10]
## 4 Unital algebras with Klein operators

In this section, we assume that $\mathcal{F}$ contains a unity, which implies that the inner product on $\mathcal{F}$ is a supertrace. We also assume that the polarization is achieved by adding an outer Klein operator $h$ to a Frobenius subalgebra $\mathcal{F}_{0}$ that is $Z_{2}$-graded with respect to it. The resulting FCS model can be formulated succinctly in terms of a single master field $Z \in \mathcal{H} \otimes \mathcal{F}$, referred to as the superconnection, allowing the inclusion of higher powers of fields into the action.

### 4.1 Trace operation and outer Klein operator

In what follows, we shall assume $\mathcal{F}$ to be unital, which implies that the inner product is equivalent to the nondegenerate graded cyclic supertrace operation

$$
\begin{equation*}
\operatorname{STr}_{\mathcal{F}}(a):=(1, a)_{\mathcal{F}}, \quad a \in \mathcal{F}, \tag{4.1}
\end{equation*}
$$

whose graded cyclicity follows from the fact that

$$
\begin{equation*}
\operatorname{STr}_{\mathcal{F}}\left(a b^{ \pm}\right)=\left(1, a b^{ \pm}\right)_{\mathcal{F}}=\left(a, b^{ \pm}\right)_{\mathcal{F}}= \pm\left(b^{ \pm}, a\right)_{\mathcal{F}}= \pm \operatorname{STr}_{\mathcal{F}}\left(b^{ \pm} a\right) \tag{4.2}
\end{equation*}
$$

for all $a \in \mathcal{F}$ and $b^{ \pm} \in \mathcal{F}^{ \pm}$.
We furthermore assume that $\mathcal{F}$ contains an idempotent element $h$, referred to as the Klein operator of the $Z_{2}$-graded algebra, such that

$$
\begin{equation*}
h a^{ \pm} h= \pm a^{ \pm}, \quad h^{2}=1, \quad a^{ \pm} \in \mathcal{F}^{ \pm} \tag{4.3}
\end{equation*}
$$

Inserting this operator into the supertrace yields the nondegenerate (cyclic) trace operation

$$
\begin{equation*}
\operatorname{Tr}_{\mathcal{F}}(a):=\operatorname{STr}(h a), \quad a \in \mathcal{F} . \tag{4.4}
\end{equation*}
$$

In view of (3.50), the polarization conditions (3.44) and (3.45), which ensure that

$$
\begin{equation*}
\mathcal{F}_{0}:=\mathcal{F}_{+}^{+} \oplus \mathcal{F}_{+}^{-} \tag{4.5}
\end{equation*}
$$

is an associative subalgebra of $\mathcal{F}$, can be solved by taking

$$
\begin{equation*}
\mathcal{F}=\mathcal{F}_{0} \oplus h \mathcal{F}_{0}, \quad \mathcal{F}_{-}^{+}=h \mathcal{F}_{+}^{+}, \quad \mathcal{F}_{-}^{-}=h \mathcal{F}_{+}^{-} \tag{4.6}
\end{equation*}
$$

that is, by taking $h$ to be outer with respect to $\mathcal{F}_{0}$, and requiring that $\mathcal{F}_{0}$ equipped with the inner product

$$
\begin{equation*}
(a, b)_{\mathcal{F}_{0}}:=(a, b)_{\mathcal{F}}, \quad a, b \in \mathcal{F}_{0}, \tag{4.7}
\end{equation*}
$$

or, equivalently, the trace operation

$$
\begin{equation*}
\operatorname{Tr}_{\mathcal{F}_{0}}(a b):=\operatorname{Tr}_{\mathcal{F}}(a b), \quad a, b \in \mathcal{F}_{0} \tag{4.8}
\end{equation*}
$$

is a Frobenius algebra. In other words, we assume that $1 \in \mathcal{F}_{0}$ and that $(\cdot, \cdot)_{\mathcal{F}_{0}}$ is nondegenerate, after which we can define the element $h$ via (4.6).

### 4.2 Superconnection

In view of (4.5) and (4.6), we introduce the superconnection

$$
\begin{equation*}
Z=h X+P, \quad X=\mathcal{A}+\mathcal{B}, \quad P=h(\overline{\mathcal{U}}+\overline{\mathcal{V}}) \tag{4.9}
\end{equation*}
$$

where thus both $X, P \in \mathcal{F}_{0}$, and the superdifferential

$$
\begin{equation*}
q=h d \tag{4.10}
\end{equation*}
$$

Thus, by letting $\pi_{h}$ denote the automorphism of $\mathcal{F}$ that sends $h$ to $-h$ while acting as the identity on $\mathcal{F}_{0}$, the action (3.49) takes the compact form

$$
\begin{align*}
S & =\int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H} \otimes \mathcal{F}}\left(\frac{1}{2} Z \star q Z+\frac{1}{3} Z \star Z \star Z\right)-\frac{1}{4} \oint_{\partial \mathcal{M}} \operatorname{Tr}_{\mathcal{H} \otimes \mathcal{F}}\left[h \pi_{h}(Z) \star Z\right]  \tag{4.11}\\
& =\int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H} \otimes \mathcal{F}_{0}}\left(P \star F^{X}+\frac{1}{3} P \star P \star P\right),
\end{align*}
$$

where

$$
\begin{equation*}
F^{X}=d X+h X h \star X . \tag{4.12}
\end{equation*}
$$

As for the global definition of the theory, we recall that the structure group is generated by a subalgebra of the algebra gauged by $X$, and that

$$
\begin{equation*}
\left.P\right|_{\partial \mathcal{M}}=0 \tag{4.13}
\end{equation*}
$$

### 4.3 Component formulation

Using (4.9) and defining

$$
\begin{equation*}
\mathcal{U}=h \overline{\mathcal{U}}, \quad \mathcal{V}=h \overline{\mathcal{V}} \tag{4.14}
\end{equation*}
$$

such that $P=\mathcal{U}+\mathcal{V}$, the action (3.49) can be written as

$$
\begin{equation*}
S=\int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H} \otimes \mathcal{F}_{0}}\left[\mathcal{U} \star \mathcal{D} \mathcal{B}+\mathcal{V} \star\left(\mathcal{F}-\mathcal{B} \star \mathcal{B}+\mathcal{U} \star \mathcal{U}+\frac{1}{3} \mathcal{V} \star \mathcal{V}\right)\right] \tag{4.15}
\end{equation*}
$$

with $\mathcal{F}$ and $\mathcal{D B}$ from (3.48). The form of this action resembles that of the action (2.37) for the 4D FCS higher spin gravity model reviewed in Section 2, though the Frobenius algebra and the attendant trace operation used in (4.15) is general. The action can be given explicitly by splitting

$$
\begin{equation*}
e_{I}=\left(e_{i}, e^{i}\right), \quad f_{P}=\left(f_{p}, f^{p}\right), \quad e^{i}=h e_{i}=e_{i} h, \quad f^{p}=h f_{p}=-f_{p} h \tag{4.16}
\end{equation*}
$$

where $\left(e_{i}, f_{p}\right)$ is a basis for $\mathcal{F}_{0}$ with product rules

$$
\begin{equation*}
e_{i} e_{j}=e_{k} t^{k}{ }_{i j}, \quad f_{p} f_{q}=-e_{i} s^{i}{ }_{p q}, \quad e_{i} f_{p}=-s_{i p}{ }^{q} f_{q}, \quad f_{q} e_{i}=f_{p} s_{i}^{p}{ }_{q}, \tag{4.17}
\end{equation*}
$$

subject to associativity conditions given by (3.15) with majuscule indices replaced by minuscule indices. Thus

$$
\begin{equation*}
\mathcal{F}_{+}^{+}=\bigoplus_{i=1}^{\frac{1}{2} N^{+}} \mathrm{C} \otimes e_{i}, \quad \mathcal{F}_{+}^{-}=\bigoplus_{p=1}^{\frac{1}{2} N^{-}} \mathrm{C} \otimes f_{p} \tag{4.18}
\end{equation*}
$$

$$
\begin{equation*}
\mathcal{F}_{-}^{+}=\bigoplus_{i=1}^{\frac{1}{2} N^{+}} \mathrm{C} \otimes e^{i}, \quad \mathcal{F}_{-}^{-}=\bigoplus_{p=1}^{\frac{1}{2} N^{-}} \mathrm{C} \otimes f^{p} \tag{4.19}
\end{equation*}
$$

and the fields can be expanded as

$$
\begin{align*}
& \mathcal{A}=\sum_{i} A^{i} e_{i}, \quad \mathcal{B}=\sum_{p} B^{p} f_{p},  \tag{4.20}\\
& \mathcal{V}=\sum_{i} V^{i} e_{i}, \quad \mathcal{U}=\sum_{p} U^{p} f_{p} . \tag{4.21}
\end{align*}
$$

