



Einstein–Gauss–Bonnet black holes in de Sitter spacetime and the quasilocal formalism

Yves Brihaye^{a,*}, Eugen Radu^b

^a Physique-Mathématique, Université de Mons-Hainaut, Mons, Belgium

^b Institut für Physik, Universität Oldenburg, Postfach 2503, D-26111 Oldenburg, Germany

ARTICLE INFO

Article history:

Received 8 January 2009

Received in revised form 7 May 2009

Accepted 5 June 2009

Available online 10 June 2009

Editor: A. Ringwald

ABSTRACT

We propose to compute the action and global charges of the asymptotically de Sitter solutions in Einstein–Gauss–Bonnet theory by using the counterterm method in conjunction with the quasilocal formalism. The general expression of the counterterms and the boundary stress tensor is presented for spacetimes of dimension $d \leq 7$. We apply this technique for several different solutions in Einstein–Gauss–Bonnet theory with a positive cosmological constant. Apart from known solutions, we consider also $d = 5$ vacuum rotating black holes with equal magnitude angular momenta. These solutions are constructed numerically within a nonperturbative approach, by directly solving the Einstein–Gauss–Bonnet equations with suitable boundary conditions.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

One of the most fruitful approaches in computing conserved quantities in general relativity is to employ the quasilocal formalism [1]. The basic idea here is to enclose a given region of spacetime with some surface, and to compute all relevant (conserved and/or thermodynamic) quantities with respect to that surface. For a spacetime that is either asymptotically anti-de Sitter (AdS) [2–4] or asymptotically flat [5–7] it is possible to extend the quasilocal surface to spatial infinity without difficulty, provided one incorporates appropriate boundary terms in the action to remove divergences. The boundary terms are built up with curvature invariants of the boundary metric and thus obviously they do not alter the bulk equations of motion. Therefore, this approach has the nice feature that it is not necessary to embed the boundary geometry in a reference background (see also [9–11] for other applications of this formalism for a different asymptotic structure of spacetime).

The situation is much more involved for asymptotically de Sitter (dS) spacetimes, because of the absence of spatial infinity and a globally timelike Killing vector in this case. In the prescription proposed in [8], these obstacles are avoided by computing the quasilocal tensor of Brown and York [1] (augmented by suitable boundary counterterms), on the Euclidean surfaces at future/past timelike infinity \mathcal{I}^\pm . This allows also a discussion of the thermodynamics of the asymptotically dS solutions outside the cosmological horizon, the boundary counterterms regularising the (tree-level) gravitation action as well. The efficiency of this approach has been demon-

strated in a broad range of examples, including configurations with gravitating matter fields [12–15].

The results in [8–15] concern the case of Einstein gravity with a positive cosmological constant. However, for a spacetime dimension $d > 4$, the Einstein gravity presents a natural generalisation – the so-called Lovelock theory, constructed from vielbein, the spin connection and their exterior derivatives without using the Hodge dual, such that the field equations are second order [16,17]. Following the Ricci scalar, the next order term in the Lovelock hierarchy is the Gauss–Bonnet (GB) one, which contains quadratic powers of the curvature. As discussed in the literature, this term appears as the first curvature stringy correction to general relativity [18,19], when assuming that the tension of a string is large as compared to the energy scale of other variables.

In principle, there are no obstacles in computing the action and global charges of EGB solutions in dS spacetime by using a quasilocal formalism similar to that proposed in [8] for the Einstein gravity. At any given dimension one can write down only a finite number of counterterms that do not vanish at future/past timelike infinity. This feature does not depend upon the bulk theory is Einstein or GB. However, the presence in this case of a new length scale (the GB coupling constant) implies a complicated expression for the coefficients of the boundary counterterms and makes the procedure technically much more involved.

The corresponding problem for an asymptotically AdS spacetime has been discussed in the recent paper [20] (see also [21]). The main purpose of this work is to generalize the boundary counterterms and the quasilocal stress energy tensor there to a positive value of the cosmological constant and thus to extend the prescription in [8,12] to the case of EGB theory. Our results are valid for configurations with $d \leq 7$, although a general counterterm ex-

* Corresponding author.

E-mail address: Yves.Brihaye@umh.ac.be (Y. Brihaye).

pression is also conjectured. In the second part of this Letter we apply this general formalism to several different asymptotically dS black holes. Apart from known static solutions, we consider also rotating black holes with two equal magnitude angular momenta in EGB theory formulated in five spacetime dimensions. These solutions are constructed numerically within a nonperturbative approach, by directly solving the EGB equations with suitable boundary conditions. They also provide a nontrivial generalization in EGB theory of a particular class of the known Myers–Perry–dS₅ black holes [22].

Most of the notation and sign conventions used in this Letter are similar to those in Ref. [12].

2. The general formalism

2.1. The action and field equations

We consider the EGB model with a positive cosmological constant $\Lambda = (d-2)(d-1)/2\ell^2$, coupled with some matter fields with a Lagrangian density \mathcal{L}_M

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}} d^d x \sqrt{-g} \left(R - 2\Lambda + \frac{\alpha}{4} L_{GB} + \mathcal{L}_M \right), \quad (2.1)$$

where

$$L_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}, \quad (2.2)$$

is the GB term, while R , $R_{\mu\nu}$ and $R_{\mu\nu\sigma\tau}$ are the Ricci scalar, the Ricci tensor and the Riemann tensor associated with the bulk metric $g_{\mu\nu}$. For $d=4$, L_{GB} is a topological invariant and thus does not contribute to the equations of motion; in higher dimensions it is the most general quadratic expression which preserves the property that the equations of motion involve only second order derivatives of the metric. The constant α in (2.1) is the GB coefficient with dimension $(length)^2$ and is positive in the string theory. We shall therefore restrict in this work to the case $\alpha > 0$, although the counterterm expression does not depend on this choice.

The variation of the action (2.1) with respect to the metric tensor results in the gravity equations of the model

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{\alpha}{4}H_{\mu\nu} = 2T_{\mu\nu}, \quad (2.3)$$

where

$$H_{\mu\nu} = 2(R_{\mu\sigma\kappa\tau}R_{\nu}^{\sigma\kappa\tau} - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2R_{\mu\sigma}R^{\sigma}_{\nu} + RR_{\mu\nu}) - \frac{1}{2}L_{GB}g_{\mu\nu}, \quad (2.4)$$

and $T_{\mu\nu}$ is the energy-momentum tensor of the matter fields.

For a well-defined variational principle, one has to supplement the action (2.1) with the Gibbons–Hawking surface term [23]

$$I_b^{(E)} = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}^+} d^{d-1}x \sqrt{\gamma} K, \quad (2.5)$$

and its counterpart for the GB gravity [19]

$$I_b^{(GB)} = -\frac{\alpha}{16\pi G} \int_{\partial\mathcal{M}^+} d^{d-1}x \sqrt{\gamma} (sJ - 2G_{ab}K^{ab}), \quad (2.6)$$

where γ_{ab} is the induced metric on the boundary with the outward-pointing normal vector n^a and K is the trace of the extrinsic curvature K_{ab} of the boundary. The factor s in (2.6) is $s=+1$ for a spacelike normal vector n^a and $s=-1$ for a timelike normal vector (the case considered here), see e.g. [24]. Other

quantities in (2.6) are G_{ab} – the Einstein tensor of the metric γ_{ab} and J – the trace of the tensor

$$J_{ab} = \frac{1}{3}(2K K_{ac}K_b^c + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2 K_{ab}). \quad (2.7)$$

Also, the case of interest in this Letter corresponds to a spatial boundary future/past timelike infinity.¹ Therefore $\partial\mathcal{M}^\pm$ are spatial Euclidean boundaries at early and late times, while $\int_{\partial\mathcal{M}^+} d^{d-1}x$ indicates an integral over the late time boundary minus an integral over the early time boundary. In what follows, to simplify the picture, we will consider the \mathcal{I}^+ boundary only, dropping the \pm indices (similar results hold for \mathcal{I}^-).

