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Higher-derivative harmonic oscillators: stability of classical dynamics and adiabatic invariants

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Abstract The status of classical stability in higher-derivative systems is still subject to discussions. In this note, we argue that, contrary to general belief, many higher-derivative systems are classically stable. The main tool to see this property are Nekhoroshev's estimates relying on the action-angle formulation of classical mechanics. The latter formulation can be reached provided the Hamiltonian is separable, which is the case for higher-derivative harmonic oscillators. The Pais–Uhlenbeck oscillators appear to be the only type of higher-derivative harmonic oscillator with stable classical dynamics. A wide class of interaction potentials can even be added that preserve classical stability. Adiabatic invariants are built in the case of a Pais–Uhlenbeck oscillator slowly changing in time; it is shown indeed that the dynamical stability is not jeopardised by the time-dependent perturbation.

1 Introduction

Variational principles based on action functionals of the form $S[x] = \int L(x, \dot{x}) \, dt$ have a special status in the sense that Newtonian mechanics can be recovered from Lagrangians $L(x, \dot{x})$ depending only on position and velocity. Still, Lagrangians depending on higher time derivatives of the position, i.e, higher-derivative (HD) Lagrangians, are also worth of interest. Let us mention three areas in which HD models are encountered:

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1. Explicit construction of classically stable and unstable HD dynamical systems: A class of classical HD harmonic oscillators was proposed in Sect. II of [1] and is still actively studied nowadays under the name of Pais–Uhlenbeck (P–U) oscillator, see e.g. Refs. [2–13] and references therein for recent contributions to the field;

2. Renormalizability of HD field theories: In their pioneering work [1], Pais and Uhlenbeck addressed the issue of renormalizability in field theory through the inclusion of HD terms. HD gravities, like Weyl gravity, are promising renormalisable models of quantum gravity, see the seminal paper [14,15] and recent references in [16,17]. These HD models bring in the Einstein–Hilbert term upon radiative corrections, see e.g. [18]. They are also interesting in the context of cosmology and supergravity, see [19–22] and Refs. therein;

3. HD effective dynamics of voluntary human motions: The underlying dynamics of such motions is expected to involve HD variational principles such as minimal jerk, see e.g. Refs. [23–27]. In this case, higher derivative terms may be thought of as a way to account for an intrinsic nonlocality (in time) of planified motion: Human motor control may indeed add memory effects to standard Newtonian dynamics, which may be translated into a HD effective action.

A key feature of classical HD dynamics is that the energy has no definite sign, as it is readily observed from the general structure of HD Hamiltonians [28]. The presence of HD terms may lead to unbounded trajectories at the classical level – an explicit case is built in [2] – and to loss of unitarity at the quantum level [10]. However several cases are known for

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which classical trajectories are bounded and unitarity is preserved at the quantum level [10]. Having these recent results in mind, we think that providing a general method in order to assess classical stability of HD models is worth of interest. This is the main goal of the present paper, in which we focus on the case of HD harmonic oscillators. In the case of perturbed harmonic oscillators, an important body of works concerning their stability has been produced [29,30] that seems to have gone unnoticed by the HD community, so far. In a very specific case of a P–U oscillator with at most two time derivatives in the Lagrangian, Pagani et al. proved stability under a general class of cubic and quartic interactions [31].

The present paper is organised as follows. The Lagrangian (Sect. 2) and Hamiltonian (Sect. 3) formulations of HD harmonic oscillators are reviewed and the necessary conditions for their classical trajectories to be bounded are established. The dynamics is then formulated in terms of the action-angle coordinates in Sect. 4 and adiabatic invariants are computed. Finally, classical stability against time-dependent perturbations is discussed by using Nekhoroshev's estimates [32,33].

2 Lagrangian formulation

In this section we review the Lagrangian formulation of HD classical systems with finitely many degrees of freedom, in essentially the way that was presented long ago by Ostrogradki [28]. We then review the Pais–Uhlenbeck parametrisation [1] of HD Lagrangians.

