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Tangent circle graphs and ‘orders’

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Abstract

Consider a horizontal line in the plane and let $\gamma(A)$ be a collection of n circles, possibly of different sizes all tangent to the line on the same side. We define the tangent circle graph associated to $\gamma(A)$ as the intersection graph of the circles. We also define an irreflexive and asymmetric binary relation P on A; the pair (a, b) representing two circles of $\gamma(A)$ is in P iff the circle associated to a lies to the right of the circle associated to b and does not intersect it. This defines a new nontransitive preference structure that generalizes the semi-order structure. We study its properties and relationships with other well-known order structures, provide a numerical representation and establish a sufficient condition implying that P is transitive. The tangent circle preference structure offers a geometric interpretation of a model of preference relations defined by means of a numerical representation with multiplicative threshold; this representation has appeared in several recently published papers.
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1. Introduction

There are many examples of pairs *order/graph* that have been studied in the literature. The pair *interval order/interval graph* is investigated extensively in [13]. Similarly, *semi-orders* and *indifference graphs* form another, famous, pair [16,18] (see also [17]), not to mention the pair *weak order/equivalence relation*. Such pairs have often been studied separately in the past and often by members of different communities, namely in the context of order theory and that of graph theory.

Preference structures as defined by Roubens and Vincke [20] provide an appropriate framework for a systematic study of complementary pairs order/graph.

A complete preference structure, according to [20], is a pair (P, I) of relations on a set A with P asymmetric, I, reflexive and symmetric, $P \cap I = \emptyset$ and $P \cup P^{-1} \cup I = A \times A$ (P^{-1} denotes the inverse of relation P). P and I are, respectively, interpreted as strict preference and indifference. When a preference is complete, knowing P is tantamount

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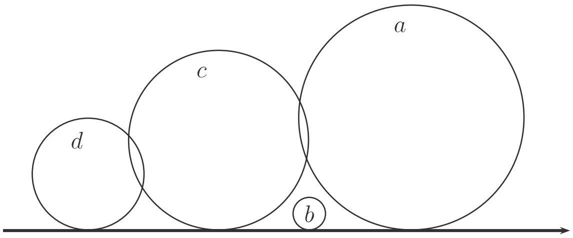


Fig. 1. A geometric representation of a tangent circle preference structure.

to knowing *I* since $I = (A \times A) \backslash (P \cup P^{-1})$. We suppose throughout this paper that the set *A* is finite. Usual preference structures include the cases in which:

- *P* is an interval order and *I* its associated interval graph;
- *P* is a semi-order and *I* its associated indifference graph;
- *P* is a weak order and *I* its associated equivalence relation;
- *P* is a complete order and $I = \{(a, a), a \in A\}$.

In that framework, it is quite natural to interpret the relation *I*, that is symmetric, as a nonoriented graph and the asymmetric relation *P* as an order, as soon as *P* is also transitive.

Many interesting pairs order/graph have been identified, using that framework (see [3]). In many cases, only the graph *I* had been studied beforehand. This is not surprising since, starting from a graph *I* that is a co-comparability graph (i.e. its complement can be transitively oriented [15, p. 149]), there is at least one partial order *P* such that (P, I) is a complete preference structure. Focussing on the pair (P, I), instead of studying only *I* (or *P*), can be fruitful even in case *I* determines *P* (i.e. the transitive orientation of the complement of *I* is essentially unique) since it often yields new characterizations, in terms of both *P* and *I*, of particular types of graphs *I* (see [1,4] for several examples of such characterizations).

As is implicit in Roubens and Vincke's definition of a preference structure, neither *I* nor *P* really need to be transitive. Historically, the first example of a preference structure that is not transitive is Luce's semi-order, in which indifference is not transitive [16] (see also [20,17]). Interval graphs also lack the transitivity property [11,12]. However, the corresponding asymmetric relations *P* are transitive, both in the case of the semi-order and the interval order.

Although nontransitive preferences may seem unnatural, intransitivity is observed in several decisional contexts; Fishburn [13] offers a survey of different cases and argues that intransitivity, even of strict preference, does not necessarily imply irrationality on behalf of the decision maker.

In this paper, we define and study a simple example of a complete preference structure, that we call the *tangent circle preference structure*, in which neither the symmetric nor the asymmetric part is, in general, transitive. Yet this structure is intuitively appealing since it admits a geometric interpretation that makes it look close to be an order although it is actually not transitive (it enjoys a weaker form of transitivity that we shall describe below).

2. Definitions and motivation

A tangent circle order is the abstract relational structure describing the ordering of a set of circles of various sizes all tangent to an horizontal, oriented, straight line drawn in the Euclidean plane. Without loss of generality, we shall assume throughout that the line is oriented from the left to the right and that all the circles lie above this axis (see Fig. 1). Sticking to the framework proposed in [20], we define a *tangent circle preference structure*(TCPS) as composed of an asymmetric binary relation *P*, called a *tangent circle* "*order*",[1] (TCO) and a reflexive symmetric relation *I*, called a *tangent circle graph* (TCG). Their definition is quite similar to that of interval order and interval graph: intervals have to be substituted by circles tangent to a line (note that we use the word "circle" in the sense of "closed disc"). In the sequel we shall slightly abuse words by using indifferently, when needed, the concept of relation and that of graph.