Eqs. (2.3) present many interesting solutions possessing new features as compared to the pure Einstein gravity case (for a review, see the recent work [27]). In this section we shall consider the issue of computing the action and global charges of the EGB solutions in dS spacetime, several examples of such configurations being discussed in the next sections. Here we have found convenient to write²

$$\ell_c = \ell \sqrt{\frac{1+U}{2}}, \quad \text{with } U = \sqrt{\frac{\alpha(d-3)(d-4)}{\ell^2}} + 1, \quad (2.8)$$

which results in a compact form for the expression below; physically, ℓ_c corresponds to an effective dS length scale in EGB theory.

2.2. The counterterms and the boundary stress tensor

The GB term in (2.1) does not change the general formalism to compute the conserved charges and the action of asymptotically dS solutions developed in [8,12]. Therefore, we only recapitulate the basic steps here, emphasizing the new features which emerge for $\alpha \neq 0$.

In general, the action (2.1) (together with the boundary terms (2.5), (2.6)) is divergent when evaluated on a solution to the equations of motion (2.3). In the counterterm approach, the remedy is to supplement the initial action (2.1) by a boundary counterterm part I_{ct} depending only on geometric invariants of the boundary metric (therefore the bulk equations of motion remain the same). I_{ct} regularizes the tree gravitational action I_{cl} and the boundary stress tensor. Crucial to the success of the counterterm prescription is that the divergencies are universal, so that a single choice of the counterterms suffices to render finite the action of all asymptotically dS solutions.

For $d < 8$ solutions, our proposal for the boundary counterterm action is³

$$I_{ct} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{\gamma} \left\{ -\left(\frac{d-2}{\ell_c}\right) \left(\frac{2+U}{3}\right) + \frac{\ell_c \Theta(d-4)}{2(d-3)} (2-U) R - \frac{\ell_c^3 \Theta(d-6)}{2(d-3)^2(d-5)} \left[U \left(R_{ab} R^{ab} - \frac{d-1}{4(d-2)} R^2 \right) \right] \right\}$$

¹ For the black hole solutions in this Letter, this corresponds to evaluate various quantities for some fixed radius larger than the radius of the cosmological horizon and then sending this radius to infinity.

² For the sake of simplicity, we have restricted ourselves to the case of solutions with a well defined Einstein gravity limit. However, the results in this section can easily be generalized to the branch of solutions diverging as $\alpha \rightarrow 0$, in which case several terms in (2.9) have an opposite sign.

³ Note that in odd spacetime dimensions, for some boundary geometries, there is an additional logarithmic divergence that cannot be cancelled without including an explicit cutoff dependence in the counterterm action, which should be supplemented with new extra terms. This feature occurs already for Einstein gravity theory, leading to a conformal anomaly similar to what has been obtained in the context of AdS spacetime [2]. However, this is not the case of the solutions discussed in next sections.

$$-\frac{d-3}{2(d-4)}(U-1)L_{\text{GB}}\Bigg\}, \quad (2.9)$$

where R , R^{ab} and L_{GB} are the curvature, the Ricci tensor and the GB term associated with the induced metric γ . Also, $\Theta(x)$ is the step-function with $\Theta(x) = 1$ provided $x \geq 0$, and zero otherwise. One can easily see that as $\alpha \rightarrow 0$ (thus $U \rightarrow 1$), the known counterterm expression in the Einstein gravity [8,12] is recovered.

Following Ref. [20], we conjecture the general form of I_{ct} in d spacetime dimensions:

$$I_{\text{ct}} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{\gamma} \left\{ \sum_{k \geq 1} \Theta(d-2k) \times (f_1(U)L_E + f_2(U)\mathcal{L}_{(k-1)}) \right\}, \quad (2.10)$$

where L_E is the corresponding k -th part of the counterterm Lagrangian for a theory with only Einstein gravity in the bulk (with the length scale ℓ in front of it replaced by the new effective dS radius ℓ_c) and $\mathcal{L}_{(k-1)}$ is the $(k-1)$ term in the Lovelock hierarchy. The functions $f_1(U)$, $f_2(U)$ are first order polynomials in U , whose expression can easily be derived from those given in [20] for $\Lambda < 0$. The series (2.10) truncates for any fixed dimension, with new terms entering at every new even value of d .

Once we know the expression of the boundary counterterm, the computation of the conserved charges is performed in a similar way to the $\alpha = 0$ limit [8,12]. The (Euclidean) boundary metric on equal time surfaces can be written, at least locally, in a ADM-like general form

$$ds^2 = \gamma_{ab} dx^a dx^b = N_\rho^2 d\rho^2 + \sigma_{ab}(d\psi^a + N^a d\rho)(d\psi^b + N^b d\rho), \quad (2.11)$$

where N_ρ and N^a are the lapse function and the shift vector respectively, while ψ^a are angular variables parametrizing a closed surfaces Σ . The physical significance of the coordinate ρ in (2.11) depends on the considered situation; e.g. for the black hole solutions discussed in the next sections, ρ is the coordinate associated with the asymptotic Killing vector that is timelike inside the static patch of dS, but spacelike at \mathcal{I}^\pm .

Varying the total action with respect to the boundary metric γ_{ab} results in the following boundary stress-energy tensor

$$T_{ab} = \frac{2}{\sqrt{\gamma}} \frac{\delta}{\delta\gamma^{ab}} (I + I_b^{(E)} + I_b^{(\text{GB})} + I_{\text{ct}}), \quad (2.12)$$

with the following expression valid for $d < 8$:

$$\begin{aligned} 8\pi G T_{ab} = & K_{ab} - \gamma_{ab} K + \frac{\alpha}{2} \left(Q_{ab} - \frac{1}{3} Q \gamma_{ab} \right) + \frac{d-2}{\ell_c} \gamma_{ab} \left(\frac{2+U}{3} \right) \\ & + \frac{\ell_c \Theta(d-4)}{d-3} (2-U) \left(R_{ab} - \frac{1}{2} \gamma_{ab} R \right) \\ & + \ell_c^3 \Theta(d-6) \left\{ -\frac{U}{(d-3)^2(d-5)} \left(-\frac{1}{2} \gamma_{ab} \left(R_{cd} R^{cd} \right. \right. \right. \\ & \left. \left. \left. - \frac{(d-1)}{4(d-2)} R^2 \right) - \frac{(d-1)}{2(d-2)} R R_{ab} + 2R^{cd} R_{cadb} \right. \right. \\ & \left. \left. - \frac{d-3}{2(d-2)} \nabla_a \nabla_b R + \nabla^2 R_{ab} - \frac{1}{2(d-2)} \gamma_{ab} \nabla^2 R \right) \right. \\ & \left. + \frac{U-1}{2(d-3)(d-4)(d-5)} H_{ab} \right\} + \dots \end{aligned} \quad (2.13)$$

where [24,25]

$$\begin{aligned} Q_{ab} = & s(2K K_{ac} K_b^c - 2K_{ac} K^{cd} K_{db} + K_{ab} (K_{cd} K^{cd} - K^2)) \\ & + 2K R_{ab} + R K_{ab} - 2K^{cd} R_{cadb} - 4R_{ac} K_b^c, \end{aligned} \quad (2.14)$$

and H_{ab} given by (2.4), this time written in terms of the boundary metric γ_{ab} , however. All terms in (2.13), except the first four, come from the variation of the counterterms in (2.9). The boundary stress-energy tensor T_{ab} measure the response of the spacetime to changes of the boundary metric and encodes the notion of conserved global charges.

