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Cell decomposition and dimension function in the theory of closed ordered differential fields ☆

Thomas Brihaye, Christian Michaux, Cédric Rivière *

Université de Mons-Hainaut, Belgium

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ABSTRACT

In this paper we develop a differential analogue of o-minimal cell decomposition for the theory *CODF* of closed ordered differential fields. Thanks to this differential cell decomposition we define a well-behaving dimension function on the class of definable sets in *CODF*. We conclude this paper by proving that this dimension (called δ-dimension) is closely related to both the usual differential transcendence degree and the topological dimension associated, in this case, with a natural differential topology on ordered differential fields.

0. Introduction

This paper is dedicated to the study of definable sets in the theory *CODF* of closed ordered differential fields. This theory was introduced by M. Singer in 1978 (see [14]) as the model completion of the theory of ordered differential fields. Singer's definition of *CODF* clearly shows that this theory has quantifier elimination in the natural language of ordered differential rings. Our approach here is to develop an analogue of o-minimality in the differential context of ordered differential fields.

O-minimal structures were introduced in the late 80s (see [8,4,17]) and are defined as totally ordered structures whose one dimensional definable sets are the ones obtained using only the order and the equality (i.e. these sets are finite unions of points and open intervals). Lots of interesting structures have been proved to be o-minimal: real closed fields, ordered abelian divisible groups, the field of real numbers with the exponential function and others (see for example [20,15,18,19,13] for the latter and other examples of o-minimal expansions of $\mathbb{R}$).

O-minimal structures possess very nice geometric properties; in particular the Cell Decomposition Theorem (see [17, Chapter 3 (2.11)]) states that every n-dimensional definable set can be decomposed into a finite union of "elementary" definable pieces called cells. As a consequence of this theorem, a well-behaving notion of dimension can be associated with any definable set.

The goals of this paper are: firstly, to prove a differential analogue of the Cell Decomposition Theorem and secondly, to define a natural notion of dimension for definable sets in *CODF*.

Our first step is to define a reasonable notion of cells in *CODF* (called δ-cells), keeping in mind that the subfield of constant elements (i.e. element having derivative zero) of a closed ordered differential field M is a dense (w.r.t. the order

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* Corresponding author.
E-mail addresses: thomas.brihaye@umh.ac.be (T. Brihaye), christian.michaux@umh.ac.be (C. Michaux), cedric.riviere@umh.ac.be (C. Rivière).

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topology on M) definable subset of M. This simple question requires in fact quite a lot of work and leads to the introduction of jet-spaces in *CODF* and a notion of equivalence (called δ-equivalence) on the "derivation free" definable sets in *CODF*. Although this notion of δ-equivalence gives rise to some technical problems, it allows us to define a natural "differential" topology (called δ-topology) on any model M of *CODF* which can be interpreted as the trace on the infinite jet-space $J_\omega(M) := \{(x, x', \ldots, x^{(n)}, \ldots) \mid x \in M\}$ of the product topology induced on M^ω by the order topology.

We are then able to introduce a notion of *δ-cell* generalizing the notion of o-minimal cell. Even if these δ-cells do not behave as well as their o-minimal analogues (e.g. there exist (0)-δ-cells which are infinite!), it appears that they share quite a lot of interesting properties with them. In particular we define a *δ-dimension* on δ-cells which, as in the o-minimal case, resumes to summing 1's and 0's.

Furthermore we prove a theorem of differential cell decomposition which generalizes the Cell Decomposition Theorem for o-minimal structures. As a consequence of this result we extend the definition of the δ-dimension to any definable set in *CODF* and show that this δ-dimension is a *dimension function* in the sense of the axioms given par L. van den Dries in [16]. We also remark that the δ-dimension is equal to the differential transcendence degree of a *generic point* in an elementary extension of M and to the *topological dimension* associated with the δ-topology.