[1] Although *P* is not stricto sensu an order, we shall call it so and drop the quotes in the rest of the paper.

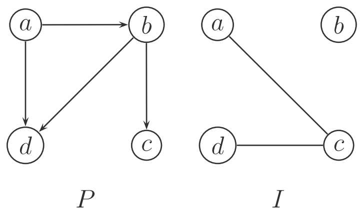


Fig. 2. Graphs associated with the example pictured in Fig. 1.

Nondirected graphs represent symmetric relations (the loops will not appear on the figures even in case the represented relation is reflexive); asymmetric relations will be assimilated to directed graphs.

Definition 1. A tangent circle preference structure (TCPS) on a set A is a pair of binary relations (P, I) on A such that it is possible to associate a circle $\gamma(a)$ to each element a of A, all such circles lying above an horizontal line of the plane, tangent to it, and

(i) for all a,b $\in A$, $(a, b) \in P$ iff the circle $\gamma(a)$ does not intersect the circle $\gamma(b)$ and totally lies to the right of $\gamma(b)$,
(ii) for all a,b $\in A$, $(a, b) \in I$ iff the circles $\gamma(a)$ and $\gamma(b)$ have a nonempty intersection.

Clearly, P is asymmetric ($(a, b) \in P$ implies $(b, a) \notin P$), I is reflexive (for all $a \in A$, $(a, a) \in I$) and symmetric ($(a, b) \in I$ implies $(b, a) \in I$) and $P \cup I$ is complete ($P \cup P^{-1} \cup I = A \times A$). The relation P (resp. I) may be associated a directed (resp. nondirected) graph on the set A; those graphs will be denoted by (A, P) and (A, I), respectively. The edges of the graph (A, I) are all the pairs (a, b) that belong to the relation I except for the loops, i.e. for the edges (a, a), $a \in A$. (A, I) is the intersection graph (see [18, or15, p. 9]) of the associated set of circles $\gamma(A)$. The graphs (A, P) and (A, I) corresponding to the tangent circle preference structure (P, I) represented in Fig. 1 are shown in Fig. 2. From the example it is readily seen that neither I nor P is in general transitive: (a, c) and (c, d) belong to I but (a, d) does not; (a, b) and (b, c) belong to P but (a, c) does not.

Beside the fact that TCPS is a simple example of an intransitive preference structure the asymmetric part of which is close to being an order (as will be shown in Section 4), there is a stronger motivation for studying tangent circle orders: they admit a numerical representation involving a threshold, like semi-orders and interval orders. Complete preference structures that admit a representation using a general form of a threshold function have been studied independently by Agaev and Aleskerov [5] on the one hand and by Abbas and Vincke [2,4] on the other hand. These authors have considered relations (P, I) that can be represented by means of a function $g : A \longrightarrow \mathbb{R}$ and a symmetric nonnegative threshold function $Q : A \times A \longrightarrow \mathbb{R}^+$ such that:

$$aPb \text{ iff } g(a) > g(b) + Q(a, b), \tag{1}$$

$$aIb \text{ iff } |g(a) - g(b)| \leqslant Q(a, b). \tag{2}$$

In the above, the notation aPb (resp. aIb) stands for $(a, b) \in P$ (resp. $(a, b) \in I$) and both will be used indifferently in the sequel. The case in which $Q(a, b) = q(a) + q(b)$ for some nonnegative function $q : A \longrightarrow \mathbb{R}^+$ corresponds to a pair interval order/interval graph, the interval $[g(a) \pm q(a)]$ representing a (see [11,12]). The semi-order/indifference graph is a particular case of the latter, in which it is possible to find some g such that a constant threshold q can be used. Tangent circle orders and graphs correspond (see Section 3) to another simple particularization of the function $Q(a, b)$, namely the case of a *multiplicative threshold function* $Q(a, b) = q(a).q(b)$ for some function q. Identifying preference structures with a multiplicative threshold function as TCPS was a result obtained in the doctoral dissertation of Abbas [1], in which a number of properties of this structure were also established; these results remained unpublished. Since then, the TCPS has been used in at least two papers [14,10]; preference structures with multiplicative threshold functions have recently been studied by Aleskerov and Masatlıoğlu [6], apparently without the authors being aware of any geometric interpretation. The aim of the present paper is to make the related results contained in Abbas' thesis more easily available for further reference.

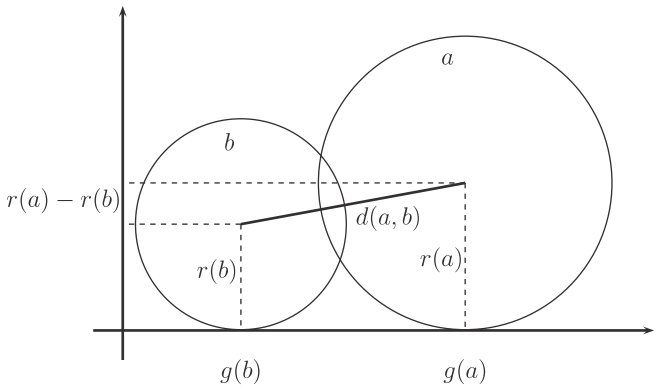


Fig. 3. A representation in the Euclidean plane.