Following [8,12], let us suppose that ξ^i is a Killing vector generating an isometry of the boundary geometry (2.11). Then it is straightforward to show that $T_{ij} \xi^j$ is divergenceless and one can define a conserved quantity Ω_ξ associated with ξ^i as follows

$$\Omega_\xi = \oint_\Sigma d^n \psi \sqrt{\sigma} n^i T_{ij} \xi^j, \quad (2.15)$$

where n^i is a unit vector normal on a surface of fixed ρ . The physical interpretation of this relation is the same for any theory of gravity: it means that a collection of observers, on the hypersurface with the induced metric σ_{ij} , would all measure the same value of Ω_ξ provided this surface has an isometry generated by ξ^i . As mentioned above, a dS spacetime has no globally timelike Killing vector, which makes difficult to define a mass for the solutions with this asymptotics. However, for all cases of interest (e.g. the black holes solutions in the next sections), there is a Killing vector that is timelike inside a static patch, while it is spacelike outside the cosmological horizon and therefore at \mathcal{I}^\pm . (Moreover, any spacetime that is asymptotically dS will have such an asymptotic symmetry generator.) The total mass/energy of solutions is evaluated with respect to this Killing vector.

Proceeding further, one can define a Hawking temperature T_H (e.g. by computing the corresponding surface gravity) and entropy S for the cosmological horizon by using the saddle point approximation to the gravitational partition function (namely the generating functional analytically continued to the Euclidean spacetime). In the semiclassical approximation, the dominant contribution to the path integral will come from the neighborhood of saddle points of the action, that is, of classical solution; the zeroth order contribution to $\log Z$ is given by $-I_{\text{cl}}$. A tree-level evaluation of the path integral with a GB term may be carried out along the lines described e.g. in Ref. [12] for the Einstein gravity case. Therefore, we find the entropy of the cosmological horizon (with $\beta = 1/T_H$)

$$S = \beta(E - \mu_i \mathfrak{C}_i) - I_{\text{cl}}, \quad (2.16)$$

which is found upon application of the Gibbs–Duhem relation to the partition function, with chemical potentials \mathfrak{C}_i and conserved charges μ_i , while E is the total mass/energy, evaluated according to (2.15). Also, all solutions should satisfy the first law of thermodynamics for the cosmological horizon

$$dS = \beta(dE - \mu_i d\mathfrak{C}_i), \quad (2.17)$$

which provides a test of the general formalism.

3. Applications: Known solutions

3.1. dS spacetime in EGB theory

As the simplest illustration of the above formalism, we consider the case of empty dS spacetime. This solution has a simple form in a large number of coordinate systems. For example, there is a static frame centered on each observer (timelike geodesic) in dS. Moreover, when a black hole exists, there is still a static frame centered about the black hole. Since different parametrizations emphasize different features, it is of interest to consider dS spacetime in alternative coordinate systems corresponding to different classes of observers.

Starting with an inflationary coordinate system, the dS solution reads

$$ds^2 = -dt^2 + e^{2t/\ell_c} d\vec{x}^2, \quad (3.1)$$

which solves the EGB equations (2.3) with $T_{\mu\nu} = 0$ (i.e. no matter fields). The properties of this solution are similar to the case of Einstein gravity (see e.g. [26]); the equal time surfaces here are flat, while t runs from $-\infty$ to $+\infty$. One can easily verify that the counterterms (2.9) removes all divergencies of the total action for $d \leq 7$, and leads to $I_{cl} = 0$. The total mass/energy of this solution is also vanishing, since $T_{ij} = 0$.

The situation is different for a static coordinate system, the corresponding line element being

$$ds^2 = \frac{dr^2}{F(r)} + r^2 d\Omega_{d-2}^2 - F(r) dt^2, \quad (3.2)$$

where (here we shall consider only the branch of solutions with a smooth Einstein gravity limit)

$$F(r) = 1 + \frac{2r^2}{\alpha(d-3)(d-4)} \left(1 - \sqrt{1 + \left(\alpha(d-3)(d-4) \frac{1}{\ell^2} \right)} \right) \\ = 1 - \frac{r^2}{\ell_c^2}.$$

This spacetime has a cosmological horizon at $r_c = \ell_c$ (where $F(r_c) = 0$), with an associated temperature $T_H^c = (2\pi\ell_c)^{-1}$. The topology of this solution for large constant $r > r_c$, is an Euclidean cylinder $R \times S^{d-2}$ and t is the coordinate along the cylinder. \mathcal{I}^\pm are located outside the future/past cosmological horizons, where r is timelike and t is spacelike. The relationship between the coordinate patches (3.1) and (3.2) and their Penrose diagrams are presented in Ref. [28].

The general formalism in Section 32 is applied working outside of the cosmological horizon, where $F(r) < 0$. The gravitational mass/energy is the charge associated with the Killing vector $\partial/\partial t$ – now spacelike outside the cosmological horizon. As expected, the total energy found by using the counterterm prescription vanishes for an even dimensional spacetime and has a nonzero value for an odd d :

$$M_0 = \frac{V_{d-2}}{8\pi G} \frac{(d-2)!!^2}{(d-1)!} \left(\frac{(d-2)U-2}{d-4} \right) \ell_c^{d-3} \delta_{2p+1,d}, \quad (3.3)$$

where V_{d-2} is the area of the unit S^{d-2} sphere and $p \geq 2$ is an integer. For solutions in Einstein gravity ($\alpha = 0$), M_0 is usually interpreted as the Casimir energy in the context of dS/CFT correspondence. Also, it reduces to the expression obtained in Ref. [12] when $U = 1$.

From (2.16) one finds the following expression for the entropy of dS spacetime in EGB theory:

$$S = \frac{V_{d-2}}{4G} \ell_c^{d-4} \left(\ell_c^2 + \frac{\alpha}{2} (d-2)(d-3) \right), \quad (3.4)$$

which in the limit of small α can be written in the simple form

$$S = S_0 + S_c \quad \text{with} \quad S_0 = \frac{\ell_c^{d-2} V_{d-2}}{4G}, \\ S_c = \alpha \frac{V_{d-2}}{4G} d(d-2)(d-3) \ell_c^{d-4}. \quad (3.5)$$

From the study of (3.1), (3.2) we conclude that, similar to the case of Einstein gravity, the horizon and entropy of the dS space in EGB theory have an obvious observer dependence.

3.2. Reissner–Nordström–dS–GB black hole

These solutions are found for a matter Lagrangian density $\mathcal{L}_M = -\mathcal{F}^2$, with the Maxwell field strength tensor $\mathcal{F} = dA$, where the (pure electric-) gauge potential is

$$A = A_t dt = \left(\sqrt{\frac{d-2}{2(d-3)}} \frac{Q}{r^{d-3}} + \Phi \right) dt, \quad (3.6)$$

where Φ is a constant. Working again in a static coordinate system, the line element is still given by (3.2), with a different expression for $F(r)$, however:

$$F(r) = 1 + \frac{2r^2}{\alpha(d-3)(d-4)} \\ \times \left(1 - \sqrt{1 + \alpha(d-3)(d-4) \left(\frac{M}{r^{d-1}} - \frac{Q^2}{r^{2(d-2)}} + \frac{1}{\ell^2} \right)} \right). \quad (3.7)$$

As argued below, M and Q in the above expression are constants proportional to the gravitational mass/energy E and the total electric charge \mathbf{Q} , respectively. The $Q = 0$ limit of this metric corresponds to the EGB generalization of the McVittie solution describing a Schwarzschild black hole embedded in dS spacetime [14].