The inner product matrices are taken to be

$$
\Sigma_{I J}=\left[\begin{array}{cc}
0 & \delta_{i}^{j}  \tag{4.22}\\
\delta_{j}^{i} & 0
\end{array}\right], \quad \Omega_{P Q}=\left[\begin{array}{cc}
0 & -\delta_{p}^{q} \\
\delta_{q}^{p} & 0
\end{array}\right]
$$

such that

$$
\Theta_{I J}=\left[\begin{array}{cc}
0 & \delta_{i}^{j}  \tag{4.23}\\
-\delta_{j}^{i} & 0
\end{array}\right], \quad \Xi_{P Q}=\left[\begin{array}{cc}
0 & -\delta_{p}^{q} \\
-\delta_{q}^{p} & 0
\end{array}\right]
$$

In summary so far, starting from the Ansatz (3.1) for a gauge invariant action, including boundary terms, and assuming that the resulting $Z_{2_{2}}$-graded quasi-Frobenius algebra $\mathcal{F}$ (as in Section 3.2) in addition
i) obeys the polarization conditions (3.44) and (3.45) under the assumption that Eq. (3.41) holds;
ii) contains a unity (as in Section 4.1); and
iii) is $Z_{2}$-graded by means of a Klein operator $h \in \mathcal{F}$ leading to the decomposition (4.6);
we arrive at the action (4.15) with master fields in $\mathcal{H} \otimes \mathcal{F}_{0}$, where $\mathcal{F}_{0}$ is the proper Frobenius subalgebra of $\mathcal{F}$ defined in (4.5).

## 5 3-grading

In this section we shall consider models in which the $Z_{2}$-grading is extended into a 3 -grading that allows the truncation of top-forms off-shell to achieve equations of motion that do not contain any algebraic zero-form constraints. We shall then describe a general scheme to obtain the 3 -grading by assuming that the $Z_{2}$-grading of $\mathcal{F}_{0}$ is achieved by an inner Klein operator $\gamma \in \mathcal{F}_{0}$.

### 5.1 On-shell free differential algebra

As shown in Section 3, the $Z_{2}$-grading suffices for constructing globally defined actions including top-forms leading to equations of motion with zero-form constraints. The system can be constrained algebraically off-shell as to remove the top-forms and hence the zero-form constraints on-shell, provided that the algebra admits a three grading defined by

$$
\begin{equation*}
\mathcal{F}^{(0)}:=\mathcal{F}^{+}, \quad \mathcal{F}^{(-1)} \oplus \mathcal{F}^{(+1)}:=\mathcal{F}^{-} \tag{5.1}
\end{equation*}
$$

and $\mathcal{F}^{(k)} \equiv 0$ for $k= \pm 2, \pm 3, \ldots$, such that

$$
\begin{equation*}
\mathcal{F}^{(k)} \star \mathcal{F}^{\left(k^{\prime}\right)} \subseteq \mathcal{F}^{\left(k+k^{\prime}\right)} \tag{5.2}
\end{equation*}
$$

Defining the Z -valued superdegree map

$$
\begin{equation*}
\operatorname{deg}_{\mathcal{E}}:=\operatorname{deg}_{\mathcal{M}}+\operatorname{deg}_{\mathcal{F}}, \tag{5.3}
\end{equation*}
$$

all top-forms as well as a subset of the next-to-top and zero-forms can be set to zero off-shell by imposing

$$
\begin{equation*}
\operatorname{deg}_{\mathcal{E}}(\boldsymbol{A}), \operatorname{deg}_{\mathcal{E}}(\boldsymbol{B}) \in\{1,3, \ldots, 2 n-1\} \tag{5.4}
\end{equation*}
$$

It follows that the curvatures in (3.35) obey

$$
\begin{equation*}
\operatorname{deg}_{\mathcal{E}}\left(\boldsymbol{R}^{A}\right), \operatorname{deg}_{\mathcal{E}}\left(\boldsymbol{R}^{B}\right) \in\{2,4, \ldots, 2 n\} \tag{5.5}
\end{equation*}
$$

Thus, the truncation is consistent with the equations of motion, and leads to a free differential algebra on-shell, since

$$
\begin{equation*}
\boldsymbol{R}_{[0]}^{B}=-\boldsymbol{B}_{[0]} \star \boldsymbol{B}_{[0]}=-\boldsymbol{B}_{[0]}^{(+1)} \star \boldsymbol{B}_{[0]}^{(+1)} \equiv 0 \tag{5.6}
\end{equation*}
$$

by (5.2). Hence, since the algebra is free universally, it follows by a general lemma that the action is gauge invariant.

### 5.23 -grading from inner Klein operator of $\mathcal{F}_{0}$

Let us assume that $\mathcal{F}_{0}$, which is a unital Frobenius algebra by the assumptions made so far, contains an inner Klein operator $\gamma$ that is compatible with $h$ in the sense that

$$
\begin{equation*}
[h, \gamma]=0, \quad \gamma a^{ \pm}= \pm a^{ \pm} \gamma \quad \text { for all } \quad a^{ \pm} \in \mathcal{F}_{0}^{\sigma}, \quad \gamma^{2}=1 \tag{5.7}
\end{equation*}
$$

We can then introduce the following 3-grading

$$
\begin{equation*}
\mathcal{F}=\bigoplus_{q= \pm 1,0} \mathcal{F}^{(q)}, \quad \mathcal{F}^{( \pm 1)}=\frac{1}{2}(1 \pm \gamma) \mathcal{F}^{-}, \quad \mathcal{F}^{(0)}=\mathcal{F}^{+} \tag{5.8}
\end{equation*}
$$

and decompose

$$
\begin{equation*}
\mathcal{F}^{(0)}=\mathcal{F}^{(-0)} \oplus \mathcal{F}^{(+0)}, \quad \mathcal{F}^{( \pm 0)}=\frac{1}{2}(1 \pm \gamma) \mathcal{F}^{+} \tag{5.9}
\end{equation*}
$$

such that

$$
\begin{equation*}
\mathcal{F}^{(\sigma 0)}=\mathcal{F}^{(\sigma 1)} \mathcal{F}^{(-\sigma 1)}, \quad \sigma= \pm \tag{5.10}
\end{equation*}
$$

Thus, in effect, $\mathcal{F}$ has the following two by two block structure:

$$
\mathcal{F}=\left[\begin{array}{ll}
\mathcal{F}^{(+0)} & \mathcal{F}^{(+1)}  \tag{5.11}\\
\mathcal{F}^{(-1)} & \mathcal{F}^{(-0)}
\end{array}\right]=\left[\begin{array}{l}
\frac{1}{2}(1+\gamma) \mathcal{F}^{+} \\
\frac{1}{2}(1+\gamma) \mathcal{F}^{-} \\
\frac{1}{2}(1-\gamma) \mathcal{F}^{-} \\
\frac{1}{2}(1-\gamma) \mathcal{F}^{+}
\end{array}\right]
$$

In particular,

$$
\mathcal{F}_{0}=\left[\begin{array}{ll}
\mathcal{F}_{0}^{(+0)} & \mathcal{F}_{0}^{(+1)}  \tag{5.12}\\
\mathcal{F}_{0}^{(-1)} & \mathcal{F}_{0}^{(-0)}
\end{array}\right]=\left[\begin{array}{ll}
\frac{1}{2}(1+\gamma) \mathcal{F}_{0}^{+} & \frac{1}{2}(1+\gamma) \mathcal{F}_{0}^{-} \\
\frac{1}{2}(1-\gamma) \mathcal{F}_{0}^{-} & \frac{1}{2}(1-\gamma) \mathcal{F}_{0}^{+}
\end{array}\right]
$$

Thus, upon expanding

$$
\begin{gather*}
X=\left[\begin{array}{cc}
\frac{1}{2}(1+\gamma) \mathcal{A} & \frac{1}{2}(1+\gamma) \mathcal{B} \\
\frac{1}{2}(1-\gamma) \mathcal{B} & \frac{1}{2}(1-\gamma) \mathcal{A}
\end{array}\right] \equiv\left[\begin{array}{cc}
A & B \\
\widetilde{B} & \widetilde{A}
\end{array}\right],  \tag{5.13}\\
P=\left[\begin{array}{ll}
\frac{1}{2}(1+\gamma) \mathcal{V} & \frac{1}{2}(1+\gamma) \mathcal{U} \\
\frac{1}{2}(1-\gamma) \mathcal{U} & \frac{1}{2}(1-\gamma) \mathcal{V}
\end{array}\right] \equiv\left[\begin{array}{cc}
V & U \\
\widetilde{U} & \widetilde{V}
\end{array}\right], \tag{5.14}
\end{gather*}
$$

the action assumes the form

$$
\begin{gather*}
S=\int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H} \otimes \mathcal{F}_{0}}\left[V \star\left(F-B \star \widetilde{B}+\frac{1}{3} V \star V+U \star \widetilde{U}\right)+\widetilde{U} \star D B\right.  \tag{5.15}\\
\left.+\widetilde{V} \star\left(\widetilde{F}-\widetilde{B} \star B+\frac{1}{3} \widetilde{V} \star \widetilde{V}+\widetilde{U} \star U\right)+U \star \widetilde{D} \widetilde{B}\right]
\end{gather*} .
$$