This paper is organized as follows. In Section 3, we establish that a TCPS is the preference structure that admits a numerical representation involving a multiplicative threshold function. The tangent circle preference structure, although not transitive, verifies a weak variant of the transitivity property that we call *hemi-transitivity*. This property is defined and proved to hold for TCPS in Section 3. We also examine in that section, additional conditions that imply the transitivity (in the true sense) of P in a TCPS.

In Section 4, we analyze the inter-relations of TCPS with other preference structures including interval orders/graphs, semi-orders/indifference graphs and trapezoid orders/graphs. Semi-orders obviously are included in the class of TCPS (they correspond to a TCPS that can be represented by circles having constant radius).[2]

Finally, we establish necessary conditions for a complete preference structure with acyclic P to be a TCPS. Minimal forbidden configurations of P and I are exhibited; if such a configuration is observed in a preference relation, the latter cannot be a TCPS. Unfortunately, we are not able either to prove or disprove that the absence of such configurations is sufficient for having a TCPS.

The reader is referred to [22] for more precision on preference modelling and to [7,15] for graph theoretic definitions omitted above.

3. Numerical representation

Let (P, I) be a TCPS on the finite set A. By Definition 1, each element a of A can be associated a circle $\gamma(a)$; all circles of $\gamma(A)$ are tangent to a line, on the same side of this line. We suppose, without loss of generality, that the line is the first coordinate axis of the Euclidean plane endowed with a Cartesian system of coordinates. Let $(g(a), r(a))$ denote the coordinates of the center of the circle $\gamma(a)$, the radius length of which is $r(a)$. The collection of circles $\gamma(A)$ represents the TCPS (P, I) iff the following conditions are fulfilled: $\forall a, b \in A$,

$$aPb \text{ iff } [g(a) > g(b) \text{ and } d(a,b) > r(a) + r(b)]$$

and

$$aIb \text{ iff } d(a,b) \leqslant r(a) + r(b),$$

where $d(a,b) = \sqrt{(g(a) - g(b))^2 + (r(a) - r(b))^2}$ is the Euclidean distance between the center $(g(a), r(a))$ of circle $\gamma(a)$ and the center $(g(b), r(b))$ of circle $\gamma(b)$ (see Fig. 3). We have

$$[g(a) > g(b) \text{ and } d(a,b) > r(a) + r(b)] \text{ iff } \quad g(a) - g(b) > 2\sqrt{r(a)r(b)}$$

[2] Note that another type of "circle order" has been studied in [21]; this structure models the inclusion relation of circles in the plane and has little to do with the structure analyzed here.

and

$$d(a,b) \leqslant r(a) + r(b) \text{ iff } \quad |g(a) - g(b)| \leqslant 2\sqrt{r(a)r(b)}.$$

Hence, we get, $\forall a, b \in A$:

$$aPb \text{ iff } \ g(a) - g(b) > 2\sqrt{r(a)r(b)}, \tag{3}$$

$$aIb \text{ iff } |g(a) - g(b)| \leqslant 2\sqrt{r(a)r(b)}. \tag{4}$$

Conversely, if we start from a preference structure (P, I) and if there are functions g and r that verify conditions (3) and (4), then (P, I) is a TCPS that can be represented by the family of circles $\{\gamma(a),\ a \in A\}$ defined by

$$(x - g(a))^2 + (y - r(a))^2 \leqslant r^2(a).$$

Substituting $r(a)$ by $\frac{1}{2}q^2(a)$ in (3) and (4), we have proved Theorem 1 below, which is a characterization of a TCPS in terms of a multiplicative threshold function q.

Theorem 1. *Let P and I be two binary relations on the same set A.* (P, I) *is a tangent circle preference structure (TCPS) on A if and only if there exist a real-valued function g on A and a nonnegative function q on A such that,* $\forall a, b \in A$, *we have*

$$aPb \textit{ iff } g(a) - g(b) > q(a)q(b), \tag{5}$$

$$aIb \textit{ iff } |g(a) - g(b)| \leqslant q(a)q(b). \tag{6}$$

Remark 1. The numerical representation of a TCPS according to conditions (5) and (6) is by no means unique; in particular, g may be multiplied by any positive number and q by the square root of the same number. Whenever g and q verify (5) and (6), a geometric representation by circles tangent to the first coordinate axis of the Euclidean plane is obtained by associating to any $a \in A$, the circle centered in $(g(a), \frac{1}{2}q^2(a))$ with radius length $r(a) = \frac{1}{2}q^2(a)$.