A discussion of the solution (3.6), (3.7) appeared in Ref. [29] (see also Ref. [13] for an extended analysis of the limiting case $\alpha = 0$, including also multi-black hole generalizations). Here we briefly review its basic properties. One can easily verify that the metric has a curvature singularity at the origin $r = 0$. In general, the metric (3.2) presents Killing horizons at the radii where $F(r)$ vanishes. Of interest are the outer black hole horizon at $r = r_h$ and the cosmological horizon $r = r_c$ corresponding to the largest root of $F(r)$. The Hawking temperature associated to each of the horizons is $T_H^{h,c} = |F'(r_{h,c})|/(4\pi)$, where a prime denotes the derivative with respect to the radial coordinate. The two horizons are not in thermal equilibrium because the time periods in the Euclidean section required to avoid a conical singularity at both do not match in general. An extremal black hole is found by imposing $F(r_h) = F'(r_h) = 0$ which fixes M, Q as functions of ℓ, α and r_h (a similar relation is found when considering instead the cosmological horizon). The constant Φ in (3.6) is usually fixed such that $A_t(r_c) = 0$, and thus it corresponds to the electrostatic difference between the cosmological horizon and infinity.

The computation of the mass, action and entropy of a RNdS black hole is a direct application of the method described in the previous section. The gravitational mass/energy is the charge associated with the Killing vector $\partial/\partial t$. The total mass/energy found by using the counterterm prescription described in the previous section is

$$E = -\frac{V_{d-2}}{16\pi G} (d-2)M + M_0, \quad (3.8)$$

with M_0 the Casimir term given by (3.3). The negative sign implies that the black hole lowers the total bulk energy with respect to the total energy of the pure dS spacetime [8].

The computation of the total electric charge is similar to that performed in [13] for the case without a GB term. The results there show that the total electric charge evaluated at future/past infinity is

$$\mathbf{Q} = \frac{Q V_{d-2}}{8\pi G} \sqrt{2(d-3)(d-2)}. \quad (3.9)$$

From (2.16) one finds the entropy of the cosmological horizon (note that both S_0 and S_c have a nontrivial dependence on α):

$$S = S_0 + S_c \quad \text{with } S_0 = \frac{V_{d-2}}{4G} r_c^{d-2},$$

$$S_c = \alpha \frac{V_{d-2}}{4G} \frac{1}{2} r_c^{d-4} (d-2)(d-3). \quad (3.10)$$

One can easily verify that the first law of thermodynamics (2.17) also holds, with $\mu_i = \Phi$, $\mathfrak{C}_i = \mathbf{Q}$.

It would be interesting to study the properties of the Reissner–Nordström–de Sitter solution (3.6), (3.7) in the inflationary coordinate system (3.1). This problem has been considered in Ref. [13] in the absence of a GB term in the action. Interestingly, the same general picture have been found there for black holes in both coordinate systems (3.1) and (3.2), which shows the complex relation between different classes of observers in dS spacetime. For example, the mass of the black holes in an inflationary coordinate system is still given by (3.8), with $M_0 = 0$ however.⁴ We expect that a similar result will be found in the presence of a GB term. However, in the absence of an explicit form of the Reissner–Nordström–dS–GB black hole in the inflationary coordinate system,⁵ any progress in this direction appears to require a separate numerical study of these solutions.

4. Rotating EGB black holes with positive cosmological constant

4.1. The metric ansatz and known limits

The computation of the global charges and entropy of a rotating black holes in EGB theory represents another nontrivial application of the general formalism in Section 2. Unfortunately, no exact solutions are available in this case, and one has to solve numerically the field equations.

To simplify the general picture, we consider here the vacuum case in $d = 5$ dimensions only, although the inclusion of a U(1) field is straightforward in principle. A general spinning black hole solution is characterized in this case by two angular momenta and its mass/energy, and can be found by solving a set of seven partial differential equations. However, the numerical problem is greatly simplified by taking the *a priori* independent two angular momenta to be equal in order to factorize the angular dependence [31,32]. The asymptotic expressions and the explicit computation of the action and boundary stress tensor also simplifies drastically in this case.

To construct these solutions, we use the same metric ansatz employed in Ref. [33] for the corresponding problem with $\Lambda < 0$:

$$ds^2 = \frac{dr^2}{f(r)} + g(r) d\theta^2 + h(r) \sin^2 \theta (d\varphi - w(r) dt)^2$$

$$+ h(r) \cos^2 \theta (d\psi - w(r) dt)^2$$

$$+ (g(r) - h(r)) \sin^2 \theta \cos^2 \theta (d\varphi - d\psi)^2 - b(r) dt^2, \quad (4.1)$$

where $\theta \in [0, \pi/2]$, $(\varphi, \psi) \in [0, 2\pi]$, r and t being the radial and time coordinates. This ansatz has a residual degree of freedom which is fixed by taking $g(r) = r^2$.

The equations satisfied by the functions b, f, h, w result directly from (2.3). We refrain to write them because they are very long and not particularly enlightening. They present however two exact solutions which are important in what follows. The dS₅ generalization [22] of the Myers–Perry rotating black holes [34] with

equal magnitude angular momenta (hereafter MPdS₅) is found for $\alpha = 0$ (no GB term) and has

$$f(r) = 1 - \frac{r^2}{\ell^2} - \frac{2\hat{M}\mathcal{E}}{r^2} + \frac{2\hat{M}\hat{a}^2}{r^4},$$

$$h(r) = r^2 \left(1 + \frac{2\hat{M}\hat{a}^2}{r^4} \right), \quad w(r) = \frac{2\hat{M}\hat{a}}{r^2 h(r)},$$

$$g(r) = r^2, \quad b(r) = \frac{r^2 f(r)}{h(r)}, \quad (4.2)$$

where \hat{M} and \hat{a} are two constants related to the solution's mass and angular momentum, while $\mathcal{E} = 1 + \hat{a}^2/\ell^2$.

For $g(r) = h(r) = r^2$, $w(r) = 0$ and $f(r) = b(r) = 1 + r^2/\alpha(1 - \sqrt{1 + 2\alpha(M/r^4 + 1/\ell^2)})$, one recovers the Schwarzschild–dS₅ solution with a Gauss–Bonnet term. The slowly rotating generalisation of this solution⁶ is found for small values of the rotation parameter a , and reads

$$w(r) = \frac{2aU^2}{\ell_c^2(U-1)} \left(\sqrt{1 + \frac{2M\ell_c^2(U-1)}{r^4 U^2}} - 1 \right), \quad (4.3)$$

the other metric function remaining unchanged to this order in a .

4.2. Boundary conditions and global charges

We want the generic line element (4.1) to describe a nonsingular, asymptotically de Sitter spacetime outside a cosmological horizon located at $r = r_c > 0$, with $f(r_c) = 0$. Here $f(r_c) = 0$ is only a coordinate singularity. The regularity assumption implies that all curvature invariants at $r = r_c$ are finite. Outside the cosmological horizon r and t changes the character (*i.e.* r becomes a timelike coordinate for $r > r_c$). A nonsingular extension across this null surface can be found just as at the event horizon of a black hole. These configurations possess also an event horizon located at some intermediate value of the radial coordinate $0 < r_h < r_c$, all curvature invariants being also finite as $r \rightarrow r_h$.

