On expansions of $(\mathbf{Z}, +, 0)$

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Abstract

Call a (strictly increasing) sequence (r_n) of natural numbers *regular* if it satisfies the following condition: $r_{n+1}/r_n \to \theta \in \mathbf{R}^{>1} \cup \{\infty\}$ and, if θ is algebraic, then (r_n) satisfies a linear recurrence relation whose characteristic polynomial is the minimal polynomial of θ. Our main result states that $(\mathbf{Z}, +, 0, R)$ is superstable whenever R is enumerated by a regular sequence. We give two proofs of this result. One relies on a result of E. Casanovas and M. Ziegler and the other on a quantifier elimination result. We also show that $(\mathbf{Z}, +, 0, <, R)$ is NIP whenever R is enumerated by a regular sequence that is ultimately periodic modulo m for all $m > 1$.

Keywords: expansions of $(\mathbf{Z}, +)$, regular sequences, superstability, quantifier elimination, decidability
2010 MSC: 03B25, 03C10, 03C35, 03C45

Introduction

Recently, stability properties of expansions $\mathscr{Z}_R = (\mathbf{Z}, +, 0, R)$ of $(\mathbf{Z}, +, 0)$ by a unary predicate R for a subset of the integers have attracted the attention of many researchers. Motivated by a question of A. Pillay on the induced structure on non-trivial centralizers in the free group on two generators, D. Palacín and R. Sklinos proved in [15] that for any natural number $q > 1$, the structure $\mathscr{Z}_{\Pi_q}$ is superstable of Lascar rank ω, where $\Pi_q = \{q^n \mid n \in \mathbf{N}\}$ (this was also proved independently and using different methods by B. Poizat in [18, Théorème 25]). They also showed the same result for $R = \{n! \mid n \in \mathbf{N}\}$ and more generally for sets enumerated by sequences (r_n) such that $r_{n+1}/r_n \to \infty$ and that are *congruence periodic*, namely ultimately periodic modulo m for all $m > 1$. They used results of E. Casanovas and M. Ziegler [3] on stable expansions by a unary predicate. In another direction, when R is the set $\mathbf{P}$ of prime numbers, I. Kaplan and S. Shelah showed in [11], assuming Dickson's Conjecture ([11, Conjecture 1.1]), that $\mathscr{Z}_{\mathbf{P} \cup -\mathbf{P}}$ is unstable and supersimple of Lascar rank 1.

In this paper, we investigate such expansions $\mathscr{Z}_R$ with R interpreting a subset of the natural numbers, generalizing the above results of D. Palacín and R. Sklinos. Call a sequence (r_n) *regular* if it satisfies the following condition: $r_{n+1}/r_n \to \theta \in \mathbf{R}^{>1} \cup \{\infty\}$ and, if θ is algebraic, (r_n) follows a linear recurrence relation whose characteristic polynomial is the minimal polynomial of θ. For such regular sequences, we show the superstability of the corresponding expansion.

Theorem A (Theorem 1.5)**.** *Assume that R is enumerated by a regular sequence. Then* $\mathrm{Th}(\mathscr{Z}_R)$ *is superstable of Lascar rank ω.*

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[1] Research Director at the FRS-FNRS, this material is based upon work supported by the National Science Foundation under Grant No. DMS-1440140 while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2017.

We give two proofs of this theorem. The first one follows the strategy of D. Palacín and R. Sklinos. The second one is based on a quantifier elimination result for such expansions.

Expansions of Presburger arithmetic by regular sequences have been already studied and shown to be tame. At the end of the paper, we give a proof of an announced result in [1, 2] that expansions of Presburger arithmetic by Π_q, $\{n! \mid n \in \mathbf{N}\}$ or the set of Fibonacci numbers are NIP.

Theorem B (Corollary 2.34)**.** *Assume that R is interpreted by a congruence periodic regular sequence. Then* $\mathrm{Th}(\mathbf{Z}, +, 0, <, R)$ *is NIP.*

Let us now outline the content of the paper. From now on, we assume that R is interpreted by a regular sequence.

In Section 1, we give the first proof of Theorem A by applying the result of E. Casanovas and M. Ziegler. In our context, their result [3, Proposition 3.1] reads as follows. An expansion of the form $\mathscr{Z}_R$ is superstable whenever R is bounded and the induced structure on R by $\mathscr{Z}_R$ is superstable (see Definition 1.6 and Theorem 1.7, and the comments after these). In fact, we only need to check that the induced structure on R by equations in $\mathscr{Z}_R$ is superstable. So we analyze sets of the form $X_{\bar{a}} = \{(r_{n_1}, \ldots, r_{n_k}) \in R^k \mid a_1 r_{n_1} + \cdots + a_k r_{n_k} = 0\}$, where $k \geq 1$ and $\bar{a} \in \mathbf{Z}$.

We show, in Proposition 1.15, that for all $\bar{a} \in \mathbf{Z}^k$ there is $c_{\bar{a}} \in \mathbf{N}$ such that if $(r_{n_1}, \ldots, r_{n_k}) \in X_{\bar{a}}$, then

$$\max\{|n_i - n_j| \mid 1 \leq i, j \leq k\} \leq c_{\bar{a}},$$

unless there exists $I \subsetneq \{1, \ldots, k\}$ such that $\sum_{i \in I} a_i r_{n_i} = 0$. The proof of this proposition relies on the following property: sets of the form $\{r_n \in R \mid a'_0 r_n + a'_1 r_{n+1} + \cdots + a'_\ell r_{n+\ell} = 0\}$ are either finite or R, where $\bar{a}' \in \mathbf{Z}^{\ell+1}$ and $\ell \in \mathbf{N}$. The analysis of the sets $X_{\bar{a}}$ allows us to show that the induced structure on R is definably interpreted in the superstable structure $(\mathbf{N}, S, S^{-1}, 0)$, where $S(n) = n + 1$, $S^{-1}(n + 1) = n$ and $S^{-1}(0) = 0$. Thus, the induced structure is superstable.

Recall that a subset of $\mathbf{N}$ is *piecewise syndetic* if it contains arbitrarily long sequences with bounded gaps. We use again Proposition 1.15 to show that we cannot bound the length of expansions in base R of natural numbers. In other words, we show that any set of the form

$$\{z \in \mathbf{Z} \mid z = a_1 r_{n_1} + \cdots + a_k r_{n_k} \text{ for some } (r_{n_1}, \ldots, r_{n_k}) \in R^k\} \cap \mathbf{N}$$

is not piecewise syndetic. This allows us to prove that R is bounded, see Subsection 1.2 and Theorem 1.21 therein.

In Section 2, we give the second proof of Theorem 1.5, using a quantifier elimination result in a language extending $\mathcal{L}_R = \{0, +, R\}$.

We first axiomatize the theory of $\mathscr{Z}_R$ in an enriched language $\mathcal{L}$: to $\mathcal{L}_R$ we add new predicates interpreted in $\mathscr{Z}_R$ by certain existentially defined sets. For instance,

$$\{z \in \mathbf{Z} \mid z = a_1 r_{n_1} + \cdots + a_k r_{n_k} \text{ for some } (r_{n_1}, \ldots, r_{n_k}) \in R^k\}$$

will be quantifier-free $\mathcal{L}$-definable. We let T_R be this $\mathcal{L}$-axiomatization of $\mathscr{Z}_R$.

Theorem C (Theorem 2.1 and Corollary 2.3)**.** *The $\mathcal{L}$-theory T_R has quantifier elimination and is complete.*

This quantifier elimination result allows us to prove that T_R is superstable by counting types. As a consequence, we recover the superstability of $\mathrm{Th}(\mathscr{Z}_R)$.

We conclude this paper by Subsections 2.3 and 2.4, where we respectively show that, when R is interpreted by a congruence periodic regular sequence, T_R is decidable (when θ can be computed effectively and the congruence periodicity is effective) and $\mathrm{Th}(\mathbf{Z}, +, 0, <, R)$ is NIP. This last result relies on a quantifier elimination result of the second author in [16] for expansions of Presburger arithmetic by so-called sparse predicates, introduced by A. L. Sëmenov.

Independently of our work [17], G. Conant published a paper [4] on sparsity notions and stability for sets of integers. There he defines the notion of a *geometrically sparse* set R (see [4, Definition 6.2]). For such a set R, he proves superstability of $(\mathbf{Z}, +, 0, R)$ and calculates its Lascar rank [4, Theorem 7.1]. So there is an overlap between his result and our Theorem 1.5 (see also [12]); we give an account of this overlap at the end of Section 1. We also point out that, in the first version of this paper, our main result had an extra hypothesis on regular sequences, namely that they were congruence periodic. This hypothesis was necessary to understand the trace, on R, of congruence relations. However, G. Conant showed that in some cases, the analysis of the trace of congruence relations is not necessary and we decided to incorporate this in Theorem 1.5. This is explained after the statement of Theorem 1.7.

Notation and convention

In this section, we fix some notations and conventions for the rest of this paper. The set of natural numbers, of integers and of real numbers will be denoted respectively $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$. When X is one of the above sets and $a \in X$, the notations $X^{>a}$, $X^{\geq a}$ and $X_\infty^{\geq a}$ refer respectively to the sets $\{x \in X \mid x > a\}$, $\{x \in X \mid x \geq a\}$ and $X^{>a} \cup \{\infty\}$. For a natural number n, the set $\{1, \ldots, n\}$ will be denoted $[n]$. The cardinality of a set A will be denoted by $|A|$. Likewise, the length of a tuple $\bar{x}$ will be denoted $|\bar{x}|$.

Capital letters I, J and K will refer to (usually non-empty) sets of indices. Capital letters will refer to sets and small letters will refer to elements of a given set. For a tuple $\bar{a}$ of length n and $I \subset [n]$, $\bar{a}_I$ refers to the tuple $(a_i \mid i \in I)$. For $n \in \mathbf{N}^{>0}$, we let $\mathfrak{P}([n])$ be the set of (ordered) partitions $\bar{I} = (I_1, \ldots, I_\ell)$ of $[n]$.

A first order language will be denoted by the letter $\mathcal{L}$, possibly with a subscript. An $\mathcal{L}$-structure will be referred to by a round letter and its domain by the corresponding capital letter. For instance $\mathscr{M}$ is an $\mathcal{L}$-structure whose domain is M. For an element a of M and $A \subset M$, the notations $\mathrm{acl}^{\mathcal{L}}(a/A)$, $\mathrm{tp}^{\mathcal{L}}(a/A)$ mean respectively the algebraic closure and the type of a over A in $\mathscr{M}$. If $R \in \mathcal{L}$ is a n-ary predicate symbol, the set $\{\bar{a} \in M^n \mid \mathscr{M} \models R(\bar{a})\}$ will be denoted $R(M^n)$ or simply R when there is no confusion.

We make the following (usual) abuse of notations. When R is a unary predicate symbol, expressions of the form $\exists x \in R\, \varphi(x)$ and $\forall x \in R\, \varphi(x)$ respectively mean $\exists x\, (R(x) \wedge \varphi(x))$ and $\forall x\, (R(x) \rightarrow \varphi(x))$. An expression of the form $x > c$, where $c \in \mathbf{N}$, is an abbreviation for $\bigwedge_{i=0}^{c} x \neq i$.

For each $n \in \mathbf{N}^{>1}$, let D_n be a unary predicate. We let $\mathcal{L}_g = \{+, -, 0, D_n \mid n > 1\}$ and $\mathcal{L}_S = \{S, S^{-1}, c\}$, where S and S^{-1} are unary function symbols and c is a constant symbol. An abelian group $(G, +, -, 0)$ will always be expanded to an $\mathcal{L}_g$-structure as follows: for each $n \in \mathbf{N}^{>1}$, the symbol D_n is interpreted as the set $\{x \in G \mid (G, +, -, 0) \models \exists y\, x = ny\}$.

1. Expansion of $(\mathbf{Z}, +, -, 0)$ by a regular sequence

In this section, we consider expansions of $(\mathbf{Z}, +, -, 0)$ by a unary predicate R interpreting an infinite subset of $\mathbf{N}$. We let (r_n) be the unique (strictly increasing) enumeration of $R(\mathbf{Z})$. When there is no risk of confusion, we will also denote (r_n) by R. The main result of this section is the superstability of the expansion $\mathscr{Z}_R = (\mathbf{Z}, +, -, 0, R)$ when (r_n) is a *regular sequence*, defined below.

Definition 1.1. Let $R = (r_n)$ be a sequence of natural numbers that satisfy a *linear recurrence relation*: there are $a_0, \ldots, a_{k-1} \in \mathbf{Q}$, with $k \in \mathbf{N}^{\geq 1}$ minimal, such that for all $n \in \mathbf{N}$,

$$r_{n+k} = \sum_{i=0}^{k-1} a_i r_{n+i}.$$

The polynomial P_R defined by $P_R(X) = X^k - \sum_{i=0}^{k-1} a_i X^i$ is called the *characteristic polynomial* and the numbers $r_0, \ldots, r_{k-1}$ the *initial conditions*.

Definition 1.2. Let $R = (r_n)$ be a sequence of natural numbers. We say that R is *regular* if it satisfies the following property: $r_{n+1}/r_n \to \theta \in \mathbf{R}_\infty^{\geq 1}$ and, if θ is algebraic (over $\mathbf{Q}$), then R satisfies a linear recurrence relation whose characteristic polynomial P_R is the minimal polynomial of θ.

Remark 1.3. Let $R = (r_n)$ be defined by a recurrence relation whose characteristic polynomial is P_R. A. Fiorenza and G. Vincenzi, in [6, 7] provide a necessary and sufficient condition on P_R and the initial conditions of R for the existence of $\lim_{n\to\infty} r_{n+1}/r_n$ (see [6, Theorem 2.3]).

We define an action of $\mathbf{Z}[X]$ on the set of sequences of integers. We let X act as the shift σ: for all $n \in \mathbf{N}$, $\sigma(s_n) = s_{n+1}$. Likewise, X^i acts as σ^i. We extend this by linearity: if $Q(X) = \sum_{i=0}^d a_i X^i$, then Q acts as $\sum_{i=0}^d a_i \sigma^i$. This action has the following property: for all $Q, Q' \in \mathbf{Z}[X]$, QQ' acts as the action of Q' followed by the action of Q. Let $Q\cdot$ denote the action of Q. Note that $P_R \cdot (r_n) = (0)$.

We will need the following well-known result on linear recurrence relations.

Proposition 1.4. *Let $R = (r_n)$ be a linear recurrence relation and let $Q \in \mathbf{Q}[X]$, $Q(X) = \sum_{i=0}^d a_i X^i$. The following are equivalent*

1. *P_R divides Q (in $\mathbf{Q}[X]$);*
2. *$Q \cdot (r_n) = (0)$, that is for all $n \in \mathbf{N}$, $a_0 r_n + a_1 r_{n+1} + \cdots + a_d r_{n+d} = 0$.*

Proof. By the euclidean algorithm (in $\mathbf{Q}[X]$, see [10, Theorem 2.14]), we have $Q = Q_1 P_R + Q_2$, for some $Q_1, Q_2 \in \mathbf{Q}[X]$ with $\deg(Q_2) < \deg(P_R)$.

First assume that P_R divides Q. Then $Q = Q_1 P_R$. Thus, $Q \cdot (r_n) = (Q_1 P_R) \cdot (r_n) = Q_1 \cdot (P_R \cdot (r_n)) = Q_1 \cdot (0) = (0)$. This implies that $Q \cdot (r_n) = (0)$.

Second assume that $Q \cdot (r_n) = (0)$. Since $P_R \cdot (r_n) = (0)$, we get that $Q_2 \cdot (r_n) = (0)$, which contradicts the minimality of P_R, unless $Q_2 = 0$. So P_R must divide Q. □

Here is a list of examples of regular sequences (we will come back to some of these examples at the end of this section):

- $(n!)$;
- (q^n), where $q \in \mathbf{N}^{>1}$;
- the sequence $(\lfloor \pi^n \rfloor)$, where $\lfloor x \rfloor$ denotes the integer part of x;
- the Fibonacci sequence as defined by $r_0 = 1$, $r_1 = 2$ and $r_{n+2} = r_{n+1} + r_n$ for all $n \in \mathbf{N}$;
- the sequence given by the linear recurrence relation $r_{n+2} = 5r_{n+1} + 7r_n$, $r_1 = 1$ and $r_0 = 0$.

Now, let us state the main result of this section.

Theorem 1.5. *Assume $R(\mathbf{Z})$ is enumerated by a regular sequence. Then $\mathrm{Th}(\mathscr{Z}_R)$ is superstable of Lascar rank ω.*

The proof of this theorem follows the same strategy as D. Palacín and R. Sklinos in [15] which is based on the following result of E. Casanovas and M. Ziegler [3].

Definition 1.6. Let $\mathscr{M}$ be an $\mathcal{L}$-structure and $A \subset M$, $A \neq \emptyset$.

1. We say that $\mathscr{M}$ *does not have the finite cover property* (in short: $\mathscr{M}$ is nfcp) if for all formulas $\varphi(x, \bar{y})$, there exists $k \in \mathbf{N}$ such that for all $\bar{m}_i \in M^{|\bar{y}|}$, $i \in I$, if the set $X = \{\varphi(x, \bar{m}_i) \mid i \in I\}$ is k-consistent, then X is consistent. Similarly, we say that $\mathscr{M}$ *has* nfcp *over* A if for all formulas $\varphi(x, \bar{y}, \bar{z})$, there exists $k \in \mathbf{N}$ such that for all $\bar{a}_i \in A^{|\bar{y}|}$ $\bar{m}_i \in M^{|\bar{z}|}$, $i \in I$, if the set $X = \{\varphi(x, \bar{a}_i, \bar{m}_i) \mid i \in I\}$ is k-consistent, then X is consistent.

2. To each $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, we associate a new n-ary predicate $R_{\varphi,n}$ and we denote by $\mathcal{L}_{\mathrm{ind}}$ the language
$$\{R_{\varphi,n} \mid \varphi(x_1, \ldots, x_n) \text{ is an } \mathcal{L}\text{-formula}\}.$$
The *induced structure* on A (by $\mathscr{M}$), denoted A_{ind}, is the $\mathcal{L}_{\mathrm{ind}}$-structure whose domain is A and $R_{\varphi,n}(A) = \varphi(M^n) \cap A^n$. Similarly, we define A^0_{ind} to be the induced structure on A by

equations ($\mathcal{L}_{\text{ind}}$ is replaced by $\mathcal{L}^0_{\text{ind}}$, which contains the symbols $R_{\varphi,n}$, where $\varphi(x_1,\ldots,x_n)$ is a boolean combination of equations[2] in $\mathcal{L}$).

3. Let R be a unary predicate not in $\mathcal{L}$ and let $\mathcal{L}_R$ be the language $\mathcal{L}\cup\{R\}$. Let $\mathscr{M}_A$ denote the $\mathcal{L}_R$-expansion of $\mathscr{M}$, with $R(M)=A$.
   (a) We say that A is *small* if there is an $\mathcal{L}_R$-structure $\mathscr{N}_{R(N)}$ elementary equivalent to $\mathscr{M}_A$ such that: for all finite subsets B of N, any type in $\mathcal{L}$ over $B\cup R(N)$ is realized in $\mathscr{N}$.
   (b) We say that A is *bounded* if for all $\mathcal{L}_R$-formulas $\varphi(\bar{x})$ there is an $\mathcal{L}$-formula $\psi(\bar{x},\bar{y})$ such that φ is equivalent (in $\mathscr{M}$) to the formula $Q_1y_1{\in}R\ldots Q_ny_n{\in}R\,\psi(\bar{x},\bar{y})$, where $Q_i\in\{\exists,\forall\}$.

The main result of [3] states that the stability of $\mathscr{M}_A$ is equivalent to that of $\mathscr{M}$ and A_{ind}, under the assumptions of nfcp for $\mathscr{M}$ and smallness for A, see [3, Theorem A] for a precise statement. To prove [3, Theorem A], E. Casanovas and M. Ziegler first show that when $\mathscr{M}$ is nfcp, if A is small then A is bounded. Next they show an analogue of [3, Theorem A] under the sole assumption of boundness of A. This is actually the result used by D. Palacín and R. Sklinos in [15].

Theorem 1.7 ([3, Proposition 3.1])**.** *Let $\mathscr{M}$ be an $\mathcal{L}$-structure and let $A\subset M$. Suppose that A is bounded. Then for all $\lambda\geq|\mathcal{L}|$, if $\mathscr{M}$ and A_{ind} are λ-stable, then $\mathscr{M}_A$ is λ-stable.*

Let $R(\mathbf{Z})\subset\mathbf{N}$, enumerated by a regular sequence (r_n). Recall that $\mathscr{Z}$ has quantifier elimination in $\mathcal{L}_g$ (see [19, Theorem15.2.1]) and is superstable (see [19, Theorem 15.4.4]). So, in order to show that $\mathscr{Z}_R$ is superstable, we only need to show that R is small and that R_{ind} is superstable. For the latter, the study of R_{ind} is reduced to the study of the trace on $R(\mathbf{Z})$ of equations of the form $a_1x_1+\cdots+a_nx_n=0$ or divisibility relations of the form $D_m(a_1x_1+\cdots+a_nx_n)$. Actually, we can further reduce the analysis of R_{ind} to the analysis of R^0_{ind}, using the following observation of G. Conant (see [4, Section 5]). For $\mathscr{N}$ an $\mathcal{L}$-structure, let $\mathscr{N}^1$ be the expansion of $\mathscr{N}$ by predicates for *all* subsets of N. G. Conant observed that R_{ind} is an expansion of R^0_{ind} by unary predicates [4, Corollary 5.7]. As a consequence of this observation, if R^0_{ind} is definably interpreted in a structure $\mathscr{N}$ whose expansion $\mathscr{N}^1$ is superstable, then R_{ind} is superstable. We apply this by showing that R^0_{ind} is definably interpreted in the structure $\mathscr{N}=(\mathbf{N},S)$, where $S(n)=n+1$, whose expansion $\mathscr{N}^1$ has been shown to be superstable [4, Proposition 5.9]. This is done as follows. We interpret $R(\mathbf{Z})$ by $\mathbf{N}$. Given an equation $a_1x_1+\cdots+a_kx_k=0$, where $\bar{a}\in\mathbf{Z}^k$, we interpret the set of solutions $(r_{n_1},\ldots,r_{n_k})$ in $R(\mathbf{Z})^k$ as the set $\{(n_1,\ldots,n_k)\in\mathbf{N}^k\mid a_1r_{n_1}+\cdots+a_kr_{n_k}=0\}$ and we show that it is $\mathcal{L}_S$-definable. We actually prove that sets of the form

$$N_{\bar{a}}=\left\{(n_1,\ldots,n_k)\in\mathbf{N}^k\;\middle|\;a_1r_{n_1}+\cdots+a_kr_{n_k}=0\text{ and for all }I\subset[k],\sum_{i\in I}a_in_i\neq0\right\}.$$

are $\mathcal{L}_S$-definable and explain why this is enough to conclude. In order to show that the sets $N_{\bar{a}}$ are definable, we first show that functions of the form $\mathsf{f}:\mathbf{N}\to\mathbf{Z}:n\mapsto b_0r_n+b_1\sigma(r_n)+\cdots+b_d\sigma^d(r_n)$, where $\bar{b}\in\mathbf{Z}$, called *operators* on R, behave predictably: either $\mathsf{f}(n)=0$ for all $n\in\mathbf{N}$ or $\mathsf{f}(n)\neq0$ for all but finitely many $n\in\mathbf{N}$, see Proposition 1.9. So for any operator f, the set $\{n\in\mathbf{N}\mid\mathsf{f}(n)=0\}$ is $\mathcal{L}_S$-definable. Then, we show that the set $N_{\bar{a}}$ is controlled by finitely many operators, see Proposition 1.15 and Remark 1.16. This control has the effect to reduce the definability of $N_{\bar{a}}$ to the definability of sets of the form $\{n\in\mathbf{N}\mid\mathsf{f}(n)=0\}$.