This result for the 3 -graded models is to be compared with the action for the $Z_{2}$-graded model given in (4.15). The equations of motion at $\partial \mathcal{M}$, viz. $d X+h X h \star X \approx 0$, resulting from the action above take the form

$$
\begin{array}{ll}
d A+A \star A-B \star \widetilde{B} \approx 0, & d \widetilde{A}+\widetilde{A} \star \widetilde{A}-\widetilde{B} \star B \approx 0 \\
d B+A \star B-B \star \widetilde{A} \approx 0, & d \widetilde{B}+\widetilde{A} \star \widetilde{B}-\widetilde{B} \star A \approx 0 \tag{5.17}
\end{array}
$$

In summary, the presence of the extra Klein operator $\gamma$ yields a refined 3-grading in which $\mathcal{F}^{(0)}$ is replaced by two blocks, namely $\frac{1}{2}(1 \pm \gamma) \mathcal{F}^{(0)}$, where $\mathcal{F}_{0}$ is the proper Frobenius subalgebra of $\mathcal{F}$ defined in (4.5). The resulting action (5.15) is of the same form as the original action in (2.37) but with more general master fields belonging to $\frac{1}{2}(1 \pm \gamma)$ projections of $\mathcal{H} \otimes \mathcal{F}_{0}$.

## 6 Examples

In this section we shall provide examples based on unital $Z_{2}$-graded Frobenius algebras including 3-graded and not 3-graded cases.

### 6.1 3-graded matrix algebra

Unital $Z_{2}$-graded Frobenius algebras with Klein operator $h$ of the form $\mathcal{F}=\mathcal{F}_{0} \oplus h \mathcal{F}_{0}$ can be obtained by taking

$$
\begin{equation*}
\mathcal{F}_{0}=\operatorname{mat}_{N}(\mathrm{c}):=\bigoplus_{i, j=1}^{N} \mathrm{c} \otimes m_{i, j}, \quad m_{i, j} m_{k, l}:=\delta_{j k} m_{i, l}, \quad h m_{i, j} h:=\sigma_{i} \sigma_{j} m_{i, j}, \tag{6.1}
\end{equation*}
$$

where $\sigma_{i} \in\{ \pm 1\}$, and

$$
\begin{equation*}
\operatorname{Tr}_{\mathcal{F}} m_{i, j}:=\delta_{i, j}, \quad \operatorname{Tr}_{\mathcal{F}} h m_{i, j}:=0 . \tag{6.2}
\end{equation*}
$$

The decomposition (4.5) of $\mathcal{F}_{0}$ into eigenspaces of the adjoint action of $h$ is given by

$$
\begin{equation*}
\mathcal{F}_{+}^{+}=\bigoplus_{i, j=1}^{N} \mathrm{C} \otimes e_{i, j}, \quad \mathcal{F}_{+}^{-}=\bigoplus_{i, j=1}^{N} \mathrm{C} \otimes f_{i, j}, \tag{6.3}
\end{equation*}
$$

where

$$
\begin{equation*}
e_{i, j}:=\frac{1}{2}\left(1+\sigma_{i} \sigma_{j}\right) m_{i, j}, \quad f_{i, j}:=\frac{1}{2}\left(1-\sigma_{i} \sigma_{j}\right) m_{i, j}, \tag{6.4}
\end{equation*}
$$

have traces

$$
\begin{equation*}
\operatorname{Tr}_{\mathcal{F}} e_{i, j}=\delta_{i, j}, \quad \operatorname{Tr}_{\mathcal{F}} f_{i, j}=0 \tag{6.5}
\end{equation*}
$$

The analogous decomposition of $h \mathcal{F}_{0}$ leads to the subspaces

$$
\begin{equation*}
\mathcal{F}_{-}^{+}=\bigoplus_{i, j=1}^{N} \mathrm{C} \otimes h e_{i, j}, \quad \mathcal{F}_{-}^{-}=\bigoplus_{i, j=1}^{N} \mathrm{C} \otimes h f_{i, j}, \tag{6.6}
\end{equation*}
$$

whose basis elements have traces

$$
\begin{equation*}
\operatorname{Tr}_{\mathcal{F}} h e_{i, j}=0, \quad \operatorname{Tr}_{\mathcal{F}} h f_{i, j}=0 \tag{6.7}
\end{equation*}
$$

The corresponding master fields

$$
\begin{equation*}
\mathcal{A}=\sum_{i, j=1}^{N} A^{i, j} e_{i, j}, \quad \mathcal{B}=\sum_{i, j=1}^{N} B^{i, j} f_{i, j}, \tag{6.8}
\end{equation*}
$$

$$
\begin{equation*}
\mathcal{V}=\sum_{i, j=1}^{N} V^{i, j} e_{i, j}, \quad \mathcal{U}=\sum_{i, j=1}^{N} U^{i, j} f_{i, j} \tag{6.9}
\end{equation*}
$$

We note that for a given choice of $\sigma_{i}$ one has

$$
\begin{equation*}
N^{\sigma}=\sum_{i, j=1}^{N} \frac{1}{2}\left(1+\sigma \sigma_{i} \sigma_{j}\right) \tag{6.10}
\end{equation*}
$$

such that if all $\sigma_{i}$ are equal then the model consists of only odd forms.
The 3 -grading results from the fact that the outer action of $h$ on $\mathcal{F}_{0}$ is equivalent to the inner adjoint action of

$$
\begin{equation*}
\gamma=\sum_{i=1}^{N} \sigma_{i} m_{i, i} \tag{6.11}
\end{equation*}
$$

viz. $h m_{i, j} h=\gamma m_{i, j} \gamma$. Hence, the above decomposition of $\mathcal{F}$ can be written as

$$
\begin{gather*}
\mathcal{F}_{+}^{ \pm}=\frac{1}{4}(1+\gamma) \mathcal{F}_{0}(1 \pm \gamma)+\frac{1}{4}(1-\gamma) \mathcal{F}_{0}(1 \mp \gamma) .  \tag{6.12}\\
\mathcal{F}_{-}^{ \pm}=\frac{1}{4}(1+\gamma) h \mathcal{F}_{0}(1 \pm \gamma)+\frac{1}{4}(1-\gamma) h \mathcal{F}_{0}(1 \mp \gamma) \tag{6.13}
\end{gather*}
$$

The 3-grading can be used to project the model in order to solve the zero-form constraints, as discussed in Section 5. To this end, we permute the basis such that

$$
\mathcal{F}_{0}=\left[\begin{array}{cc}
\operatorname{mat}_{N_{1}}(\mathrm{C}) & N_{1} \otimes N_{2}^{*}  \tag{6.14}\\
N_{2} \otimes N_{1}^{*} & \operatorname{mat}_{N_{2}}(\mathrm{C})
\end{array}\right] \cong \operatorname{mat}_{N}(\mathrm{C}), \quad \gamma:=\left[\begin{array}{cc}
1_{N_{1}} & 0 \\
0 & -1_{N_{2}}
\end{array}\right],
$$

where thus $N=N_{1}+N_{2}, N^{+}=\left(N_{1}\right)^{2}+\left(N_{2}\right)^{2}$ and $N^{-}=2 N_{1} N_{2}$. The graded inner product on $\mathcal{F}$ now reads

$$
\begin{equation*}
\left(a_{0}+a_{0}^{\prime} h, b_{0}+b_{0}^{\prime} h\right)_{\mathcal{F}}=\operatorname{Tr}_{\operatorname{mat}_{N}(\mathrm{C})} \gamma\left(a_{0} b_{0}+a_{0}^{\prime} h b_{0}^{\prime} h\right), \quad a_{0}, a_{0}^{\prime}, b_{0}, b_{0}^{\prime} \in \mathcal{F}_{0} \tag{6.15}
\end{equation*}
$$

where we note that $h b_{0}^{\prime} h \in \mathcal{F}_{0}$. The decomposition under the 3 -grading now reads

$$
\mathcal{F}^{(+0)}=\left[\begin{array}{c|c}
\operatorname{mat}_{N_{1}}(\mathrm{C}) & 0  \tag{6.16}\\
\hline 0 & 0
\end{array}\right], \quad \mathcal{F}^{(+1)}=\left[\begin{array}{c|c}
0 & N_{1} \otimes N_{2}^{*} \\
\hline 0 & 0
\end{array}\right]
$$

$$
\mathcal{F}^{(-1)}=\left[\begin{array}{c|c}
0 & 0  \tag{6.17}\\
\hline N_{2} \otimes N_{1}^{*} & 0
\end{array}\right], \quad \mathcal{F}^{(-0)}=\left[\begin{array}{c|c}
0 & 0 \\
\hline 0 & \operatorname{mat}_{N_{2}}(\mathrm{C})
\end{array}\right]
$$

obeying (5.10). The resulting model, with action (5.15) with $\operatorname{Tr}_{\mathcal{H} \otimes \mathcal{F}_{0}}$ replaced by $\operatorname{Tr}_{\mathcal{H}} \operatorname{Tr}_{\text {mat }}^{N}$, represents a straightforward extension of the original FCS model with $(A, \widetilde{A} ; B, \widetilde{B})$ valued in subspaces of $\mathcal{H} \otimes \operatorname{mat}_{N}(\mathrm{C})$ in accordance with (6.16) and (6.17) idem $(V, \widetilde{V} ; U, \widetilde{U})$.