Restricting to nonextremal solutions, the following expansion holds near the event horizon with the parameters f_1^h, b_1^h, w_h^h and h_h^h , where $(f_1^h, b_1^h, h_h^h) > 0$:

$$f(r) = f_1^h(r - r_h) + O(r - r_h)^2, \quad h(r) = h_h^h + O(r - r_h),$$

$$b(r) = b_1^h(r - r_h) + O(r - r_h)^2, \quad w(r) = w_h^h + O(r - r_h). \quad (4.4)$$

A similar expansion holds for cosmological horizon, the corresponding parameters there being f_1^c, b_1^c, h_h^c , and w_h^c .

Both the event and the cosmological horizon have their own surface gravity $\kappa^{h,c}$, the associated Hawking temperatures being

$$T_H^{h,c} = \frac{|\kappa^{h,c}|}{2\pi} = \frac{\sqrt{b_1^{h,c} f_1^{h,c}}}{4\pi}. \quad (4.5)$$

Another quantities of interest are the area $A_H^{h,c}$ of the black hole/cosmological horizon

$$A_H^{h,c} = \sqrt{h_h^{h,c}} (r_h^{h,c})^2 V_3, \quad (4.6)$$

where $V_3 = 2\pi^2$ denotes the area of the unit three-dimensional sphere.

⁴ This result has been interpreted in [13] as providing support for the putative dS/CFT correspondence, since the general features of the CFT dual to a black hole should not depend on the dS slicing choice.

⁵ The main obstacle here is the absence of a simple closed form expression of the Reissner–Nordström–GB (or even Schwarzschild–GB) black hole in an isotropic coordinate system for the $\Lambda = 0$ case. For $\alpha = 0$, this form of the solution is used to construct cosmological configurations by using the prescription in [30].

⁶ By using the results derived in [35] for $\Lambda < 0$, one can find a different set of closed form EGB asymptotically dS₄ rotating black hole solutions with only one non-vanishing angular momentum (where the rotation parameter appears as a small quantity), the effects of an U(1) field being also included. A different approximation of the rotating black hole solution of the $d = 5$ EGB equations with a cosmological constant have been presented in closed form in [36].

The Killing vector $\chi = \partial/\partial t + \Omega_\varphi \partial/\partial \varphi + \Omega_\psi \partial/\partial \psi$ is orthogonal to and null on both horizons. For the solutions within the ansatz (4.1), the event horizon's angular velocities are all equal, $\Omega_\psi^{h,c} = \Omega_\varphi^{h,c} = \Omega_H^{h,c} = w(r)|_{r=r_{h,c}}$.

A direct computation reveals that the solution admits at large r a power series expansion of the form:

$$\begin{aligned} f(r) &= 1 - \frac{r^2}{\ell_c^2} + \sum_{k \geq 1} f_{2k} \left(\frac{\ell_c}{r} \right)^{2k}, \\ b(r) &= 1 - \frac{r^2}{\ell_c^2} + \sum_{k \geq 1} b_{2k} \left(\frac{\ell_c}{r} \right)^{2k}, \\ h(r) &= r^2 \left(1 + \sum_{k \geq 1} h_{2k} \left(\frac{\ell_c}{r} \right)^{2k} \right), \\ w(r) &= \frac{1}{r} \sum_{k \geq 1} w_{2k+1} \left(\frac{\ell_c}{r} \right)^{2k+1}, \end{aligned} \quad (4.7)$$

where the coefficients f_{2k} , b_{2k} , h_{2k} , w_{2k+1} with $k > 1$ are determined by f_2 , b_2 and w_3 . Specifically, we find $f_4 = h_4 = b_2 - f_2$, $b_6 = (b_2(f_2 + b_2(U - 2)) + (3U - 2)w_3^2)/(2U)$, $w_7 = -(f_2 + U(2b_2 - 3f_2))w_3/(2U)$, for the lowest order nonvanishing terms. Their expression becomes more complicated for higher k , with no general pattern becoming apparent.

The mass/energy⁷ E and angular momenta of these solutions evaluated at future/past timelike infinity by using the counterterm formalism are fixed by the constants f_2 , b_2 and w_3 , and read

$$E = \frac{V_3}{16\pi G} \ell_c^2 U (4b_2 - f_2), \quad J_\varphi = J_\psi = J = -\frac{V_3}{8\pi G} \ell_c^3 U w_3. \quad (4.8)$$

The entropy of these solutions associated with the cosmological horizon is found from the relation (2.17) with $\mu_i = \Omega_{\psi,\phi}^c$, $\mathcal{C} = J$:

$$\begin{aligned} S &= S_0 + S_{GB}, \quad \text{with } S_0 = \frac{A_H^c}{4G}, \\ S_{GB} &= \alpha \frac{V_3}{4G} \sqrt{h_h^c} \left(4 - \frac{h_h^c}{(r_h^c)^2} \right). \end{aligned} \quad (4.9)$$

4.3. The numerical method

Finding numerical solutions of a field theory model in a dS spacetime is a notoriously difficult task. Therefore, before describing the properties of the solutions, we shall give some details on the numerical methods we have used.⁸ The EGB field equations were solved by employing a collocation method for boundary-value ordinary differential equations, equipped with an adaptive mesh selection procedure [37]. Typical mesh sizes include 10^3 – 10^4 points. The solutions have a typical relative accuracy of 10^{-8} . In constructing rotating EGB–dS black holes, we make use of the existence of the MPdS₅ and Schwarzschild–GB–dS closed form solutions, and employ them as starting configurations, increasing gradually Ω_H^h or α , respectively.

However, when trying to find black hole solutions with $\Lambda > 0$ for $r \in [r_h, \infty]$ by imposing a regular horizon at $r = r_h$, one has

to tackle the technical difficulty that there also appear a cosmological horizon.⁹ That is, the metric functions f, b admit a zero at an intermediate value of the variable r , say at $r = r_c > r_h$. Of course, the value of r_c is not known *a priori* as a function of Λ, α . However, the condition of a regular horizon should be imposed both at $r = r_h$ and $r = r_c$. In our approach, we impose by hand the values of r_h, r_c and solve the equations first for $r \in [r_h, r_c]$ as a boundary value problem. At the same time, we compute the value of Λ corresponding to this cosmological horizon by using the fictitious equation $d\Lambda/dr = 0$. In a second step, we finally integrate the equations for $r \in [r_c, \infty]$ as an initial value problem with this value of the cosmological constant. This assures that the metric functions and their first and second derivatives are continuous at $r = r_c$.

In this approach, the set of boundary condition we imposed at r_h, r_c is

$$\begin{aligned} f &= 0, \quad b = 0, \quad b' = 1, \quad G(g_{ij}, g'_{ij}) = 0, \\ w &= w_h \quad \text{at } r = r_h, \end{aligned} \quad (4.10)$$

$$\text{and } f = 0, \quad b = 0, \quad G(g_{ij}, g'_{ij}) = 0 \quad \text{at } r = r_c, \quad (4.11)$$

where $G(g_{ij}, g'_{ij})$ is a complicated expression in terms of the metric function and their first derivatives which occurs from the condition for a regular horizon. In the above expression, the arbitrary rescaling of time is used to set $b'(r_h) = 1$, keeping in mind that the function $b(r)$, $w(r)$ have to be renormalized at the end of the second step according to

$$b(r) \rightarrow \tilde{b}(r) = b(r)\mu^2, \quad w(r) \rightarrow \tilde{w}(r) = w(r)\mu, \quad (4.12)$$

where the constant μ is chosen in such a way that the $\tilde{b}(r)$ approaches the asymptotic (4.7).