The boundness of R will be a consequence of Propositions 1.9 and 1.15: we deduce from them that a set of the form $a+d\mathbf{N}$ cannot be covered by finitely many sets of the form $\{z+\mathsf{f}_1(n_1)+\cdots+\mathsf{f}_k(n_k)\mid\bar{n}\in\mathbf{N}\}$, where $z\in\mathbf{Z}$ and $\mathsf{f}_1,\ldots,\mathsf{f}_k$ are operators on $R(\mathbf{Z})$. This is done in Section 1.2.

[2] By an equation $\varphi(x_1,\ldots,x_n)$, we mean an atomic formula of the form $t(x_1,\ldots,x_n)=t'(x_1,\ldots,x_n)$ where t and t' are $\mathcal{L}$-terms.

1.1. The induced structure on a regular sequence

Throughout this section we fix $R(\mathbf{Z}) \subset \mathbf{N}$ and we assume that (r_n) is a regular sequence and $\theta = \lim_{n\to\infty} r_{n+1}/r_n \in \mathbf{R}^{>1}_{\infty}$.

Definition 1.8. Let $Q \in \mathbf{Z}[X]$, $Q(X) = \sum_{i=0}^{d} a_i X^i$, $\bar{a} \in \mathbf{Z}^{d+1}$. The *operator* associated to Q, denoted f_Q or simply f, is the function $\mathsf{f} : \mathbf{N} \to \mathbf{Z} : n \mapsto a_0 r_n + a_1\sigma(r_n) \cdots + a_d\sigma^d(r_n)$.

The aim of this section is to understand the structure of subsets of $\mathbf{N}^k$ of the form

$$N_{\bar{\mathsf{f}},z} = \{\bar{n} \in \mathbf{N}^k \mid \mathsf{f}_1(n_1) + \cdots + \mathsf{f}_s(n_k) = z\}, \tag{1}$$

where f_i is an operator, $i \in [s]$ and $z \in \mathbf{Z}$.

1.1.1. Operators on a regular sequence

We start with the case of a single operator, that is with sets of the form $N_{\mathsf{f},z}$.

Proposition 1.9. *Let* f *be an operator. Then* $N_{\mathsf{f},0}$ *is either finite or* $\mathbf{N}$.

The proof of this proposition follows from the following lemmas.

Lemma 1.10. *Suppose that* $\theta = \infty$*. Let* $Q \in \mathbf{Z}[X] \setminus \{0\}$*. Then* $N_{\mathsf{f}_Q,0}$ *is finite.*

Proof. Assume that $Q(X) = \sum_{i=0}^{d} a_i X^i$ and let $\mathsf{f} = \mathsf{f}_Q$. Then $\mathsf{f}(n) = 0$ if and only if $a_0 r_n/r_{n+d} + \cdots + a_{d-1} r_{n+d-1}/r_{n+d} + a_d = 0$. Thus, as $r_{n+i}/r_{n+d} \to 0$ for all $0 \leq i < d$, for all n sufficiently large, $\mathsf{f}(n) \neq 0$. $\square$

Lemma 1.11. *Suppose that* $\theta \in \mathbf{R}^{>1}$*. Let* $Q \in \mathbf{Z}[X] \setminus \{0\}$ *and suppose that* $Q(\theta) \neq 0$*. Then* $N_{\mathsf{f}_Q,0}$ *is finite.*

Proof. Assume that $Q(X) = \sum_{i=0}^{d} a_i X^i$ and let $\mathsf{f} = \mathsf{f}_Q$. Let $u = Q(\theta)$ and let $\|u\| = \sum_{i=0}^{d} |a_i|$. Since $N_{\mathsf{f},0} = N_{-\mathsf{f},0}$, we may assume that $u > 0$. Choose $\epsilon > 0$ such that $\epsilon\|u\| < u$ and let $k \in \mathbf{N}$ be such that for all $n \geq k$ and $i \in [d]$, $|r_{n+i} - \theta^i r_n| < \epsilon r_n$. Then, for all $n \geq k$, $|a_i r_{n+i} - a_i\theta^i r_n| < \epsilon|a_i| r_n$ (whenever $a_i \neq 0$). By our choice of ϵ we have $0 < r_n(u - \epsilon\|u\|) < a_0 r_n + \cdots + a_m r_{n+d}$, for all $n \geq k$. $\square$

Remark 1.12. We choose to make an ϵ-style proof of the previous lemma to stress the fact that, when (r_{n+1}/r_n) converges to θ effectively, we can bound effectively the size of $N_{\mathsf{f},0}$. We can adapt this kind of proof to show that the size of any set of the form $N_{\mathsf{f},z}$ can be bounded effectively if finite. This will be needed in Section 2.3, where we look at the decidability of $\mathscr{Z}_R$.

Lemma 1.13. *Suppose that* R *satisfies a linear recurrence. Then for any* $Q \in \mathbf{Z}[X]$*,* $N_{\mathsf{f}_Q,0}$ *is either finite or* $\mathbf{N}$*. Furthermore,* $N_{\mathsf{f}_Q,0} = \mathbf{N}$ *if and only if* $Q(\theta) = 0$*.*

Proof. Notice that, by assumption, $\mathsf{f}(n)/r_n \to Q(\theta)$. Thus, if $Q(\theta) \neq 0$, $N_{\mathsf{f},0}$ is finite. Otherwise, P_R divides Q and in this case, by Proposition 1.4, $N_{\mathsf{f},0} = \mathbf{N}$. $\square$

We end this section with by a description of sets of the form $N_{\mathsf{f},z}$ when $z \neq 0$.

Proposition 1.14. *Let* f *be an operator and* $z \in \mathbf{Z}\setminus\{0\}$*. Then* $N_{\mathsf{f},z}$ *is finite.*

Proof. Assume that $\mathsf{f} = \mathsf{f}_Q$, where $Q(X) = a_0 + a_1 X + \cdots + a_d X^d$. We may assume that $Q \neq 0$. Since R is regular, we have that $u = \lim_{n\to\infty} \mathsf{f}(n)/r_{n+d}$ is either a_d (when $\theta = \infty$) or $a_0\theta^{-d} + \cdots + a_d = \theta^{-d}Q(\theta)$ (when $\theta \in \mathbf{R}^{>1}$). Notice that if $u \neq 0$, then $N_{\mathsf{f},z}$ is finite since $\lim_{n\to\infty} z/r_{n+d} = 0$. Now, if $u = 0$, then θ is algebraic: $Q(\theta) = 0$. Thus, R satisfies a linear recurrence relation, and since P_R divides Q, we have that $N_{\mathsf{f},0} = \mathbf{N}$, by Proposition 1.4. We conclude that $N_{\mathsf{f},z} = \emptyset$. $\square$

1.1.2. Equations and the induced structure

Let $Q_1, \dots, Q_s \in \mathbf{Z}[X]$ be operators and let $\mathsf{f}_i = \mathsf{f}_{Q_i}$ for all $i \in [s]$. Let $z \in \mathbf{Z}$. We consider the question whether z is in the image of the sum of these operators. This amounts to determine when the equation $\mathsf{f}_1(x_1) + \cdots + \mathsf{f}_s(x_s) = z$ has a solution in $\mathbf{N}^s$. We call a tuple $\bar{n} \in \mathbf{N}^s$ a *non-degenerate solution* of $\mathsf{f}_1(x_1) + \cdots + \mathsf{f}_s(x_s) = z$ when the following conditions hold:

1. $\mathsf{f}_1(n_1) + \cdots + \mathsf{f}_s(n_s) = z$;
2. for all $I \subsetneq [s]$, $\sum_{i \in I} \mathsf{f}_i(n_i) \neq 0$.

We now explain how to decompose $N_{\bar{\mathsf{f}},z}$ into sets of non-degenerate solutions.

Let $\bar{I} = (I_1, \dots, I_k) \in \mathfrak{P}([n])$. To this partition we associate the following system of equations:

$$\begin{cases} \sum_{i \in I_1} \mathsf{f}_i(n_i) = z, \\ \sum_{i \in I_2} \mathsf{f}_i(n_i) = 0, \\ \vdots \\ \sum_{i \in I_k} \mathsf{f}_i(n_i) = 0. \end{cases} \tag{2}$$

Let

$$N^{\mathrm{nd}}_{\bar{\mathsf{f}},z,\bar{I}} = \Big\{ \bar{n} \in \mathbf{N}^s \,\Big|\, \bar{n}_{I_1} \text{ is a non-degenerate solution of } \sum_{i \in I_1} \mathsf{f}_i(n_i) = z \text{ and for all } j \in [k]^{>1} \\ \bar{n}_{I_j} \text{ is a non-degenerate solution of} \sum_{i \in I_j} \mathsf{f}_i(n_i) = 0 \Big\}.$$

When $\bar{I} = ([s])$, we use $N^{\mathrm{nd}}_{\bar{\mathsf{f}},z}$ instead of $N^{\mathrm{nd}}_{\bar{\mathsf{f}},z,\bar{I}}$. In this setting, we decompose $N_{\bar{\mathsf{f}},z}$ as

$$N_{\bar{\mathsf{f}},z} = \bigcup_{\bar{I} \in \mathfrak{P}([s])} N^{\mathrm{nd}}_{\bar{\mathsf{f}},z,\bar{I}}. \tag{3}$$

This decomposition will prove to be quite useful as the set of non-degenerate solutions of $\mathsf{f}_1(x_1) + \cdots + \mathsf{f}_s(x_s) = z$ is easily understood. For instance, Proposition 1.15 implies that for some constant m depending only on $\bar{\mathsf{f}}$ and z, if $\bar{n}$ is a non-degenerate solution, then $\max\{|n_i - n_j| \mid i, j \in [s]\} \leq m$.

Proposition 1.15. *Let $\mathsf{f}_1, \dots, \mathsf{f}_s$ be operators and $z \in \mathbf{Z}$. Then, there exist $k \in \mathbf{N}$ and $\bar{m}_1, \dots, \bar{m}_k \in \mathbf{Z}^s$ such that for all $\bar{\ell} \in \mathbf{N}^s$, if $\bar{\ell} \in N^{\mathrm{nd}}_{\bar{\mathsf{f}},z}$ then for some $i \in [k]$, $\ell_j = \ell_1 + m_{ij}$ for all $j \in [s]$.*

Proof. Assume that $\mathsf{f}_j = \mathsf{f}_{Q_j}$ where $Q_j(X) = \sum_{i=0}^{d_j} a_{ji} X^i$ and $a_{jd_j} \neq 0$. Suppose, towards a contradiction, that the proposition is false:

($\star$) for all $k \in \mathbf{N}$ and $\bar{m}_1, \dots, \bar{m}_k \in \mathbf{Z}^s$, there exists $\bar{\ell} \in N^{\mathrm{nd}}_{\bar{\mathsf{f}},z}$ such that for all $i \in [k]$, $\ell_j \neq \ell_1 + m_{ij}$ for some $j \in [s]$.

From this, we construct two sequences $(\bar{\ell}_i) \subset \mathbf{N}^s$ and $(\bar{m}_i) \subset \mathbf{Z}^s$ that will help us reach a contradiction.

Start with any $\bar{\ell}_1 \in N^{\mathrm{nd}}_{\bar{\mathsf{f}},z}$ and define $\bar{m}_1$ as $m_{1j} = \ell_{1j} - \ell_{11}$ for all $j \in [s]$. Assuming $\bar{\ell}_i$ and $\bar{m}_i$ are constructed, we let $\bar{\ell}_{i+1}$ be a non-degenerate solution obtained from ($\star$) with $k = i$ and $\bar{m}_1, \dots, \bar{m}_i$. We define $\bar{m}_{i+1}$ as $m_{(i+1)j} = \ell_{(i+1)j} - \ell_{(i+1)1}$ for all $j \in [s]$.

We may assume, up to a permutation of $\bar{\mathsf{f}}$ and passing to a sub-sequence using the pigeonhole principle, that

1. for all $i \in \mathbf{N}$, $m_{ij} \leq m_{i(j+1)}$ for all $j < s$.

Again, using the pigeonhole principle, we may assume that

2. there is $j^* \in [s]$ such that for all $i \in \mathbf{N}$, $m_{ij^*} = 0$;
3. for all $i \in \mathbf{N}$, $m_{is} < m_{(i+1)s}$.

We now decompose the tuples $\bar{m}_i$, $i \in \mathbf{N}$, in two parts according to whether the differences $m_{is} - m_{ij} = \ell_{is} - \ell_{ij}$ are bounded. Let $J \subset [s]$ be of maximal size such that for all $j \in J$, $\max\{m_{is} - m_{ij} \mid i \in \mathbf{N}\} < \infty$. Notice that $s \in J$ and $j^* \notin J$. Applying the pigeonhole principle several times, we may assume, without loss of generality, that

4. for all $j \in J$, there exists $k_j \in \mathbf{N}$ such that $m_{is} - m_{ij} = k_j$ for all $i \in \mathbf{N}$;
5. for all $j \notin J$, $m_{is} - m_{ij} \to \infty$.

Let $j_0 = \min J$ and, for all $i \in \mathbf{N}$ and $j \in J$, rewrite ℓ_{ij} as $\ell_{ij_0} + (m_{ij} - m_{ij_0}) = \ell_{ij_0} + (k_{j_0} - k_j)$ (note that $k_{j_0} - k_j \geq 0$). Set $Q'_j(X) = \sum_{n=1}^{d_j} a_{jn} X^{k_{j_0} - k_j + n}$ and $\mathsf{f}'_j = \mathsf{f}_{Q'_j}$. We have, for all $j \in J$,

$$\begin{aligned} \mathsf{f}_j(\ell_{ij}) &= \mathsf{f}_j(\ell_{ij_0} + (m_{ij} - m_{ij_0})) \\ &= \mathsf{f}_j(\ell_{ij_0} + (k_{j_0} - k_j)) \\ &= \mathsf{f}'_j(\ell_{ij_0}). \end{aligned}$$

Define $Q(X) = \sum_{j \in J} Q'_j(X)$, let d be the degree of Q and a_d be the coefficient of X^d in Q. For all $i \in \mathbf{N}$,

$$\begin{aligned} \sum_{j \in J} \mathsf{f}_j(\ell_{ij}) &= \sum_{j \in J} \mathsf{f}'_j(\ell_{ij_0}) \\ &= \mathsf{f}_Q(\ell_{ij_0}). \end{aligned}$$

Notice that by non-degeneracy and the fact that J is a proper (non-empty) subset of $[s]$, $Q \neq 0$ (in particular $a_d \neq 0$).

Since, for all $n \in \mathbf{N}$, $\lim_{k \to \infty} r_n / r_{n+k} = 0$, for all $j \notin J$,

$$u_j = \lim_{i \to \infty} \frac{\mathsf{f}_j(\ell_{i1} + m_{ij})}{r_{\ell_{1i} + m_{ij_0} + d}} = \lim_{i \to \infty} \sum_{n=0}^{d_j} \frac{a_{jn} r_{\ell_{i1} + m_{ij} + n}}{r_{\ell_{1i} + m_{ij_0} + d}} = 0.$$

(Indeed, by 4 and 5, for all $j \notin J$, $m_{ij_0} - m_{ij} = m_{is} - m_{ij} - k_{j_0} \to \infty$.)

Let us now perform a similar calculation. Recall that for all $k \in \mathbf{N}^{>0}$,

$$\lim_{n \to \infty} r_n / r_{n+k} = \begin{cases} \theta^{-k} & \text{if } \theta \in \mathbf{R}^{>1} \\ 0 & \text{if } \theta = \infty, \end{cases}$$

So we have that

$$\begin{aligned} u_J &= \lim_{i \to \infty} \frac{\sum_{j \in J} \mathsf{f}_j(\ell_{i1} + m_{ij})}{r_{\ell_{i1} + m_{ij_0 + d}}} \\ &= \lim_{i \to \infty} \frac{\mathsf{f}_Q(\ell_{ij_0})}{r_{\ell_{ij_0 + d}}} \\ &= \begin{cases} \theta^{-d} Q(\theta) & \text{if } \theta \in \mathbf{R}^{>1} \\ a_d & \text{if } \theta = \infty. \end{cases} \end{aligned}$$

Thus,

$$\lim_{i \to \infty} \sum_{j=1}^{s} \frac{\mathsf{f}_j(\ell_{i1} + m_{ij})}{r_{\ell_{i1} + m_{ij_0} + d}} = u_J + \sum_{j \notin J} u_j = u_J = \lim_{i \to \infty} \frac{z}{r_{\ell_{i1} + m_{ij_0} + d}} = 0,$$

where that last equality comes from the fact that, by 3 and 4, the sequence $(r_{\ell_{i1}+m_{ij_0}+d})$ is not bounded.

Since $u_J = 0$ and $a_d \neq 0$, we must have $\theta \in \mathbf{R}^{>1}$. In particular, $u_J = \theta^d Q(\theta)$. So $Q(\theta) = 0$. Since R is regular and $Q \neq 0$, R satisfies a linear recurrence relation and P_R divides Q. So by Proposition 1.9 $N_{\mathsf{f}_Q,0} = \mathbf{N}$, in contradiction with the assumption that $\bar{\ell}_i$ is non-degenerate for all $i \in \mathbf{N}$ and the fact that J is a proper non-empty subset of $[s]$. ☐

For an operator f_Q, $Q(X) = \sum_{i=0}^{d} a_i X^i$, let

$$N^{\circ}_{\mathsf{f}_Q,z} = \{n \in N_{\mathsf{f}_Q,z} \mid \text{for all } I \subsetneq [d], \sum_{i \in I} a_i r_{n+i} \neq 0\}.$$

For an s-tuple $\bar{\mathsf{f}}$ of operators and $\bar{n} \in \mathbf{N}^s$, we let $\mathsf{f}_{\bar{n}}(\ell) = \sum_{j=1}^{s} \mathsf{f}_j(\ell + n_j)$.

Remark 1.16. Let $\mathsf{f}_1, \dots, \mathsf{f}_s$ be operators and $z \in \mathbf{Z}$. By Proposition 1.14, there exist k and $\bar{m}_1, \dots, \bar{m}_k \in \mathbf{Z}^s$ such that, letting $m_i = \min\{m_{i1}, \dots, m_{is}\}$ and $\bar{n}_i = \bar{m}_i - m_i$:

$\bar{\ell} \in N^{\mathrm{nd}}_{\bar{\mathsf{f}},z}$ if and only if for some $i \in [k]$, $\ell_1 + m_i \in N^{\circ}_{\mathsf{f}_{\bar{n}_i},z}$ and $\ell_j = \ell_1 + m_{ij}$ for all $j \in [s]$.

Corollary 1.17. *Let $\mathsf{f}_1, \dots, \mathsf{f}_s$ be operators and $z \in \mathbf{Z}$. Let $k \in \mathbf{N}$ and $\bar{m}_1, \dots, \bar{m}_k \in \mathbf{Z}^s$ be given by Proposition 1.15. Then $N^{\mathrm{nd}}_{\bar{\mathsf{f}},z}$ is infinite if and only if $z = 0$ and $N^{\circ}_{\mathsf{f}_{\bar{m}_i},0}$ is infinite for some $i \in [k]$.*

Proof. This follows from Propositions 1.14 and 1.15. ☐

The following corollary states that operators are ultimately injective functions, unless $N_{\mathsf{f},0}$ is infinite.

Corollary 1.18. *Let $Q \in \mathbf{Z}[X]$. Then exactly one of the following holds:*

- $N_{\mathsf{f}_Q,0} = \mathbf{N}$*;*
- $N_{(\mathsf{f}_Q,-\mathsf{f}_Q),0} \backslash \{(n,n) \mid n \in \mathbf{N}\}$ *is finite.*

Proof. Assume $N_{\mathsf{f}_Q,0}$ is finite. Let us then show that $N_{(\mathsf{f}_Q,-\mathsf{f}_Q),0} \backslash \{(n,n) \mid n \in \mathbf{N}\}$ is finite. Since $N_{\mathsf{f}_Q,0}$ is finite, $N_{(\mathsf{f}_Q,-\mathsf{f}_Q),0} \backslash N^{\mathrm{nd}}_{(\mathsf{f}_Q,-\mathsf{f}_Q),0}$ is finite, so we only need to show that $N^{\mathrm{nd}}_{(\mathsf{f}_Q,-\mathsf{f}_Q),0} \backslash \{(n,n) \mid n \in \mathbf{N}\}$ is finite. By Proposition 1.15, this amounts to show that for all $k \in \mathbf{N}^{>0}$, the operator $\mathsf{f}_k(n) = \mathsf{f}_Q(n) - \mathsf{f}_Q(n+k)$ is such that $N_{\mathsf{f}_k,0}$ is finite. Assume on the contrary that $N_{\mathsf{f}_k,0} = \mathbf{N}$ for some $k \in \mathbf{N}^{>0}$. This implies that R satisfies a linear recurrence relation and in that case P_R divides $Q(X)(1 - X^k)$. But since $\theta > 1$, we must have that P_R divides Q, in contradiction with the fact that $N_{\mathsf{f}_Q,0}$ is finite. ☐

As a corollary of Proposition 1.15 and the following result, we obtain the superstability of R^0_{ind}.