### 6.2 3-graded Clifford algebra

For $N=\widetilde{N}=2^{n-1}$, the 3 -graded matrix FCS model introduced in the previous section is equivalent to a model with

$$
\begin{equation*}
\mathcal{F}_{0}=\mathcal{C} \ell_{2 n} \tag{6.18}
\end{equation*}
$$

the Clifford algebra generated by $2 n$ elements $\gamma_{i}(i=1, \ldots, 2 n)$ obeying

$$
\begin{equation*}
\left\{\gamma_{i}, \gamma_{j}\right\}=2 \delta_{i j} \tag{6.19}
\end{equation*}
$$

The trace operation can be defined in the basis consisting of totally antisymmetric elements

$$
\begin{equation*}
\gamma^{i_{1} \ldots i_{p}}:=\gamma^{\left[i_{1}\right.} \cdots \gamma^{\left.i_{p}\right]} \tag{6.20}
\end{equation*}
$$

as the projection onto the identity, i.e.

$$
\begin{equation*}
\operatorname{Tr}_{\mathcal{C} \ell_{2 n}} \gamma^{i_{1} \ldots i_{p}}=\delta_{p, 0} \tag{6.21}
\end{equation*}
$$

The 3-grading is achieved by the inner Klein operator

$$
\begin{equation*}
\gamma=i^{n} \gamma_{1} \cdots \gamma_{2 n} \tag{6.22}
\end{equation*}
$$

The resulting model thus consists of odd forms, not containing top-forms, valued in $\frac{1}{4}(1+$ $\gamma) \mathcal{C} \ell_{2 n}(1+\gamma)$ and $\frac{1}{4}(1-\gamma) \mathcal{C} \ell_{2 n}(1-\gamma)$, both isomorphic to mat ${ }^{2 n-2}(C)$, and even forms, with constrained zero-form and $2 n$-form content, valued in $\frac{1}{4}(1+\gamma) \mathcal{C} \ell_{2 n}(1-\gamma)$ and $\frac{1}{4}(1-$
$\gamma) \mathcal{C} \ell_{2 n}(1+\gamma)$, both isomorphic to $2^{2 n-2} \otimes\left(2^{2 n-2}\right)^{*}$. In particular, on

$$
\begin{equation*}
\mathcal{M}=\mathcal{X}_{5} \times \mathcal{Z}_{4} \tag{6.23}
\end{equation*}
$$

as in Section 2, it contains a Konstein-Vasiliev phase in which the two-form is given by an expectation value proportional to the closed and central element

$$
\begin{equation*}
J \in \Omega_{[2]}\left(\mathcal{Z}_{4}\right) \otimes \mathcal{H}, \quad d J=0, \quad J \star f=f \star J, \quad f \in \Omega(\mathcal{M}) \otimes \mathcal{H} \tag{6.24}
\end{equation*}
$$

Fixing gauges for the resulting fluctuations in the forms of positive degrees is equivalent to performing the consistent truncation 14

$$
\begin{array}{cl}
A=\frac{1}{2}(1+\gamma) W, & B=\frac{1}{2}(1+\gamma) C \gamma_{2 n}, \\
\widetilde{B}=\frac{1}{2}(1-\gamma) \gamma_{2 n} J, & \widetilde{A}=\frac{1}{2}(1-\gamma) \gamma_{2 n} W \gamma_{2 n} \tag{6.26}
\end{array}
$$

where the reduced master field ${ }^{15}$

$$
\begin{equation*}
C, W \in \frac{1}{2}(1+\gamma) \mathrm{C}_{\mathcal{C} \ell_{2 n}}(\gamma) \otimes \mathcal{H}, \quad \frac{1}{2}(1+\gamma) \mathrm{C}_{\mathcal{C}_{2 n}}(\gamma) \cong \operatorname{mat}_{2^{2 n-2}}(\mathrm{C}) \tag{6.27}
\end{equation*}
$$

which yields

$$
\begin{equation*}
d W+W \star W+C \star J=0, \quad d C+W \star C-C \star W=0 \tag{6.28}
\end{equation*}
$$

Modulo reality and other kinematic conditions ${ }^{16}$, we identify the above model as an FCS extension of the bosonic Konstein-Vasiliev model with gauge algebra $h u\left(2^{2 n-2}, 0\right)$ [16].

[^11]
### 6.3 Twisted group algebra of $Z_{2} \times Z_{2 n}$

An example of a unital $Z_{2}$-graded Frobenius algebra that does not admit any 3 -grading is provided by a twisting of the group algebra ${ }^{17}$ of $Z_{2} \times Z_{2 n}$ viz.

$$
\begin{equation*}
\mathcal{F}=\mathrm{C}\left[\mathrm{Z}_{2} \times \mathrm{Z}_{2 n}, \alpha\right] \tag{6.29}
\end{equation*}
$$

where the group is generated by two elements $a$ and $b$ subject to the conditions

$$
\begin{equation*}
a^{2 n}=I=b^{2}, \quad a b=b a \tag{6.30}
\end{equation*}
$$

and the co-cycle $\alpha$ is chosen such that

$$
\begin{equation*}
e_{a^{k}} e_{b}=(-1)^{k} e_{b} e_{a^{k}}, \quad k=0,1, \ldots, 2 n-1 . \tag{6.31}
\end{equation*}
$$

As for $\operatorname{Tr}_{\mathcal{F}}$, we take the operation in (A.8) ${ }^{18}$, and the $Z_{2}$ grading can be achieved by taking

$$
\begin{equation*}
\mathcal{F}=\mathcal{F}_{0} \oplus h \mathcal{F}_{0}, \quad h=e_{b}, \quad \mathcal{F}_{0}=\mathrm{C}\left[\mathrm{Z}_{2 n}\right] \tag{6.32}
\end{equation*}
$$

Turning to the master fields, they are given by

$$
\begin{equation*}
\mathcal{A}=\sum_{k=1}^{n} A^{(2 k-2)} e_{a^{2 k-2}}, \quad \mathcal{B}=\sum_{k=1}^{n} B^{(2 k-1)} e_{a^{2 k-1}} \tag{6.33}
\end{equation*}
$$

idem $\mathcal{V}$ and $\mathcal{U}$.
$Z_{2} \times Z_{4}$ model. To exhibit the structure, let us take $n=2$. The master fields can now be expanded as

$$
\begin{equation*}
\mathcal{A}=\sum_{\sigma= \pm} A_{\sigma} e_{\sigma}, \quad \mathcal{B}=\sum_{\sigma= \pm} B_{\sigma} f_{\sigma} \tag{6.34}
\end{equation*}
$$

[^12]idem $\mathcal{U}$ and $\mathcal{V}$, where the basis elements
\[

$$
\begin{equation*}
e_{\sigma}=\frac{1}{2}\left(e_{I}+\sigma e_{a^{2}}\right), \quad f_{\sigma}=e_{a} \epsilon_{\sigma} \tag{6.35}
\end{equation*}
$$

\]

obey

$$
\begin{equation*}
e_{\sigma} e_{\sigma^{\prime}}=\delta_{\sigma \sigma^{\prime}} e_{\sigma}, \quad e_{\sigma} f_{\sigma^{\prime}}=\delta_{\sigma \sigma^{\prime}} f_{\sigma}, \quad f_{\sigma} f_{\sigma^{\prime}}=\sigma \delta_{\sigma \sigma^{\prime}} e_{\sigma} \tag{6.36}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Tr}_{\mathcal{F}} e_{\sigma}=4, \quad \operatorname{Tr}_{\mathcal{F}} f_{\sigma}=0 \tag{6.37}
\end{equation*}
$$