2.1 Generalities

Let $L(x^{(0)}, x^{(1)}, \ldots, x^{(N)})$ be a Lagrangian depending on the N first derivatives of the dynamical variables $x(t) := x^{(0)}(t)$, $x^{(1)}(t) := \dot{x}(t)$ etc. possibly upon adding total derivatives in order to lower as much as possible the order of derivatives of $x(t)$. The action, evaluated between time t_1 and time t_2, is $S[x] = \int_{t_1}^{t_2} L \, dt$. Hamilton's variational principle $\delta S = 0$ implies the equations of motion
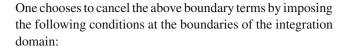
$$0 = \frac{\delta L}{\delta x} \equiv \sum_{j=0}^{N} \left(-\frac{d}{dt}\right)^j \frac{\partial L}{\partial x^{(j)}}, \qquad (1)$$

together with the vanishing of the boundary terms

$$\sum_{i=0}^{N-1} \delta x^{(i)} p_i \Big|_{t_1}^{t_2} = 0,$$

$$p_i := \frac{\delta L}{\delta x^{(i+1)}} \equiv \sum_{j=0}^{N-i-1} \left(-\frac{d}{dt}\right)^j \frac{\partial L}{\partial x^{(i+j+1)}},$$

$$i \in \{0, \ldots, N-1\}. \qquad (2)$$

One chooses to cancel the above boundary terms by imposing the following conditions at the boundaries of the integration domain:

$$\delta x^{(i)}(t_2) = 0 = \delta x^{(i)}(t_1), \quad \forall i \in \{0, \ldots, N-1\}.$$

It amounts to declaring that the initial data needed for solving the equation (1) is given by the values of $x^{(0)}$, $x^{(1)}$, $\ldots$, $x^{(N-2)}$ and $x^{(2N-1)}$ at initial time t_1. Indeed, provided one assumes the regularity condition

$$\frac{\partial^2 L}{\partial x^{(N)} \partial x^{(N)}} \neq 0,$$

the above initial data give a well-posed Cauchy problem for the ordinary differential equation (1).

2.2 Toy model

The simple HD harmonic oscillator

$$L_{\text{toy}} = \frac{\lambda}{2} \left(x^{(N)}\right)^2 - \frac{(-)^{N+1}}{2} \lambda \beta^{2N} x^2, \qquad (3)$$

with $N \in \mathbb{N}_0$ and $\lambda, \beta \in \mathbb{R}^+$ can be used to illustrate some features of HD dynamics. It reduces to the standard harmonic oscillator for $N = 1$, in which case λ is a mass-parameter. The case $N = 2$ has already been used as a toy model in [34].

The equation of motion (1) reads

$$x^{(2N)} + \beta^{2N} x = 0, \qquad (4)$$

and its classical solution is given by

$$x(t) = \sum_{j=0}^{2N-1} A_j \, e^{\beta_j t} \quad \text{with} \quad \beta_j = \beta \, e^{i \theta_j} \quad \text{and}$$

$$\theta_j = \frac{\pi}{2N} + \frac{j\pi}{N}. \qquad (5)$$

It can be observed from $x(t)$ that the allowed trajectories go beyond a standard periodic motion since, up to an appropriate choice of A_j, there may appear:

- unbounded when $\text{Re } \beta_j > 0$. This occurs for j such that $0 < 2j + 1 < N$ and $3N < 2j + 1 \leqslant 4N - 1$;
- damped when $\text{Re } \beta_j < 0$. This occurs for j such that $N < 2j + 1 < 3N$;
- periodic with period $2\pi/\beta$ if $\text{Re } \beta_j = 0$: This can happen for N odd and the two values $j = (N-1)/2$ and $j = (3N-1)/2$.

The standard case $N = 1$ is the only value for which all the possible trajectories are periodic. When $N > 1$, damping or "blowing-up" phenomena occur at time scales of order $\beta \cos(\pi/2N)$.

2.3 General case: Pais–Uhlenbeck oscillator

Lagrangian (3) always allows for unbounded classical motions. It can be showed that adding intermediate derivatives $x^{(0<i<N)}$ may reduce and even suppress such instabilities.[1] Lagrangian (3) may be generalised as follows

$$L(x^{(0)}, x^{(1)}, \ldots, x^{(N)}) = \tfrac{1}{2} \sum_{j=0}^{N} a_j (x^{(j)})^2 , \quad a_j \in \mathbb{R} , \quad (6)$$

and the corresponding equation of motion reads

$$\sum_{j=0}^{N} (-)^j a_j x^{(2j)} = 0 . \quad (7)$$

The characteristic polynomial of the above differential equation is

$$p(\omega) = \sum_{j=0}^{N} (-)^j a_j \omega^{2j} . \quad (8)$$

If the signs of a_i are alternating, i.e. $a_j = (-)^{j+1} b_j$ with $b_j \in \mathbb{R}_0^+$, then $p(\omega)$ can be factorised under the form $p(\omega) \sim \Pi_{j=1}^{N} \left(\omega^2 - (i\omega_j)^2 \right)$ with $\omega_j \in \mathbb{R}_0$ and all the trajectories $x(t)$ will be bounded. The further choice $a_0 < 0$ is such that a standard potential energy is recovered for $N = 1$.