Proposition 1.19 ([4, Proposition 5.9])**.** *Let $\mathscr{N}$ be the structure $(\mathbf{N}, S, S^{-1}, 0)$, where $S(n) = n+1$, $S^{-1}(n+1) = n$ and $S^{-1}(0) = 0$. Then $\mathscr{N}^1$ is superstable of U-rank* 1.

Corollary 1.20. *Let R be a regular sequence. Then R^0_{ind} is definably interpreted in $\mathscr{N}$.*

Proof. We interpret the domain of R^0_{ind} as $\mathbf{N}$. Let $a_1, \dots, a_s \in \mathbf{Z}\backslash\{0\}$. We need to interpret in $\mathscr{N}$ the set of s-tuples of elements in R that satisfy the equation $a_1 x_1 + \dots + a_s x_s = 0$. For all $i \in [s]$, let f_i be the operator $n \mapsto a_i r_n$. We interpret $\{\bar{x} \in R^s \mid a_1 x_1 + \dots + a_n x_n = 0\}$ as $N_{\bar{\mathsf{f}},0}$ in $\mathscr{N}$. Let us show that $N_{\bar{\mathsf{f}},0}$ is definable in $\mathscr{N}$. As explained at the beginning of Section 1.1, the set

$$N_{\bar{\mathsf{f}},0} = \bigcup_{\bar{I} \in \mathfrak{P}([s])} N^{\mathrm{nd}}_{\bar{\mathsf{f}},0,\bar{I}}.$$

So we need only to show that $N^{\mathrm{nd}}_{\bar{\mathsf{f}},0,\bar{I}}$ is definable in $\mathscr{N}$ for all $\bar{I} \in \mathfrak{P}([s])$. We focus on the case $\bar{I} = ([s])$, the general case being similar. By Remark 1.16, to show that $N_{\bar{\mathsf{f}},0}$ is definable, we only need to show that $N^{\circ}_{\mathsf{f}_{\bar{n}_i},0}$ is definable for all $i \in [k]$. But as $N^{\circ}_{\mathsf{f}_{\bar{n}_i},0}$ is either empty or cofinite in $N_{\mathsf{f}_{\bar{n}_i},0}$, we only need to show that the latter is definable. But we know that, by Proposition 1.9, the set $N_{\mathsf{f}_{\bar{n}_i},0}$ is either finite or $\mathbf{N}$, hence definable. ☐

1.2. Every $\mathcal{L}_R$-formula is bounded

Throughout this section, we will use the following notations. Let $\bar{\mathsf{f}}$ be a tuple of k operators. Define $\mathrm{Im}(\bar{\mathsf{f}})$ as

$$\{a \in \mathbf{Z} \mid a = \mathsf{f}_1(n_1) + \cdots + \mathsf{f}_k(n_k) \text{ for some } \bar{n} \in \mathbf{N}^k\}.$$

Notice that $\mathrm{Im}(\bar{\mathsf{f}}) = \{a \in \mathbf{Z} \mid N_{\bar{\mathsf{f}},a} \neq \emptyset\}$. Similarly define $\mathrm{Im}^+(\bar{\mathsf{f}})$ as $\mathrm{Im}(\bar{\mathsf{f}}) \cap \mathbf{N}$.

This section is devoted to the proof of the following theorem.

Theorem 1.21. *Let $a, d \in \mathbf{N}$, $d > 0$. Then, the set $a + d\mathbf{N}$ cannot be covered by finitely many sets of the form $z + \mathrm{Im}^+(\bar{\mathsf{f}})$, where $\bar{\mathsf{f}}$ is a tuple of k operators, $k \in \mathbf{N}$ and $z \in \mathbf{Z}$.*

Recall that a set $A \subset \mathbf{N}$ is called *piecewise syndetic* if there exists $d \in \mathbf{N}^{>0}$ such that for all $k \in \mathbf{N}$, there exists $a_1 < \cdots < a_k \in A$ such that $a_{i+1} - a_i \leq d$ for all $i \in [k-1]$. A key property of piecewise syndetic sets is the so-called Brown's Lemma.

Theorem 1.22 (Brown's Lemma [13, Theorem 10.37])**.** *Let $A \subset \mathbf{N}$ be piecewise syndetic. If $A = A_1 \cup \cdots \cup A_n$, then there exists $i \in [n]$ such that A_i is piecewise syndetic.*

In the next proposition, we show that the image of arbitrary linear combinations of operators is not piecewise syndetic.

Proposition 1.23. *Let $\bar{\mathsf{f}}$ be a tuple of k operators. Then $\mathrm{Im}^+(\bar{\mathsf{f}})$ is not piecewise syndetic.*

Before giving a proof of Proposition 1.23, let us show how it is used to prove Theorem 1.21.

Proof of Theorem 1.21. Since $a + d\mathbf{N}$ is piecewise syndetic, if it were covered by sets of the form $z + \mathrm{Im}^+(\bar{\mathsf{f}})$, then one of them would also be piecewise syndetic, by Brown's Lemma. But this would imply that a set of the form $\mathrm{Im}^+(\bar{\mathsf{f}})$ is piecewise syndetic since any translate of a piecewise syndetic set is again piecewise syndetic. This contradicts Proposition 1.23. $\square$

We will need the following lemma.

Lemma 1.24. *Let $\mathsf{f}_1, \ldots, \mathsf{f}_k$ be operators and $e \in \mathbf{N}^{>0}$. Let X_e be the set $\{a \in \mathrm{Im}^+(\bar{\mathsf{f}}) \mid \exists a' \in \mathrm{Im}^+(\bar{\mathsf{f}}), |a - a'| = e\}$. Then there exist a finite set Z of integers such that*

$$X_e \subset \bigcup_{z \in Z} \bigcup_{I \subsetneq [k]} (z + \mathrm{Im}^+(\bar{\mathsf{f}}_I)).$$

Proof. We first identify Z. Let $a \in X_e$. By definition, there is $a' \in \mathrm{Im}^+(\bar{\mathsf{f}})$ such that $e = a - a'$ or $e = a' - a$, that we shorten by $e = \pm(a - a')$. Since both a and a' are in $\mathrm{Im}^+(\bar{\mathsf{f}})$, we can find $\bar{n}, \bar{n}' \in \mathbf{N}^k$ such that

$$a = \sum_{i=1}^{k} \mathsf{f}_i(n_i) \text{ and } a' = \sum_{i=1}^{k} \mathsf{f}_i(n_i').$$

Since $e = \pm(a - a')$ we can find $I, I' \subset [k]$ such that

$$e = \pm\left(\sum_{i \in I} \mathsf{f}_i(n_i) - \sum_{i \in I'} \mathsf{f}_i(n_i')\right),$$

$(\bar{n}_I, \bar{n}'_{I'}) \in N^{\mathrm{nd}}_{\bar{\mathsf{f}}_I \cup -\bar{\mathsf{f}}_{I'}, e} \cup N^{\mathrm{nd}}_{-\bar{\mathsf{f}}_I \cup \bar{\mathsf{f}}_{I'}, e}$ and

$$0 = \sum_{i \notin I} \mathsf{f}_i(n_i) - \sum_{i \notin I'} \mathsf{f}_i(n_i').$$

We thus let Z be the set

$$\left\{ \sum_{i \in I} \mathsf{f}_i(n_i), \sum_{i \in I'} \mathsf{f}_i(n_i') \;\middle|\; I, I' \subset [k], (\bar{n}_I, \bar{n}'_{I'}) \in N^{\mathrm{nd}}_{\bar{\mathsf{f}}_I \cup -\bar{\mathsf{f}}_{I'}, e} \cup N^{\mathrm{nd}}_{-\bar{\mathsf{f}}_I \cup \bar{\mathsf{f}}_{I'}, e} \right\} \cup \{0\}.$$

By Corollary 1.17, we have that the sets $N^{\mathrm{nd}}_{-\bar{\mathsf{f}}_I \cup \bar{\mathsf{f}}_{I'},e}$ and $N^{\mathrm{nd}}_{\bar{\mathsf{f}}_I \cup -\bar{\mathsf{f}}_{I'},e}$ are finite. Hence Z is finite.

Let us show that

$$a \in \bigcup_{z \in Z} \bigcup_{I \subsetneq [k]} (z + \mathrm{Im}^+(\bar{\mathsf{f}}_I)).$$

We distinguish three cases.

1. $I = [n]$. In that case, $a \in Z$.
2. $\emptyset \neq I \subsetneq [n]$. In that case, $[n] \setminus I$ is a proper subset of $[n]$ and

$$a = z + \sum_{i \in [n] \setminus I} \mathsf{f}_i(n_i),\ z = \sum_{i \in I} \mathsf{f}_i(n_i) \in Z.$$

3. $I = \emptyset$. Since $e > 0$, we have that $I' \neq \emptyset$. Now if $I' = [n]$, we have $a = 0 \in Z$. So let us assume that $I' \subsetneq [n]$. In that case, $a = \sum_{i \in [n] \setminus I'} \mathsf{f}_i(n'_i)$. Since $[n] \setminus I'$ is a proper subset of $[n]$, a has the required form. □

We now prove Proposition 1.23 by induction on the length of the tuple $\bar{\mathsf{f}}$.

Proof of Proposition 1.23. Let f be an operator. By Lemma 1.24, we have that X_e is finite for all $e \in \mathbf{N}^{>0}$. This implies that $\mathrm{Im}^+(\mathsf{f})$ cannot be piecewise syndetic.

Let $k > 1$ and assume that the proposition holds for all tuple $\bar{\mathsf{f}}$ of length $\leq k$. Let $\mathsf{f}_1, \dots, \mathsf{f}_{k+1}$ be operators such that $\mathrm{Im}^+(\bar{\mathsf{f}})$ is infinite. Suppose, towards a contradiction that $\mathrm{Im}^+(\bar{\mathsf{f}})$ is piecewise syndetic. Assume $d \in \mathbf{N}^{>0}$ witnesses the fact that $\mathrm{Im}^+(\bar{\mathsf{f}})$ is piecewise syndetic. Recall that we defined X_e as $\{a \in \mathrm{Im}^+(\bar{\mathsf{f}}) \mid \exists a' \in \mathrm{Im}^+(\bar{\mathsf{f}}), |a - a'| = e\}$. Even though $X_1 \cup \dots \cup X_d$ may not equal $\mathrm{Im}^+(\bar{\mathsf{f}})$, it is this subset that will play a key role in the rest of the proof, as it is the "syndetic part of $\mathrm{Im}^+(\bar{\mathsf{f}})$ with respect to d". Indeed, the set $X_1 \cup \dots \cup X_d$ is itself piecewise syndetic so that by Brown's Lemma, there exists $i \in [d]$ such that X_i is also piecewise syndetic. But by lemma 1.24 we know that X_i is contained in a finite union of sets of the form $z + \mathrm{Im}^+(\bar{\mathsf{f}}')$, where $\bar{\mathsf{f}}'$ is of length $\leq k$. But this implies, by Brown's Lemma and the fact that a set containing a piecewise syndetic set is itself piecewise syndetic, the existence of a piecewise syndetic set of the form $z + \mathrm{Im}^+(\bar{\mathsf{f}}')$, where $\bar{\mathsf{f}}'$ is of length $\leq k$. This contradicts our induction hypothesis. So $\mathrm{Im}^+(\bar{\mathsf{f}})$ is not piecewise syndetic, which is what we wanted. □

Corollary 1.25. *Let R be enumerated by regular sequence. Then R is bounded.*

Proof. The proof follows [15, Lemma 3.4 and Lemma 3.5] and is based on [3, Proposition 2.1]. That proposition states, for an $\mathcal{L}$-structure $\mathscr{M}$ and a small subset A, that A is bounded if $\mathscr{M}$ is both stable and nfcp over A. The proof of [3, Proposition 2.1] is an induction on the number of quantifiers in $\mathcal{L}_A$-formulas. Palacín and Sklinos [15, Lemma 3.5] noticed that the smallness of A and nfcp over A can be weakened to the following statement:

($\star$) for any $\mathcal{L}$-formula $\varphi(\bar{x}, y, \bar{z})$, there exists $k \in \mathbf{N}$ such that

$$\mathscr{M}_A \models \forall \bar{x} \left(\left(\forall \bar{z}_0 \in A \dots \forall \bar{z}_k \in A \exists y \bigwedge_{j<k} \varphi(\bar{x}, y, \bar{z}_j) \right) \to \exists y \forall \bar{z} \in A \varphi(\bar{x}, y, \bar{z}) \right).$$

Let us now explain how one proceeds in the case of a regular sequence.

First let us show that, for any $\mathcal{L}_g$-formula $\varphi(\bar{x}, y, \bar{z})$, any consistent set the form

$$\Gamma(y) = \{\varphi(\bar{b}, y, \bar{\alpha}) \mid \bar{\alpha} \in R^n\},$$

where $\bar{b} \in \mathbf{Z}$ and n is the length of the tuple $\bar{z}$, is realized by some $c \in \mathbf{Z}$.

Using quantifier elimination, the consistency of $\Gamma(y)$ and the properties of the congruence relations, we can assume that φ is of the form

$$\bigwedge_{i\in I_1} t_i(\bar{x},y,\bar{z})=0\wedge\bigwedge_{i\in I_2} t_i(\bar{x},y,\bar{z})\neq 0\wedge\bigwedge_{i\in I_3} D_{n_i}(y+t_i'(\bar{x},\bar{z})),$$

where $t_i(\bar{x},y,\bar{z})$ and $t_j'(\bar{x},\bar{z})$ are terms for all $i\in I_1\cup I_2$ and $j\in I_3$. We may further assume that $I_1=\emptyset$ (otherwise it is clear that $y\in\mathbf{Z}$). Given $i\in I_2$, the term $t_i(\bar{b},y,\bar{z})$ is equal to $m_i y+z_i+a_1z_1+\cdots+a_{\ell_i}z_{\ell_i}$, where $m_i,z\in\mathbf{Z}$, $\bar{a}_i\in\mathbf{Z}^{\ell_i}$ and $\ell_i\in\mathbf{N}$. Notice that we may assume that $m_i=m$ for all $i\in I_2$ (otherwise, we multiply the inequation $t_i(\bar{x},y,\bar{z})\neq 0$ by $\prod_{j\neq i}m_j$ for all $i\in I_2$).

Thus $\Gamma(y)$ expresses the fact that y is in a coset of a subgroup of $\mathbf{Z}$, say $c+d\mathbf{Z}$ for some $c,d\in\mathbf{N}$, and my is not in the set

$$X=\bigcup_{i\in I_2}\{z_i+a_1z_1+\cdots+a_{\ell_i}z_{\ell_i}\mid \bar{z}\in R^{\ell}\}.$$

But, by Theorem 1.21, X does not cover $mc+md\mathbf{N}$ (use operators of the form $n\mapsto ar_n$ to apply the theorem). So there is $s\in\mathbf{N}$ such that $m(c+ds)$ is not in X, which is what we wanted. This shows [15, Lemma 3.4] for regular sequences.

From this and the fact that $\mathscr{Z}$ has nfcp, we deduce directly that $(\star)$ holds for regular sequences. Thus R is bounded. $\square$

Remark 1.26. G. Conant and C. Laskowski recently showed in [5, Theorem 2.8] that *any* subset of a weakly minimal group is bounded, at the cost of adding constants in the language[3]. This applies in particular to $\mathscr{Z}$ and thus eliminates the necessity of Corollary 1.25 in the proof of our Theorem 1.5. However, the material of this section – specifically Proposition 1.23 – is needed in Section 2 towards the proof of Theorem 2.1, namely in Propositions 2.15 and 2.16.

1.3. Main theorem

We are now able to prove the main theorem of this section.

Theorem 1.5. *Let R be enumerated by a regular sequence. Then* $\mathrm{Th}(\mathscr{Z}_R)$ *is superstable of Lascar rank ω.*

Proof. By Proposition 1.19 and Corollary 1.20, we get that R^0_{ind} is superstable. Furthermore, by Corollary 1.25 R is bounded. So, we deduce from Theorem 1.7 that $\mathscr{Z}_R$ is superstable. Since $\mathscr{Z}_R$ is a proper expansion of $\mathscr{Z}$, it must have Lascar rank $\geq\omega$ by [15, Theorem 1]. So what remains to be shown is that the rank is $\leq\omega$. For $R=\Pi_q$, this is done in [15, Theorem 2] and the only property that we need to check here is that the U-rank of R_{ind} is 1. But, by Proposition 1.19, $\mathscr{N}$ has U-rank 1 and by Corollary 1.20, R_{ind} is definably interpreted in $\mathscr{N}$.

For convenience of the reader, we give now a sketch of [15, Theorem 2]. Let $\mathscr{G}\succ\mathscr{Z}_R$ be a monster model. Since $\mathscr{G}$ is a superstable group, it has a unique generic type $p\in S_1^{\mathcal{L}_R}(\emptyset)$ in the connected component of $\mathscr{G}$. Since the type p has maximal U-rank (in the sense of the $\mathcal{L}_R$-theory of $\mathscr{Z}_R$), it suffices to show that $U(p)\leq\omega$. By definition, this amounts to show that any forking extension of p has finite U-rank.

First let us show that if $u\in\mathrm{acl}^{\mathcal{L}_R}(R(G),B)$, then $U(u/B)<\omega$ (†). Let $\bar{c}\in R(G)^n$ be such that $u\in\mathrm{acl}^{\mathcal{L}_R}(R(G),\bar{c})$. By Proposition 1.19 and Corollary 1.20, $U(R(G))\leq n$. Then by Lascar's inequality $U(u/B)\leq U(\bar{c}/B)\leq n<\omega$.

Now consider a forking extension of p, say $\mathrm{tp}^{\mathcal{L}_R}(b/B)$ with $|B|<|G|$ and suppose that $b\notin\mathrm{acl}^{\mathcal{L}_R}(R(G),B)$.

Let $q=\mathrm{tp}(a/B)$ be a non-forking extension of p, $U(q)\geq\omega$. By (†) $a\notin\mathrm{acl}^{\mathcal{L}_R}(R(G),B)$ and hence $a\notin\mathrm{acl}^{\mathcal{L}_g}(R(G),B)$. So, in the $\mathcal{L}_g$-theory of $\mathscr{Z}$, $U(a/R(G),B)=1$. Thus, $\mathrm{tp}^{\mathcal{L}_g}(a/R(G),B)$

[3]A group G is weakly minimal when its $\{+,0\}$-theory is superstable of U-rank 1.

is a generic type. This is also true for $\mathrm{tp}^{\mathcal{L}_g}(b/R(G), B)$. So there exists $g \in G$ such that $\mathrm{tp}^{\mathcal{L}_g}(b+g/R(G), B) = \mathrm{tp}^{\mathcal{L}_g}(a/R(G), B)$. This implies, by [15, Corollary 3.7], $\mathrm{tp}^{\mathcal{L}_R}(b+g/B) = \mathrm{tp}^{\mathcal{L}_R}(a/B)$. So $\mathrm{tp}^{\mathcal{L}_R}(b/B)$ is also generic and hence non-forking, a contradiction. $\square$

As we mentioned in the introduction, there is an overlap between Theorem 1.5 and [4, Theorem 7.1]. We recall that a sequence (r_n) is *geometrically sparse* [4, Definition 6.2] if there exists a sequence $(\lambda_n) \subset \mathbf{R}^{\geq 1}$ such that $X = \{\lambda_m/\lambda_n \mid n < m\}$ is closed and discrete and $\sup_{n\in\mathbf{N}} |r_n - \lambda_n| < \infty$.

In our comparison between our theorem and [4, Theorem 7.1], we will use the following observation about sparse sequences (r_n) such that $r_{n+1}/r_n \to \theta \in \mathbf{R}^{>1}$: for such a sequence, there exists $\tau \in \mathbf{R}^{\geq 1}$ such that $r_n/\theta^n \to \tau$. Indeed, since $\sup_{n\in\mathbf{N}} |r_n - \lambda_n| < \infty$ and $r_{n+1}/r_n \to \theta$, we have

$$\lim_{n\to\infty} \frac{\lambda_n}{r_n} = 1 \text{ and } \lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = \theta.$$

But, as X is closed and discrete, if the sequence $(\lambda_{n+1}/\lambda_n)$ converges, then it is ultimately constant and so ultimately equal to θ. Hence (λ_n/θ^n) converges to some limit $\tau \in \mathbf{R}^{\geq 1}$.