One disadvantage of this method is that the solutions cannot be studied systematically for fixed Λ . For the same reason, we have found difficult to study families of solutions obtained by varying α while Ω_H^h is fixed.

To summarise, in our approach the input parameters are the black hole event horizon radius r_h , the cosmological horizon radius r_c , the black hole event horizon velocity Ω_H^h and the GB coupling parameter α . The value of the cosmological constant, the metric functions and their derivatives at $r = r_c$ and the global charges emerge from the numerical output.

4.4. Numerical results

A systematic study of the properties of these rotating black holes appears to be a difficult task and is beyond the purposes of this work. In practice we have solved the equations numerically for several values of r_h, r_c and Ω_H^h and a range of the Gauss–Bonnet coupling constant α .

When increasing from zero the angular velocity Ω_H^h , we have found numerical evidence for the existence of nontrivial generalizations of any static Schwarzschild–GBdS configuration we considered; the shape of the metric function $w(r)$ we found for small values of Ω_H^h is in good agreement with (4.3). We reach the same conclusion when considering instead GB counterparts of the Einstein gravity rotating solution (4.2), by slowly increasing the parameter α . As a general remark, the qualitative features of all solutions we have constructed are rather similar to the MPdS₅ case. For $\alpha > 0$, we have noticed only quantitative difference in the values on the cosmological horizon and at infinity, for a number of parameters of interest.

In order to limit the amount of numerical investigation, we have studied in details mainly the case $r_h = 1$, $r_c = 3$. For the

⁷ In the expression of E , we have subtracted the Casimir energy of the pure dS₅ space as given by (3.3).

⁸ To our knowledge, this is the first attempt in the literature to numerically construct EGB rotating solutions in a dS background. The approach and the numerical methods here are quite different from those employed e.g. in [33] for rotating EGB solutions with AdS asymptotics or for rotating Einstein–Maxwell black hole solutions [31,32].

⁹ We do not consider in this work the behaviour of solutions inside the black hole event horizon $r < r_h$.

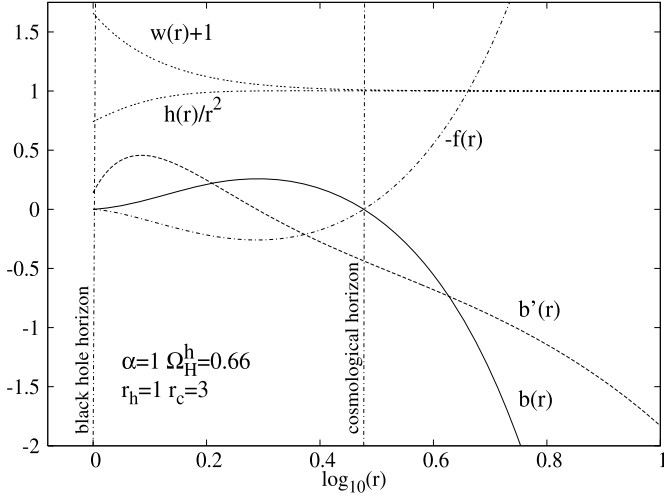


Fig. 1. The profiles for a generic $d=5$ rotating black hole solution in EGB theory with positive cosmological constant.

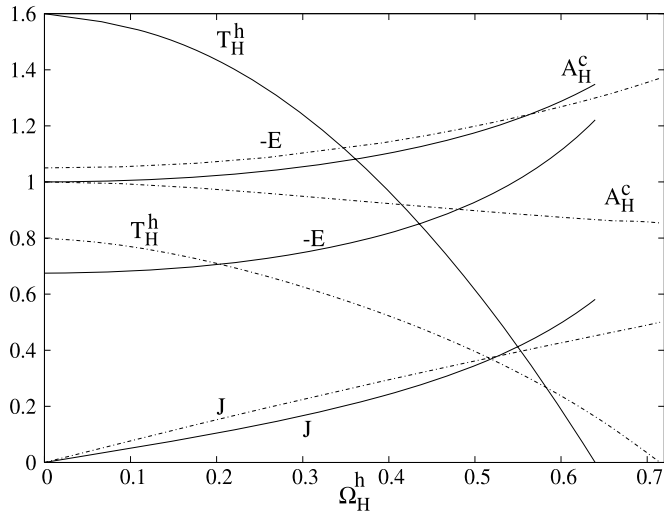


Fig. 2. Several parameters are compared for the Myers-Perry-dS₅ (dotted curves) and Einstein-Gauss-Bonnet (continuous line) rotating black hole solutions. The input parameters here were $r_h = 1$, $r_c = 3$, $\alpha = 0.5$ and $\ell^2 = 10$.

non-rotating limit, this corresponds to $\ell^2 = 10$ and $-(8\pi G)E/V_3 = 3M/4 = 3(5\alpha + 9)/40$, $4\pi T_H^h = 8/(5(\alpha + 1))$.

The profiles of the metric functions of a typical EGB-dS rotating black hole solution corresponding to $\alpha = 1$, $\Omega_H^h = 0.66$, $\ell \simeq 3.2$ are presented on Fig. 1. One can see that the rotation leads to nonconstant values for $h(r)/r^2$ and $b(r) \neq f(r)$, especially in the region close to the black hole horizon. Also, the metric functions and their derivatives are continuous at the cosmological horizon (although to simplify the plot we presented there only the profile of $b'(r)$).

Several parameters characterizing the solutions are represented in Fig. 2 as a function of the angular velocity at the black hole horizon. The data corresponding to MPdS₅ solution is represented by the dotted lines and results from analytical calculations. In contrast, the curves corresponding to the EGB theory are represented by continuous lines and result from our numerical calculation. (The energy E and angular momentum J are represented in the units of $V_3/(4\pi)$. The Hawking temperature is represented in units $1/(4\pi)$ and horizon area A_H in units V_3 , while we have set also $G = 1$ in all data.) Along with the case of the MPdS₅ solutions, the EGB

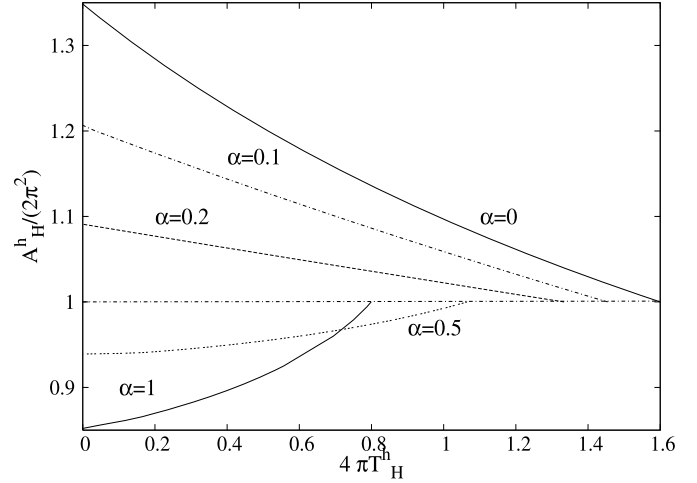


Fig. 3. The black hole event horizon area is plotted as function of the black hole temperature for several values of the GB coupling constant α . These solutions have $r_h = 1$, $r_c = 3$ and $\ell^2 = 10$.