Therefore, replacing $(x^{(j)})^2$ by $(-)^j x\, x^{(2j)}$ up to total derivatives, Lagrangian (6) can be rewritten as

$$L(x^{(0)}, x^{(1)}, \ldots, x^{(N)}) = -\tfrac{1}{2} x \sum_{j=0}^{N} b_j x^{(2j)} . \quad (9)$$

Lagrangian (9) is nothing but the P–U oscillator [1], originally written under the equivalent form

$$L = -\tfrac{1}{2} x\, F_N \left(\frac{d}{dt} \right) x \quad \text{where} \quad F_N(D) = \prod_{i=0}^{N-1} \left(1 + \frac{D^2}{\omega_i^2} \right) \quad (10)$$

where the frequencies ω_i are assumed to be real and distinct. It has been shown in [1] that equal or imaginary frequencies lead to unbounded trajectories so these cases will not be considered in the following.

The solution of the equation of motion related to (10), $F_N \left(\frac{d}{dt} \right) x = 0$, reads $x(t) = \sum_{k=1}^{N} A_k \sin(\omega_k t + \varphi_k)$ with $A_k, \varphi_k \in \mathbb{R}$. All classical trajectories are therefore bounded. Note that $x(t)$ may describe the motion of a given mass in an N−body coupled harmonic oscillator whose normal modes have frequencies ω_i: The formal analogy between the P–U oscillator and the dynamics of an N-body spring-mass system has been explored in the $N = 2$ case in Ref. [36].

An equivalent writing of (10) makes use of the oscillator variables

$$Q_i := \prod_{0=j\neq i}^{N-1} \left(1 + \frac{1}{\omega_j^2} \frac{d^2}{dt^2} \right) x \quad (11)$$

and shows that (10) can formally be written as a Lagrangian describing N decoupled harmonic oscillators:

$$L = -\frac{1}{2} \sum_{i=0}^{N-1} \eta_i\, Q_i \left(\omega_i^2 + \frac{d^2}{dt^2} \right) Q_i , \quad (12)$$

where it can be deduced from [1], see [37], that

$$\eta_i = \frac{1}{\omega_i^2} \frac{\sum_{\substack{0 \leq m_0 < \cdots < m_{N-2} \leq N-1 \\ m_j \neq i}} \omega_{m_0}^2 \cdots \omega_{m_{N-2}}^2}{\prod_{\substack{m=0 \\ m \neq i}}^{N-1} (\omega_m^2 - \omega_i^2)} . \quad (13)$$

The signs of η_i are alternating, which is a typical signature of HD dynamics, eventually leading to a total energy whose sign is undefined. The equation of motion for the Q_i's reads

$$\ddot{Q}_i + \omega_i^2 Q_i = 0 , \quad i = 0, \ldots, N-1 . \quad (14)$$

Generalization of the P–U oscillator in more than one spatial dimension is straightforward and will be left for future works. New integrals of motion such as HD angular momenta are naturally expected: We refer the interested reader to [38–40] for explicit definitions.

3 Hamiltonian formalism

The Ostrograski construction that we have reviewed in Sect. 2.1 canonically leads to Hamiltonians that are not separable in the variables p_i (2) and

$$q^i := x^{(i)} , \quad \forall i \in \{0, \ldots, N-1\} . \quad (15)$$

In this section we show in some particular cases that a suitable change of canonical variables allows to recast the P–U Hamiltonian under a separable form.

3.1 Ostrogradski's approach

As in the standard $N = 1$ case, Hamilton's variational principle naturally leads to a symplectic structure and an Hamiltonian function H. Provided that the regularity condition (3) holds one can invert the last relation (2) defining the momentum p_{N-1} in order to express $x^{(N)}$ in terms of p_{N-1} and the $x^{(i)}$'s, $i \in \{0, \ldots, N-1\}$:

$$\frac{\partial^2 L}{\partial x^{(N)} \partial x^{(N)}} \neq 0 \implies x^{(N)} = \mathscr{V}(x^{(0)}, \ldots, x^{(N-1)}, p_{N-1}) . \quad (16)$$

---

[1] Adding degrees of freedom coupled to x is another way of addressing the problem, see. e.g. [35].