Now, let us discuss the overlap between Theorem 1.5 and [4, Theorem 7.1]:

- the case where $r_{n+1}/r_n \to \infty$ is completely covered by [4, Theorem 7.1] (as a consequence of [4, Proposition 6.3]);

- the case where $r_{n+1}/r_n \to \theta$ and θ is algebraic, we provide more examples of expansions by recurrence relations than [4, Theorem 7.1]: in addition to our hypotheses, θ needs to be either a Pisot number or a Salem number in order to be geometrically sparse. In fact, we can show by direct calculations that the sequence defined by $r_{n+2} = 5r_{n+1} + 7r_n$, $r_1 = 1$ and $r_0 = 0$, is regular but not geometrically sparse. Indeed, first notice that for all $n \in \mathbf{N}$, $r_n = \alpha(\lambda_+^n - \lambda_-^n)$, where $\alpha = 1/\sqrt{53}$, $\lambda_\pm = (5 \pm \sqrt{53})/2$. Then assume that there is a sequence (λ_n) such that $\sup\{|r_n - \lambda_n| \mid n \in \mathbf{N}\} = k \in \mathbf{R}$. Let $\kappa_n = \lambda_n/\lambda_+^n$ and note that $\kappa_n \to \alpha$. Now we have $|r_n - \lambda_n| = |(\alpha - \kappa_n)\lambda_+^n - \alpha\lambda_-^n|$ $(*)$. We want to show that $\{\lambda_m/\lambda_n \mid n \leq m\}$ cannot be both closed and discrete. Assume towards a contradiction that $\{\lambda_m/\lambda_n \mid n \leq m\}$ is closed and discrete. In that case, λ_{n+1}/λ_n is ultimately equal to λ_+. Thus κ_{n+1}/κ_n is ultimately equal to 1. This in turn implies that $\kappa_n = \alpha$ for all sufficiently large $n \in \mathbf{N}$. But in this case, $(*) = |\alpha\lambda_-^n|$ for all sufficiently large $n \in \mathbf{N}$, in contradiction with the boundness of $(*)$;

- for the case where $r_{n+1}/r_n \to \theta$ and θ is transcendental, the overlap is less precise and we did not manage make a clear distinction between the two results. However, if (r_n) is geometrically sparse, that is $\sup_{n\in\mathbf{N}} |r_n - \lambda_n| < \infty$ for some sequence $(\lambda_n) \subset \mathbf{R}^{\geq 1}$ such that $X = \{\lambda_m/\lambda_n \mid n < m\}$ is closed and discrete, then the sequence $(r_n + n)$ is not geometrically sparse but satisfies Theorem 1.5. Since there exists $\tau \in \mathbf{R}^{\geq 1}$ such that $r_n/\theta^n \to \tau$, we may assume $\lambda_n = \tau\theta^n$.
  Assume, towards a contradiction, that there is a sequence (λ'_n) such that $\sup_{n\in\mathbf{N}} |r_n + n - \lambda'_n| < \infty$ and $X' = \{\lambda'_m/\lambda'_n \mid n \leq m\}$ is closed in discrete. Now let $\kappa_n = \tau\theta^n + n - \lambda'_n$. Notice that (κ_n) is bounded since we assumed (r_n) geometrically sparse. So we have that $\lambda'_{n+1}/\lambda'_n \to \theta$. Since X' is closed in discrete, this last sequence is ultimately constant: for all $n \in \mathbf{N}$ sufficiently large,
  $$\theta = \frac{\tau\theta^{n+1} + n + 1 - \kappa_{n+1}}{\tau\theta^n + n - \kappa_n}.$$
  So, for all sufficiently large $n \in \mathbf{N}$, $n(\theta - 1) = 1 - \kappa_{n+1} + \theta\kappa_n$, a contradiction.

The assumption on θ when it is algebraic cannot be removed. Indeed for all $a, b \in \mathbf{N}$ with $b > 0$, if R is enumerated by $(a + bn)$, then $\mathscr{Z}_R$ is unstable[4] and satisfy the linear recurrence $r_{n+2} = 2r_{n+1} - r_n$.

[4]This is also true for any sequence (r_n) such that there exists $k \in \mathbf{N}$ such that for all $n \in \mathbf{N}$, $|r_{n+1} - r_n| \leq k$.

2. The theory T_R

In this section, we axiomatize, in a language $\mathcal{L} \supset \mathcal{L}_g$, the theory T_R of $\mathscr{Z}_R = (\mathbf{Z}, +, -, 0, R)$, where $R(\mathbf{Z})$ is enumerated by a regular sequence (r_n). We show that T_R has quantifier elimination in $\mathcal{L}$ and has a prime model (and hence T_R is complete). Using this quantifier elimination result, we then prove, by means of counting of types, that T_R is superstable. As a consequence we deduce that the $\mathcal{L}_R$-theory $\mathrm{Th}(\mathscr{Z}_R)$ is superstable. We then close this section with a decidability result and point out the parallel between our results and those corresponding to expansions of Presburger arithmetic. We furthermore use a quantifier elimination result of F. Point to show that expansions of Presburger arithmetic by regular sequences are NIP.

From now on, we fix a infinite set $R(\mathbf{Z}) \subset \mathbf{N}$ that is enumerated by a regular sequence. We know then from the previous section that

1. (Proposition 1.15) for all $Q_1, \dots, Q_s \in \mathbf{Z}[X]$, there is $k = k(\bar{Q}) \in \mathbf{N}$ and a finite set $E = E_{\bar{Q}} \subset \mathbf{Z}^k$ such that for all $\bar{\ell} \in \mathbf{N}^s$, if $\bar{\ell} \in N^{\mathrm{nd}}_{\bar{\mathsf{f}},0}$ then for some $\bar{m} \in E$, $l_i = l_1 + m_i$ for all $i \in [k]$, where $\mathsf{f}_i = \mathsf{f}_{Q_i}$ for all $i \in [s]$;
2. (Corollary 1.18) for all $Q \in \mathbf{Z}[X]$, either $N_{\mathsf{f}_Q,0} = \mathbf{N}$ or there is $e = e(Q)$ such that for all $n, m > e$, if $\mathsf{f}_Q(n) = \mathsf{f}_Q(m)$ then $n = m$. Let Triv be the set of $Q \in \mathbf{Z}[X]$ such that $N_{\mathsf{f}_Q,0} = \mathbf{N}$. Note that $\mathrm{Triv} = \{0\}$ unless θ is algebraic, in which case Triv is the ideal of $\mathbf{Z}[X]$ generated by P_R.

Our choice of $\mathcal{L}$ will allow us to express the two above properties in a first order way.

2.1. Axiomatization and quantifier elimination

Let us define the language in which we axiomatize $\mathscr{Z}_R$. As mentioned in the introduction, $\mathcal{L}_g$ is the language $\{+, -, 0, D_n \mid n \in \mathbf{N}^{>1}\}$ and $\mathcal{L}_S$ is the language $\{S, S^{-1}, c\}$. These new symbols are interpreted in $\mathscr{Z}$ as follows: c is interpreted as r_0, for all $n \in \mathbf{N}$, $S(r_n) = r_{n+1}$, $S^{-1}(r_{n+1}) = r_n$, $S^{-1}(r_0) = r_0$ and $S(z) = z = S^{-1}(z)$ for all $z \in \mathbf{Z} \setminus R(\mathbf{Z})$. To each $Q \in \mathbf{Z}[X]$, we let $\mathsf{f} = \mathsf{f}_Q$ be the $\mathcal{L}_g \cup \mathcal{L}_S$-term $\sum_{i=0}^d n_i S^i(x)$ $(\star)$, where $Q(X) = \sum_{i=0}^d n_i X^i$ and $S^0(x) = x$. Notice that such terms are similar to the operators of the previous section: in fact a term of the form $(\star)$ composed with the function $n \mapsto r_n$ will be an operator in the sense of Definition 1.8. This explains why we decided to keep the same notations. Furthermore, in this section, the symbol f will always denote a term of the form f_Q.

We now work in $\mathcal{L}_g \cup \{1\} \cup \mathcal{L}_S$, where 1 is a constant symbol that is interpreted in $\mathscr{Z}_R$ by the integer 1. For $n, m \in \mathbf{N}$ we let $\mathbf{Z}[X]^{n \times m}$ be the set of $n \times m$ matrices with entries in $\mathbf{Z}[X]$.

Let $[Q] = (Q_{ij}) \in \mathbf{Z}[X]^{n \times m}$ and let $\varphi_{[Q]}(\bar{x}, \bar{y})$ be the formula

$$\bigwedge_{i \in [n]} \sum_{j \in [m]} \mathsf{f}_{Q_{ij}}(x_j) = y_i.$$

Notice that, working in $\mathscr{Z}_R$, the formula $\exists \bar{x} \in R\ \varphi_{[Q]}(\bar{x}, \bar{y})$ expresses the fact that

$$\bigcap_{i \in [n]} N_{\bar{\mathsf{f}}_i, y_i} \neq \emptyset.$$

Let $D = \{(P_i, \ell_i, k_i) \mid i \in [m]\}$ be a (finite) subset of $\mathbf{Z}[X] \times \mathbf{N} \times \mathbf{N}$ such that if $(P, \ell, k) \in D$, then $k < \ell$. We call such a set D a *set of divisibility conditions* and define $\varphi_D(\bar{x})$ as the formula

$$\bigwedge_{i \in [m]} D_{\ell_i}(\mathsf{f}_{P_i}(x_i) + k_i).$$

To $[Q]$ and D as above, we associate an n-ary predicate $\mathrm{Im}_{[Q],D}(\bar{y})$. When D is empty, we write $\mathrm{Im}_{[Q]}$ instead of $\mathrm{Im}_{[Q],D}$. In $\mathscr{Z}_R$, the predicate $\mathrm{Im}_{[Q],D}(\bar{y})$ is interpreted as follows: for all $z \in \mathbf{Z}^{|\bar{y}|}$, $\mathrm{Im}_{[Q],D}(\bar{z})$ if and only if $\exists \bar{x} \in R(\varphi_{[Q]}(\bar{x}, \bar{y}) \wedge \varphi_D(\bar{x}))$. So this symbol states that $\bar{z}$ is in the image of sums of operators, and a witness of this fact satisfies certain divisibility conditions.

Finally let $\mathcal{L}$ be the language

$$\mathcal{L}_g \cup \{1\} \cup \mathcal{L}_S \cup \{R\} \cup \{\mathrm{Im}_{[Q],D} \mid [Q] \text{ and } D \text{ as above}\},$$

and we let $\mathscr{Z}_{R,\mathcal{L}}$ be the $\mathcal{L}$-expansion of $\mathscr{Z}_R$ described above.

We fix an axiomatization T_1 of $\mathrm{Th}(\mathbf{Z}, +, -, 0, 1, D_n \mid 1 < n \in \mathbf{N})$ (see [19, Chapter 15, Section 15.1]) and we let T_2 be the following universal axiomatization of $\mathrm{Th}(R, S, S^{-1}, c)$:

$$T_2 = \{\forall x(x \neq c \to S(S^{-1}(x)) = x), \forall x(S^{-1}(S(x)) = x), \forall x(S(x) \neq c), S^{-1}(c) = c\}.$$

We will denote by T_2^R the theory obtained by relativizing to the predicate R the quantifiers appearing in each element of T_2. We will frequently use the fact that, modulo T_1, a formula of the form $\neg D_n(x)$ is equivalent to

$$\bigvee_{k=1}^{n-1} D_n(x+k).$$

Let $\mathscr{M}$ be an $\mathcal{L}$-structure. Let $Q_1, \dots, Q_n \in \mathbf{Z}[X]$. We say that $\bar{a} \in M^n$ is a non-degenerate solution of $\sum_{i=1}^n \mathsf{f}_{Q_i}(x_i) = y$ if it is a solution of this equation and no proper sub-sum is equal to 0. This can be expressed by the following first-order formula $N_{\bar{Q}}^{\mathrm{nd}}(\bar{x}, y)$

$$\sum_{i=1}^n \mathsf{f}_{Q_i}(x_i) = y \wedge \bigwedge_{I \subsetneq [n]} \sum_{i \in I} \mathsf{f}_{Q_i}(x_i) \neq 0.$$

Let T_R be the following set of axioms.

(Ax.1) T_1;

(Ax.2) T_2^R;

(Ax.3) $c = r_0$ (that is c equals the term $\underbrace{1 + \dots + 1}_{r_0 \text{ times}}$);

(Ax.4) $\forall x(\neg R(x) \to S(x) = x)$;

(Ax.5) for all $\ell_1, \dots, \ell_n$, all $0 \leq k_i \leq \ell_i$ and $\bar{Q} \in \mathbf{Z}[X]^n$, if

$$\left\{ \bar{z} \in R(\mathbf{Z})^n \;\middle|\; \mathscr{Z}_{\mathcal{L},R} \models \bigwedge_{i \in [n]} D_{\ell_i}(\mathsf{f}_{Q_i}(z_i) + k_i) \right\} = \{\bar{w}_1, \dots, \bar{w}_m\}$$

then we add the axiom

$$\forall \bar{x} \in R \left(\bigwedge_{i \in [n]} D_{\ell_i}(\mathsf{f}_{Q_i}(x_i) + k_i) \to \bigvee_{i \in [m]} \bigwedge_{j \in [n]} x_i = w_{ij} \right);$$

(Ax.6) for all $[Q], D$ as above, we add the axiom

$$\forall \bar{y} \left(\mathrm{Im}_{[Q],D}(\bar{y}) \leftrightarrow \exists \bar{x} \in R(\varphi_D(\bar{x}) \wedge \varphi_{[Q]}(\bar{x}, \bar{y})) \right);$$

(Ax.7) for all $Q \in \mathrm{Triv}$, we add the axiom

$$\forall x \in R \;\; \mathsf{f}_Q(x) = 0$$

and for all $Q \notin \mathrm{Triv}$ we add

$$\forall x, y \in R \left(x > e \wedge y > e \wedge x \neq y \to \mathsf{f}_Q(x) \neq \mathsf{f}_Q(y) \right),$$

where $e = e(Q)$ (see 2 on page 14);

(Ax.8) for every $Q_1, \ldots, Q_s \in \mathbf{Z}[X]$, we add the axiom

$$\forall \bar{x} \in R \left(N^{\mathrm{nd}}_{\bar{Q}}(\bar{x}, 0) \rightarrow \bigvee_{\bar{m} \in E} \bigwedge_{i \in [s]} x_i = S^{m_i}(x_1) \right),$$

where $E = E_{\bar{Q}}$ (see 1 on page 14).

Note that $\mathscr{Z}_{R,\mathcal{L}}$ is a model of T_R. Indeed, (Ax.7) follows from Corollary 1.18 and (Ax.8) follows from Proposition 1.15. In particular T_R is consistent. Also note that all axioms but the defining axioms for the congruences D_n and the predicates $\mathrm{Im}_{[Q],D}$ are universal.

The main result of this section is the following theorem.

Theorem 2.1. *The $\mathcal{L}$-theory T_R has quantifier elimination.*

Notice that the sequence $R = \{2^n + n \mid n \in \mathbf{N}\}$ does not satisfy (Ax.8): considering the term $\mathsf{f}(x) = S^2(x) - 3S(x) + 2x$, one can find infinitely many (non-degenerate) solutions of the equation $\mathsf{f}(x_1) - \mathsf{f}(x_2) = 0$. In view of Theorem 2.20 this is not surprising because the structure $(\mathbf{Z}, +, 0, 1, R, S)$ is known to be unstable: $\mathbf{N}$ is definable by the formula $\exists y \in R\, y \neq 1 \wedge (2y - S(y) = x)$. However, we do not know if there exists a sequence R such that $\mathscr{Z}_R$ is (super)stable and $(\mathbf{Z}, +, 0, 1, R, S)$ unstable.

To establish quantifier elimination, we use the following criterion. Given two models $\mathscr{M}_0 \subset \mathscr{M}$ of an arbitrary theory, we say that $\mathscr{M}_0$ is *1-e.c.* in $\mathscr{M}$ if any quantifier-free definable subset of M, defined with parameters in M_0, has a non-empty intersection with M_0.

Proposition 2.2 ([14, Corollary 3.1.12])**.** *Let T be an $\mathcal{L}$-theory such that*

1. *(T has algebraically prime models) for all $\mathscr{M} \models T$ and all $\mathscr{A} \subset \mathscr{M}$, there exists a model $\overline{\mathscr{A}}$ of T such that for all $\mathscr{N} \models T$, any embedding $f : \mathscr{A} \to \mathscr{N}$ extends to an embedding $\bar{f} : \overline{\mathscr{A}} \to \mathscr{N}$;*
2. *(T is 1-e.c.) for all $\mathscr{M}_0, \mathscr{M} \models T$, if $\mathscr{M}_0 \subset \mathscr{M}$ then $\mathscr{M}_0$ is 1-e.c. in$\mathscr{M}$.*

Then T has quantifier elimination.

The proof of Theorem 2.1 will be a consequence of Proposition 2.2 and the work done in the following subsections. In Section 2.1.1, we prove several direct consequences of T_R regarding equations of the form $\mathsf{f}_1(x_1) + \cdots + \mathsf{f}_n(x_n) = a$. In Section 2.1.2, we give a detailed construction of algebraically prime models of T_R. Finally, we show in Section 2.1.3 that T_R is 1-e.c.

Corollary 2.3. *The $\mathcal{L}$-structure $\mathscr{Z}_{R,\mathcal{L}}$ is a prime model of T_R. In particular T_R is complete.*

Proof. Since $\mathscr{Z}_{R,\mathcal{L}}$ is an algebraically prime model and T_R has quantifier elimination, $\mathscr{Z}_{R,\mathcal{L}}$ is a prime model. Therefore, T_R is complete. □

2.1.1. Equations in T_R

Definition 2.4. A term f is said to be *trivial* if $\mathsf{f} = \mathsf{f}_Q$ for some $Q \in \mathrm{Triv}$.

Definition 2.5. Let $\mathscr{M} \models T_R$ and $a, b \in R$. The orbit of a is the set $\{S^k(a) \mid k \in \mathbf{Z}\}$ and is denoted by $\mathrm{Orb}(a)$. We say that a and b are in the same orbit if and only if $b \in \mathrm{Orb}(a)$.

The relation "a and b are in the same orbit" is an equivalence relation.

Lemma 2.6. *Let $\mathscr{M} \models T_R$. Let $\bar{\mathsf{f}}$ be a n-tuple of non-trivial terms, $n > 1$, and let $b_1, \ldots, b_k \in R$, $1 < k \leq n$, be in different orbits.*

1. *If $k > n/2$, then for all $c_{k+1}, \ldots, c_n \in R$,*

$$\sum_{i=1}^{k} \mathsf{f}_i(b_i) + \sum_{i=k+1}^{n} \mathsf{f}_i(c_i) \neq 0;$$

2. *If $k \leq n/2$, then for all $c_{k+1}, \dots, c_n \in R$, the elements $b_1, \dots, b_k, c_{k+1}, \dots, c_n$ do not form a non-degenerate solution of the equation $\sum_{i=1}^n \mathsf{f}_i(x_i) = 0$. Moreover, if $\sum_{i=1}^k \mathsf{f}_i(b_i) + \sum_{i=k+1}^n \mathsf{f}_i(c_i) = 0$, then for all $i \in [k]$ there exists a non-empty $P_i \subset \{k+1, \dots, n\}$ such that $P_i \cap P_{i'} = \emptyset$ for all $i \neq i' \in [k]$ and for all $i \in [k]$ $b_i, (c_j)_{j \in P_i}$ is a non-degenerate solution of*

$$\mathsf{f}_i(x_i) + \sum_{j \in P_i} \mathsf{f}_j(x_j) = 0.$$

Proof. Let $c_{k+1}, \dots, c_n \in R$. It is clear from (Ax.8) that $b_1, \dots, b_k, c_{k+1}, \dots, c_n$ cannot be a non-degenerate solution of

$$\sum_{i=1}^n \mathsf{f}_i(x_i) = 0,$$

since, for instance, b_1 is not in the same orbit as b_2. Suppose $b_1, \dots, b_k, c_{k+1}, \dots, c_n$ is degenerate. Then one shows by induction on n that there exists a partition $(P_1, \dots, P_\ell)$ of $[n]$ such that for all $j \in [\ell]$ $(b_i)_{i \in P_j \cap [k]}, (c_i)_{i \in P_j \cap \{k+1, \dots, n\}}$ is a non-degenerate solution of

$$\sum_{i \in P_j \cap [k]} \mathsf{f}_i(x_i) + \sum_{i \in P_j \cap \{k+1, \dots, n\}} \mathsf{f}_i(x_i) = 0.$$

Since $b_1, \dots, b_k$ are in different orbits, we must have, by (Ax.8), $|P_j \cap [k]| \leq 1$ for all $j \in [\ell]$. Also, since all terms involved are non-trivial, we must have $|P_j \cap \{k+1, \dots, n\}| > 0$ for all $j \in [\ell]$. This implies in particular that $k \leq n/2$ and finishes the proof of the lemma. □

We now show that (Ax.8) is true for non-homogeneous equations.

Proposition 2.7. *Let $\mathscr{M} \models T_R$, $\bar{Q} \in \mathbf{Z}[X]^n$ and $a \in M$, $a \neq 0$. Then there exist $\bar{b}_1, \dots, \bar{b}_k \in R$ such that*

$$\mathscr{M} \models \forall \bar{x} \in R \left(N_{\bar{Q}}^{\mathrm{nd}}(\bar{x}, a) \to \bigvee_{j=1}^k \bigwedge_{i=1}^n x_i = b_{ji} \right).$$

Proof. Let $\mathsf{f}_i = \mathsf{f}_{Q_i}$ for all $i \in [n]$. Assume there exist infinitely many distinct non-degenerate solutions $\bar{b}_i \in M^n$, $i \in \mathbf{N}$, of the equation

$$\mathsf{f}_1(x_1) + \cdots + \mathsf{f}_n(x_n) = a.$$

We will reach a contradiction using (Ax.8) applied to the equation

$$\sum_{i=1}^n \mathsf{f}_i(x_i) - \sum_{i=n+1}^{2n} \mathsf{f}_{i-n}(x_i) = 0,$$

which we denote by $\varphi(\bar{x})$.

We have that for all $i \in \mathbf{N}$, the tuple $(\bar{b}_0, \bar{b}_i)$ is a solution of $\varphi(\bar{x})$. We may assume that there exists a partition $I = (I_1, \dots, I_\ell)$ of $[2n]$ such that for all $i \in \mathbf{N}$ and all $j \in [\ell]$, $(b_{0k} \mid k \in I_j \cap [n])$, $(b_{ik} \mid k + n \in I_j \setminus [n])$ is a non-degenerate solution of the equation

$$\sum_{i \in I_j \cap [n]} \mathsf{f}_i(x_i) - \sum_{i \in I_j \setminus [n]} \mathsf{f}_{i-n}(x_i) = 0.$$

(Notice that for each $j \in [\ell]$ $I_j \cap [n] \neq \emptyset$ and $I_j \setminus [n] \neq \emptyset$ by non-degeneracy and the fact that $a \neq 0$.)