black holes exist up to a maximal value of $\Omega_H^h = \Omega_{\max}^h$. For $\alpha = 0$ one finds

$$\Omega_{H(\max)}^h = \sqrt{\frac{2}{r_c^2 + 2r_h^2} \frac{r_c r_h^3 (r_c^2 + r_h^2)}{r_c^2 r_h^4 + r_c^2 + 2r_h^2}}, \quad \frac{1}{\ell^2} = \frac{1}{r_c^2 + 2r_h^2}. \quad (4.13)$$

For the cases we have investigated, when fixing the values of r_h, r_c , this maximal value gets larger when the Gauss-Bonnet coupling constant increases. In this limit, the solution approaches an extremal black hole, i.e. the functions $f(r), b(r)$ reach a double root at the black hole event horizon. However, these function still present a simple zero at the cosmological horizon. For the values of the parameters adopted here we find e.g. $\Omega_H^h \approx 0.64$ for $\alpha = 0$ and $\Omega_H^h \approx 0.72$ for $\alpha = 1$.

For the Einstein gravity black holes, the event horizon area A_H^h increases with increasing Ω_H^h while the Hawking temperature T_H^h decreases. The entropy $S(T_H)$ turns out to be a decreasing function of the temperature, for fixed event horizon radius. We have found that when the GB parameter α is large enough, the scenario is completely different. For instance, both T_H^h and A_H^h decreases while Ω_H^h increases. The entropy is an increasing function of T_H^h . In Fig. 3 we show this behaviour for several values of α . There the parameter varying on the $A_H^h(T_H^h)$ curves is Ω_H^h .

As with other rotating black holes, these solutions present also an ergoregion inside of which the observers cannot remain stationary, and will move in the direction of rotation. The ergoregion is the region bounded by the black hole event horizon, located at $r = r_h$ and the stationary limit surface, or the ergosurface, with $r = r_E < r_c$. The Killing vector $\partial/\partial t$ becomes null on the ergosurface, i.e. $g_{tt} = -b(r_E) + r_E^2 w(r_E)^2 = 0$. For the ansatz (4.1), the ergosurface does not intersect the horizon. We observe indeed that, for rotating solution with fixed Ω_H^h , the value r_E decreases slightly and get closer to r_h when α increases. For example, with $\Omega_H^h = 0.66$, we get $r_E/r_h \approx 1.34$ and $r_E/r_h \approx 1.27$ respectively for $\alpha = 0.1$ and $\alpha = 1$. In principle, there is also a second value of r , located outside the cosmological horizon, where the Killing vector $\partial/\partial t$ becomes null again. However, for all solutions we could construct, the metric component g_{tt} there is dominated by $b(r)$ and thus the “cosmological” ergo-radius hardly differs from $r = r_c$.

Another qualitative difference between Einstein and EGB black holes resides in the magnitude of the ratio $\rho \equiv f'/b'|_{r=r_h}$. For instance, for $\alpha = 0$, we have $\rho < 1$ for all values of the angular momentum. However, when α got sufficiently large, one can find solutions with $\rho > 1$.

5. Further remarks

In this work we have presented the boundary counterterm that removes the divergences of the action and conserved quantities of the solutions in EGB theory with a positive cosmological constant for a spacetime dimension $d \leq 7$. Similar to the case of Einstein gravity, the counterterm is built up with curvature invariants of the boundary metric. Their coefficients, however, present an explicit dependence of the dimensionless factor $\alpha^2 \Lambda$.

Here one should say that the expression of the counterterm proposed in this Letter was obtained by demanding cancellation of divergencies for a number of solutions in EGB theory, which was also the approach used in initial work on the boundary counterterm in Einstein gravity [2,3]. However, for asymptotically dS solutions in the Einstein gravity, there exist an algorithmic procedure for constructing I_{ct} in a rigorous way, and so its determination is unique for $\alpha = 0$ [12]. This procedure involves solving the Einstein equations (written in Gauss–Codacci form) in terms of the extrinsic curvature functional of the boundary and its derivatives to isolate the divergent parts. All divergent contributions are independent of the boundary normal and can be expressed in terms of intrinsic boundary data. In principle, this approach can be extended to asymptotically dS solutions in EGB theory, the only obstacle we can see at this stage being the huge complexity of the required computation. A more promising direction would be to look for the expression of I_{ct} in the linear order in α , by generalising the work in [21] to the $\Lambda > 0$ case.

For asymptotically AdS solutions, an alternative regularization prescription for any Lovelock theory has been proposed in [38]. This approach uses boundary terms with explicit dependence on the extrinsic curvature K_{ab} , also known as counterterms. It would be interesting to generalize the approach in [38] to dS asymptotics and to compare the results with those found here.

In the second part of this work, the general formalism has been applied for several different asymptotically dS black hole solutions in EGB theory. Apart from several known solutions, we have considered also rotating black holes with two equal-magnitude angular momenta in $d = 4 + 1$ EGB theory with a positive cosmological constant. Although the numerical difficulties associated with the existence of a cosmological horizon prevented us from a systematic study of the parameter space, we have presented arguments for the existence of nontrivial generalization in EGB theory of a particular class of the known MPdS₅ black holes.

As avenue for future research, it would be interesting to consider the status of “the maximal mass conjecture” in EGB theory, by using the mass definition proposed in this work. Formulated in [8] for Einstein gravity, this conjecture states that any asymptotically dS spacetime cannot have a mass larger than the pure dS case without inducing a cosmological singularity. Here we mention only the fact that all rotating black holes we have constructed in Section 4 satisfy this conjecture.

The conserved charges of the rotating solutions in this Letter have been evaluated on a Euclidean surface at future timelike infinity. In principle, by using the results in Section 2, a similar computation can be performed for a spatially finite boundary inside the cosmological event horizon. The corresponding problem for Kerr–dS rotating black holes in Einstein gravity has been considered in Ref. [39]. The results in that work show that quasilocal angular momentum is independent on the radius of the boundary, which does not hold for the total mass of the solutions.

The relevance of the results discussed in this Letter in a dS/CFT context is another interesting open question. For the AdS/CFT case, the higher derivatives curvature terms can be viewed as the corrections of large N expansion of the boundary CFT in the strong coupling limit, see e.g. [40]. For the asymptotically dS case, any

progress in this direction is likely to require first a better understanding of the conjectured dS/CFT correspondence [41] with $\alpha = 0$.

Acknowledgements

Y.B. thanks the Belgian FNRS for financial support. The work of E.R. was supported by a fellowship from the Alexander von Humboldt Foundation.