Details about singular HD Lagrangians can be found in [41, 42]. The Hamiltonian function H is defined as

$$
\begin{aligned}
H(\{q^i, p_i\}_{i=0,\ldots,N-1}) &:= \sum_{j=0}^{N-2} p_i q^{i+1} \\
&\quad + p_{N-1}\mathscr{V} - L(q^0, q^1, \ldots, q^{N-1}, \mathscr{V}) .
\end{aligned}
\tag{17}
$$

The symplectic structure, as already apparent from the structure of the boundary terms in the variation δS in the equation above (2), is given by the two-form $\boldsymbol{\Omega} = \sum_{i=0}^{N-1} dp_i \wedge dq^i$. In particular, the Poisson–Ostrogradski bracket between any two functions in the phase space $T^*\mathbb{Q}$ locally coordinatized by the $2N$ variables $(q^i, p_i)_{i=0,\ldots,N-1}$ is given by

$$
\{f, g\} = \sum_{i=0}^{N-1} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right) .
\tag{18}
$$

The symplectic structure $\boldsymbol{\Omega}$ amounts to writing $\{q^i, p_j\} = \delta^i_j$, $\{q^i, q^j\} = 0 = \{p_i, p_j\}$, $\forall\, i, j \in \{0, \ldots, N-1\}$, and the field equations are given as usual by $\dot{q}^i = \{q^i, H\} \equiv \frac{\partial H}{\partial p_i}$, $\dot{p}_i = \{p_i, H\} \equiv -\frac{\partial H}{\partial q^i}$. The original equation of motion is reproduced from $\dot{p}_0 = \{p_0, H\}$, while all the other equations above reproduce the relations (2) between the momenta- and the position-like variables.

Starting from Lagrangian (6) and applying Ostrogradski's procedure one gets

$$
p_i = \sum_{j=0}^{N-i-1} (-)^j a_{i+j+1} x^{(i+2j+1)}, \quad i = 0, \ldots, N-1
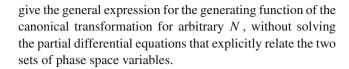\tag{19}
$$

and the Hamiltonian reads

$$
H = \sum_{j=0}^{N-2} p_j q^{j+1} + \frac{p_{N-1}^2}{2a_N} - \frac{1}{2} \sum_{j=0}^{N-1} a_j (q^j)^2 .
\tag{20}
$$

The equations of motion are explicitly given by

$$
\begin{aligned}
\dot{q}^i &= q^{i+1}, \quad i = 0, \ldots, N-2 \\
\dot{q}^{N-1} &= \frac{p_{N-1}}{a_N} \\
\dot{p}_0 &= a_0 q_0 \\
\dot{p}_j &= -p_{j-1} + a_j q_j, \quad j = 1, \ldots, N-1 .
\end{aligned}
\tag{21}
$$

3.2 Link with Pais–Uhlenbeck variables

The equivalence of Ostrogradski and P–U formalisms at Hamiltonian level is not obvious in the sense that the canonical transformation between the Ostrogradski and the P–U variables has not been explicitly given in the general case, as far as we could see. It was shown for $N = 2$ in [31]. Here we show it for the $N = 3$ case and leave the explicit form of the canonical transformation relating the Ostrogradski to the P–U variables for future works. Below we simply

give the general expression for the generating function of the canonical transformation for arbitrary N, without solving the partial differential equations that explicitly relate the two sets of phase space variables.

3.2.1 The case N = 3

A convenient parametrisation in this case is

$$
L = -\frac{1}{2} x^2 + \frac{A_1}{2\Omega^6} \dot{x}^2 - \frac{A_2}{2\Omega^6} \ddot{x}^2 + \frac{1}{2\Omega^6} \dddot{x}^2 ,
\tag{22}
$$

with

$$
A_1 := \sum_{i<j} (\omega_i \omega_j)^2, \quad A_2 := \sum_{i=0}^{2} \omega_i^2 ,
$$

$$
\Omega^6 := (\omega_0 \omega_1 \omega_2)^2 \quad \text{and} \quad \omega_2 > \omega_1 > \omega_0 .
\tag{23}
$$

Note that Lagrangian (22) can be recast under the form (12) provided that

$$
\sum_{i=0}^{2} \omega_i^2 \eta_i = 1 , \quad \sum_{i=0}^{2} \omega_i^4 \eta_i = 0 = \sum_{i=0}^{2} \omega_i^6 \eta_i .
\tag{24}
$$