By (Ax.8) for all $j \in [\ell]$, there is a finite set $E_j \subset \mathbf{Z}^{|I_j|}$ such that for all $i \in \mathbf{N}$

$$\bigvee_{\bar{m} \in E_j} \left(\bigwedge_{k \in I_j \cap [n]} b_{0k_0} = S^{m_k}(b_{0k}) \wedge \bigwedge_{k \in I_j \setminus [n]} b_{0k_0} = S^{m_k}(b_{ik}) \right),$$

where $k_0 = \min I_j$.

But this is a contradiction since the set defined by the formula

$$\bigvee_{\bar{m} \in E_j} \left(\bigwedge_{k \in I_j \cap [n]} b_{0k_0} = S^{m_k}(b_{0k}) \wedge \bigwedge_{k \in I_j \setminus [n]} b_{0k_0} = S^{m_k}(x_k) \right)$$

is finite for all $j \in [\ell]$. □

As a corollary, we obtain a uniform bound on the number of non-degenerate solutions.

Corollary 2.8. *Let $\bar{Q} \in (\mathbf{Z}[X] \setminus \mathrm{Triv})^n$ and $\mathscr{M} \models T_R$. Then there exist $k \in \mathbf{N}$ such that*

$$\mathscr{M} \models \forall y \left(y \neq 0 \to \exists \bar{x}_1, \ldots, \bar{x}_k \in R \, \forall \bar{z} \in R \left(N^{\mathrm{nd}}_{\bar{Q}}(\bar{z}, y) \to \bigvee_{j=1}^{k} \bigwedge_{i=1}^{n} z_i = x_{ji} \right) \right).$$

Proof. Let $\mathsf{f}_i = \mathsf{f}_{Q_i}$ for all $i \in [n]$. If the corollary is false, we can find a sequence $(a_i \mid i \in \mathbf{N}^{>0})$ in $M \setminus \{0\}$ such that $\mathsf{f}_1(x_1) + \cdots + \mathsf{f}_n(x_n) = a_i$ has at least $i+1$ non-degenerate solutions in $R(M)^n$. By Proposition 2.7, we may assume that $a_i \neq a_{i'}$ whenever $i \neq i'$. For all $i \in \mathbf{N}$, let $(\bar{b}_{ij} \mid j \in [i+1])$ be a sequence of pairwise distinct non-degenerate solutions of $\mathsf{f}_1(x_1) + \cdots + \mathsf{f}_n(x_n) = a_i$. Let U be a non principal ultrafilter over $\mathbf{N}$. Let $\bar{b}^U_j = [(\bar{b}_{ij} \mid i \in \mathbf{N}^{>0})]_U$. We have that $\bar{b}^U_j \neq \bar{b}^U_{j'}$ whenever $j \neq j'$. So $\mathscr{M}^U$ has infinitely many non-degenerate solutions of $\mathsf{f}_1(x_1) + \cdots + \mathsf{f}_n(x_n) = [(a_i)]_U$, a contradiction with Proposition 2.7 since $\mathscr{M}^U \models T_R$. □

Proposition 2.9. *Let $\mathscr{M}, \mathscr{M}_0 \models T_R$ such that $\mathscr{M}_0 \subset \mathscr{M}$. Let $\bar{\mathsf{f}}$ be a tuple of n non-trivial terms, $b_1, \ldots, b_n \in R(M) \setminus R(M_0)$ in different orbits and $a \in M_0$, $a \neq 0$. Then*

$$\sum_{i=1}^{n} \mathsf{f}_i(b_i) + a \notin R(M).$$

Proof. Suppose, towards a contradiction, that $\sum_{i=1}^{n} \mathsf{f}_i(b_i) + a = b_{n+1} \in R(M)$. By (Ax.6), there exists $b_{n+2}, \ldots, b_{2n+2} \in R(M_0)$ such that $\sum_{i=1}^{n} \mathsf{f}_i(b_i) - b_{n+1} = \sum_{i=1}^{n} \mathsf{f}_i(b_{n+1+i}) - b_{2n+2} = -a$. By Lemma 2.6 applied to

$$\sum_{i=1}^{n} \mathsf{f}_i(x_i) - x_{n+1} - \sum_{i=1}^{n} \mathsf{f}_i(x_{n+1+i}) + x_{2n+2} = 0,$$

and $b_1, \ldots, b_n$, for all $i \in [n]$, there exists $J_i \subset \{n+1, \ldots, 2n+2\}$ such that $b_i, (b_j)_{j \in J_i}$ is a non-degenerate solution to the corresponding equation. We furthermore have that $J_i \neq \emptyset$ for all $i \in [n]$ and $J_i \cap J_{i'} = \emptyset$ for all $i \neq i' \in [n]$. Furthermore, since $b_k \in M_0$ for all $k > n+1$, $k \notin J_i$ for all $i \in [n]$. So $n+1 \in J_i$ for all $i \in [n]$. This implies that $n = 1$. So, $\mathsf{f}_1(b_1) - b_2 = 0$. But this contradicts the fact that $a \neq 0$. □

In the following lemma, we express the fact that the set of solutions of an equation can be decomposed as a union of non-degenerate sets of solutions of sub-equations, as we did in Section 1 before the statement of Proposition 1.15. As the proof is a straightforward verification, we leave the details to the reader.

Lemma 2.10. *Let $\bar{Q} \in \mathbf{Z}[X]^n$. Then*

$$T_R \models \forall y \forall \bar{x} \in R \left(\sum_{i=1}^{n} \mathsf{f}_i(x_i) = y \leftrightarrow \bigvee_{\bar{I} \in \mathfrak{P}([n])} \left(N^{\mathrm{nd}}_{\bar{Q}_{I_0}}(\bar{x}_{I_0}, y) \wedge \bigwedge_{j=1}^{|\bar{I}|-1} N^{\mathrm{nd}}_{\bar{Q}_{I_j}}(\bar{x}_{I_j}, 0) \right) \right). \quad \square$$

We apply this to understand sets defined by Im predicates. Let $\mathscr{M} \models T$. Let $[Q] \in \mathbf{Z}[X]^{n\times m}$, $[Q'] \in \mathbf{Z}[X]^{n\times m'}$ and D a set of divisibility conditions of size m. Let $\bar{a} \in M^n$. We want to understand the set N defined by

$$\mathrm{Im}_{[Q],D}\left(a_1 + \sum_{i=1}^{m'} \mathsf{f}_{Q'_{1i}}(x_i), \ldots, a_n + \sum_{i=1}^{m'} \mathsf{f}_{Q'_{ni}}(x_i)\right). \tag{4}$$

This will be needed to show that T_R is 1-e.c. We want to show that the formula (4) expresses two things for a tuple $\bar{b} \in R(M)^{m'}$:

1. there exists $J' \subset [m']$ such that $\bar{b}_{J'}$ belongs to a finite set depending only on $[Q], [Q'], D$ and $\bar{a}$;
2. for all $J \subset [m']$, $\bar{b}_J$ satisfies a finite number of recurrence relations and congruence relations again depending only on $[Q], [Q'], D$ and $\bar{a}$. Also, when $J_1, J_2 \subset [m]$ have a non-empty intersection, the above conditions on $\bar{b}_{J_1}$ and $\bar{b}_{J_2}$ must be consistent.

To do so, for all $J_{01}, \ldots, J_{0n} \subset [m]$ and $J_{11}, \ldots, J_{1n} \subset [m']$, we let $N^{\mathrm{nd}}_{[Q],[Q'],\bar{J}_0,\bar{J}_1}(\bar{a})$ be the set defined by the formula

$$\exists \bar{z} \in R\, \varphi_D(\bar{z}) \wedge \bigwedge_{i\in[n]} \left(\sum_{j\in J_{0i}} \mathsf{f}_{Q_{ij}}(z_j) = a_i + \sum_{j\in J_{1i}} \mathsf{f}_{Q'_{ij}}(x_i) \wedge \bar{z}_{J_{0i}}\bar{x}_{J_{1i}} \text{ is non-degenerate}\right).$$

Notice that by Proposition 2.7, the set $N^{\mathrm{nd}}_{[Q],[Q'],\bar{J}_0,\bar{J}_1}(\bar{a})$ is finite if $a_i \neq 0$ for some $i \in [n]$.

Recall that by (Ax.6), the formula (4) is satisfied by some $\bar{b} \in R(M)^{m'}$ if and only if there is $\bar{z} \in R(M)^m$ such that the following system of equations and congruence relations is satisfied:

$$\begin{cases} \mathsf{f}_{Q_{11}}(z_1) + \cdots + \mathsf{f}_{Q_{1m}}(z_m) = a_1 + \mathsf{f}_{Q'_{11}}(b_1) + \cdots + \mathsf{f}_{Q_{1m'}}(b_{m'}) \\ \vdots \\ \mathsf{f}_{Q_{n1}}(z_1) + \cdots + \mathsf{f}_{Q_{nm}}(z_m) = a_n + \mathsf{f}_{Q'_{n1}}(b_1) + \cdots + \mathsf{f}_{Q_{nm'}}(b_{m'}) \\ D_{\ell_1}(\mathsf{f}_{P_1}(z_1) + k_1), \ldots, D_{\ell_m}(\mathsf{f}_{P_m}(z_m) + k_m). \end{cases}$$

Now, for each $i \in [n]$, choose, according to Lemma 2.10, $\bar{J}_i = (J_{i0}, \ldots, J_{i\ell_1}) \in \mathfrak{P}([m])$ and $\bar{J'}_i = (J'_{i0}, \ldots, J'_{i\ell_2}) \in \mathfrak{P}([m'])$ such that for all $i \in [n]$, the following equalities hold in a non-degenerate way

$$\sum_{j\in J_{i0}} \mathsf{f}_{Q_{ij}}(z_j) = a_i + \sum_{j\in J'_{i0}} \mathsf{f}_{Q_{ij}}(b_j),$$

$$\sum_{j\in J_{is_1}} \mathsf{f}_{Q_{ij}}(z_j) = \sum_{j\in J'_{is_2}} \mathsf{f}_{Q_{ij}}(b_j),$$

for all (s_1, s_2) in some fixed $K_i \subset [m] \times [m']$ and for the s_1 and s_2 that do not appear in K_i

$$0 = \sum_{j\in J'_{is_2}} \mathsf{f}_{Q_{ij}}(b_j) \text{ and } \sum_{j\in J_{is_1}} \mathsf{f}_{Q_{ij}}(z_j) = 0.$$

This decomposition of each equation in the system shows that $\bar{b}_{J'}$ is in $N^{\mathrm{nd}}_{[Q],[Q'],\bar{J}_0,\bar{J}_1}(\bar{a})$, where $J' = \cup_{i\in[n]} J'_{i0}$. For the homogeneous equations above, we may apply (Ax.8) to obtain the desired relations between $\bar{z}_{J_{is_1}}$ and $\bar{b}_{J'_{is_2}}$ whenever $(s_1, s_2) \in K_i$.

To summarize, we state the following corollary, which is an explicit statement of the above discussion.

Corollary 2.11. *Let $[Q] \in \mathbf{Z}[X]^{n\times m}$ and D a set of divisibility of size m. Let $[Q'] \in \mathbf{Z}[X]^{n\times m'}$. Let $\mathscr{M} \models T_R$ and $\bar{a} \in M^n$. Then for all $\bar{b} \in R(M)^{m'}$*

$$\mathscr{M} \models \mathrm{Im}_{[Q],D}\left(a_1 + \sum_{i=1}^{m'} \mathsf{f}_{Q'_{1i}}(b_i), \dots, a_n + \sum_{i=1}^{m'} \mathsf{f}_{Q'_{ni}}(b_i)\right)$$

if and only if for all $i \in [n]$ there are $(J_{i0}, \dots, J_{is}) \in \mathfrak{P}([m])$ and $(J'_{i0}, \dots, J'_{is'}) \in \mathfrak{P}([m'])$ and $K_i \subset [s] \times [s']$ such that for all $s_1 \in [n]$ there is at most one $s_2 \in [n]$ such that $(s_1, s_2) \in K_i$ and

$$\begin{aligned}
\mathscr{M} \models{} & \bar{b}_{J'} \in N^{\mathrm{nd}}_{[Q],[Q'],\bar{J}_0,\bar{J}'_0}(\bar{a}) & (5)\\
& \wedge \bigwedge_{i\in[n]} \bigwedge_{(s_1,s_2)\in K_i} \left(\sum_{j\in J_{is_1}} \mathsf{f}_{Q_{ij}}(S^{k_{ij}}(b_{j^*})) = \sum_{j\in J'_{is_2}} \mathsf{f}_{Q'}(S^{k'_{ij}}(b_{j^*})) \right. & (6)\\
& \left. \wedge \bigwedge_{j\in J'_{is_2}} b_j = S^{k'_{ij}}(b_{j^*}) \bigwedge_{j\in J_{is_1}} D_{\ell_j}(\mathsf{f}_{P_j}(S^{k_{ij}}(b_{j^*})) + k'_j) \right) & (7)\\
& \wedge \bigwedge_{i\in[n]} \bigwedge_{(s_1,s_2)\notin K_i} \left(0 = \sum_{j\in J'_{is_2}} \mathsf{f}_{Q'_{ij}}(S^{k_{ij}}(b_{j^*})) \wedge \bigwedge_{j\in J'_{is_2}} b_j = S^{k_{ij}}(b_{j^*}) \right) & (8)\\
& \wedge \bigwedge_{i\in[n]} \bigwedge_{(s_1,i',s'_1,s'_2)\in K'_i} \left(\sum_{j\in J_{is_1}} \mathsf{f}_{Q_{ij}}(S^{k_{ij}}(b_{j^*})) = 0 \wedge \bigwedge_{j\in J_{is_1}} D_{\ell_j}(\mathsf{f}_{P_j}(S^{k_{ij}}(b_{j^*})) + k'_j) \right) & (9)\\
& \wedge \mathrm{Im}_{[\tilde{Q}],D}(0), & (10)
\end{aligned}$$

where $J' = \cup_{i\in[n]} J'_{i0}$, for all $i \in [n]$, $j^* = \min J'_{is_2}$ and

1. *if $(s_1, s_2) \in K_i$, $\bar{k}_i$ and $\bar{k}'_i$ are given by (Ax.8) applied to the operators in (6);*
2. *if $(s_1, s_2) \notin K_i$, $\bar{k}_i$ is given by (Ax.8) applied to the operators in (8),*
3. *$(s_1, i', s'_1, s'_2) \in K'_i$ if and only if $(s_1, s_2) \notin K_i$ for all $s_2 \in [n]$, $J_{is_1} \cap J_{i's'_1} \neq \emptyset$, $(s'_1, s'_2) \in K_i$ and $j^* = \min J_{i's'_2}$. In this case, $\bar{k}_i$ is given by (Ax.8) applied to the operators in (9),*

and $[\tilde{Q}]$ is the matrix defined by $\tilde{Q}_{ij} = Q_{ij}$ if Q_{ij} does not appear in (5)-(9) and $\tilde{Q}_{ij} = 0$ otherwise. □

2.1.2. T_R has algebraically prime models

Let $\mathscr{M} \models T_R$ and $\mathscr{A} \subset \mathscr{M}$. For $X \subset M$, we let $\mathrm{div}(X)$ be the *divisible closure of X in $\mathscr{M}$*, that is the substructure generated by $\{d \mid nd \in X$ for some $n \in \mathbf{N}^{>0}\}$. We will show later on that when X is the domain of an $\mathcal{L}$-substructure, $\mathrm{div}(X)$ is the divisible closure of X in the group theoretic sense. The construction of the algebraically prime model over $\mathscr{A}$, denoted $\overline{\mathscr{A}}$, is done as follows. Let $\bar{\mathsf{f}}$ be a n-tuple of non-trivial terms. Call an n-tuple $\bar{b} \in R(M)$ $\bar{\mathsf{f}}$-*good* if

1. $b_i \notin A$ for all $i \in [n]$;
2. $\mathsf{f}_1(b_1) + \cdots + \mathsf{f}_n(b_n) \in \mathscr{A}$;
3. $b_i \notin \mathrm{Orb}(b_j)$ whenever $j \neq i$.

Let $\tilde{\mathscr{A}}$ be the substructure generated by $\mathscr{A}$ and $\bar{\mathsf{f}}$-good tuples of elements of $R(M)$, for all tuples $\bar{\mathsf{f}}$ of non-trivial terms. This structure will satisfy all axioms of T_R except the definition of the symbols D_n. So our algebraically prime model over $\mathscr{A}$ will be $\overline{\mathscr{A}} = \mathrm{div}(\tilde{\mathscr{A}})$.

Lemma 2.12. *$\overline{\mathscr{A}}$ is a model of T_R.*

Proof. We begin with a description of elements in $\tilde{\mathscr{A}}$. Assume $\tilde{\mathscr{A}} = \langle A, (b_\lambda)_{\lambda<\kappa}\rangle$, where $b_\lambda \notin \mathrm{Orb}(b_{\lambda'})$ for all $\lambda \neq \lambda'$ and each b_λ appears in a good tuple. We want to show that any $d \in \tilde{\mathscr{A}}$ can be put in the form $a + \sum_{i=1}^n \mathsf{f}_i(b_{\lambda_i})$, where $\lambda_i \neq \lambda_j$ for all $i \neq j \in [n]$ and $a \in A$. Let $t(\bar{x}, y)$ be the term $y + \sum_{i=1}^n \mathsf{f}_i(x_i)$. We show that for all $a \in A$ and $b_{\lambda_1}, \dots, b_{\lambda_n}$ in different orbits, either $S(t(\bar{b}, a)) = t(\bar{b}, a)$ or $t(\bar{b}, a) = S^k(b_\lambda)$ for some $\lambda < \kappa$ and $k \in \mathbf{Z}$. Assume $S(t(\bar{b}, a)) \neq t(\bar{b}, a)$. This implies that $b = t(\bar{b}, a) \in R(M)$. Then, since $a = b - \sum_{i=1}^n \mathsf{f}_i(b_{\lambda_i})$, either b is in the orbit of b_{λ_i} for some $i \in [n]$ or $(b, \bar{b})$ is an $(x, -\bar{\mathsf{f}})$-good tuple. This shows that $b = S^k(b_\lambda)$ for some $\lambda < \kappa$ and $k \in \mathbf{Z}$. Thus every element $\tilde{\mathscr{A}}$ is of the form $a + \sum_{i=1}^n \mathsf{f}_i(b_{\lambda_i})$.

We now do the same job for elements in $\overline{\mathscr{A}}$.

Claim 2.13. Let $d \in \overline{\mathscr{A}}$. Then there exist $a \in \tilde{\mathscr{A}}$ and $n \in \mathbf{N}^{>0}$ such that $nd = a$.

Proof. Let X be the set $\{d \mid nd \in \tilde{A} \text{ for some } n \in \mathbf{N}^{>0}\}$. We first notice that for all $\bar{d} \in X^k$ and $\bar{m} \in \mathbf{Z}^k$, there is $n \in \mathbf{N}$ such that $n(m_1 d_1 + \cdots + m_k d_k) \in \tilde{A}$ (just take n to be the product of the witnesses of the fact that $\bar{d} \in X^k$). So to conclude, it is enough to show that for all terms $t(\bar{x})$, $|x| = k$, and all $\bar{d} \in X^k$' either $t(\bar{d}) \in \tilde{A}$ or there is $\bar{m} \in \mathbf{Z}^k$ such that $t(\bar{d}) = m_1 d_1 + \cdots + m_k d_k$. This is done by induction on the complexity of terms (the complexity being here the number of occurrences of the symbols S and S^{-1}).

Let $t(\bar{x})$ be a term, $|x| = k$, and $\bar{d} \in X^k$ and assume that either $t(\bar{d}) \in \tilde{A}$ or there is $\bar{m} \in \mathbf{Z}^k$ such that $t(\bar{d}) = m_1 d_1 + \cdots + m_k d_k$. We may assume that $t(\bar{d}) = m_1 d_1 + \cdots + m_k d_k \notin \tilde{A}$, since $\tilde{\mathscr{A}}$ is closed under S and S^{-1}. We claim that $S(t(\bar{d})) = t(\bar{d})$. Indeed, since $\bar{d} \in X^k$, there exists $n \in \mathbf{N}$ such that $nt(\bar{d}) \in \tilde{A}$. Thus, if $S(t(\bar{d})) = t(\bar{d})$ is not true, then $t(\bar{d}) \in R(M)$ and this imply that $t(\bar{d})$ is (nx)-good, in contradiction with our assumption that $t(\bar{d}) \notin \tilde{A}$. □

Let us finally show that $\overline{\mathscr{A}} \models T_R$. The only axiom that requires details is (Ax.6) – the defining axioms for the divisibility predicates are true since we took the divisible closure of $\tilde{\mathscr{A}}$ and the others are universal and thus true in any substructure. So assume that $\overline{\mathscr{A}} \models \mathrm{Im}_{[Q],D}(d_1, \dots, d_n)$, where $[Q] \in (\mathbf{Z}[X])^{n\times m}$ and D is a set of divisibility conditions of size m. By Claim 2.13, we may assume that $d_1, \dots, d_n \in \tilde{\mathscr{A}}$: for all $i \in [n]$, $d_i = a_i + \sum_{j=1}^k \mathsf{f}'_{ij}(d_{\lambda_{ij}})$. So we may also assume that $d \in A$. Since $\mathscr{M} \models T_R$ and we can find $b_1, \dots, b_m \in R(M)$ such that

$$\mathscr{M} \models \bigwedge_{i\in[n]} \sum_{j\in[m]} \mathsf{f}_{Q_{ij}}(b_j) = d_i \wedge \bigwedge_{i\in[m]} D_{\ell_i}(\mathsf{f}_{Q_i}(b_i) + k_i).$$

We may assume that for all $j \in [m]$ there exists $i \in [n]$ such that $\mathsf{f}_{Q_{ij}}$ is non-trivial (if for some $j \in [m]$ the terms $\mathsf{f}_{Q_{ij}}$ are trivial for all $i \in [n]$, we may replace, by (Ax.3) and (Ax.7), b_j by any $b'_i \in R(\mathbf{Z})$ such that $\mathscr{Z}_R \models D_{\ell_j}(\mathsf{f}_j(b'_j) + k_j)$, which is possible since $\mathscr{Z}_R \subset \mathscr{A}$ and $\mathscr{Z}_R \models T_R$). Now by construction of $\overline{\mathscr{A}}$, for all $i \in [m]$, b_i is in the orbit of some b_{λ_i}. This implies that $\bar{b} \in \overline{A}^m$, as desired. □

Let us show that any embedding $f : \mathscr{A} \to \mathscr{N}$ extends to an embedding $\bar{f} : \overline{\mathscr{A}} \to \mathscr{N}$.