The rest of this paper is organised as follows: in Section 1 we recall the basic results concerning the theory *CODF* and o-minimal structures. The notion of jet-spaces is introduced in Section 2 (Definition 2.1) where we also prove a simple but important result of density on these jet-spaces (Lemma 2.2). In addition we fix some notation which will be crucial in the subsequent developments and may seem quite complicated at first hand (Definition 2.3). The third section is dedicated to the δ-topology and its elementary properties (see Definition 3.1, Proposition 3.3). In Section 4 we first introduce δ-cells (Definitions 4.1 and 4.2) and then give the statement of our theorem of cell decomposition in *CODF* (Theorem 4.9). Finally, in Section 5, we introduced a notion of δ-dimension first on δ-cells and then on any definable sets in *CODF* (Definitions 5.1 and 5.3). We conclude with a list of interesting properties satisfied by this δ-dimension (Theorems 5.14, 5.19 and 5.29, Corollary 5.27, etc.).

1. Preliminaries

For any model M of *CODF* and for any $l \leq k$, we denote by $\pi_{(j_1,\ldots,j_l)} : M^k \to M^l$ the projection onto the coordinates $j_1, \ldots, j_l$ and by π_l the projection onto the l first coordinates. We also denote by L the language $\{+, -, *, <, 0, 1\}$ of ordered rings and by L' the language $\{+, -, *, ', <, 0, 1\}$ of ordered differential rings.

1.1. O-minimal structures

O-minimal structures have become a huge domain of research since the 80s. Most of the classical results on these structures can be found in [8,9,4] and [16]. We just recall here the basic definitions and results concerning these structures.

Let $M = (M, <, \ldots)$ be a densely totally ordered structure. M is **o-minimal** if any definable subset of M is a finite union of points and open intervals (a, b) with $a, b \in M \cup \{-\infty, +\infty\}$. In other words, there is no other definable subset in M than those which are definable using $<$ and $=$. Densely linearly ordered (non-empty) sets $(Q, <)$, divisible abelian ordered groups $(G, +, <, 0)$ and real closed fields $(M, +, -, *, <, 0, 1)$ are classical examples of o-minimal structures (the o-minimality of these structures directly follows from the fact that they admit quantifier elimination in their associated language, see for example [2]). Any o-minimal structure is equipped with a natural definable topology, namely the **order topology**. A basis of open subsets of M for this topology is given by the open intervals $(a, b) \subseteq M$, i.e. this basis is uniformly defined by formulas $\varphi(a, b, X) \equiv a < X < b$, where $a, b \in M$.

In what follows, unless explicitly stated (see the δ-topology in Section 3), *all the topological objects appearing in the text refer to the order topology (or to the product topology induced by the order topology when we work in a Cartesian power of M).*

The classical tools in the study of o-minimal structures are the notion of **cells** and the **Cell Decomposition Theorem** proved by J. Knight, A. Pillay and C. Steinhorn in 1986 [4]. Cells are defined inductively as follows.

Definition 1.1. For any definable subset A of M let

$$C_{A\infty} = \{f : A \to M, f \text{ definable and continuous on A}\} \cup \{-\infty, +\infty\}$$

(where we consider $-\infty$ and $+\infty$ as constant functions on A) and define

$$(f, g)_A = \{(x, m) \in A \times M \mid f(x) < m < g(x)\}$$

where $f,\ g \in C_{A\infty}$ are such that for all $x \in A, f(x) < g(x)$.
Then

(i) a (0)-cell is a singleton $\{m\}$ of M and a (1)-cell is an open interval (a, b) with $a, b \in M \cup \{-\infty, +\infty\}$;
(ii) an $(i_1, \ldots, i_k, 0)$-cell is the graph of a continuous definable function $f : D \to M$ where D is an $(i_1, \ldots, i_k)$-cell and an $(i_1, \ldots, i_k, 1)$-cell is a set $(f, g)_D$ where D is an $(i_1, \ldots, i_k)$-cell (see Fig. 1).

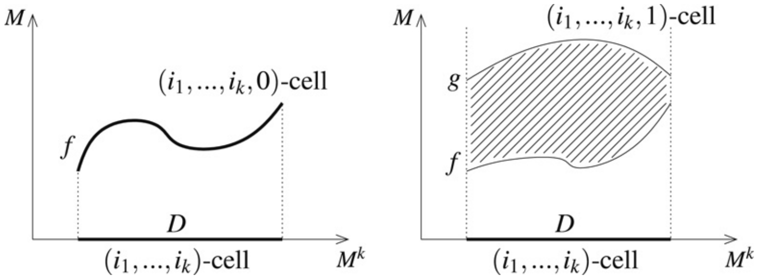

Fig. 1. Cells.