The above constraints are fulfilled with (13). The momenta associated with (22) read

$$
\begin{aligned}
p_0 &= \frac{1}{\Omega^6} \left(A_1 \dot{x} + A_2 \dddot{x} + x^{(5)} \right) , \\
p_1 &= -\frac{1}{\Omega^6} \left(A_2 \ddot{x} + x^{(4)} \right) , \quad p_2 = \frac{\dddot{x}}{\Omega^6}
\end{aligned}
\tag{25}
$$

and the Hamiltonian is given by

$$
\begin{aligned}
H &= p_0 q^1 + p_1 q^2 + \frac{\Omega^6}{2} p_2^2 + \frac{1}{2}(q^0)^2 \\
&\quad - \frac{A_1}{2\Omega^6} (q^1)^2 + \frac{A_2}{2\Omega^6} (q^2)^2 .
\end{aligned}
\tag{26}
$$

One can readily derive a generating function for the canonical transformation relating the Ostrogradski variables $(q^0, q^1, q^2, p_0, p_1, p_2)$ to the P–U phase space variables $(Q_0, Q_1, Q_2, P_0, P_1, P_2)$. It is given by

$$
\begin{aligned}
F(q^0, p_1, q^2, P_0, P_1, P_2) &= \sum_{i=0}^{2} Q_i(q^0, p_1, q^2) P_i \\
&\equiv \sum_{i=0}^{2} \tilde{Q}_i(q^0, p_1, q^2) \tilde{P}_i .
\end{aligned}
\tag{27}
$$

It leads to the following system of partial differential equations:

$$
\begin{aligned}
\frac{\partial F}{\partial P_0} &= Q^0(q^0, p_1, q^2), \quad \frac{\partial F}{\partial P_1} = Q^1(q^0, p_1, q^2), \\
\frac{\partial F}{\partial P_1} &= Q^1(q^0, p_1, q^2),
\end{aligned}
\tag{28}
$$

$$\frac{\partial F}{\partial q^0} = p_0(q^0, p_1, q^2), \quad \frac{\partial F}{\partial p_1} = -q^1(q^0, p_1, q^2),$$

$$\frac{\partial F}{\partial q^2} = p_2(q^0, p_1, q^2). \tag{29}$$

The first three equations are trivial, whereas the last three can be obtained from the relations $\sum_{i=0}^{2} \eta_i \omega_i^{2m} = \delta_1^m$, $m = 1, 2, 3$, as well as $\sum_{i=0}^{2} \eta_i \omega_i^8 = \omega_0^2 \omega_1^2 \omega_2^2$ and $\sum_{i=0}^{2} \eta_i = \frac{A_1}{\Omega^6}$. In particular, it yields

$$p_0 = P_0 + P_1 + P_2, \quad p_2 = -\frac{1}{\Omega^6} \sum_{i=0}^{2} \omega_i^4 P_i,$$

$$q^1 = \sum_{i=0}^{2} \omega_i^2 P_i. \tag{30}$$

Recalling that the η_i are of alternating sign, the Ostrogradski Hamiltonian eventually reads

$$H = \frac{1}{2} \sum_{i=0}^{2} (-)^i \left(\frac{P_i^2}{|\eta_i|} + |\eta_i| \omega_i^2 Q_i^2 \right). \tag{31}$$

Let us observe that, from the expressions (25) for the Ostrogradski momenta and the definition (11) of the P–U oscillator variables Q_i, $i = 0, 1, 2$, one has the relation $P_i = \eta_i \dot{Q}_i$, $i = 0, 1, 2$. Finally, a last canonical transformation given by the mere rescaling $\tilde{Q}_i := \sqrt{|\eta_i|} Q_i$, $\tilde{P}_i := \frac{P_i}{\sqrt{|\eta_i|}}$, gives the following P–U Hamiltonian:

$$H = \frac{1}{2} \sum_{i=0}^{2} (-)^i \left(\tilde{P}_i^2 + \omega_i^2 \tilde{Q}_i^2 \right). \tag{32}$$

3.2.2 Arbitrary N

The $N = 2$ and $N = 3$ cases show that Ostrogradski's procedure leads to P–U Hamiltonian and variables by successive canonical transformations. Moreover Lagrangian (6), once expressed as (12), leads to the separable Hamiltonian

$$H = \frac{1}{2} \sum_{i=0}^{N-1} (-)^i \left(\frac{P_i^2}{|\eta_i|} + |\eta_i| \omega_i^2 Q_i^2 \right) =: \sum_{i=0}^{N-1} (-)^i E_i \quad (33)$$

for any N, where E_i, $i = 0, 1, 2$ are positive definite quantities where

$$Q_i := \prod_{j \neq i} \left(1 + \frac{1}{\omega_j^2} \frac{d^2}{dt^2} \right) x, \tag{34}$$

$$P_i := \eta_i \dot{Q}_i \equiv (-)^i |\eta_i| \dot{Q}_i, \tag{35}$$

and where the P–U coefficients η_i can be found in [37] and in (13). Note that the frequencies ω_k are the roots of the polynomial equation $p(\omega_k) = 0$, see (8).