Lemma 2.14. *Let $f : \mathscr{A} \to \mathscr{N}$ be an $\mathcal{L}$-embedding. Then f extends to an $\mathcal{L}$-embedding $\bar{f} : \overline{\mathscr{A}} \to \mathscr{N}$.*

Proof. Let $\mathcal{L}_0$ be the language $\{+, -, 0, 1, R\} \cup \mathcal{L}_S$. We first extend f to an $\mathcal{L}_0$-embedding $\tilde{f} : \tilde{\mathscr{A}} \to \mathscr{N}$. Let q be the partial type

$$\begin{aligned}
&\{\mathsf{f}_1(x_{\lambda_1}) + \cdots + \mathsf{f}_n(x_{\lambda_n}) = f(a) \mid \mathscr{M} \models \mathsf{f}_1(b_{\lambda_1}) + \cdots + \mathsf{f}_n(b_{\lambda_n}) = a, a \in A\}\\
\cup&\{x_\lambda \neq f(a) \mid \lambda < \kappa, a \in A\}\\
\cup&\{S^k(x_{\lambda_1}) \neq x_{\lambda_2} \mid \lambda_1 \neq \lambda_2, z \in \mathbf{Z}\}\\
\cup&\{D_\ell(\mathsf{f}(x_\lambda) + k) \mid \mathscr{M} \models D_\ell(\mathsf{f}(b_\lambda + k))\}.
\end{aligned}$$

We claim that q is finitely consistent in $\mathscr{N}$. Let Δ be a finite part of q. We may assume that the conjunction of the formulas in Δ is of the form

$$\bigwedge_{i\in I_1} \mathsf{f}_{i1}(x_{\lambda_1})+\cdots+\mathsf{f}_{in}(x_{\lambda_n})=f(a_i)\wedge\bigwedge_{i\in[n]} D_{\ell_i}(\mathsf{f}_i(x_{\lambda_i})+k_i)$$
$$\wedge\bigwedge_{i\in I_2;j\in[n]} x_{\lambda_j}\neq f(a_{ij})\wedge\bigwedge_{i,j\in[n];k\in I_3} S^k(x_{\lambda_i})\neq x_{\lambda_j}.$$

By (Ax.6), there exists $\bar{b}'\in R(N)^n$ such that

$$\bigwedge_{i\in I_1} \mathsf{f}_{i1}(b'_1)+\cdots+\mathsf{f}_{in}(b'_n)=f(a_i)\wedge\bigwedge_{i\in[n]} D_{\ell_i}(\mathsf{f}_i(b'_i)+k_i).$$

Assume towards a contradiction that $\bar{b}'$ is not a realization of Δ. Then we have that, for some $i_1\in I_2$, $j_1,i_2,j_2\in[n]$ and $k\in I_3$,

$$b'_{j_1}=f(a_{i_1j_1})\vee S^k(b'_{i_2})=b'_{j_2}.$$

So, again using (Ax.6), we can find $\bar{b}''\in R(M)^n$ such that

$$\bigwedge_{i\in I_1} \mathsf{f}_{i1}(b''_1)+\cdots+\mathsf{f}_{in}(b''_n)=a_i\wedge\bigwedge_{i\in[n]} D_{\ell_i}(\mathsf{f}_i(b''_i)+k_i)\wedge(b''_{j_1}=a_{i_1j_1}\vee S^k(b''_{i_2})=b''_{j_2}),$$

in contradiction with the fact that $b_{\lambda_1},\dots,b_{\lambda_n}$ is a good tuple. Hence q is finitely consistent in $\mathscr{N}$ and so realized in an elementary extension $\mathscr{N}^*$ of $\mathscr{N}$ by some $(b'_\lambda)_{\lambda<\kappa}$. Let us show that $(b'_\lambda)_{\lambda<\kappa}$ is in $\mathscr{N}$. Let $\lambda<\kappa$. By definition, b_λ appears in a $\bar{\mathsf{f}}$-good tuple: there exist $b_{\lambda_2},\dots,b_{\lambda_n}\in R(M)\backslash A$ and $a\in A$ such that $\mathsf{f}_1(b_\lambda)+\mathsf{f}_2(b_{\lambda_2})+\cdots+\mathsf{f}_n(b_{\lambda_n})=a$. The same holds for $b'_\lambda,b'_{\lambda_2},\dots,b'_{\lambda_n}$ and $f(a)$. Furthermore, we have that $\mathscr{N}\models \mathrm{Im}_{\bar{Q}}(f(a))$, where $\mathsf{f}_i=\mathsf{f}_{Q_i}$. Since $\mathscr{N}\models T_R$, there are $d_1,\dots,d_n\in R(N)$ such that

$$\sum_{i=1}^n \mathsf{f}_i(d_i)=f(a).$$

Hence, by Lemma 2.6, b'_λ is in the orbit of d_i for some $i\in[n]$: this shows that $b'_\lambda\in N$.

Since for all $\lambda_1\neq\lambda_2$ and all $z\in\mathbf{Z}$, the formula $S^k(x_{\lambda_1})\neq x_{\lambda_2}$ is in q, we have that for all $\lambda_1\neq\lambda_2$, $b'_{\lambda_1}\notin \mathrm{Orb}(b'_{\lambda_2})$. Likewise, we have that $b'_\lambda\notin f(A)$ for all $\lambda<\kappa$. This shows that $(b'_\lambda)_{\lambda<\kappa}$ realizes the quantifier-free type of $(b_\lambda)_{\lambda<\kappa}$ over A in $\mathcal{L}_0$. Hence the map $\tilde{f}$ defined on $\tilde{\mathscr{A}}$ by $a+\sum_{i=1}^n \mathsf{f}_i(b_{\lambda_i})\mapsto f(a)+\sum_{i=1}^n \mathsf{f}_i(b'_{\lambda_i})$ is an $\mathcal{L}_0$-embedding.

Now we extend $\tilde{f}$ to an $\mathcal{L}$-embedding $\bar{f}:\overline{\mathscr{A}}\to\mathscr{N}$. Recall that for all $d\in\overline{\mathscr{A}}\backslash\tilde{\mathscr{A}}$, there exist $a\in\mathscr{A}$, $\bar{\mathsf{f}}$ a tuple of non-trivial terms, $b_{\lambda_1},\dots,b_{\lambda_n}$ and $n\in\mathbf{N}^{>0}$ such that $nd=a+\sum_{i=1}^n\mathsf{f}_i(b_{\lambda_i})$. By construction $\tilde{f}(nd)$ is divisible by n: by (Ax.1) there exists a unique d^* such that $\tilde{f}(nd)=nd^*$ (uniqueness follows from the fact that models of T_1 are torsionless). We extend $\tilde{f}$ by the rule $\bar{f}(d)=d^*$. So $\bar{f}$ respects the divisibility predicates. And since the Im predicates are definable by $\mathcal{L}_0\cup\{D_n\mid n\in\mathbf{N}^{>1}\}$-formulas, we get that $\bar{f}$ is indeed an $\mathcal{L}$-embedding □

2.1.3. T_R is 1-e.c.

We will need an analogue of Proposition 1.23.

Proposition 2.15. *Let $Q_1,\dots,Q_k\in\mathbf{Z}[X]$. Then for all $\mathscr{M}\models T_R$, $\{z\in\mathbf{N}\mid\mathscr{M}\models\exists\bar{x}\in R\,\mathsf{f}_{Q_1}(x_1)+\cdots+\mathsf{f}_{Q_k}(x_k)=z\}$ is not piecewise syndetic.*

Proof. This is an immediate consequence of Proposition 1.23 and the following observation:

$$\begin{aligned}&\{z\in\mathbf{N}\mid\mathscr{M}\models\exists\bar{x}\in R\,\mathsf{f}_{Q_1}(x_1)+\cdots+\mathsf{f}_{Q_k}(x_k)=z\}\\=&\{z\in\mathbf{N}\mid\mathscr{Z}_{R,\mathcal{L}}\models\exists\bar{x}\in R\,\mathsf{f}_{Q_1}(x_1)+\cdots+\mathsf{f}_{Q_k}(x_k)=z\}.\end{aligned}$$

□

Proposition 2.16. *Let $\mathscr{M}, \mathscr{M}_0 \models T_R$ such that $\mathscr{M}_0 \subset \mathscr{M}$. Assume that $R(M_0) = R(M)$. Then $\mathscr{M}_0$ is 1-e.c. in $\mathscr{M}$.*

Proof. Let $\varphi(x, \bar{y})$ be a quantifier-free formula such that $\mathscr{M} \models \varphi(b, \bar{a})$ for some $b \in M \backslash M_0$ and $\bar{a} \in M_0$. We will show that there exists $b_0 \in M_0$ such that $\mathscr{M}_0 \models \varphi(b_0, \bar{a})$. Let us simplify φ.

First we show that for all $\mathcal{L}$-terms $t(x, \bar{y})$, $\bar{y}$ of size n, for all $b \in M \setminus M_0$ and all $\bar{a} \in M_0^n$, there are $n \in \mathbf{Z}$ and $a \in M_0$ such that $t(b, \bar{a}) = nb + a$. It is enough to show that for all $n \in \mathbf{Z} \setminus \{0\}$, $b \in M \setminus M_0$ and $a \in M_0$, $S(nb + a) = nb + a$. But this is the case since $nb + a \notin M_0$ and $R(M) = R(M_0) \subset M_0$. In particular for all $b \in M \setminus M_0$, $n \in \mathbf{Z}$ and $a \in M_0$, $\mathscr{M} \models R(nb + a)$ if and only if $n = 0$ and $\mathscr{M} \models R(a)$.

Now we look at the atomic formulas satisfied by elements in $M \setminus M_0$ with parameters in M_0. Let $b \in M \setminus M_0$, $n_1, \ldots, n_k \in \mathbf{Z}$, $a_1, \ldots, a_k \in M_0$, $[Q] \in (\mathbf{Z}[X])^{k \times m}$ and D be a set of divisibility conditions of size m. Since $R(M) = R(M_0)$, we have $\mathscr{M} \models \mathrm{Im}_{[Q],D}(n_1 b + a_1, \ldots, n_k b + a_k)$ if and only if $n_1 = \cdots = n_k = 0$ and $\mathscr{M} \models \mathrm{Im}_{[Q],D}(\bar{a})$. Likewise, for all $n \in \mathbf{Z}$ and $a \in M_0$, we have $\mathscr{M} \models nb + a = 0$ if and only if $n = 0$ and $\mathscr{M} \models a = 0$.

Thus, after writing $\varphi(x, \bar{y})$ in its equivalent disjunctive normal form, we may select a conjunctive clause satisfied by $(b, \bar{a})$ and then assume that $\varphi(x, \bar{a})$ is of the form

$$\begin{aligned} &\bigwedge_{i \in I_1} n_i x + a'_i \neq 0 \\ \wedge &\bigwedge_{i \in I_2} \neg \mathrm{Im}_{[Q]_i, D_i}(n_{i1} x + a'_{i1}, \ldots, n_{im_i} x + a'_{im_i}) \\ \wedge &\bigwedge_{i \in I_3} D_{\ell_i}(n_i x + k_i), \end{aligned}$$

where for all $i \in I_1$, $n_i \in \mathbf{Z} \setminus \{0\}$ and $a'_i = t_i(\bar{a})$ for some $\mathcal{L}$-term $t_i(\bar{y})$, for all $i \in I_3$, $m_i \in \mathbf{N}^{>0}$, $\bar{n}_i \in (\mathbf{Z} \setminus \{0\})^{m_i}$ and $a'_{ij} = t_{ij}(\bar{a})$ for some $\mathcal{L}$-term $t_{ij}(\bar{y})$ and for all $i \in I_3$, $n_i \in \mathbf{Z} \setminus \{0\}$, $\ell_i \in \mathbf{N}^{>1}$ and $0 \leq k_i < \ell_i$.

Let us finally show that $\varphi(M_0, \bar{a})$ is not empty. By model completeness of $\mathrm{Th}(\mathbf{Z}, +, 0, 1, D_n \mid 1 < n \in \mathbf{N})$, there exists $b_0 \in M_0$ such that

$$\mathscr{M}_0 \models \bigwedge_{i \in I_1} n_i b_0 + a'_i \neq 0 \wedge \bigwedge_{i \in I_3} D_{\ell_i}(n_i b_0 + a'_i).$$

However, $\mathscr{M}_0$ may not satisfy $\varphi(b_0, \bar{a})$. But this can be overcome in the following way. Let

$$X_1 = \left\{ m \in \mathbf{N} \;\middle|\; \mathscr{M}_0 \models \bigwedge_{i \in I_3} n_i(b_0 + m) + a'_i \neq 0 \right\},$$

$$X_2 = \left\{ m \in \mathbf{N} \;\middle|\; \mathscr{M}_0 \models \bigwedge_{i \in I_2} \neg \mathrm{Im}_{[Q]_i, D_i}(n_{i1}(b_0 + m) + a'_{i1}, \ldots) \right\},$$

and

$$X_3 = \left\{ m \in \mathbf{N} \;\middle|\; \mathscr{M}_0 \models \bigwedge_{i \in I_3} D_{\ell_i}(n_i m) \right\}.$$

We want to show that the set $X = X_1 \cap X_2 \cap X_3$ is not empty. Suppose otherwise that $X = \emptyset$. This implies that $X_3 \subset \mathbf{N} \setminus (X_1 \cap X_2)$. But then, since X_3 is piecewise syndetic, $\mathbf{N} \setminus (X_1 \cap X_2)$ is piecewise syndetic. Hence, by Brown's Lemma, $\mathbf{N} \setminus X_2$ is piecewise syndetic, $\mathbf{N} \setminus X_1$ being finite. But, by Proposition 2.15, this is not possible since $\mathbf{N} \setminus X_2$ is in the image of a sum of terms of the form f_Q. $\square$

In order to establish that T_R is 1-e.c., we first show that a conjunction of Im predicates is equivalent to an Im predicate.

Lemma 2.17. *For all $[Q]_1 \in \mathbf{Z}^{n_1 \times m_1}$, ..., $[Q]_\ell \in \mathbf{Z}^{n_\ell \times m_\ell}$ and sets of divisibility conditions $D_1, \ldots, D_\ell$, there exists $[Q] \in \mathbf{Z}^{(n_1 \cdots n_\ell) \times (m_1 \cdots m_\ell)}$ and a set of divisibility conditions D such that*

$$T_R \models \forall \bar{y}_1, \ldots, \bar{y}_\ell \in R \left(\bigwedge_{i \in [\ell]} \mathrm{Im}_{[Q]_i, D_i}(\bar{y}_i) \leftrightarrow \mathrm{Im}_{[Q], D}(\bar{y}_1, \ldots, \bar{y}_\ell) \right).$$

Proof. Just take $[Q] = [Q]_1 \oplus \cdots \oplus [Q]_\ell$ ($\oplus$ denotes the direct sum) and $D = D_1 \sqcup \cdots \sqcup D_\ell$. $\square$

Theorem 2.18. *The theory T_R is 1-e.c.*

Proof. Let us show that for all $\mathscr{M}, \mathscr{M}_0 \models T_R$ such that $\mathscr{M}_0 \subset \mathscr{M}$, then $\mathscr{M}_0$ is 1-e.c. in $\mathscr{M}$. Let $\mathscr{M}, \mathscr{M}_0 \models T_R$ such that $\mathscr{M}_0 \subset \mathscr{M}$. Two cases are possible: either $R(M_0) = R(M)$ or $R(M_0) \subsetneq R(M)$. The first case has been proved in Proposition 2.16. So let us assume that we are in the second case.

By Lemma 2.12, we may assume that $\mathscr{M} = \overline{\mathscr{A}}$ where $\mathscr{A}$ is the substructure of $\mathscr{M}$ generated by $M_0 \cup R(M)$. Recall that by the proof of Lemma 2.12, any element d of $\mathscr{M}$ is such that $nd = a + \sum_{i=1}^{\ell} \mathsf{f}_i(b_i)$, where $n \in \mathbf{N}$, $a \in M_0$ and $b_1, \ldots, b_\ell \in R(M) \backslash R(M_0)$ are in different orbits. Our strategy is to establish that $\mathscr{M}_0$ is 1-e.c. in $\mathscr{M}$ from the fact that for all tuple $\bar{b}$ of elements of $R(M) \backslash R(M_0)$ in different orbits, all $\bar{a} \in M_0$ and all $\varphi(\bar{x}, \bar{y})$, $\mathscr{M} \models \varphi(\bar{b}, \bar{a})$ implies $\mathscr{M}_0 \models \exists \bar{x} \in R\, \varphi(\bar{x}, \bar{a})$.

Let us first look at terms evaluated at $d \in M$.

Claim 2.19. Let $t(x, \bar{y})$ be an $\mathcal{L}$-term, where $\bar{y}$ is a tuple of size k. Then for all $d \in M$ and $\bar{a} \in M_0^k$, there are $m \in \mathbf{Z}$ and $a \in M_0$ such that $t(d, \bar{a}) = md + a$.

Proof. Let $d \in M$, $n \in \mathbf{N}^{>0}$, $a \in M_0$ and $b_1, \ldots, b_\ell \in R(M)$ in different orbits such that $nd = a + \sum_{i=1}^{\ell} \mathsf{f}_i(b_i)$, with $\mathsf{f}_i = \mathsf{f}_{Q_i}$ for some $Q_i \in \mathbf{Z}[X]$. To prove the claim, it is enough to show that $S(md + a') = md + a'$ for all $m \in \mathbf{Z} \setminus \{0\}$ and $a' \in M_0$, unless $m = 0$ or $d \in M_0$. Without loss of generality, we assume that f_i is non-trivial for all $i \in [\ell]$. Assume $md + a' = b \in R(M)$, $m \neq 0$. Then we have that $n(md + a') = ma + a' + \sum_{i=1}^{\ell} m\mathsf{f}_i(b_i) = nb$. Thus

$$\mathscr{M} \models \mathrm{Im}_{nQ, -m\bar{Q}}(ma + na')$$

and since $\mathscr{M}_0 \subset \mathscr{M}$ and $\mathscr{M}_0 \models T_R$, there are $b', b'_1, \ldots, b'_\ell \in R(M_0)$ such that

$$nb - \sum_{i=1}^{\ell} m\mathsf{f}_i(b_i) - \left(nb' - \sum_{i=1}^{\ell} m\mathsf{f}_i(b'_i) \right) = 0.$$

But by Lemma 2.6, this implies that $b, b_1, \ldots, b_\ell \in M_0$. In particular $d \in M_0$. $\square$

Now, let $\varphi(x, \bar{y})$ an $\mathcal{L}$-formula, with $\bar{y}$ of size k, such that $\mathscr{M} \models \varphi(d, \bar{a})$, for some $d \in M \setminus M_0$ and $\bar{a} \in M_0^k$. Using the previous claim, we may assume that $\varphi(x, \bar{a})$ is of the form

$$\begin{aligned} & \bigwedge_{i \in I_1} m_i x + a'_i \neq 0 \wedge \bigwedge_{i \in I_2} D_{\ell_i}(m_i x + s_i) \\ \wedge & \bigwedge_{i \in I_3} \mathrm{Im}_{[Q]_i, D_i}(m_{i1} x + a'_{i1}, \ldots, m_{ik_i} x + a'_{ik_i}) \\ \wedge & \bigwedge_{i \in I_4} \neg \mathrm{Im}_{[Q]_i, D_i}(m_{i1} x + a'_{i1}, \ldots, m_{ik_i} x + a'_{ik_i}), \end{aligned}$$

where, for all $i \in I_1 \cup I_2$, $a'_i = t_i(\bar{a})$ for some $\mathcal{L}$-term $t_i(\bar{y})$, $m_i \in \mathbf{Z} \setminus \{0\}$, $\ell_i \in \mathbf{N}^{>1}$, $0 \leq s_i < \ell_i$ and for all $i \in I_3 \cup I_4$, $k_i \in \mathbf{N}^{>0}$, $a'_{ij} = t_i(\bar{a})$ for some $\mathcal{L}$-term $t_{ij}(\bar{y})$, $\bar{m}_i \in (\mathbf{Z} \setminus \{0\})^{k_i}$, $[Q]_i \in (\mathbf{Z}[X])^{k_i \times k'_i}$ and D_i is a set of divisibility conditions. We may also assume that $|I_3| \leq 1$ by Lemma 2.17.