Remark 2. Theorem 1 characterizes the preference structures that can be represented according to conditions (1), (2) using a threshold function $Q(a, b)$ that decomposes into a product $q(a).q(b)$. There are other well-known preference structures that can be represented in accordance with (1), (2) and for which threshold $Q(a, b)$ decomposes in another manner. The case in which it decomposes additively corresponds to interval orders/graphs [5, Theorem 9, 3, Theorem 6]). Dropping the requirement that $Q(a, b)$ is a symmetric function leads to slightly generalize definitions (1), (2) and allows us to consider a threshold function $Q(a, b)$ expressed as a difference $Q(a, b)=q(a)-q(b)$ or a ratio $Q(a, b)=q(a)/q(b)$ involving a single variable function q. Within the family of preference structures with nonnecessarily symmetric threshold function, we may consider those (P, I) for which there exist a real-valued function g on A and a strictly positive function q on A such that, for all a, b in A:

$$aPb \text{ iff } \ g(a) - g(b) > q(a) \star q(b), \tag{7}$$

$$aIb \text{ iff} \begin{cases} g(a) - g(b) \leqslant q(a) \star q(b), \\ g(b) - g(a) \leqslant q(b) \star q(a), \end{cases} \tag{8}$$

where $\star$ represents some operation on the real numbers. The following particular cases have been characterized:

(i) if $\star = +$ (addition), (P, I) is an interval order/graph structure;
(ii) if $\star = -$ (subtraction), (P, I) is a total preorder structure[3];
(iii) if $\star = .$ (multiplication), (P, I) is a TCPS.

The characterization of the first case has been mentioned above; the second result is obvious. The third characterization results from Theorem 1. As far as we know, the case in which $\star$ stands for the division operation has not been characterized.

[3] According to the terminology of Roubens and Vincke [20], (P, I) is a *total preorder preference structure* if P is a weak order and its symmetric complement I is an equivalence relation.

4. Transitivity and hemi-transitivity

We have learned from the example in Fig. 1 that, in a TCPS (P, I), neither I nor P need to be transitive. However, as shown in the next proposition, P is acyclic.

Proposition 2. *If (P, I) is a TCPS on a set A, then P is acyclic, i.e. $a_1 P a_2 \dots P a_n$ with distinct $a_1, a_2, \dots, a_n \in A$, implies Not $(a_n P a_1)$.*

Proof. Applying (5) to the pairs $a_1 P a_2 \dots a_{n-1} P a_n$ and using the fact that q is nonnegative, leads immediately to the conclusion that we cannot have $a_n P a_1$. □

Although not necessarily transitive, the strict preference relation P in a TCPS verifies a weak form of transitivity that we shall call *hemi-transitivity*.

Definition 2. A relation P defined on a set A is hemi-transitive if $\forall a, b, c, d \in A$, (aPb), (bPc) and (cPd) imply (aPc) or (bPd).

Proposition 3. *If (P, I) is a TCPS on a set A, then P is hemi-transitive.*

Proof. If relation P is not hemi-transitive, then there exist $a, b, c, d \in A$ such that: (aPb), (bPc), (cPd), (aIc) and (bId). Since (P, I) is a TCPS, we can represent it by a function g and a threshold function q that satisfy (5) and (6). This yields the following system of inequalities:

$$g(a) - g(b) > q(a)q(b), \tag{9}$$

$$g(b) - g(c) > q(b)q(c), \tag{10}$$

$$g(c) - g(d) > q(c)q(d), \tag{11}$$

$$q(a)q(c) \geqslant |g(a) - g(c)|, \tag{12}$$

$$q(b)q(d) \geqslant |g(b) - g(d)|. \tag{13}$$

Adding up (9) and (10) and comparing the result to (12) yields (14) below; (15) obtains similarly through summing up (10) and (11) and using (13).

$$q(a)q(b) + q(b)q(c) < q(a)q(c), \tag{14}$$

$$q(b)q(c) + q(c)q(d) < q(b)q(d). \tag{15}$$

It is easy to verify that the inequalities are not satisfied in the case in which $q(a)$, $q(b)$ or $q(c)$ is equal to zero. Let us suppose that $q(a)$, $q(b)$ and $q(c)$ are different from zero. No 4-tuple of strictly positive numbers can verify those inequalities. Suppose the contrary; (14) and (15) imply that

$$\frac{q(a) + q(c)}{q(a)} < \frac{q(c)}{q(b)} \quad \text{and} \quad \frac{q(c)}{q(b)} < \frac{q(d)}{q(b) + q(d)},$$

hence

$$\frac{q(a) + q(c)}{q(a)} < \frac{q(d)}{q(b) + q(d)}$$

and

$$(q(a) + q(c))(q(b) + q(d)) < q(a)q(d),$$

a contradiction. □

Note that the converse of Proposition 3 is false, i.e. there are hemi-transitive relations that are no TCO: in Section 5, an example of a trapezoid order that is no TCO is exhibited; since a trapezoid order is transitive it is a fortiori hemi-transitive and our claim follows.

Note that the hemi-transitivity property does not rule out cycles by itself, even in asymmetric relations. It has however some implications regarding cycles as noted in the following proposition.[4]

Proposition 4. *If P is hemi-transitive, P has a cycle iff it has a cycle of length* 3.

Proof. Let $a_1, a_2, a_3, \ldots, a_n$ be distinct elements of A with $n \geqslant 4$. Suppose, contrary to the hypothesis, that $a_1 P a_2 P a_3 \ldots P a_n P a_1$ is a minimal length cycle of P and assume that its length is at least 4. Consider four consecutive vertices a_1, a_2, a_3, a_4 of the cycle. We have Not $(a_1 P a_3)$ otherwise the cycle $a_1 P a_3 \ldots P a_n P a_1$ would be shorter; for similar reasons, we may not have $a_2 P a_4$. This contradicts the hemi-transitivity of P. $\square$

From the above results, it appears that TCPS, although nontransitive, are "close" to being so. A typical case in which the relation P fails to be transitive is related to the configuration of circles a, b, c in Fig. 1: one has aPb, bPc and aIc.