4 Adiabatic invariants and Nekhoroshev estimates

4.1 Action variables

The P–U Hamiltonian (33) is separable and admits elliptic trajectories in the planes (Q_i, P_i), $i \in \{0, 1, \dots, N-1\}$, these fixed-$E_i$ cycles being denoted as Γ_i. Hence a set of N action variables can be defined:

$$I_j = \frac{(-)^j}{2\pi} \oint_{\Gamma_j} P_j \, dQ_j = \frac{|\eta_j|}{2\pi} \oint_{\Gamma_j} \dot{Q}_j^2 \, dt,$$

$$j = 0, \dots, N-1. \tag{36}$$

The $(-)^j$ factor is such introduced in such a way that the action variables $\{I_i\}_{i=0,\dots,N-1}$ are all positive. It can be checked that the Hamiltonian (33) reads

$$H = \sum_{j=0}^{N-1} (-)^j I_j \omega_j, \quad \text{with} \quad I_j = \frac{E_j}{\omega_j}, \tag{37}$$

and that the relations

$$I_i = \frac{\partial E_i}{\partial \omega_i}, \quad i = 0, \dots, N-1 \tag{38}$$

holds as well. The action variable I_0 reduces to the average kinetic energy for $N = 1$, as expected.

The action variables can be expressed in terms of the classical trajectory $x(t)$ through the definition (34). When $N = 2$ for example,

$$I_0 \sim \oint_{\Gamma_0} \left[\dot{x}^2 + \frac{2}{\omega_1^2} \dot{x} \ddot{x} + \frac{1}{\omega_1^4} \ddot{x}^2 \right] dt,$$

$$I_1 \sim \oint_{\Gamma_1} \left[\dot{x}^2 + \frac{2}{\omega_0^2} \dot{x} \ddot{x} + \frac{1}{\omega_0^4} \ddot{x}^2 \right] dt. \tag{39}$$

The cross-product $\dot{x} \ddot{x}$ cannot be expressed in terms of $\ddot{x}^2$ with the use of a total derivative since $x(t)$ is not a priori periodic with frequency ω_0 or ω_1 unless $\omega_1/\omega_0 = n/m \in \mathbb{Q}$. In the latter case, after a time $T = m2\pi/\omega_0 = n2\pi/\omega_1$, the action variables can be recast under the form $\oint_{\Gamma_0} [\dot{x}^2 + c\ddot{x}^2 + d\dddot{x}^2] dt$ with c, d real coefficients. At the same time, the commensurability condition on ω_1/ω_0 implies instability of the $N = 2$ dynamics under small perturbations so it is not relevant for the present study [31].

The I_j are constant of motion provided that H does not explicitly depend on time. It is nevertheless possible that some external parameter is time-dependent: we set $\omega_i = \omega_i(t)$ in (33). Under the assumption that the $\omega_i(t)$ vary slowly enough with respect to the typical duration of a cycle – and despite the fact that no rigorous definition of "enough" can be given [43] –, the quantities I_j given by (36) are adiabatic invariants. It can be deduced from [44] that their small rate of change is given by:

$$\dot{I}_k = -\frac{\dot{\omega}_k}{\omega_k} I_k \cos 2\phi_k \tag{40}$$

$$\dot{\phi}_k = (-)^k \omega_k + \frac{\dot{\omega}_k}{2\omega_k} \sin 2\phi_k,$$

where ϕ_k are the angle variables conjugated to I_k: $\{\phi_j, I_k\} = \delta_{jk}$. The interested reader may find general computations related to time-varying harmonic oscillators in [45–47].