Since $d \in M \setminus M_0$, $nd = a + \sum_{i=1}^{\ell} \mathsf{f}_i(b_i)$, for some $n \in \mathbf{N}^{>0}$, $a \in M_0$, $b_1, \ldots, b_\ell$ in different orbits and $\mathsf{f}_1, \ldots, \mathsf{f}_\ell$ non-trivial. Since $D_{n_1 n_2}(n_1 x) \leftrightarrow D_{n_2}(x)$ and $\mathrm{Im}_{[Q],D}(\bar{y}) \leftrightarrow \mathrm{Im}_{n[Q],D}(n\bar{y})$, instead of looking at $\varphi(x, \bar{a})$ we may look at the formula $\tilde{\varphi}(\bar{x}, \bar{a})$ defined by

$$\begin{aligned}
&\bigwedge_{i\in I_1} m_i\left(a+\sum_{i=1}^{\ell}\mathsf{f}_i(x_i)\right)+na_i'\neq 0\wedge\bigwedge_{i\in I_2} D_{n\ell_i}\left(m_i\left(a+\sum_{i=1}^{\ell}\mathsf{f}_i(x_i)\right)+ns_i\right)\\
&\wedge\bigwedge_{i\in I_3}\mathrm{Im}_{n[Q]_i,D_i}\left(m_{i1}\left(a+\sum_{i=1}^{\ell}\mathsf{f}_i(x_i)\right)+na_{i1}',\ldots,m_{ik_i}\left(a+\sum_{i=1}^{\ell}\mathsf{f}_i(x_i)\right)+na_{ik_i}'\right)\\
&\wedge\bigwedge_{i\in I_4}\neg\mathrm{Im}_{n[Q]_i,D_i}\left(m_{i1}\left(a+\sum_{i=1}^{\ell}\mathsf{f}_i(x_i)\right)+na_{i1}',\ldots,m_{ik_i}\left(a+\sum_{i=1}^{\ell}\mathsf{f}_i(x_i)\right)+na_{ik_i}'\right).
\end{aligned}$$

Furthermore, we may replace

$$\bigwedge_{i\in I_2} D_{n\ell_i}\left(m_i\left(a+\sum_{i=1}^{\ell}\mathsf{f}_i(x_i)\right)+ns_i\right)$$

by

$$\bigwedge_{i\in[\ell]} D_{\ell_i'}\left(m_i'\mathsf{f}_i(x_i)+s_i'\right),$$

where for all $i \in [\ell]$, $\ell_i \in \mathbf{N}^{>1}$, $m_i' \in \mathbf{Z}$ and $0 \leq s_i' < \ell_i'$. Finally, by Lemma 2.10 and Corollary 2.11, we may assume that $\tilde{\varphi}(\bar{x}, \bar{a})$ is of the form

$$\begin{aligned}
&\bigwedge_{i\in[n]}\bigwedge_{j\in J} D_{\ell_{ij}}(\mathsf{f}_{Q_{ij}}(S^{k_j}(x_i))+k_{ij})\\
&\wedge\bigwedge_{(i,j)\in K_1}\mathsf{f}_{Q_j'}(x_i)=0\wedge\bigwedge_{(i,j)\in K_2}\mathsf{f}_{Q_j'}(x_i)\neq 0\wedge\bigwedge_{i\in I}\bar{x}_{J_i}\notin F_i,
\end{aligned}$$

where, for all $i \in I$, F_i is a finite set of $|J_i|$-tuples in M_0. But then, by (Ax.5) and (Ax.7), we may find a realization $\bar{b}_0$ of $\tilde{\varphi}(\bar{x}, \bar{a})$ in $R(M_0)$, as desired. □

2.2. Superstability

From the quantifier elimination of T_R, we deduce, by means of counting of types, that it is superstable.

Theorem 2.20. *The theory T_R is superstable.*

Proof. Let $\mathscr{C}$ be a monster model of T_R and let $A \subset C$ be a small set of parameters. We want to show that $|S_1(A)| \leq \max\{2^{\aleph_0}, |A|\}$. Without loss of generality, we may assume that A is the domain of a model. By quantifier elimination (see Theorem 2.1), any type $p(x)$ over A is determined by the set of atomic formulas it contains. Let $\mathcal{L}_1 = \mathcal{L}_g \cup \mathcal{L}_S$ and $\mathcal{L}_2$ be $\mathcal{L} \backslash \{D_n \mid n > 1\}$. Let $p_{|\mathcal{L}_i}$ denote the restriction of p to $\mathcal{L}_i$, so that $p(x) = p_{|\mathcal{L}_1}(x) \cup p_{|\mathcal{L}_2}(x)$. We may assume that $p(x)$ does not contain a formula of the form $x = a$ for some $a \in M$. We consider two cases:

(Case 1) there exist $m \in \mathbf{Z}\backslash\{0\}$ and $a \in A$ such that $R(mx + a) \in p(x)$;

(Case 2) for all $m \in \mathbf{Z}\backslash\{0\}$ and all $a \in A$, $R(mx + a) \notin p(x)$.

Claim 2.21. Let $t(x, \bar{y})$ be an $\mathcal{L}$-term, with $\bar{y}$ of size n. Then for all $d \in C \setminus A$ and $\bar{a} \in A^n$ one of the following holds:

1. if $md + a = b \in R(C)$ for some $m \in \mathbf{Z} \setminus \{0\}$ and $a \in A$, then there exist $Q \in \mathbf{Z}[X]$, $m' \in \mathbf{Z}$ and $a' \in A$ such that $t(d, \bar{a}) = \mathsf{f}_Q(b) + m'x + a'$;
2. if for all $m \in \mathbf{Z} \setminus \{0\}$ and all $a \in A$, $md + a \notin R(C)$, then there exist $m' \in \mathbf{Z}$ and $a' \in A$ such that $t(d, \bar{a}) = m'd + a'$.

Proof. Let $d \in C \setminus A$.

1. Assume $md + a = b \in R(C)$ for some $m \in \mathbf{Z} \setminus \{0\}$ and $a \in A$. It is enough to show that for all $Q \in \mathbf{Z}[X]$ $m' \in \mathbf{Z} \setminus \{0\}$ and all $a' \in A$, if $\mathsf{f}_Q(b) + m'b + a' = b' \in R(C) \setminus R(A)$, then there exists $k \in \mathbf{Z}$ such that $\mathsf{f}_Q(b) + m'b + a' = S^k(b)$. Notice that $\mathsf{f}_Q(b) + m'b + a' = b'$ is equivalent to $m\mathsf{f}_Q(b) + m'b + ma' - m'a = mb'$. Let $\mathsf{f}'(x) = m\mathsf{f}_Q(x) + m'x$, so that $\mathsf{f}'(b) - mb' = m'a - ma'$. Notice that f' is non-trivial, since $b' \in R(C) \setminus R(A)$. Since $\mathscr{A}$ is a model of T_R, we can find $b_0, b_0' \in R(A)$ such that $\mathsf{f}'(b_0) - mb_0' = m'a - ma'$. Since $b, b' \in R(C) \setminus R(A)$, this implies by Lemma 2.6 that $m'a - ma' = 0$. As f' is non-trivial and $m \in \mathbf{Z} \setminus \{0\}$, (b, b') is a non-degenerate solution of $\mathsf{f}'(x) - my = 0$. So, by (Ax.8), there exists $k \in \mathbf{Z}$ such that $b' = S^k(b)$, which is what we wanted.

2. Assume that for all $m \in \mathbf{Z}\setminus\{0\}$ and all $a \in A$, $md+a \notin R(C)$. In that case $S(md+a) = md+a$ for all $m \in \mathbf{Z} \setminus \{0\}$ and all $a \in A$. This is enough to conclude. □

Using the previous claim, we may assume in the rest of the proof that the terms (with parameters in A) that appear in formulas are of the form $\mathsf{f}_Q(mx + a) + m'x + a'$, where $m' \in \mathbf{Z}$, $a' \in A$ and

1. $m \in \mathbf{Z} \setminus \{0\}$ and $a \in A$ are fixed when we are in (Case 1);

2. $Q = 0$ when we are in (Case 2).

Claim 2.22. The number of types of the form $p_{|\mathcal{L}_1}(x)$ is at most $\max\{2^{\aleph_0}, |A|\}$.

Proof. Indeed, any formula of the form $D_n(\mathsf{f}_Q(mx + a) + m'x + a')$ is equivalent to a formula of the form $D_n(\mathsf{f}_Q(mx + a) + m'x + k)$, where $k \in \mathbf{Z}$ is such that $D_n(a' - k)$. In (Case 2), we know that a formula of the form $m'x + a' = 0$ is never in $p(x)$, unless $m' = 0$ and $a' = 0$. Let us now look at equations when we are in (Case 2).

Assume that $\mathsf{f}_Q(mx + a) + m'x + a' = 0 \in p_{|\mathcal{L}_1}(x)$, where $m' \in \mathbf{Z}$ and $a' \in A$. Then, by axiom (Ax.6), $\mathrm{Im}_{mQ,m'X}(m'a - ma')$ holds in $\mathscr{A}$. Thus there exists $b' \in R(A)$ such that $\mathsf{f}(mx+a)+m'(mx+a) = \mathsf{f}(b')+m'b'$. This implies, by axiom (Ax.7) that $\mathsf{f}(mx+a)+m'(mx+a) = 0$. Hence $m'a-ma' = 0$. So the only equations that appear in $p(x)$ are of the form $\mathsf{f}(mx+a) = 0$. □

By the previous claim, it remains to show that the number of types of the form $p_{|\mathcal{L}_2}(x)$ is at most $\max\{2^{\aleph_0}, |A|\}$. So we need to look at formulas of the form $\mathrm{Im}_{[Q],D}(\mathsf{f}_1(mx + a) + m_1x + a_1, \dots, \mathsf{f}_k(mx + a) + m_kx + a_k)$. For simplicity, we only look at the case $k = 1$. We may restrict ourselves to formulas of the form $\mathrm{Im}_{\bar{Q},D}(\mathsf{f}(mx+a)+a')$ in (Case 1) and $\mathrm{Im}_{\bar{Q},D}(m'x+a')$ in (Case 2). We want to show that in both cases, we can separate the parameters from the variable, in the same way we did for divisibility conditions. This will be enough to conclude. For (Case 1), this is a consequence of Corollary 2.11. For (Case 2), we have the following claim.

Claim 2.23. Assume we are in (Case 2). Let $\bar{Q} \in \mathbf{Z}[X]^n$ and $m \in \mathbf{Z}\setminus\{0\}$. Then there exists at most one $a_{\bar{Q}} \in A$ such that $\sum_{i=1}^n \mathsf{f}_{Q_i}(x_i) = mx + a_{\bar{Q}}$ has a non-degenerate solution in $R(C)\setminus R(A)$.

Proof. Assume that there exists another $a' \in A$ that satisfies the claim. Then we have $\mathscr{A} \models \mathrm{Im}_{\bar{\mathsf{f}},-\bar{\mathsf{f}}}(a_{\bar{Q}} - a')$. Thus, we can find tuples $\bar{b}_1, \bar{b}_2 \in R(C)\setminus R(A)$ and $\bar{b}_1', \bar{b}_2' \in R(A)$ such that

$$\sum_{i=1}^n \mathsf{f}_{Q_i}(b_{1i}) - \mathsf{f}_{Q_i}(b_{2i}) - (\mathsf{f}_{Q_i}(b_{1i}') - \mathsf{f}_{Q_i}(b_{2i}')) = 0.$$

But this can happen only if $a_{\bar{Q}} = a'$ by Lemma 2.6. □

As a consequence, we get that in (Case 2), a formula of the form $\mathrm{Im}_{\bar{Q},D}(mx + a)$ is in $p_{|\mathcal{L}_2}(x)$ if and only if some disjunction of formulas of the form

$$\mathrm{Im}_{\bar{Q}_I,D}(mx + a_{\bar{Q}_I}) \wedge \mathrm{Im}_{\bar{Q}_{[n]\setminus I},D}(a - a_{\bar{Q}_I})$$

is in $p_{|\mathcal{L}_2}(x)$. This proves that the number of types of the form $p_{|\mathcal{L}_2}(x)$ in (Case 2) is at most $\max\{|A|, 2^{\aleph_0}\}$. We conclude that $|S_1(A)| \leq \max\{|A|, 2^{\aleph_0}\}$. □

Corollary 2.24. $\mathrm{Th}(\mathscr{Z})$ *is superstable.*

2.3. Decidability

As a consequence of the fact that the theory of $\mathscr{Z}_R$ is axiomatized by T_R when $R(\mathbf{Z})$ is enumerated by a regular sequence, we get the following decidability result. First let us recall that a sequence (r_n) is *effectively congruence periodic* if for all $k \in \mathbf{N}^{>1}$, there exist effective constants $m, p \in \mathbf{N}$ such that the sequence $(r_n)_{n\geq m}$ is periodic modulo k with period p.

Theorem 2.25. *Assume that*

1. *the limit* $\theta = \lim_{n\to\infty} r_{n+1}/r_n$ *can be computed effectively and*
2. $R(\mathbf{Z})$ *is effectively congruence periodic,*

Then the $\mathcal{L}$*-theory* T_R *is decidable.*

Proof. Indeed, under these assumptions, the constants that appear in (Ax.7), (Ax.8) can be computed effectively, using the proofs of Propositions 1.9 and 1.14. Furthermore, (Ax.6) becomes effective thanks to the effective periodicity of $R(\mathbf{Z})$. Thus, T_R is recursively axiomatizable. And since T_R is complete, we may conclude that T_R is decidable. □

Examples of regular sequences that satisfy Theorem 2.25 are (q^n), $(n!)$ and the Fibonacci sequence. More generally in [16] a family of such regular sequences is described, namely regular sequences for which in addition $\theta \in \mathbf{R}$, the sequence (r_n/θ^n) has a non zero limit, and (r_n/θ^n) converges to that limit effectively (†) [16, Proposition 11]. Furthermore in the case θ is transcendental, one asks that the sequence is effectively congruence periodic. (When (r_n) satisfies a linear recurrence relation this is automatic.) In the appendix of [16], one can find a proof that if (r_n) is an A. Bertrand sequence, then condition (†) is indeed fulfilled. (The condition of being an A. Bertrand sequence is a condition on the θ-expansions of real numbers in the interval $[0, 1]$ [16, Definition, section 1].)

Previously a lot of work has been done on the corresponding expansions of Presburger arithmetic: they are decidable and admit quantifier elimination (adding to the language in particular the function $\lambda(x)$ which sends a positive x to the biggest element of the sequence smaller than x, see below). These results are mainly due to A. L. Semënov who introduced the notion of *(effectively) sparse* sequences [20]. Let us introduce some notations: $=_{pp}$ (resp. $>_{pp}, <_{pp}$) means equality (resp. $>$, $<$) for all but finitely many.

A sequence (r_n) is *sparse* if it satisfies the following properties:

1. for all $Q \in \mathbf{Z}[X]$, either $\{n \in \mathbf{N} \mid \mathsf{f}_Q(n) = 0\} = \mathbf{N}$, or $\mathsf{f}_Q >_{pp} 0$, or $\mathsf{f}_Q <_{pp} 0$
2. for all $Q \in \mathbf{Z}[X]$, if $\mathsf{f}_Q >_{pp} 0$, then there exists a natural number Δ such that $\mathsf{f}_Q(n+\Delta) - r_n > 0$ for all $n \in \mathbf{N}$.

A sparse sequence (r_n) is *effectively sparse* if the above conditions are effective.

Let $\mathscr{N}_{<,R} = (\mathbf{N}, +, -, 0, R, <)$. A. L. Semënov showed that $\mathscr{N}_{<,R}$ is model complete whenever R is enumerated by a *sparse sequence* and decidable whenever R is enumerated by an *effectively sparse sequence* [20, Theorem 3]. Another proof of [20, Theorem 3] can be found in [16]. There, one axiomatizes the theory of $\mathscr{N}_{<,R}$ for (almost) sparse sequences and proves it admits quantifier elimination [16, Proposition 9], under the assumption that the (almost) sparse sequence is congruence periodic. While this is an extra assumption in comparison to [20, Theorem 3], we believe it can be removed at the cost of adding the symbols $\mathrm{Im}_{[Q],D}$ we used in our quantifier elimination result. Then decidability follows under the assumption that the sequence is effectively sparse [16, Proposition 11].

We end this section by giving a relation between sparse and regular sequences.

Lemma 2.26. *Let* (r_n) *be a regular sequence, then* (r_n) *is sparse.*

Proof. Let $\theta = \lim\limits_{n\to\infty} r_{n+1}/r_n \in \mathbf{R}_\infty^{\geq 1}$. It is known that (r_n) is sparse whenever $\theta = \infty$ [20, § 3] or $\lim\limits_{n\to\infty} r_n/\theta^n \in \mathbf{R}^{>0}$ [16, §4].

Assume that $\theta \in \mathbf{R}^{>1}$. We use the same kind of reasoning as in Section 1 (see for instance Lemma 1.11), but we consider polynomials $Q(X) \in \mathbf{Q}[X]$ (instead of in $\mathbf{Z}[X]$) and in addition we whether the operators are strictly positive.

Note that conditions such as $\mathsf{f}_Q(n) > 0$ for all sufficiently large $n \in \mathbf{N}$ are equivalent to $Q(\theta) > 0$. So the first condition of the definition of sparse sequence is satisfied. For the second condition assume that $Q(\theta) > 0$. The condition $\mathsf{f}_Q(n+\Delta) - r_n > 0$ for all $n \in \mathbf{N}$ sufficiently large is equivalent to $\theta^{\Delta} Q(\theta) - 1 > 0$. Since $Q(\theta) > 0$, we can then find $\Delta_0 \in \mathbf{N}$ and $n_0 \in \mathbf{N}$ such that $\mathsf{f}_Q(n+\Delta_0) - r_n > 0$ for all $n \geq n_0$. So, letting $\Delta = \Delta_0 + n_0$, we get that a regular sequence satisfies the second condition of the definition of a sparse sequence. □

2.4. Expansions of Presburger arithmetic and the NIP property

Let R be enumerated by a regular and congruence periodic sequence. In this last section, we will show that the theory of $\mathscr{Z}_{<,R} = (\mathbf{Z}, +, -, 0, R, <)$ is NIP. This result was announced in [1, Page 5934] and [2, Page 359] for (2^n) and the Fibonnaci sequence but without a proof. So we thought useful to provide one here.

We begin by briefly describing the setting of [16] but putting ourselves in $\mathbf{Z}$ (instead of $\mathbf{N}$) in order to remain in the same setting as in the first parts of this article. So from now on, even though all the results in [16] and [20] are stated for expansions of $\mathbf{N}$ by a (congruence periodic almost) sparse sequence, we will apply them to the corresponding expansions of $\mathbf{Z}$. In [16], the second author considered theories $T_{<,R}$ extending Presburger arithmetic and she showed on one hand that $T_{<,R}$ admits quantifier elimination [16, Proposition 9] and on the other hand that given an (almost) sparse congruence periodic sequence (r_n), the $\mathcal{L}_{<,R}$ expansion $\mathscr{Z}_{<,R}$ is a model of $T_{<,R}$. Let us briefly recall the axiomatization of $T_{<,R}$.

It is convenient to use the same language as in [16], namely

$$\mathcal{L}_{<,R} = \{+, -, <, 0, 1, \cdot/n, \lambda_R, S, S^{-1} \mid n \in \mathbf{N}^{>1}\},$$

where $\cdot/n$ is a unary function symbol interpreted as $\forall x \forall y \ (x/n = y \leftrightarrow \bigvee_{0 \leq k < n} x = ny + k)$, $n \in \mathbf{N}^{>1}$, λ_R is a unary function symbol interpreted on $\mathbf{Z}$ as the function sending $x \leq 0$ to 0 and for $x > 0$ to the biggest element of R smaller than or equal to x and S, S^{-1} are unary function symbols interpreted, as before, as the successor and predecessor functions on R. Also in this last section, $\mathcal{L}_g$ will denote the language $\{+, -, 0, 1, \cdot/n \mid n \in \mathbf{N}^{>1}\}$ (previously we used the unary relation symbols D_n instead of the unary functions $\cdot/n$.)

The $\mathcal{L}_{<,R}$-theory $T_{<,R}$ [16, page 1353] consists of a list of axioms translating the properties of $(\mathbf{Z}, +, -, <, 0, 1)$, of the sequence $(R(\mathbf{Z}), <, 1, S, S^{-1})$, and of the relationships between the two structures, namely the properties of the unary function λ_R (the sequence $R(\mathbf{Z})$ is interpreted by $\{z \in \mathbf{Z} \mid \lambda_R(z) = z\}$):

$$\forall x \ \big((\lambda_R(\lambda_R(x)) = \lambda_R(x) \ \wedge \ (x \leq 0 \rightarrow \lambda_R(x) = 0 \ \wedge \forall x \geq 1 \ (\lambda_R(x) \leq x < S(\lambda_R(x)) \ \wedge$$

$$\forall y (\lambda_R(x) \leq y < S(\lambda_R(x)) \rightarrow \lambda_R(y) = \lambda_R(x))\big),$$

how the sequence $R(\mathbf{Z})$ behaves with respect to the congruences and finally the properties of an (almost) sparse sequence (called axiom (6) in [16]). In the following we will make explicit use of this last scheme of axioms and so we will state it below explicitly. We use the notations of [16] and in particular the $\mathcal{L}_{<,R}$-terms that we will describe are the analogs of the $\mathcal{L}_g \cup \mathcal{L}_S$-terms f_Q, for $Q(X) \in \mathbf{Z}[X]$ in Section 2.1.

Given an $\mathcal{L}_{<,R}$-term $T(x)$ of the form $\sum_{j \geq 0} m_j S^{-j}(x)/n_j$ with $m = (m_j)$ and $n = (n_j)$ with $m_j \in \mathbf{Z}, n_j \in \mathbf{N}, m_0 \neq 0$, we associate finitely many expressions: $A(m,n)(x) - d(x)$, where $d(x) \in \mathbf{Z}$, $A(m,n)(x) = \sum_{j \geq 0} \frac{m_j}{n_j} S^{-j}(x)$ and $d(x) = \sum_{j \geq 0} \frac{m_j}{n_j} d_j$ where $S^{-j}(x) \equiv_{n_j} d_j$, $0 \leq d_j < n_j$. So even though in $A(m,n)(x)$ coefficients in $\mathbf{Q}$ may occur, we only apply division by a non-zero natural number n_j when the numerator is divisible by n_j.

Given any such $A(m,n)(x)$ abbreviated by $A(x)$, there is a constant $k(m,n)$ and a natural number s such that the following axiom holds:

whenever for some $x \in R$, $x > k(m,n)$ and $A(x) > 0$, then we have either:

$$\forall y \in R\ (y > k(m,n) \to S^{s-1}(y) < A(y) < S^s(y)), \quad \text{or} \tag{11}$$
$$\forall y \in R\ (y > k(m,n) \to S^{s-1}(y) = A(y)). \tag{12}$$

Moreover, we can describe the behaviour of $A(x)$ under a *small* perturbation: there is a constant $k'(m,n) \geq k(m,n)$ and $j_0 \in \mathbf{N}$ such that for every $y \in R, y > k'(m,n)$ and all $j \geq j_0$:

$$S^{s-1}(y) \leq A(y) < S^s(y) \to S^{s-1}(y) \leq A(y) + S^{-j}(y) < S^s(y), \tag{13}$$
$$S^{s-1}(y) < A(y) < S^s(y) \to S^{s-1}(y) \leq A(y) - S^{-j}(y) < S^s(y), \tag{14}$$
$$S^{s-1}(y) = A(y) \to S^{s-2}(y) \leq A(y) - S^{-j}(y) < S^{s-1}(y). \tag{15}$$

We will refer to this scheme of axioms (when $A(m,n)(x)$ varies) by (S_{sparse}).