A condition under which P is not only hemi-transitive but also transitive is given in the next proposition. This condition, which, obviously, is not a necessary one, shows that intransitivity is possible only if the ratio of the radius lengths of the largest and the smallest circles is larger than 4 (remember that $r(a) = \frac{1}{2}q^2(a)$).

Proposition 5. *Let (g, q) be a numerical representation of a TCPS (P, I) on a set A. If the values of q belong to the interval $[t, 2t]$ for some $t \in \mathbb{R}_0^+$, then P is transitive.*

Proof. Suppose on the contrary that P is not transitive. Then, because P is acyclic and $P \cup I$ is complete, there exist $a, b, c \in A$ such that aPb, bPc and aIc. For any numerical representation (g, q) of the TCPS, we have

$$g(a) - g(b) > q(a)q(b),$$

$$g(b) - g(c) > q(b)q(c),$$

$$q(a)q(c) \geqslant |g(a) - g(c)|.$$

Adding up these inequalities and taking into account that $g(a) > g(c)$ yields $q(a)q(c) > q(a)q(b) + q(b)q(c)$. Either $q(a) \leqslant q(c)$ or $q(c) < q(a)$. In the former case, we derive $q(c) > 2q(b)$; in the latter, we get $q(a) > 2q(b)$; in both cases, a contradiction. $\square$

5. TCPS and other preference structures

We establish relationships between TCPS and other preference structures and mention some open questions.

5.1. TCPS and interval order/graph

We call a pair (P, I) in which P is an interval order and I the corresponding interval graph an *interval preference structure* (IPS). The example of nondirected graph shown on Fig. 4(a) is the intersection graph of the family of (tangent) circles shown on Fig. 4(b); it is obvious that it is neither a comparability nor a co-comparability graph (the graph and its complement are both a cycle of length 5, hence they cannot be transitively oriented). As a consequence, the associated TCPS is not an interval preference structure. The converse question is an open problem: we do not know whether there is an interval order/graph that is not a TCPS. The relationship between TCPS and IPS is an interesting subject for further study. There is a relationship that is obvious: the orthogonal projection of a family of circles representing a TCPS onto the line to which they are all tangent yields a family of intervals and the corresponding interval order/graph. The latter is not unique in general since it depends on the particular representation of the TCPS by circles. What can be said for sure is that the intersection graph of the intervals contains as a subgraph the intersection graph of the circles

[4] This proposition is a pure consequence of hemi-transitivity; it has no relevance for a TCPS since, in a TCPS, P is acyclic (Proposition 2).

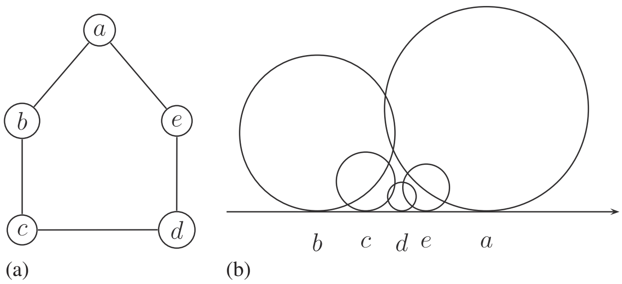


Fig. 4. A TCPS with an indifference relation that is neither an interval graph nor a trapezoid graph.

(if two circles intersect, their orthogonal projections do as well). Conversely, at least one TCPS can be associated to any IPS. Let $(\overline{P}, \overline{I})$ be an interval preference structure. A numerical representation of $(\overline{P}, \overline{I})$ can be described by a pair of functions, the one specifying the left-hand side endpoint of each interval, the other, the corresponding interval length. Alternatively (as described in [4]), we consider a pair of functions $(\overline{g}, \overline{r})$ on the set A; $\overline{g}$ gives the position of the center of the interval and $\overline{r}$ specifies the half-length or radius length of the corresponding interval. Hence $a \in A$ is associated the interval $[\overline{g}(a) \pm \overline{r}(a)]$. We thus have, for all $a, b \in A$:

$$a\overline{P}b \text{ iff } \overline{g}(a) - \overline{g}(b) > \overline{r}(a) + \overline{r}(b), \tag{16}$$

$$a\overline{I}b \text{ iff } |\overline{g}(a) - \overline{g}(b)| \leqslant \overline{r}(a) + \overline{r}(b). \tag{17}$$