As an illustration of the above relations, suppose that the time-dependent parameter is a small perturbation of the frequencies:

$$\omega_i(t) = \varpi_i(1 + \epsilon\, g_i(t)), \quad \text{with} \quad |\epsilon| \ll 1, \tag{41}$$

the only dependence on time being contained in the real functions g_i. At the lowest order in ϵ, Eqs. (40) become

$$\dot{I}_k = -\epsilon\, \dot{g}_k I_k \cos 2\varphi_k,$$

$$\dot{\phi}_k = (-)^k \varpi_k (1 + \epsilon\, g_k) + \frac{\epsilon}{2} \dot{g}_k \sin 2\varphi_k,$$

$$\varphi_k := (-)^k \varpi_k t + \alpha_k \tag{42}$$

For $g_i(t)$ arbitrary and $\bar{I}_k$ the values of the action variables for the unperturbed system, the simple shape

$$I_k = \bar{I}_k(1 - \epsilon\, g_k) \tag{43}$$

solves Eq. (42) at the first order in ϵ when $\varphi_k = n\,\pi, n \in \mathbb{Z}$. It may therefore be used to estimate the trend of the modifications induced by g_i on the action variables. In the special case $g_i(t) = \bar{g}\, \sin(\Omega_i t + \beta_i)$, the solution of (42) up to first order in ϵ is given by

$$I_k = \bar{I}_k \left[1 - \frac{\epsilon}{2} \bar{g} \left(\frac{\Omega_k}{2\omega_k^+} \sin 2(\omega_k^+ t + \alpha_k^+) \right. \right.$$
$$\left. \left. + \frac{\Omega_k}{2\omega_k^-} \sin 2(\omega_k^- t + \alpha_k^-) \right) \right],$$

$$\phi_k = \alpha_k + (-)^k \varpi_k t + (-)^{k+1} \epsilon\, \frac{\varpi_k}{\Omega_k} \bar{g}\, \cos(\Omega_k t + \beta_k)$$
$$+ (-)^{k+1} \frac{\epsilon}{4} \bar{g} \left(\frac{\Omega_k}{2\omega_k^+} \cos 2(\omega_k^+ t + \alpha_k^+) \right.$$
$$\left. + \frac{\Omega_k}{2\omega_k^-} \cos 2(\omega_k^- t + \alpha_k^-) \right),$$

$$\omega_k^\pm := \varpi_k \pm \frac{(-)^k}{2} \Omega_k, \quad \alpha_k^\pm := (-)^k (\alpha_k \pm \frac{\beta_k}{2}). \tag{44}$$

In the case where the perturbations $g_k(t)$ are slow, as is prescribed by the theory of adiabatic invariants, one has that the frequencies Ω_k are small compared to the characteristic frequencies ϖ_k of the system and one finds that the variation of the action variables starts at second order, while the variation of the angle variables starts earlier.

4.2 Nekhoroshev estimates

The P–U Hamiltonian (37) with the time-dependent perturbation (41) can be formally written under the form

$$H = h(\boldsymbol{I}) + \epsilon f(\boldsymbol{I}, \boldsymbol{\phi}), \tag{45}$$

where

$$h(\boldsymbol{I}) = \boldsymbol{\varpi} \cdot \boldsymbol{I}, \quad f(\boldsymbol{I}, \boldsymbol{\phi}) = \boldsymbol{I} \cdot \boldsymbol{g}(\boldsymbol{\phi}), \tag{46}$$

and where the vectors $(\boldsymbol{I})_k = I_k$, $(\boldsymbol{\phi})_k = \phi_k$, $(\boldsymbol{g})_k = g_k$ and $(\boldsymbol{\varpi})_k = (-)^k \varpi_k$ have been introduced.

Nekhoroshev's theorem [32,33] states that if the nearly integrable Hamiltonian (45) is analytic and the unperturbed part $h(\boldsymbol{I})$ is steep (or convex, or quasiconvex) on some domain, then there is a threshold $\epsilon_0 > 0$ and positive constants R, T, a and b such that whenever $|\epsilon| < \epsilon_0$, for all initial actions variables $\boldsymbol{I}(0)$ in the domain (and far enough from the boundary), one has

$$\max\{|I_k(t) - I_k(0)|\} < R\,\epsilon^b \quad \text{for times} \quad |t| < T \exp \varepsilon^{-a}. \tag{47}$$