Remark 2.27. The axiom scheme (S_{sparse}) has the following consequence. Let $M \models T_{<,R}$. Recall that two strictly positive elements $g, h \in M$ are in the same archimedean class if there are $n, m \in \mathbf{N}^0$ such that $g \leq nh \leq mg$. We observe that if the sequence (r_{n+1}/r_n) is unbounded, then $r_{n+1}/r_n \to \infty$. Indeed, if (r_{n+1}/r_n) is unbounded, then for all $m \in \mathbf{N}^{>0}$, for all but finitely many elements x of R, by axioms (11) and (12), we have $mx < S(x)$. So we get that either there exists $n \in \mathbf{N}^{>0}$ such that for all positive y but finitely many $\lambda_R(y) \leq y \leq n\lambda_R(y)$ or for all $m \in \mathbf{N}^{>0}$ for all but finitely many y, $m\lambda_R(y) \leq S(\lambda_R(y))$. In other words for elements y bigger than $\mathbf{Z}$, either y and $\lambda_R(y)$ are in the same archimedean class or never.

Lemma 2.28. *Let $R(\mathbf{Z})$ be enumerated by a sparse sequence (r_n), then $\mathscr{Z}_{<,R}$ satisfies the scheme (S_{sparse}).*

Proof. Let $\mathsf{f}_Q(n) = a_0 r_n + \cdots + a_d r_{n+d}$ with $Q(X) = \sum_{i=0}^d a_i X^i \in \mathbf{Q}[X]$. Note that, for $m \in \mathbf{N}$, $\mathsf{f}_{QX^m}(n) = a_0 r_{n+m} + \cdots + a_d r_{n+d+m}$, $\mathsf{f}_{QX^m-1}(n) = a_0 r_{n+m} + \cdots + a_d r_{n+d+m} - r_n$.

Assume that $\mathsf{f}_Q >_{pp} 0$. Since (r_n) is sparse, there exists $\Delta \in \mathbf{N}$ such that $\mathsf{f}_Q(n+\Delta) - r_n > 0$ for all n. Then choose Δ minimal with the property that $\mathsf{f}_{QX^\Delta - 1} >_{pp} 0$. This implies in particular that we do not have $\mathsf{f}_{QX^{\Delta-1}-1} >_{pp} 0$. But $\mathsf{f}_{QX^{\Delta-1}-1}$ is another operator and so since the sequence is sparse, we have that either $\mathsf{f}_{QX^{\Delta-1}-1}(n) = 0$ for all n, or that $\mathsf{f}_{QX^{\Delta-1}-1} <_{pp} 0$. So for almost all n, we have that $r_0 \leq a_0 r_{n+\Delta} + \cdots + a_d r_{n+d+\Delta} < r_{n+1}$. In case $a_0 r_{n+\Delta} + \cdots + a_d r_{n+d+\Delta} - r_n > 0$, by re-applying the same argument we get that for some Δ', we have that $a_0 r_{n+\Delta+\Delta'} + \cdots + a_d r_{n+d+\Delta+\Delta'} - r_{n+\Delta'} - r_n > 0$, namely setting $m = n + \Delta'$, we get: $a_0 r_{m+\Delta} + \cdots + a_d r_{m+d+\Delta} - r_{m-\Delta'} > r_m$. Finally, we consider $r_{n+1} - a_0 r_{n+\Delta} + \cdots + a_d r_{n+d+\Delta}$, since this is strictly positive there exists Δ'' such that $r_{n+\Delta''+1} - (a_0 r_{n+\Delta+\Delta''} + \cdots + a_d r_{n+d+\Delta+\Delta''}) - r_n > 0$. Therefore, $a_0 r_{n+\Delta+\Delta''} + \cdots + a_d r_{n+d+\Delta+\Delta''} + r_n < r_{n+\Delta''+1}$. Again setting $m = n + \Delta''$, we get: $a_0 r_{m+\Delta} + \cdots + a_d r_{m+d+\Delta} + r_{m-\Delta''} < r_{m+1}$. □

Let T be a complete $\mathcal{L}$-theory. Then T is NIP if all (partitioned) formulas $\varphi(x; y)$ are NIP [21, Definition 2.10]. (In a partitioned formula, one indicates the parameters (in this case y)). Also, for convenience, in this section we will adopt the following conventions: we will use single letters to possibly denote tuples of variables and since we deal with ordered structures, we will use the notation $|x|$ for $\max\{x, -x\}$, even though we previously used it for denoting either the cardinality of a set or the length of a tuple.

Lemma 2.29 ([21, Lemma 2.9])**.** *Assume that T admits quantifier elimination. Then T is NIP if and only if all atomic formulas $\varphi(x; y)$ are NIP.*

Lemma 2.30 ([21, Proposition 2.8])**.** *Let $\mathscr{M} \models T$. The partitioned formula $\varphi(x; y)$ is NIP if and only if for any indiscernible sequence $(a_i \mid i < \omega_1)$, where the length of a_i is equal to the length of x and tuple b in M there is some end segment $I \subset \omega_1$ such that for any $i \in I$, the truth value of $\varphi(a_i; b)$ is constant.*

We will use Hahn representation theorem for divisible ordered abelian groups [9, Section 4.5]. Let G be an abelian totally ordered group and let $\bar{G}$ be its divisible closure. Given an element $g \in G \setminus \{0\}$ there is a unique convex subgroup V maximal for the property of not containing g; it is called a *value* for g. There is also a smallest convex subgroup V^+ containing g and the quotient V^+/V is an archimedean ordered group (which by Hölder's theorem, embeds in $(\mathbf{R}, +, <, 0)$). The set of all values in G forms a chain denoted by $\Gamma(G)$; we set $\Gamma(G) = \{V_\gamma \mid \gamma \in \Gamma\}$ and for $V = V_\gamma$, we denote V^+ by V^γ. Note that $\Gamma(\bar{G}) = \Gamma(G)$. Denote by $R_\gamma = V^\gamma/V_\gamma$ and let $\bar{R}_\gamma$ be the $\mathbf{Q}$-vector-subspace generated by R_γ in $\mathbf{R}$. One can decompose $\bar{G}$ as a direct sum $\bar{G} = \bar{V}^\gamma \oplus D_\gamma$, where $\bar{V}^\gamma$ is the divisible closure of V^γ in $\bar{G}$ and D_γ is some direct summand. Denote by π_γ the projection of $\bar{G}$ to $\bar{V}^\gamma$ and let $\rho_\gamma : \bar{V}^\gamma \to \bar{R}_\gamma$. Then one sends g to the function $\hat{g} : \Gamma(G) \to \mathbf{R} : \gamma \mapsto \rho_\gamma \pi_\gamma(g) = \hat{g}(\gamma)$. One verifies that $\text{supp}(g) = \{\gamma \in \Gamma(G) \mid \hat{g}(\gamma) \neq 0\}$ is an anti-well ordered subset of $\Gamma(G)$. Denote by $V(\Gamma(G), \bar{R}_\gamma)$ the lexicographically ordered group of functions f from $\Gamma(G)$ to $\mathbf{R}$ with anti-well-ordered support, such that $f(\gamma) \in \bar{R}_\gamma$, for any $\gamma \in \Gamma(G)$. Then G embeds in $V(\Gamma(G), \bar{R}_\gamma)$ by the map $g \mapsto \hat{g}$ [9, Theorem 4C]. Define a map $v : G \to \Gamma(G)$ which sends $g \in G \setminus \{0\}$ to $\max(\text{supp}(\hat{g}))$. This is a valuation map on G as defined in [8, Chapter 4, section 4] except that there one takes the opposite order on $\Gamma(G)$. It is constant on an archimedean class, namely if satisfies the following: for all $g, h \in G^{>0}$, $v(g) = v(h)$ if and only if $g \leq nh \leq mg$ for some $n, m \in \mathbf{N}^{>0}$ (in other words, g and h are in the same archimedean class).

In order to show that $T_{<,R}$ is NIP, we need some preparatory work in order to evaluate terms of the form $\lambda_R(x \pm y)$ for $x, y > 0$.

Lemma 2.31. *Let $\mathscr{M} \models T_{<,R}$, $d \in M$ and $(c_i \mid i \in \omega_1)$ be a non-constant indiscernible sequence in M such that $d > 0$ and $c_i > 0$ for all $i \in \omega_1$. Then there exist $i_0 \in \omega_1$, $\ell \in \mathbf{Z}$ such that one of the following holds for all $i \geq i_0$:*

- $\lambda_R(c_i \pm d) = S^\ell(\lambda_R(c_i))$;
- $\lambda_R(d \pm c_i) = S^\ell(\lambda_R(d))$;
- $\lambda_R(|d - c_i|) = S^\ell(\lambda_R(|d - c_{i_0}|))$;
- $\lambda_R(|d - c_i|) = S^\ell(\lambda_R(|c_{i+1} - c_i|))$;
- $\lambda_R(|d - c_{i+1}|) = S^\ell(\lambda_R(|c_{i+1} - c_i|))$.

Proof. The proof has two main ingredients. The first one is that in a model of a NIP theory, an indiscernible sequence remains eventually indiscernible over a parameter (see [21, Claim in the proof of Proposition 2.11]). The second one is the following consequence of axioms (11) and (12) in scheme (S_{sparse}): for all $n, m \in \mathbf{N}^{>0}$ there exist $\ell, \ell' \in \mathbf{Z}$ such that for all $x \in M$, if x is bigger than some natural number then $S^\ell(\lambda_R(x)) \leq \frac{1}{n}\lambda_R(x) \leq \frac{m}{n} S(\lambda_R(x)) \leq S^{\ell'}(\lambda_R(x))$.

Let $(c_i \mid i \in \omega_1)$ be a non-constant indiscernible sequence. We will apply the first ingredient to certain sequences of the form $(t(c_i) \mid i \in \omega_1)$, where $t(x)$ is a $\mathcal{L}_{<,R}$-terms, and to the NIP theory $\text{Th}(\mathbf{Z}, +, -, 0, <)$. For ease of notations, we may assume that $(c_i \mid i \in \omega_1)$ is indiscernible over d in $\{+, -, 0, <\}$. In particular we have that either $d > c_i$ for all $i \in \omega_1$ or $d < c_i$ for all $i \in \omega_1$. By comparing $v(d)$ to $v(c_0)$, we obtain that the sequence $(c_i \mid i \in \omega_1)$ falls into the following cases for some $m, n \in \mathbf{N}^{>0}$:

1. $c_i \leq n(c_i \pm d) \leq mc_i$ for all $i \in \omega_1$;
2. $d \leq n(d \pm c_i) \leq md$ for all $i \in \omega_1$;
3. $|c_{i+1} - c_i| \leq n|d - c_{i+1}| \leq m|c_{i+1} - c_i|$ for all $i \in \omega_1$;
4. $|d - c_0| \leq n|d - c_i| \leq m|d - c_0|$ for all $i \in \omega_1$;
5. $|c_{i+1} - c_i| \leq n|d - c_i| \leq m|c_{i+1} - c_i|$ for all $i \in \omega_1$.

Cases 1 and 2 occur when $v(d) \neq v(c_0)$ and case 2 together with the remaining cases when $v(d) = v(c_0)$. In case $v(d) = v(c_0)$, we further compare $v(c_1 - c_0)$, $v(d - c_1)$ and $v(d - c_0)$. Let us check it in details.

1. $v(d) < v(c_0)$. Then $v(c_0 \pm d) = v(c_0)$, so that $c_0 \leq n(c_0 \pm d) \leq mc_0$ for some $n, m \in \mathbf{N}^{>0}$. By indiscernibility over d, we get that $c_i \leq n(c_i \pm d) \leq mc_i$ for all $i \in \omega_1$;

2. $v(d) > v(c_0)$. Then $v(d \pm c_0) = v(d)$, so that $d \leq n(d \pm c_0) \leq md$ for some $n, m \in \mathbf{N}^{>0}$. By indiscernibility over d, we get that $d \leq n(d \pm c_i) \leq md$ for all $i \in \omega_1$;

3. $v(d) = v(c_0)$. Then, as $v(d + c_0) = v(d)$, we have that $d \leq n(d + c_0) \leq md$ for some $n, m \in \mathbf{N}^{>0}$. Hence $d \leq n(d + c_i) \leq md$ for all $i \in \omega_1$. Furthermore, one of the following holds, using indiscernibility as before:

   (a) $v(c_1 - c_0) = v(d - c_1)$. Then there are $n, m \in \mathbf{N}^{>0}$ such that $|c_1 - c_0| \leq n|d - c_1| \leq m|c_1 - c_0|$ and so for all $i \in \omega_1$, $|c_{i+1} - c_i| \leq n|d - c_{i+1}| \leq m|c_{i+1} - c_i|$;

   (b) $v(c_1 - c_0) \neq v(d - c_1)$. Then $v(d - c_0) = \max\{v(d - c_1), v(c_1 - c_0)\}$. So,

      i. either $v(d-c_0) = v(d-c_1)$, in which case there are $n, m \in \mathbf{N}^{>0}$ such that $|d-c_0| \leq n|d - c_1| \leq m|d - c_0|$ and so for all $i \in \omega_1$, $|d - c_0| \leq n|d - c_i| \leq m|d - c_0|$;

      ii. or $v(d - c_0) = v(c_1 - c_0)$, in which case there are $n, m \in \mathbf{N}^{>0}$ such that $|c_1 - c_0| \leq n|d - c_0| \leq m|c_1 - c_0|$ and so for all $i \in \omega_1$, $|c_{i+1} - c_i| \leq n|d - c_i| \leq m|c_{i+1} - c_i|$.

Now given $g, h \in M^{>0}$ in the same archimedean class, more precisely such that $h \leq ng \leq mh$, for some $n, m \in \mathbf{N}^{>0}$, observe that there are $\ell, \ell' \in \mathbf{N}$ such that $\lambda_R(g) \in \{S^k(\lambda_R(h)) \mid \ell \leq k \leq \ell'\}$. This follows from axioms (11) and (12). Indeed, there are $\ell, \ell' \in \mathbf{N}$ such that to $\frac{m}{n}S(\lambda_R(h)) \leq S^{\ell}(\lambda_R(h))$ and $S^{\ell'}(\lambda_R(h)) \leq \frac{1}{n}\lambda_R(h)$, for h bigger than some natural number.

Applying the discussion above, with g of the form $|d \pm c_i|$ and h equal to either c_i, d, $d - c_0$, $|c_{i+1} - c_i|$, or $|c_i - c_{i-1}|$, we get:

- $\lambda_R(c_i \pm d) \in \{S^k(\lambda_R(c_i)) \mid \ell \leq k \leq \ell'\}$;
- $\lambda_R(d \pm c_i) \in \{S^k(\lambda_R(d)) \mid \ell \leq k \leq \ell'\}$;
- $\lambda_R(|d - c_i|) \in \{S^k(\lambda_R(|d - c_0|)) \mid \ell \leq k \leq \ell'\}$;
- $\lambda_R(|d - c_i|) \in \{S^k(\lambda_R(|c_{i+1} - c_i|)) \mid \ell \leq k \leq \ell'\}$;
- $\lambda_R(|d - c_{i+1}|) \in \{S^k(\lambda_R(|c_{i+1} - c_i|)) \mid \ell \leq k \leq \ell'\}$.

In case $\lambda_R(|d - c_i|)$ and $\lambda_R(d + c_i)$ belong to a finite set, for i sufficiently big, we get a constant value since these sequences are monotone.
Let us focus on the case where $\lambda_R(|d - c_i|) \in \{S^k(\lambda_R(|c_{i+1} - c_i|)) \mid \ell \leq k \leq \ell'\}$. The cases where $\lambda_R(c_i \pm d) \in \{S^k(\lambda_R(c_i)) \mid \ell \leq k \leq \ell'\}$ and $\lambda_R(|d - c_{i+1}|) \in \{S^k(\lambda_R(|c_{i+1} - c_i|)) \mid \ell \leq k \leq \ell'\}$ are similar.

First let us assume that $|d-c_i| = d-c_i$ for all $i \in \omega_1$. Recall that $\lambda_R(d-c_i) = S^k(\lambda_R(|c_{i+1}-c_i|))$ means $S^k(\lambda_R(|c_{i+1} - c_i|)) \leq d - c_i < S^{k+1}(\lambda_R(|c_{i+1} - c_i|))$. So, for $k \in [\ell, \ell']$, let $(t_k(c_i)) \mid i \in \omega_1)$ be the sequence defined by $t_k(c_i) = S^k(\lambda_R(|c_{i+1} - c_i|)) + c_i$. For all $k \in [\ell, \ell']$, the sequence $(t_k(c_i) \mid i \in \omega_1)$ is indiscernible. Hence for each $k \in [\ell, \ell']$, we may assume that $(t_k(c_i) \mid i \in \omega_1)$ is $\{<\}$-indiscernible over d, for $i > i_k$. Let $k_0 \in [\ell, \ell']$ maximal such that $t_{k_0}(c_i) \leq d$ for all $i > i_0 = \max\{i_k \mid \ell \leq k \leq \ell'\}$. Then we have that for all $i > i_0$ that $d < t_{k_0+1}(c_i)$. Thus, $\lambda_R(d - c_i) = S^{k_0}(\lambda_R(|c_{i+1} - c_i|))$ for all $i > i_0$.

Second, if $|d - c_i| = c_i - d$ for all $i \in \omega_1$, we can repeat the argument using the sequences $(t'_k(c_i) \mid i \in \omega_1)$ defined by $t'_k(c_i) = c_i - S^k(\lambda_R(|c_{i+1} - c_i|))$, for all $i \in \omega_1$ and $k \in [\ell, \ell']$. □

Theorem 2.32. *The theory $T_{<,R}$ is NIP.*

Proof. Since $T_{<,R}$ has quantifier elimination by [16, Proposition 9], we may apply Lemma 2.29 and consider atomic formulas. Let us first look at $\mathcal{L}_{<,R}$-terms, when we evaluate them along an indiscernible sequence and a parameter. Let $(a_i \mid i \in \omega_1)$ be a non-constant indiscernible sequence and $b \in M$ some tuple of parameters.

Claim 2.33. Let $t(x;y)$ be an $\mathcal{L}_{<,R}$-term. Then there exists $i_0, i_1, \ldots, i_n \in \omega_1$ and $\mathcal{L}_{<,R}$-terms $t_1(x, x_1, \ldots, x_{2n})$, $t_2(y, y_1, \ldots, y_n)$ such that for all $i \geq i_0$:

$$t(a_i;b) = t_1(a_i, a_{i+1}, \ldots, a_{i+n}, a_{i-1}, \ldots, a_{i-n}) + t_2(b, a_{i_1}, \ldots, a_{i_n}).$$

Proof of Claim. We proceed by induction on the number of occurrences of the symbol λ_R in $t(x;y)$. If $t(x;y)$ is an $\mathcal{L}_g$-term, we can write it as $t_1(x) + t_2(y)$, see [16, Lemma 4]. So what remains to be shown is that the claim holds for terms of the form

$$\lambda_R(t_1(a_i, a_{i+1}, \ldots, a_{i+n}, a_{i-1}, \ldots, a_{i-n}) + t_2(b, a_{i_1}, \ldots, a_{i_n})),$$

where $t_1(x, x_1, \ldots, x_{2n})$ and $t_2(y, y_1, \ldots, y_n)$ are $\mathcal{L}_{<,R}$-terms. Let $d = t_2(b, a_{i_1}, \ldots, a_{i_n})$ and $c_i = t_1(a_i, a_{i+1}, \ldots, a_{i+n}, a_{i-1}, \ldots, a_{i-n})$. Without loss of generality, we may assume that $d \neq 0$ and $(c_i \mid i \in \omega_1)$ is non-constant. Then $(c_i \mid i \in \omega_1)$ is indiscernible and one of the following holds:

1. $c_i + d \leq 0$ for all $i \in \omega_1$ sufficiently large. In that case, $\lambda_R(c_i + d) = 0$ for all $i \in \omega_1$ sufficiently large;
2. $c_i + d > 0$ for all $i \in \omega_1$ sufficiently large. In this case, we apply Lemma 2.31 to conclude. □

Finally let us check that the truth value of $\varphi(a_i;b)$ is eventually constant, when $\varphi(x;y)$ is an atomic formula. By Claim 2.33, we may assume that $\varphi(x;y)$ is of the form $t_1(\bar{x}) < t_2(\bar{y})$ or $t_1(\bar{x}) = t_2(\bar{y})$ for t_1 and t_2 two $\mathcal{L}_{<,R}$-terms. Here we used $\bar{x}$ instead of x since the length of $\bar{x}$ is possibly bigger than the length of x (and likewise for y). Let $c_i = t_1(\bar{a})$ and $d = t_2(\bar{b})$. Then $(c_i \mid i \in \omega_1)$ is indiscernible and there exists $i_0 \in \omega_1$ such that $(c_i \mid i \geq i_0)$ is indiscernible over d in the language $\{<\}$. In particular the truth value of $t_1(\bar{x}) < t_2(\bar{b})$ or $t_1(\bar{x}) = t_2(\bar{b})$ is constant over $(c_i \mid i \geq i_0)$. □

Corollary 2.34. *Let $R(\mathbf{Z})$ be enumerated by a congruence periodic regular sequence. Then the theory* $\mathrm{Th}(\mathscr{Z}_{<,R})$ *is NIP.*

Proof. Since $R(\mathbf{Z})$ is enumerated by a sparse sequence (see Lemma 2.26), by Lemma 2.28, the theory $T_{<,R}$ is equal to the theory of $\mathscr{Z}_{<,R}$. □

Acknowledgments

The authors would like to thank Gabriel Conant for helpful comments on a previous version of this paper and in particular for spotting a mistake in a previous draft and for enlightening exchanges. Finally, we would like to warmly thank the referee for their very detailed and careful report.

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