In components, the boundary equations of motion, viz. $\left.F_{X}\right|_{\partial \mathcal{M}}=0$, with $F_{X}:=d X+h X h \star$ $X=0$ and $X:=\mathcal{A}+\mathcal{B}$, read

$$
\begin{align*}
F_{\sigma} & :=d A_{\sigma}+A_{\sigma} \star A_{\sigma} \approx \sigma B_{\sigma} \star B_{\sigma},  \tag{6.38}\\
D_{\sigma} B_{\sigma} & :=d B_{\sigma}+\left[A_{\sigma}, B_{\sigma}\right]_{\star} \approx 0, \tag{6.39}
\end{align*}
$$

which is a Cartan integrable system containing the zero-form constraint

$$
\begin{equation*}
B_{[0] \sigma} \star B_{[0] \sigma} \approx 0 \tag{6.40}
\end{equation*}
$$

The FCS action (4.15) is given by

$$
\begin{equation*}
S=S_{+}+S_{-}, \tag{6.41}
\end{equation*}
$$

where

$$
\begin{equation*}
S_{\sigma}=\int_{\mathcal{M}} \operatorname{Tr}_{\mathcal{H}}\left(\sigma U_{\sigma} \star D_{\sigma} B_{\sigma}+V_{\sigma} \star\left(F_{\sigma}-\sigma B_{\sigma} \star B_{\sigma}+\sigma U_{\sigma} \star U_{\sigma}+\frac{1}{3} V_{\sigma} \star V_{\sigma}\right)\right) \tag{6.42}
\end{equation*}
$$

In order for the action to be real and non-degenerate, and the zero-form constraint to have a nontrivial solution space, we can impose the reality condition

$$
\begin{array}{ll}
\left(A_{\sigma}\right)^{\dagger}=-A_{-\sigma}, & \left(B_{\sigma}\right)^{\dagger}=B_{-\sigma} \\
\left(V_{\sigma}\right)^{\dagger}=-V_{-\sigma}, & \left(U_{\sigma}\right)^{\dagger}=\sigma U_{-\sigma} \tag{6.44}
\end{array}
$$

which implies that $\left(S_{\sigma}\right)^{\dagger}=S_{-\sigma}$ and that (6.38) and (6.39) follow from the variational principle. As for (6.40), nontrivial solution spaces arise due to the fact that $B_{[0] \sigma}$ is a complex element in $\mathcal{H}$, e.g. by using star product realizations of Fock space endomorphisms 19.

Four-dimensional self-dual branch. Taking $\partial \mathcal{M}_{4}=\mathcal{X}_{4} \times \mathcal{Z}_{4}$ and $\mathcal{H}$ to be the fourdimensional bosonic higher spin algebra augmented with outer Klein operators ( $k, \bar{k}$ ) and using the notation and results of [20, 19, 1], a branch describing self-dual configurations arises as follows: In holomorphic gauge, the constraint (6.40) is solved by

$$
\begin{equation*}
B_{[0]+}=\Psi(y, \bar{y}) \star \kappa_{y} k, \quad \Psi \star \Psi=0 \tag{6.45}
\end{equation*}
$$

where $\kappa_{y}=2 \pi \delta^{2}(y)$ is an inner Klein operator and $\Psi$ is a nilpotent (complex) Fock space endomorphism in the adjoint representation of $h s(4 ; \mathrm{C})$. For example, using the star product algebra realization of $\left|m_{+}, m_{-}\right\rangle\left\langle n_{+}, n_{-}\right|$where $m_{ \pm}, n_{ \pm} \in Z+\frac{1}{2}$ are eigenvalues of $E \pm J$ (for details, see [19]), one may expand $\Psi$ by taking $m_{+}=\frac{1}{2} \bmod 4$ and $n_{+}=\frac{5}{2} \bmod 4$ (without any need to constrain $m_{-}, n_{-}$). More generally, the solution space of (6.45) decomposes into $h s(4 ; \mathrm{C})$ orbits; in this sense, oe may think of $\Psi$ as an higher spin generalization of a pure spinor

The master field equations in positive degrees can be solved by setting all forms in degrees greater than two to zero, and taking20

$$
\begin{equation*}
B_{[2]+}=\frac{i}{8} k \kappa_{y} \star \kappa_{z} d z^{\alpha} d z_{\alpha}, \quad A_{[1]+}=d z^{\alpha} v_{\alpha}(z) \star \Psi \tag{6.46}
\end{equation*}
$$

where the two-form is closed and central and obeys $B_{[2]+} \star B_{[2]+}=0$, and $v=d z^{\alpha} v_{\alpha}(z)$ obeys

$$
\begin{equation*}
d v=\frac{i \pi}{2} \delta^{2}(z) d z^{\alpha} d z_{\alpha} \tag{6.47}
\end{equation*}
$$

that can be achieved by taking $v_{\alpha}$ to have a simple pole at $z^{\alpha}=0$. As shown in [19], this sin-

[^13]gularity can be removed by a (unitary) vacuum gauge function $21 . \mathcal{X}_{4} \rightarrow S O(2,3) / S O(1,3)$; the symbol of $L^{-1} \star\left(A_{[1]+}+d\right) \star L$ in Vasiliev's normal order is analytic on $\mathcal{Z}_{4}$ (minus the point at infinity) over a finite region of $\mathcal{X}_{4}$. Thus there exists a field dependent gauge function that takes the configuration to Vasiliev's gauge, i.e. $z^{\alpha} A_{\alpha+}=0$ in normal order, in which vierbein, Lorentz connection, Fronsdal fields and Weyl tensors can be defined in a manifestly Lorentz covariant basis after a field redefinition.

Since the two-form is holomorphic, only the dotted Weyl curvatures of the linearized Fronsdal fields are sourced by the Weyl zero-form. Apart from the zero-form constraints and the modified reality condition, the mechanism leading to self-dual linearized curvatures in the current model is the same as that spelled out in [20] 22: The calculation of the linearized sources for the Fronsdal curvatures follows the same steps as in Vasiliev's original work [2], but since there is no anti-holomorphic term in $B_{[2]+}$ the source terms containing $\left.\Phi\right|_{z^{\alpha}=\bar{z}^{\dot{\alpha}}=\bar{y}^{\dot{\alpha}}=0}$ are not present. It follows that the linearized curvatures of the Fronsdal fields are self-dual, though the zero-forms in $\left.\Phi\right|_{z^{\alpha}=\bar{z}^{\dot{\alpha}}=\bar{y}^{\dot{\alpha}}=0}$ are nonetheless part of the spectrum, playing the role of additional matter fields. Thus, the spectrum of dynamical fields on $\mathcal{X}_{4}$ consists of a complexified scalar and a tower of self-dual complexified gauge fields.

## 7 Conclusions

Generalizing the minimal Frobenius-Chern-Simons action of [1], we have constructed, under a mild set of assumptions, the most general cubic action for a set of even and odd forms on an odd-dimensional manifold $\mathcal{X} \times \mathcal{Z}$ where $\mathcal{X}$ is open and commutative and $\mathcal{Z}$ closed and noncommutative; in the global formulation, the are on $\mathcal{X}$ whereas all fields are assumed to be globally defined on $\mathcal{Z}$. The underlying symmetry group is based on the direct product $\mathcal{H} \otimes \mathcal{F}$ of two associative algebras with non-degenerate invariant inner products. As for $\mathcal{H}$, it has been assumed to be unital and trivially graded, which means that its inner product

[^14]is a trace operation; in concrete models its role is to realize the higher spin algebra and its representations. The algebra $\mathcal{F}$, on the other hand, has been assumed to be finitedimensional and $Z_{2}$-graded; in the unital case, it is thus a $Z_{2}$-graded Frobenius algebra, while we refer to it as being quasi-Frobenius in the non-unital case. In the latter case, the resulting action has the appearance of a matter-coupled Chern-Simons-like action, as even and odd forms must be treated on unequal footing. In the unital case, and under the additional assumption that the $Z_{2}$-grading can be achieved by an inner Klein operator, the even and odd forms can be assempled into a single superconnection, resulting in pure Chern-Simons-like action, that we refer to as a Frobenius-Chern-Simons action.

Furthermore, we have distinguished between $Z_{2}$-graded Frobenius algebras and 3-graded versions, and shown that in the latter case constraints on zero-form master fields can be avoided. In particular, the original model of [1], which admits a perturbative description in terms of real Fronsdal tensors in $A d S_{4}$, is based on a 3-graded Frobenius algebra given by a twisting of the group algebra based on $\left(Z_{2}\right)^{3}$. As a simple modification of it, we have shown that a twisting of the group algebra of $Z_{2} \times Z_{4}$ yields a $Z_{2}$-graded Frobenius algebra that leads to a model with zero-form constraints that admits a perturbative description in terms of self-dual complex Fronsdal tensors in $A d S_{4}$. Another class of models arise from the 3-graded matrix algebras. A special case are the Clifford algebras, which lead to an interesting off-shell extension of a bosonic subclass of the Konstein-Vasiliev models, namely those that accommodate the Clifford algebras as an internal symmetry.