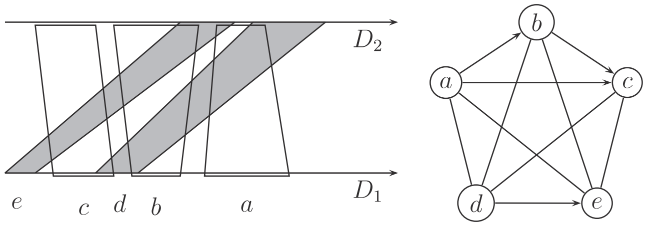
A natural way of associating a TCPS (P, I) to the IPS $(\overline{P}, \overline{I})$ is the following. To each $a \in A$, we associate the circle the center of which is placed in $(\overline{g}(a), \overline{r}(a))$ and the radius is $\overline{r}(a)$. For all $a, b \in A$, we have aPb iff $\overline{g}(a) - \overline{g}(b) > 2\sqrt{\overline{r}(a)\overline{r}(b)}$. Comparing the latter condition to (16) and in view of the relation between the geometric and the arithmetic mean, we conclude that $\overline{P} \subseteq P$, which was evident from geometric considerations. In view of completeness, we have the converse inclusion $\overline{I} \supseteq I$ for the symmetric parts. It is not difficult to convince oneself that the TCPS associated to the IPS $(\overline{P}, \overline{I})$ may depend on the numerical representation of $(\overline{P}, \overline{I})$; it may happen, depending on the radius length of the intervals in the chosen representation, that pairs that are indifferent in the interval order, are represented by nonintersecting circles while for other choices of a numerical representation of the same interval order, they remain indifferent in the TCPS.

5.2. TCPS and semi-order/indifference graph

A semi-ordered preference structure (SOPS) (see e.g. [16,17,19,20]) is defined as a complete preference structure (P, I) with P, a semi-order, and its symmetric complement I, an indifference graph [18]. A semi-order is an interval order that can be represented by intervals of constant length (provided A is finite). An indifference graph is the intersection graph of intervals of constant length. From what we have said about interval orders in the previous section and since the equal length intervals representing a semi-order can be seen as the projections of circles of equal radius on their common tangent, we infer that any semi-order is a TCPS. Aleskerov and Masatlıoğlu [6] have recently established conditions for a numerical representation with multiplicative thresholds to be that of an IPS or a SOPS.

5.3. TCPS and trapezoid order/graph

Trapezoid orders and trapezoid graphs have been studied e.g. in [8,9]. A trapezoid order is the intersection of two interval orders; its geometric interpretation is closely related to that of the tangent circle order. Consider two parallel horizontal lines D_1 and D_2 in $\mathbb{R}^2$. To each $a \in A$, we associate the pair $(i(a), j(a))$ where $i(a)$ is an interval of D_1 and $j(a)$ is an interval of D_2. The trapezoid $t(a)$ associated to a is the convex hull in $\mathbb{R}^2$ of the intervals $i(a)$ and $j(a)$. The relation P on A is a *trapezoid order* if for all $a, b \in A$, aPb iff the trapezoid $t(a)$ entirely lies to the right of the trapezoid $t(b)$. Alternatively, P is the intersection of the interval orders P_1 and P_2 determined, respectively, by the families of intervals $\{i(a), a \in A\}$ and $\{j(a), a \in A\}$ on D_1 and D_2 (oriented e.g. from left to right). The intersection graph of the trapezoids is called a *trapezoid graph*.



(a) (b)

Fig. 5. A trapezoid preference structure that is no TCPS.

Tangent circle graphs (resp. orders) are not in general trapezoid graphs (resp. orders). Minimal examples showing this are provided in Figs. 4 and 5. Fig. 4 shows an example of a TCPS that is not a pair trapezoid order/graph since the complement of the tangent circle graph—shown in Fig. 4(b)—cannot be transitively oriented, while a trapezoid order is transitive.

Conversely, Fig. 5 shows a family of trapezoids that is not a TCPS. The mixed graph in Fig. 5(b) represents both the corresponding trapezoid order (directed arcs) and the trapezoid graph (nondirected arcs). The proof that this structure cannot be represented by tangent circles is elementary but technical. The idea of the proof is as follows. Consider the sub-graph of the graph shown in Fig. 5(b) induced by the nodes a, b, d and e. There is essentially only one configuration of tangent circles that can represent it; in particular, we must have $g(a) > g(d) > g(b) > g(e)$. It is not possible to add a circle representing c to that configuration. Suppose the contrary and let (g, q) be a numerical representation of (P, I). If $g(c) < g(b)$, then we cannot have dIc, because aPc and dPe. Otherwise, $g(b) \leqslant g(c)$ is also impossible because of bPc.

6. Forbidden configurations

In this section, we investigate necessary conditions for a complete preference structure to be a TCPS. These conditions are expressed in terms of *forbidden configurations*. We describe two infinite families of such configurations and identify the minimal ones, i.e. those that do not properly contain any other. Unfortunately, we can neither prove that these configurations characterize a TCPS nor that they do not, i.e. that there exist other forbidden configurations.

Forbidden configurations can best be detected by looking at the mixed graph $G = (A, P, I_0)$, that has both arcs (those belonging to P) and edges (belonging to I_0 that is I without the loops). Note that "configuration" should not be confused with "induced subgraph" of G; they usually are graphs *included* in a subgraph in the sense that their arcs and edges are a subset of the arcs and edges connecting a subset of vertices of A. The first lemma below will help us to construct a first family of forbidden configurations and identify the minimal ones. We denote by $x\hat{P}y$ the fact that there exists a directed path of P-arcs from x to y.