This result has been particularized to several explicit examples, among which the harmonic oscillator in Refs. [29,30]. The fact that some components of $\boldsymbol{\varpi}$ are negative is allowed by the formalism of the latter references. From the study [29] in particular, it can be deduced that unstable behaviours in the P–U oscillator appears at exponentially large time in ϵ, i.e. the dynamics is classically stable. Since $\epsilon_0 \sim N^{-2N}$ [29], the more the dynamics contains HD, the more the perturbation must be small to preserve stability. Moreover, there must exist two real positive constants σ, τ such that $|\boldsymbol{\varpi} \cdot \boldsymbol{n}| \geq \sigma \left[\sum_{j=0}^{N-1} |n_j| \right]^{-\tau}$ for all $\boldsymbol{n} \in \mathbb{Z}_0^N$ otherwise ϵ_0 becomes arbitrarily large and the system is unstable. In other words the frequencies must define a nonresonant harmonic oscillator. Such an instability can only occur for $N > 1$ one-dimensional dynamics; it is trivially avoided when $N = 1$ because no energy transfer between the different components E_i are de facto absent in this case.

5 Concluding comments

Higher-derivative action principles generally lead to unstable classical dynamics. However, all the classical trajectories allowed by the Pais–Uhlenbeck oscillator (10) with distinct and nonresonant frequencies are bounded: it is an explicit realisation of a stable classical theory with higher-derivatives. Therefore the problem can be formulated in action-angle variables formalism, allowing a computation of adiabatic invariants and a proof of the classical stability based on Nekhoroshev estimates. Although the Pais–

Uhlenbeck oscillator has been widely studied as a prototypal higher-derivative physical theory, it is the first time, to our knowledge, that such results are obtained. Emphasis has been put on harmonic potentials in the present study. Other types of potentials or higher-derivative terms may also lead to stable classical dynamics. For example it is shown in Ref. [31] that a $N = 2$ Pais–Uhlenbeck oscillator with cubic and quartic potential terms is stable too, except for very special values of the parameters. The stable models of [31] should give, after generalisation to field theory Lagrangians, further examples (compared to the one reviewed in [10]) of well-behaved dynamical systems with infinite number of degrees of freedom.

One can recover a Lagrangian function from a Hamiltonian function provided that the regularity condition $\det \frac{\partial^2 H}{\partial p_{N-1} \partial p_{N-1}} \neq 0$ is satisfied – which is the case for a free system that is regular when the added interactions in the perturbed Hamiltonian are algebraic functions in the P–U variables Q's, as we assumed implicitly here. This is clear for N odd since the latter variables do not imply Ostrogradski's last momentum p_{N-1}. In the even N cases where the Q variables involve p_{N-1}, the added perturbations cannot ruin the regularity property of the unperturbed P–U Hamiltonian as long as the perturbation is of polynomiality degree in Q higher than two, as we assume here. However, it is notoriously difficult (see e.g. Section 8 of [10]) to find the analytical expression of the perturbed Lagrangian corresponding to a given Hamiltonian perturbation. It it usually nonlocal, even for simple (e.g. quartic) Hamiltonian perturbation [10], which suggests considering as a starting point the $N \to \infty$ limit studied by Pais and Uhlenbeck, as it would encompass all the possible Hamiltonian perturbations.

It is worth making comments about quantization. In a first approximation a Bohr–Sommerfeld quantization rule can be applied since action variables exist. The fact that the energy spectrum is unbounded both from below and above in higher-derivative theories does not a priori forbids well-behaved quantum dynamics. In fact, a quantization technique was proposed in [6] that keeps the higher-derivative dynamics stable at the quantum level. In fact, we propose that the positive-definite quantities suggested in [6] and that are responsible for a stable dynamics at the quantum level are nothing but the action variables. Indeed, even for a perturbatively perturbed classical motion, as long as the trajectories are bounded, the action variables are positive-definite quantities and conserved to the approximation given. More recently, it was conjectured in [10] that indeed, when all the classical trajectories of a given higher-derivative model are bounded, its quantum dynamics only contains so-called benign ghosts, i.e. negative-energy quantum states with a normalisable wave function and preserved unitarity of the evolution. The present work aimed at clarifying the conditions for higher-derivative models to exhibit bounded classical dynamics; hence it is a step toward the identification of quantum models with unitary quantum dynamics, to which the Pais–Uhlenbeck oscillator belongs. Further issues about quantization of higher-derivative Lagrangians are discussed for example in [48–50].

Finally it has to be noticed that the necessary conditions for adiabatic invariants and Nekhoroshev estimates to be computed are the separability of the higher-derivative Hamiltonian and the existence of bounded classical trajectories. Both conditions are met in the Pais–Uhlenbeck oscillator case after appropriate choice of canonical variables, but we believe that other classes of higher-derivative systems may be studied by resorting to the methods we have presented.

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