References

- [1] J.D. Brown, J.W. York, Phys. Rev. D 47 (1993) 1407, arXiv:gr-qc/9209012.
- [2] V. Balasubramanian, P. Kraus, Commun. Math. Phys. 208 (1999) 413, arXiv:hep-th/9902121.
- [3] R. Emparan, C.V. Johnson, R.C. Myers, Phys. Rev. D 60 (1999) 104001, arXiv:hep-th/9903238.
- [4] M. Henningson, K. Skenderis, JHEP 9807 (1998) 023, arXiv:hep-th/9806087.
- [5] P. Kraus, F. Larsen, R. Siebelink, Nucl. Phys. B 563 (1999) 259, arXiv:hep-th/9906127.
- [6] R.B. Mann, D. Marolf, Class. Quantum Grav. 23 (2006) 2927, arXiv:hep-th/0511096.
- [7] D. Astefanesei, E. Radu, Phys. Rev. D 73 (2006) 044014, arXiv:hep-th/0509144; D. Astefanesei, R.B. Mann, C. Stelea, Phys. Rev. D 75 (2007) 024007, arXiv:hep-th/0608037; R.B. Mann, D. Marolf, R. McNees, A. Virmani, Class. Quantum Grav. 25 (2008) 22, arXiv:0804.2079 [hep-th]; R.B. Mann, D. Marolf, A. Virmani, Class. Quantum Grav. 23 (2006) 6357, arXiv:gr-qc/0607041.
- [8] V. Balasubramanian, J. de Boer, D. Minic, Phys. Rev. D 65 (2002) 123508, arXiv:hep-th/0110108.
- [9] R.G. Cai, N. Ohta, Phys. Rev. D 62 (2000) 024006, arXiv:hep-th/9912013.
- [10] E. Radu, Mod. Phys. Lett. A 17 (2002) 2277, arXiv:gr-qc/0211035.
- [11] B. Kleihaus, J. Kunz, E. Radu, JHEP 0606 (2006) 016, arXiv:hep-th/0603119.
- [12] A.M. Ghezelbash, R.B. Mann, JHEP 0201 (2002) 005, arXiv:hep-th/0111217.
- [13] D. Astefanesei, R.B. Mann, E. Radu, JHEP 0401 (2004) 029, arXiv:hep-th/0310273.
- [14] T. Shiromizu, D. Ida, T. Torii, JHEP 0111 (2001) 010, arXiv:hep-th/0109057.
- [15] M.H. Dehghani, H. KhajehAzad, Can. J. Phys. 81 (2003) 1363, arXiv:hep-th/0209203; A.M. Ghezelbash, D. Ida, R.B. Mann, T. Shiromizu, Phys. Lett. B 535 (2002) 315, arXiv:hep-th/0201004; R. Clarkson, A.M. Ghezelbash, R.B. Mann, Phys. Rev. Lett. 91 (2003) 061301, arXiv:hep-th/0304097; A.M. Ghezelbash, R.B. Mann, Phys. Rev. D 72 (2005) 064024, arXiv:hep-th/0412300; Y. Brihaye, B. Hartmann, E. Radu, Phys. Rev. D 74 (2006) 025009, arXiv:hep-th/0511305; Y. Brihaye, E. Radu, D.H. Tchrakian, Phys. Rev. D 75 (2007) 024022, arXiv:gr-qc/0610087.
- [16] D. Lovelock, J. Math. Phys. 12 (1971) 498.
- [17] A. Mardones, J. Zanelli, Class. Quantum Grav. 8 (1991) 1545.
- [18] D.J. Gross, E. Witten, Nucl. Phys. B 277 (1986) 1; R.R. Metsaev, A.A. Tseytlin, Phys. Lett. B 191 (1987) 354; C.G. Callan, R.C. Myers, M.J. Perry, Nucl. Phys. B 311 (1988) 673.
- [19] R.C. Myers, Phys. Rev. D 36 (1987) 392.
- [20] Y. Brihaye, E. Radu, JHEP 0809 (2008) 006, arXiv:0806.1396 [gr-qc].
- [21] J.T. Liu, W.A. Sabra, arXiv:0807.1256 [hep-th].
- [22] G.W. Gibbons, H. Lu, D.N. Page, C.N. Pope, Phys. Rev. Lett. 93 (2004) 171102.
- [23] G.W. Gibbons, S.W. Hawking, Phys. Rev. D 15 (1977) 2752.
- [24] E. Gravanis, S. Willison, Phys. Rev. D 75 (2007) 084025, arXiv:gr-qc/0701152.
- [25] S.C. Davis, Phys. Rev. D 67 (2003) 024030, arXiv:hep-th/0208205; E. Gravanis, S. Willison, Phys. Lett. B 562 (2003) 118, arXiv:hep-th/0209076.
- [26] D. Klemm, L. Vanzo, JCAP 0411 (2004) 006, arXiv:hep-th/0407255.
- [27] C. Charmousis, Lect. Notes Phys. 769 (2009) 299, arXiv:0805.0568 [gr-qc].
- [28] S.W. Hawking, G.F.R. Ellis, The Large Structure of Space–Time, Cambridge University Press, Cambridge, 1973. Chapter 5.
- [29] T. Torii, H. Maeda, Phys. Rev. D 71 (2005) 124002, arXiv:hep-th/0504127.
- [30] D. Kastor, J.H. Traschen, Phys. Rev. D 47 (1993) 5370, arXiv:hep-th/9212035.
- [31] J. Kunz, F. Navarro-Lerida, A.K. Petersen, Phys. Lett. B 614 (2005) 104, arXiv:gr-qc/0503010.
- [32] J. Kunz, F. Navarro-Lerida, J. Viebahn, Phys. Lett. B 639 (2006) 362, arXiv:hep-th/0605075; J. Kunz, F. Navarro-Lerida, Phys. Lett. B 643 (2006) 55, arXiv:hep-th/0610036; J. Kunz, F. Navarro-Lerida, E. Radu, Phys. Lett. B 649 (2007) 463, arXiv:gr-qc/0702086.
- [33] Y. Brihaye, E. Radu, Phys. Lett. B 661 (2008) 167, arXiv:0801.1021 [hep-th].

- [34] R.C. Myers, M.J. Perry, *Ann. Phys.* 172 (1986) 304.
- [35] H.C. Kim, R.G. Cai, *Phys. Rev. D* 77 (2008) 024045, arXiv:0711.0885 [hep-th].
- [36] S. Alexeyev, N. Popov, M. Startseva, A. Barrau, J. Grain, *J. Exp. Theor. Phys.* 106 (2008) 709, arXiv:0712.3546 [gr-qc].
- [37] U. Ascher, J. Christiansen, R.D. Russell, *Math. Comput.* 33 (1979) 659;
U. Ascher, J. Christiansen, R.D. Russell, *ACM Trans.* 7 (1981) 209.
- [38] G. Kofinas, R. Olea, *Phys. Rev. D* 74 (2006) 084035, arXiv:hep-th/0606253;
O. Miskovic, R. Olea, *JHEP* 0710 (2007) 028, arXiv:0706.4460 [hep-th];
G. Kofinas, R. Olea, *JHEP* 0711 (2007) 069, arXiv:0708.0782 [hep-th];
R. Olea, *JHEP* 0704 (2007) 073, arXiv:hep-th/0610230.
- [39] M.H. Dehghani, *Phys. Rev. D* 65 (2002) 104030, arXiv:hep-th/0201128.
- [40] A. Fayyazuddin, M. Spalinski, *Nucl. Phys. B* 535 (1998) 219, arXiv:hep-th/9805096;
O. Aharony, A. Fayyazuddin, J.M. Maldacena, *JHEP* 9807 (1998) 013, arXiv:hep-th/9806159;
S. Nojiri, S.D. Odintsov, *JHEP* 0007 (2000) 049, arXiv:hep-th/0006232.
- [41] C.M. Hull, *JHEP* 9807 (1998) 021, arXiv:hep-th/9806146;
C.M. Hull, *JHEP* 9811 (1998) 017, arXiv:hep-th/9807127;
V. Balasubramanian, P. Horava, D. Minic, *JHEP* 0105 (2001) 043, arXiv:hep-th/0103171;
E. Witten, arXiv:hep-th/0106109;
A. Strominger, *JHEP* 0110 (2001) 034, arXiv:hep-th/0106113.