In view of the above result and the fact that the four-dimensional higher spin algebra can be obtained by twisting the algebra of the group $S O(2,3) \times \mathcal{K}$, where $\mathcal{K} \cong\left(\mathrm{Z}_{2}\right)^{2}$, and factoring out ideals, it would be interesting to undertake a more thorough investigation of models based on twisted group algebras. Clearly, many Frobenius algebras may lead to novel equations of motion that are not necessarily interpretable as ordinary higher spin field equations. Instead it should be emphasised that the generalized Frobenius-Chern-Simons gauge theory presented here may have applications beyond higher spin gravity.

There are several directions for future investigations. As already mentioned, it would be interesting to seek new examples of Frobenius algebras that lead to novel spectral properties and interactions in the context of higher spin gravity, compared to the ones known until
now [21, 16, 22, 23, 18]. Of considerable interest are also generalizations of Frobenius-Chern-Simons gauge theory that includes higher than cubic interactions as well as quadratic terms. Polynomial interactions, including quiver-like interactions will be presented elsewhere. Finally, it is of great importance to establish a connection between such Frobenius-ChernSimons gauge theories and topological open string theories [24, 25], possibly by generalizing the equivalence of ordinary Chern-Simons theory and topological open strings found a long time ago by Witten [26]. To this end, the addition of quadratic and quartic and higher terms to the Hamiltonian can be shown to lead to extension of $\mathcal{F}$ into an internal $A_{\infty}$ algebra, as we hope to report on elsewhere.

Many twisted group algebras are nontrivial viewed as Hopf algebras, that is, they are cononcommutative. In this respect, it is interesting to note that Hopf algebras in the form of quantum groups provide examples of differential Poisson manifolds with nontrivial curvatures that give rise to noncommutative geometries with graded non-anticommuting line elements [27, 28]. These types of constructions may give rise to an even larger landscape of higher spin gravities provided that one is willing to deform the anti-de Sitter symmetry algebra, as makes sense for example in the application to nonrelativitic holographic dualities and in particular massive anyons. More generally, beyond the realm of differential Poisson manifolds reside the homotopy Poisson manifolds, whose quantization gives rise to a deformation of the external differential graded algera $\Omega(\mathcal{M}) \otimes \mathcal{H}$ by an external $A_{\infty}$ algebra. When combined with the aforementioned internal $A_{\infty}$ algebra, one is thus led to a topological version of the category of open string field theories proposed by Gaberdiel and Zwiebach in [29].

Acknowledgements. We thank Pierre Bieliavski and Valentin Ovsienko for discussions related to Frobenius algebras. We are very grateful to Maja Volkov for enlightening discussions on extensions of group algebras. We also value related and stimulating collaborations with Cesar Arias, Carlo Iazeolla and Alexander Torres-Gomez. N.B. is thankful to the IHÉS (Bures-sur-Yvette) and the AEI (Potsdam) for hospitality and providing an excellent environment for research. The work of E.S. is supported in part by NSF grant PHY-1214344. and PHY-1521099. P.S. is grateful to Texas A\&M University and the University of Mons for hospitality during various stages of this work. The work of P.S. is supported by Fondecyt Regular grant No 1140296 and Conicyt grant DPI 20140115 and UNAB grant DI-1382-16/R.

## A Twisted group algebras

Given a discrete group $G$, the twisted group algebra 23 [12]

$$
\begin{equation*}
\mathrm{C}[G, \alpha]=\bigoplus_{g \in G} \mathrm{C} \otimes e_{g} \tag{A.1}
\end{equation*}
$$

is the associative algebra with composition rule

$$
\begin{equation*}
e_{g} e_{g^{\prime}}=\alpha\left(g, g^{\prime}\right) e_{g g^{\prime}} \tag{A.2}
\end{equation*}
$$

where $\alpha: G \times G \rightarrow \mathrm{C} \backslash\{0\}$ is a cocycle map. Associativity implies

$$
\begin{equation*}
\alpha\left(g, g^{\prime}\right) \alpha\left(g g^{\prime}, g^{\prime \prime}\right)=\alpha\left(g, g^{\prime} g^{\prime \prime}\right) \alpha\left(g^{\prime}, g^{\prime \prime}\right) \tag{A.3}
\end{equation*}
$$

while the freedom in rescaling $e_{g} \rightarrow \beta(g) e_{g}$ by nonzero complex numbers $\beta(g)$ implies that the cocycles are defined modulo

$$
\begin{equation*}
\alpha\left(g, g^{\prime}\right) \rightarrow \beta(g) \beta\left(g^{\prime}\right) \alpha\left(g, g^{\prime}\right)\left(\beta\left(g g^{\prime}\right)\right)^{-1}, \quad \beta: G \rightarrow \mathrm{C} \backslash\{0\} . \tag{A.4}
\end{equation*}
$$

From $e_{g}\left(e_{I} e_{g^{\prime}}\right)=\left(e_{g} e_{I}\right) e_{g^{\prime}}$, where $I$ denotes the identity of $G$, it follows that $\alpha(g, I)=$ $\alpha\left(I, g^{\prime}\right)$. Hence, by making use of the freedom in $\beta(I)$ one can take

$$
\begin{equation*}
\alpha(g, I)=\alpha(I, g)=1 \tag{A.5}
\end{equation*}
$$

so that $e_{I}$ becomes the identity in the twisted group algebra, i.e.

$$
\begin{equation*}
e_{I} a=a e_{I}=a \quad \text { for all } a \in \mathbb{C}[G, \alpha] . \tag{A.6}
\end{equation*}
$$

Moreover, from $e_{g}\left(e_{g^{-1}} e_{g}\right)=\left(e_{g} e_{g^{-1}}\right) e_{g}$ it follows that

$$
\begin{equation*}
\alpha\left(g, g^{-1}\right)=\alpha\left(g^{-1}, g\right) \tag{A.7}
\end{equation*}
$$

[^15]Thus, the twisted group algebra admits the trace operation

$$
\begin{equation*}
\operatorname{Tr}_{\mathrm{C}[G, \alpha]} e_{g}:=|G| \delta_{I, g}, \quad \text { where } \quad|G|=\operatorname{dim}(G) \tag{A.8}
\end{equation*}
$$

Alternatively, the algebra $\mathrm{C}[G, \alpha]$ can be thought of as a non-commutative deformation of the algebra of C -valued functions on $G$, by defining a map $V$ that sends $\psi: G \rightarrow \mathrm{C}$ to

$$
\begin{equation*}
V_{\psi}:=\sum_{g \in G} \psi(g) e_{g} \tag{A.9}
\end{equation*}
$$

such that

$$
\begin{equation*}
V_{\psi} V_{\psi^{\prime}} \equiv V_{\psi \star \psi^{\prime}}, \tag{A.10}
\end{equation*}
$$

where the associative star product is given by

$$
\begin{equation*}
\left(\psi \star \psi^{\prime}\right)\left(g^{\prime}\right)=\sum_{g \in G} \psi(g) \alpha\left(g, g^{-1} g^{\prime}\right) \psi\left(g^{-1} g^{\prime}\right) . \tag{A.11}
\end{equation*}
$$

In this basis, the trace operation is given by evaluation at the identity $I \in G$, viz.

$$
\begin{equation*}
\operatorname{Tr}_{\mathrm{C}[G, \alpha]} V_{\psi}=|G| \psi(I) . \tag{A.12}
\end{equation*}
$$

If the twisted group algebra is $Z_{2}$-graded by means of an inner Klein operator $k$, then we may either take $h=k$ and $\mathcal{F}=\mathrm{C}[G, \alpha]$ as in Section 4, or $\gamma=k$ and $\mathcal{F}_{0}=\mathrm{C}[G, \alpha]$ as in Section 5. The discrete groups of order 4 and 8 are

$$
\begin{align*}
G_{4}: & \mathrm{Z}_{2} \times \mathrm{Z}_{2}, \quad \mathrm{Z}_{4} \\
G_{8}: & \left(\mathrm{Z}_{2}\right)^{3}, \quad \mathrm{Z}_{2} \times \mathrm{Z}_{4}, \quad \mathrm{Z}_{8}, \quad \mathrm{D}_{4}, \quad \mathrm{Q}_{8} \tag{A.13}
\end{align*}
$$