The tuple $(i_1, \ldots, i_k)$ is called the **type** of the cell (i_j is called the type in the variable X_j) and the $(1, \ldots, 1)$-cells are called **open cells** (actually they are exactly the cells which are open in M^k).

Before we give the statement of the Cell Decomposition Theorem, we recall that a **decomposition** of M^k $(k > 1)$ is a partition $\mathcal{P}$ of M^k into finitely many cells such that the projection $\pi_{k-1}(\mathcal{P})$ is still a decomposition of M^{k-1}. In the case where $k = 1$, a decomposition of M is a collection

$$\{(-\infty, m_1), (m_1, m_2), \ldots, (m_h, +\infty), \{m_1\}, \ldots, \{m_h\}\}$$

with $m_1, \ldots, m_h \in M$.

Theorem 1.2 (*Cell Decomposition Theorem (Knight–Pillay–Steinhorn)*). I_k: *For any finite collection $\mathcal{A}$ of definable (with parameters in $P \subset M$) subsets of M^k, there exists a finite decomposition of M^k into cells compatible with $\mathcal{A}$ (i.e. such that any $A \in \mathcal{A}$ is union of cells). Furthermore each cell of this decomposition is definable with parameters from P.*

II_k: *For any definable function $f : A \to M$ where $A \subseteq M^k$, there exists a finite decomposition of M^k into cells (definable over the same set of parameters as f) partitioning A such that the restriction of f to any of these cells is continuous.*

Proof. See [4] or [17, Chapter 3, (2.11)]. □

The Cell Decomposition Theorem allows to define a particularly well-behaving dimension on definable sets in an o-minimal structure. Precisely, the **dimension** of an $(i_1, \ldots, i_k)$-cell is equal to $i_1 + \cdots + i_k$ and, for each definable subset A of M^k, the dimension of A is given by the cell of maximal dimension contained in A. One can consult [17, Chapter 3] for all the properties satisfied by this dimension but most of them will be recalled in Section 5.

1.2. The theory CODF

We begin with some basic definitions and results from differential algebra, our reference being [3]. Let M be a differential field (equipped with a non-trivial derivation), the **ring of differential polynomials** on M is the ring $M[X, X', X^{(2)}, \ldots]$ which is denoted by $M\{X\}$. Its fraction field is denoted by $M\langle X\rangle$. Remark that $M\{X\}$ is a differential ring equipped with the derivation extending the one on M and sending $X^{(n)}$ to $X^{(n+1)}$. Let $f \in M\{X\}$, the **order** of f (denoted by $ord(f)$) is the highest derivative of X appearing in f (with the convention that the order of a non-zero constant polynomial is -1 and the order of the zero polynomial is $-\infty$). If $ord(f) = n$ we define the **separant** of f to be $s_f(X) = \frac{\partial f}{\partial X^{(n)}}(X)$. An element a in a differential field extending M is a **generic zero** for $f \in M\{X\}$ with $s_f \neq 0$ if $f(a) = 0$ and $g(a) \neq 0$ for all g such that $ord(g) < ord(f)$. This is equivalent to say that $a, a', \ldots, a^{(n-1)}$ are algebraically independent over M and that $a^{(n)}, a^{(n+1)}, \ldots$ are algebraic over $M(a, a', \ldots, a^{(n-1)})$.

In the same way we can define the **ring of differential polynomials in** k **variables** $M\{X_1, \ldots, X_k\}$ for each natural number k. In this case we also define the order of $f \in M\{X_1, \ldots, X_k\}$ in each variables X_i as in the "one variable case" and we denote it by $ord_{X_i}(f)$. This notion of order will be extended to any first order formula φ of the natural language of ordered differential fields in Chapter 2 (Definition 2.3).

The model-theoretic concept of an **ordered differential field**, i.e. an ordered field equipped with a derivation (no link is assumed between the order and the derivation), was first introduced by A. Robinson in [12]. In 1978 M. Singer proved that the L'-theory *ODF* of ordered differential fields has a model completion *CODF*. The models of *CODF* are called **closed ordered differential fields**.