Lemma 6. *If (P, I) is a TCPS on the set A, then, for all $a, b, c, d \in A$, aPb, $b\hat{P}c$, cPd and aIc imply bPd.*

Proof. Since P is acyclic, dPb is impossible. Suppose that there exist $a, b, c, d \in A$ such that aPb, $b\hat{P}c$, cPd, aIc and bId (see Fig. 6). Let (g, q) be a numerical representation of (P, I); (g, q) verifies (5) and (6). Hence we have

$$g(a) - g(b) > q(a)q(b), \tag{18}$$

$$g(c) - g(d) > q(c)q(d), \tag{19}$$

$$q(a)q(c) \geqslant |g(a) - g(c)|, \tag{20}$$

$$q(b)q(d) \geqslant |g(b) - g(d)|. \tag{21}$$

Multiplying (20) by (21) and substituting $q(a)q(b)q(c)q(d)$ using the product of (18) by (19), yields:

$$|g(a) - g(c)| \cdot |g(b) - g(d)| < (g(a) - g(b)) \cdot (g(c) - g(d)). \tag{22}$$

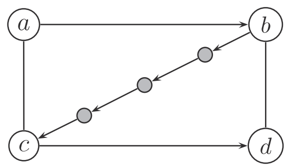


Fig. 6. Forbidden configuration of type $c(0, n, 0)$ (this instance with $n = 3$).

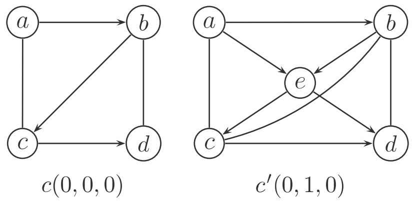


Fig. 7. Minimal forbidden configurations contained in all $c(0, n, 0)$.

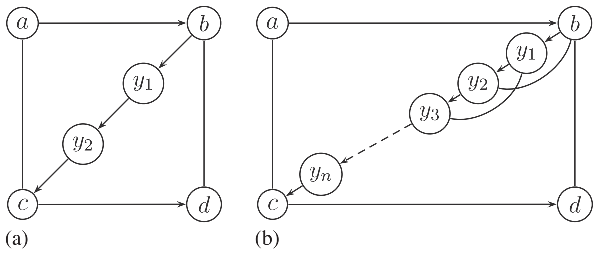


Fig. 8. Configurations used in the proof of Lemma 7.

On the other hand, using $b\hat{P}c$, we get $g(a) > g(b) > g(c) > g(d)$ which implies $g(a) - g(c) > g(a) - g(b) > 0$ and $g(b) - g(d) > g(c) - g(d) > 0$; the latter inequalities are incompatible with (22). $\square$

Let us denote by $c(0, n, 0)$ a configuration isomorphic to *aPb*, $b\hat{P}c$, *cPd*, *aIc* and *bId* (see Fig. 6); n is the number of intermediate vertices in the path of *P*-arcs between *b* and *c*; the "0" on the first and third coordinates of $c(0, n, 0)$ refer to the fact that the "paths" from *a* to *b* and from *c* to *d*, respectively, are direct (no intermediate vertices).

Observe that $c(0, 0, 0)$ (see also Fig. 7) typically is a configuration violating hemi-transitivity; $c(0, n, 0)$ is the family of configurations that cannot be found in a TCPS according to Lemma 6. When looking at minimal configurations that can be found in complete preference structures (P, I) in which *P* is acyclic but cannot be in a TCPS, the first one encountered is $c(0, 0, 0)$; it is the only one on four vertices; there is no configuration on three vertices that is forbidden in a TCPS (except of course the *P*-cycle that we exclude by looking only at acyclic *P* relations). When considering configurations on five vertices, we find a forbidden configuration that we call $c'(0, 1, 0)$ (see Fig. 7). It obtains from $c(0, 1, 0)$ by adding the arcs and edges imposed by the completeness of (P, I) and the acyclicity of *P* and avoiding in addition to create any configuration isomorphic to $c(0, 0, 0)$. We thus have found two minimal forbidden configurations; at least one of them can be found in any $c(0, n, 0)$, as shown in the next lemma.

Lemma 7. *Let (P, I) be a complete preference structure on a set A, with P acyclic. If P contains a configuration of type $c(0, n, 0)$, for some integer n, it also contains $c(0, 0, 0)$ or $c'(0, 1, 0)$.*

Proof. The proof is by induction. If $n = 2$, let a, b, y_1, y_2, c and d be the vertices of a configuration $c(0, 2, 0)$ illustrated on Fig. 8(a). We do not have bPy_2 (resp. y_1Pc), otherwise vertices a, b, y_2, c and d (resp. a, b, y_1, c and d) would induce a configuration isomorphic to $c(0, 0, 0)$.

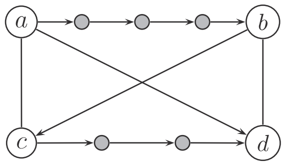


Fig. 9. Forbidden configurations of type $c(n, 0, m)$ (in this instance, $n = 3$ and $m = 2$).