The eight-dimensional 3-graded Frobenius algebra introduced in Section 2 is isomorphic to $\mathrm{C}\left[\left(\mathrm{Z}_{2}\right)^{3} ; \alpha\right]$ with cocycle factor as follows: Denoting the generators of $\left(\mathrm{Z}_{2}\right)^{3}$ by $\gamma_{i}, i=1,2,3$, obeying the group relations

$$
\begin{equation*}
\left(\gamma_{i}\right)^{2}=I, \quad \gamma_{i} \gamma_{j}=\gamma_{j} \gamma_{i}, \tag{A.14}
\end{equation*}
$$

where $I$ is the identity element, the subalgebra $\mathcal{F}_{0}$ can be spanned by

$$
\begin{equation*}
e+\tilde{e}=e_{I}, \quad e-\tilde{e}=e_{\gamma_{1} \gamma_{2}}, \quad f+\tilde{f}=e_{\gamma_{1} \gamma_{3}}, \quad f-\tilde{f}=e_{\gamma_{2} \gamma_{3}} \tag{A.15}
\end{equation*}
$$

provided $\alpha$ is chosen such that

$$
\begin{equation*}
e_{\gamma_{1} \gamma_{2}} \cdot e_{\gamma_{1} \gamma_{2}}=e_{\gamma_{1} \gamma_{3}} \cdot e_{\gamma_{1} \gamma_{3}}=-e_{\gamma_{2} \gamma_{3}} \cdot e_{\gamma_{2} \gamma_{3}}=e_{I} \tag{A.16}
\end{equation*}
$$

and

$$
\begin{align*}
& e_{\gamma_{1} \gamma_{2}} \cdot e_{\gamma_{1} \gamma_{3}}=e_{\gamma_{2} \gamma_{3}}=-e_{\gamma_{1} \gamma_{3}} \cdot e_{\gamma_{1} \gamma_{2}}  \tag{A.17}\\
& e_{\gamma_{1} \gamma_{2}} \cdot e_{\gamma_{2} \gamma_{3}}=e_{\gamma_{1} \gamma_{3}}=-e_{\gamma_{2} \gamma_{3}} \cdot e_{\gamma_{1} \gamma_{2}}  \tag{A.18}\\
& e_{\gamma_{2} \gamma_{3}} \cdot e_{\gamma_{1} \gamma_{3}}=e_{\gamma_{1} \gamma_{2}}=-e_{\gamma_{2} \gamma_{3}} \cdot e_{\gamma_{1} \gamma_{3}} \tag{A.19}
\end{align*}
$$

The Klein operator

$$
\begin{equation*}
h=e_{\gamma_{1} \gamma_{2} \gamma_{3}} \tag{A.20}
\end{equation*}
$$

and we identify

$$
\begin{equation*}
h(e-\tilde{e})=e_{\gamma_{3}}, \quad h(f+\tilde{f})=e_{\gamma_{2}}, \quad h(f-\tilde{f})=e_{\gamma_{1}} \tag{A.21}
\end{equation*}
$$

It follows that the trace operation (A.8) is equivalent to the trace used in Section 2. Alternatively, the above algebra can be viewed as the twisted product of $Z_{2} \times Z_{2}$ with another $Z_{2}$. It would be interesting to determine whether there exist further twistings leading to nonequivalent FCS models.

The star product (A.11) is the discrete counterpart of the Poincaré-Birkhoff-Witt (PBW) star product on the enveloping algebra of the Lie algebra $\mathfrak{g}$ of a Lie group $G:$ If $\psi: G \rightarrow C$ then $V_{\psi}=\int_{g \in G} d \mu(g) \psi(g) e_{g} \in \mathrm{C}[G]$, defined using the Haar measure, can be mapped via $\phi\left(e_{g}\right)=\exp _{\star}\left(i \phi^{\alpha}(g) T_{\alpha}\right) \in \operatorname{Env}(\mathfrak{g})$, where $T_{\alpha}$ are a set of generators of $\mathfrak{g}$ and $\star$ is the PBW product, to an element $\phi\left(V_{\psi}\right)=\int_{g \in G} d \mu(g) \psi(g) \exp _{\star}\left(i \phi^{\alpha}(g) T_{\alpha}\right) \in \operatorname{Env}(\mathfrak{g})$. Thus, the basic FCS model in Section 2 has an internal algebra $\mathcal{A}=\mathcal{H} \otimes \mathcal{F}$ given by the direct product of two twisted group algebras, viz. the finite-dimensional factor $\mathcal{F}=\mathrm{C}\left[\left(\mathrm{Z}_{2}\right)^{3} ; \alpha\right]$ and the infinite-
dimensional factor $\mathcal{H}=\mathcal{K} \otimes_{\alpha^{\prime}} \operatorname{Env}(\mathfrak{s o}(3,2)) / \mathcal{I} \cong \mathrm{C}\left[\mathcal{K} \times S O(2,3) ; \alpha^{\prime}\right] / \mathcal{I}$, where $\alpha^{\prime}$ encodes the (anti)commutation relations between the outer Klein operators $k$ and $\bar{k}$ generating $\mathcal{K} \cong\left(\mathrm{Z}_{2}\right)^{2}$ and the generators of $\mathfrak{s o}(3,2)$, and $\mathcal{I}$ is the singleton annihilator. This suggests that higher spin gravity can be developed further by considering internal algebras given by more general infinite-dimensional twisted group algebras.

## References

[1] N. Boulanger, E. Sezgin, and P. Sundell, "4D Higher Spin Gravity with Dynamical Two-Form as a Frobenius-Chern-Simons Gauge Theory," 1505.04957.
[2] M. A. Vasiliev, "Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions," Phys. Lett. B243 (1990) 378-382.
[3] N. Colombo and P. Sundell, "Twistor space observables and quasi-amplitudes in 4D higher spin gravity," JHEP 1111 (2011) 042 doi:10.1007/JHEP11(2011)042 [arXiv:1012.0813 [hep-th]].
[4] N. Boulanger and P. Sundell, "An action principle for Vasiliev's four-dimensional higher-spin gravity," J.Phys. A44 (2011) 495402, 1102.2219 .
[5] E. Sezgin and P. Sundell, "Geometry and Observables in Vasiliev's Higher Spin Gravity," JHEP 07 (2012) 121, 1103.2360.
[6] N. Boulanger, N. Colombo, and P. Sundell, "A minimal BV action for Vasiliev's four-dimensional higher spin gravity," JHEP 1210 (2012) 043, 1205.3339,
[7] N. Colombo and P. Sundell, "Higher Spin Gravity Amplitudes From Zero-form Charges," 1208.3880.
[8] C. Arias, R. Bonezzi, N. Boulanger, E. Sezgin, P. Sundell, A. Torres-Gomez, and M. Valenzuela, "Action principles for higher and fractional spin gravities," in International Workshop on Higher Spin Gauge Theories Singapore, Singapore, November 4-6, 2015. 2016. 1603.04454.
[9] C. Iazeolla and P. Sundell, "Biaxially symmetric solutions to 4D higher-spin gravity," J.Phys. A46 (2013) 214004, 1208.4077.
[10] J. Engquist and P. Sundell, "Brane partons and singleton strings," Nucl. Phys. B752 (2006) 206-279, hep-th/0508124.
[11] V. Didenko and E. Skvortsov, "Exact higher-spin symmetry in CFT: all correlators in unbroken Vasiliev theory," JHEP 1304 (2013) 158, 1210.7963.
[12] S. B. Conlon, "Twisted group algebras and their representations," Journal of the Australian Mathematical Society 4 (1964), no. 02, 152-173.
[13] J.-P. Serre, "Groupes finis," Cours à l'École Normale Supérieure de Jeunes Filles, (1978/1979), arXiv preprint math/0503154 (2005), first written down by M.Buhler and C.Goldstein (Montrouge 1979), then revised and put in LaTeX by N.Billerey, O.Dodane and E.Rey.
[14] M. Alexandrov, M. Kontsevich, A. Schwartz, and O. Zaboronsky, "The Geometry of the master equation and topological quantum field theory," Int. J. Mod. Phys. A12 (1997) 1405-1430, hep-th/9502010.
[15] S. E. Konshtein and M. A. Vasiliev, "Massless representations and admissibility condition for higher spin superalgebras," Nucl. Phys. B312 (1989) 402.
[16] S. E. Konstein and M. A. Vasiliev, "Extended higher spin superalgebras and their massless representations," Nucl. Phys. B331 (1990) 475-499.
[17] D. Quillen, "Superconnections and the chern character," Topology 24 (1985), no. 1, 89 $-95$.
[18] N. Boulanger, P. Sundell, and M. Valenzuela, "Three-dimensional fractional-spin gravity," JHEP 1402 (2014) 052, 1312.5700.
[19] C. Iazeolla and P. Sundell, "Families of exact solutions to Vasiliev's 4D equations with spherical, cylindrical and biaxial symmetry," JHEP 1112 (2011) 084, 1107.1217.
[20] C. Iazeolla, E. Sezgin, and P. Sundell, "Real Forms of Complex Higher Spin Field Equations and New Exact Solutions," Nucl. Phys. B791 (2008) 231-264, 0706.2983.
[21] M. A. Vasiliev, "More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions," Phys. Lett. B285 (1992) 225-234.
[22] M. A. Vasiliev, "Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)," Phys. Lett. B567 (2003) 139-151, hep-th/0304049.
[23] E. Sezgin and P. Sundell, "Supersymmetric Higher Spin Theories," J.Phys. A46 (2013) 214022, 1208.6019.
[24] C. Arias, N. Boulanger, P. Sundell, and A. Torres-Gomez, "2D sigma models and differential Poisson algebras," JHEP 08 (2015) 095, 1503.05625.
[25] R. Bonezzi, P. Sundell, and A. Torres-Gomez, "2D Poisson Sigma Models with Gauged Vectorial Supersymmetry," JHEP 08 (2015) 047, 1505.04959.
[26] E. Witten, "Chern-Simons gauge theory as a string theory," Prog. Math. 133 (1995) 637-678, hep-th/9207094.
[27] E. Beggs and S. Majid, "Semiclassical differential structures," math/0306273.
[28] S. McCurdy and B. Zumino, "Covariant Star Product for Exterior Differential Forms on Symplectic Manifolds," AIP Conf.Proc. 1200 (2010) 204-214, 0910.0459.
[29] M. R. Gaberdiel and B. Zwiebach, "Tensor constructions of open string theories. 1: Foundations," Nucl.Phys. B505 (1997) 569-624, hep-th/9705038.


[^0]:    ${ }^{1}$ The complete specification of the model requires careful choices of classes of symbols on the noncommutative twistor spaces on the base manifold and the fiber, for which we refer to [1] and [9.

[^1]:    ${ }^{2}$ It would be interesting to also consider polarizations from vector field structures on the base manifold.

[^2]:    ${ }^{3}$ The master fields are elements of $\Omega\left(\mathcal{X}_{5, \xi}\right) \otimes \Omega\left(\mathbb{C}^{4}\right)$, where $\mathcal{X}_{5, \xi}$ are coordinate charts of $\mathcal{X}_{5}$ and $\Omega\left(\mathbb{C}^{4}\right)$ consists of forms, including distributions, on a real slice of $\mathbb{C}^{4}$. The local representatives are assumed to belong to sections of a structure group such that the curvatures, covariant derivatives and Lagrange multipliers that appear in the Lagrangian obey regularity conditions in the interior of $\mathbb{C}^{4}$ and fall-off conditions at infinity as to make the action well defined. The way in which this was achieved in [1] provides $\mathcal{Z}_{4}$ with the topology of $S^{2} \times S^{2}$, whereas in general there may exist other possibilities, that we defer for future work.
    ${ }^{4}$ The transition elements of $\mathcal{E}$ consist of forms in all degrees. Moreover, in order for the model to consist of bosonic fields with integer spins, the space $\Omega\left(\mathcal{Z}_{4}\right) \otimes \mathcal{W}$, is projected in a fashion that correlates the dependencies on the generating elements of $\Omega\left(Z_{4}\right)$ and $\mathcal{W}$, that is, the dependences of the sections of $\mathcal{E}$ on base and fiber coordinates is intertwined.

[^3]:    ${ }^{5}$ The restricted spectrum of form degrees yields a model without zero-form constraints on-shell and zeroform sector identical to that of Vasiliev's original system. It is possible, however, to take $(B, \widetilde{B} ; U, \widetilde{U})$ and $A, \widetilde{A} ; V, \widetilde{V})$ to be general even and odd forms, respectively, as we shall examine in more detail in Section 3.

[^4]:    ${ }^{6}$ Beyond the semi-classical analysis, the boundary condition on $P$ follows from the Batalin-Vilkovisky master equation.

[^5]:    ${ }^{7}$ The zero-form constraints and top-forms can be treated within the context of path integral quantization using the AKSZ formalism.
    ${ }^{8}$ The positively normed Frobenius algebras are $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

[^6]:    ${ }^{9}$ Top-forms and zero-form constraints can naturally be incorporated into the AKSZ formalism.

[^7]:    ${ }^{10}$ The FCS Lagrangian (2.37) is obtained by taking $A^{I}=(A, \widetilde{A} ; V, \widetilde{V}), B^{P}=(B, \widetilde{B} ; U, \widetilde{U})$ and $\left(\Sigma_{I J}, \Omega_{P Q}, \Theta_{I J}, \Xi_{P Q}\right)=\left(\sigma_{1} \otimes \mathbb{1},-\mathrm{i} \sigma_{2} \otimes \sigma_{1}, \mathrm{i} \sigma_{2} \otimes \mathbb{1},-\sigma_{1} \otimes \sigma_{1}\right)$, and making a suitable identification of $s_{I P Q}$ and $t_{I J K}$.

[^8]:    ${ }^{11}$ We use the convention that if $f, g \in \mathcal{H}$ and $a, b \in \mathcal{F}$ then $(a f, b g)_{\mathcal{F}} \equiv(a, b)_{\mathcal{F}} f \star g$.

[^9]:    ${ }^{12}$ In the case of a cubic action being considered here, the gluing compatibility condition for a generalized bundle holds identically for any choice of structure group, see [4, 5, 6].

[^10]:    ${ }^{13}$ Instead of working with patches one may use partitions of unity.

[^11]:    ${ }^{14}$ It is important that the truncation does not affect the zero-form sector.
    ${ }^{15}$ We use a notation in which $\mathrm{C}_{\mathcal{A}}(x)$ denotes the centralizer of an element $x$ in an associative algebra $\mathcal{A}$.
    ${ }^{16}$ Whether there exist consistent truncations to the Konstein-Vasiliev models with husp or ho algebras is left for future work.

[^12]:    ${ }^{17}$ An outline of twisted group algebras is given in Appendix A,
    ${ }^{18}$ In terms of the basis for $\mathbb{C}\left[\mathbb{Z}_{2 n}\right]$ consisting of the projectors $p_{l}:=\frac{1}{2 n} \sum_{k=0}^{2 n-1} e^{i \frac{k l \pi}{n}} e_{a^{k}}, l=0,1, \ldots, 2 n-1$, $p_{i} p_{j}=\delta_{i, j} p_{i}$, we have $\operatorname{Tr}_{\mathcal{F}} e_{i}=1$. Thus, expanding $x \in \mathbb{C}\left[\mathbb{Z}_{2} \times \mathbb{Z}_{2 n}, \alpha\right]$ as $x=\sum_{l=0}^{2 n-1} e_{l}\left(x_{l}+h \tilde{x}_{l}\right), x_{l}, \tilde{x}_{l} \in \mathbb{C}$, its trace $\operatorname{Tr}_{\mathcal{F}} x=\sum_{l=0}^{2 n-1} x_{l}$.

[^13]:    ${ }^{19}$ For a related truncation of three-dimensional fractional spin gravity, see Section 4.5 of [18].
    ${ }^{20}$ Keeping the antiholomorphic component of the two-form activates the three-form, as $B_{[2]+} \star B_{[2]+}$ is now proportional to $d z^{2} d \bar{z}^{2} \kappa \star \bar{\kappa}$.

[^14]:    ${ }^{21}$ The gauge function is defined on a subset of $\mathcal{X} 4$. The sterographic coordinate system $x^{\mu} \in \mathbb{R}^{1,3} \backslash\{x$ : $\left.x^{2}=1\right\}$ with metric $d x^{2} /\left(1-x^{2}\right)^{2}$ covers the coset once. We take $\mathcal{X}_{4}$ to be $\mathbb{R}^{1,3}$ with points at infinity such that $\partial \mathcal{X}_{4}=0$, and allow the gauge fields (but not the curvatures) to blow up on the surface $\left\{x: x^{2}=1\right\}$.
    ${ }^{22}$ In 20 the zero-form is unconstrained and proper reality conditions, viz. $B_{[0]}^{\dagger}=B_{[0]}, A_{[1]}^{\dagger}=-A_{[1]}$ and $\widetilde{B}_{[2]}^{\dagger}=-\widetilde{B}_{[2]}$ are imposed, which requires either $(2,2)$ of $(4,0)$ Lorentz signature in order for $\widetilde{B}_{[2]}$ to be holomorphic.

[^15]:    ${ }^{23}$ Twisted group algebras are also known as extensions of standard group algebras by an abelian group so that they are group algebras of extended groups; for example, p. 31 in e.g. [13].