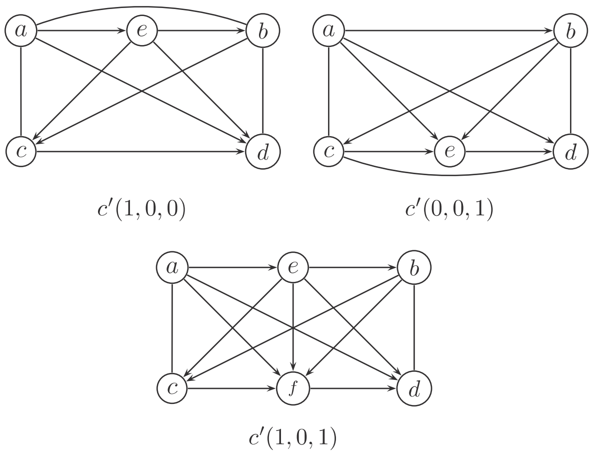


Fig. 10. Minimal forbidden configurations of type $c'(1, 0, 0)$, $c'(0, 0, 1)$, $c'(1, 0, 1)$.

Suppose that the thesis is verified whenever (P, I) contains a configuration $c(0, p, 0)$ with $2 \leqslant p \leqslant n - 1$; we show that it is also valid if (P, I) contains $c(0, n, 0)$. Denote by $b, y_1, y_2, \ldots, y_n$ and c, the sequence of vertices of the directed path of P from b to c in a configuration $c(0, n, 0)$ with $n \geqslant 3$ (see Fig. 8(b)). If bPy_2 or y_1Py_3, it is easily seen that (P, I) contains a configuration of the type $c(0, n - 1, 0)$ and by the induction hypothesis, we get the conclusion. If neither bPy_2 nor y_1Py_3, then, since P has no cycle, bIy_2 and y_1Iy_3 and there is a $c(0, 0, 0)$ configuration on the vertices b, y_1, y_2, y_3. $\square$

The proof of the next lemma is similar to that of Lemma 6 and is left to the reader. As in Lemma 6, $a\hat{P}b$ (resp. $c\hat{P}d$) denotes the fact that there is a directed path in the graph (A, P) from a to b (resp. from c to d).

Lemma 8. *If (P, I) is a TCPS on the set A, then, for all $a, b, c, d \in A$, aPd, bPc, aIc, $a\hat{P}b$ and $c\hat{P}d$ imply bPd.*

The latter lemma establishes that another family of configurations are forbidden, namely those of $c(n, 0, m)$ type illustrated in Fig. 9. Looking for minimal forbidden configurations in the same spirit as after Lemma 6, one finds two additional ones, on five vertices, and one more on six vertices. Those configurations that we call, respectively, $c'(1, 0, 0)$, $c'(0, 0, 1)$ and $c'(1, 0, 1)$ are shown in Fig. 10. They derive, respectively, from $c(1, 0, 0)$, $c(0, 0, 1)$ and $c(1, 0, 1)$ by adding the arcs and edges that result from the assumption that (P, I) is complete, P acyclic and avoiding to create any configuration isomorphic to $c(0, 0, 0)$.

In a similar way as for Lemma 7, one proves the next result, the proof of which is left to the reader.

Lemma 9. *Let (P, I) be a complete preference structure on a set A, with P acyclic. If P contains a configuration of type $c(n, 0, m)$, for some integer $n, m \geqslant 2$, it also contains $c(0, 0, 0)$, $c'(1, 0, 0)$, $c'(0, 0, 1)$ or $c'(1, 0, 1)$.*

Lemmas 7 and 9 show that there are at least five minimal configurations that have to be searched for if one wants to establish that a complete preference structure (P, I), with P acyclic, is not a TCPS. Those configurations are $c(0, 0, 0)$, $c'(0, 1, 0)$, $c'(1, 0, 0)$, $c'(0, 0, 1)$ and $c'(1, 0, 1)$; the first or the second one may be found as soon as a preference

Table 1
Summary of the results

Numerical representation	Multiplicative threshold	Theorem 1
Properties	P acyclic	Proposition 2
	P hemi-transitive	Proposition 3
Relationship with other preference structures	TCPS$\not\Rightarrow$IPS	Section 5.1
	TCPS$\not\Rightarrow$SOPS	
	SOPS $\Rightarrow$ TCPS	Section 5.2
	TCPS$\not\Rightarrow$Trapezoid PS	
	Trapezoid PS$\not\Rightarrow$TCPS	Section 5.3
Characterization	Forbidden configurations of P and I (incomplete)	Section 6

structure (with P acyclic) contains a $c(0, n, 0)$ configuration, for any n; the first or one of the latter three, is present as soon as the preference contains a $c(n, 0, m)$ configuration, for any value of n and m. Unfortunately, an algorithm for detecting those configurations can only lead to a negative conclusion and cannot be used to establish that a preference structure is a TCPS, since we do not know the complete list of minimal forbidden configurations; we do not even know if such a list is finite (there might exist minimal forbidden configurations on sets of vertices of arbitrarily large cardinality).

7. Conclusion

Our findings regarding the tangent circle preference structure are summarized in Table 1.

We hope to have convinced the reader that TCPS is an interesting nontransitive preference structure in the sense that it enjoys weak forms of transitivity (and in particular, P is acyclic). The main motivation for studying this structure however is that it provides an appealing geometric interpretation for preference structures that can be represented using multiplicative threshold functions.

TCPS certainly deserves further study since it raises a few challenging questions. In particular, besides the characterization of TCPS in terms of numerical representation, we emphasize that an "abstract" definition is still lacking, be it in terms of relational properties or in terms of forbidden configurations. The relationship between TCPS and interval orders/graphs also deserves further investigation.

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