Let us recall Singer's axiomatization for *CODF*:

Definition 1.3. Let M be an ordered differential field, then $M \models CODF$ if

(i) M is a real closed field[1];

[1] In what follows, *RCF* will denote the L-theory of real closed fields.

(ii) for all $f, g_1, \ldots, g_m$ in $M\{X\}$ with $n = ord(f) > ord(g_i) > -\infty$ $(i = 1, \ldots, m)$,

$$\underbrace{\exists \overline{X}\left(f(\overline{X}) = 0 \wedge s_f(\overline{X}) \neq 0 \wedge \bigwedge_{i=1}^{m} g_i(\overline{X}) > 0\right)^L}_{(*)} \Rightarrow \underbrace{\exists X\left(f(X) = 0 \wedge \bigwedge_{i=1}^{m} g_i(X) > 0\right)}_{(*')}$$

where we use the superscript L to denote that we consider the formula as an L-formula (i.e. we consider each differential polynomial appearing in the formula as an ordinary polynomial in the variables $X_0, X_1, \ldots, X_n$).

The scheme of axioms in (ii) just says that if the system $(*)$ of (ordinary) polynomial equations and inequations above has a solution $(x_0, \ldots, x_n)$ in M^{n+1} then the *differential* system $(*')$ has a solution x in M.

As a direct consequence of the construction of *CODF*, Singer obtained the following important result:

Theorem 1.4 ([14]). *CODF has quantifier elimination in the language* $L' = \{+, -, *, ', <, 0, 1\}$.

2. Jet-spaces in *CODF* and associated notation

Definition 2.1. Let M be a differential field, the n-**jet-space** of M is the subset of M^{n+1} defined by

$$J_n(M) := \{(x, x', \ldots, x^{(n)}) \mid x \in M\}.$$

More generally for each natural number k and each k-tuple $(n_1, \ldots, n_k) \in \mathbb{N}^k$ we define the $(n_1; \ldots; n_k)$-jet-space of M^k to be the set[2]:

$$\begin{aligned} J_{(n_1;\ldots;n_k)}(M^k) &:= \{(x_1, x_1', \ldots, {x_1}^{(n_1)}; \ldots; x_k, x_k', \ldots, {x_k}^{(n_k)}) \mid (x_1; \ldots; x_k) \in M^k\} \\ &= J_{n_1}(M) \times \cdots \times J_{n_k}(M). \end{aligned}$$

Remark that this definition naturally extends to any subset A of M^k and we then can speak of the $(n_1; \ldots; n_k)$-jet-space associated with A (denoted by $J_{(n_1;\ldots;n_k)}(A)$).

This notion of jet-space can be defined in any differential fields but, in the case of closed ordered differential fields, they have the following interesting property.[3]

Lemma 2.2. *If* $M \models$ *CODF then for each k-tuple* $(n_1, \ldots, n_k)$ *of positive integers, the jet-space* $J_{(n_1;\ldots;n_k)}(M^k)$ *is dense (and co-dense when* $n_1 + \cdots + n_k > 0$*) in* $M^{(n_1+1)+\cdots+(n_k+1)}$ *w.r.t. the order topology.*

Proof. Remark first that, since density is preserved by direct product of topological spaces, it suffices to prove the result for $k = 1$. Hence let us fix a natural number n in order to show that $J_n = \{(x, x', \ldots, x^{(n)}) \mid x \in M\}$ is dense in M^{n+1}.

For this let $(m_0, \ldots, m_n)$ be an element of M^{n+1} and consider the algebraic system

$$f(\overline{X}) = X_{n+1} = 0 \wedge s_f(\overline{X}) \neq 0 \wedge \bigwedge_{i=0}^{n} (m_i - \epsilon_i < X_i < m_i + \epsilon_i)$$

where $\overline{X} = (X_0, \ldots, X_{n+1})$, $s_f(\overline{X}) = \frac{\partial f}{\partial X_{n+1}}(\overline{X})$ and $\epsilon_i > 0$.

This system has an algebraic solution $(m_0, \ldots, m_n, 0)$ and, since $s_f(X)$ is the constant polynomial 1, the axiomatization of *CODF* provides a differential solution of the form $(m, m', \ldots, m^{(n)}, 0)$ (with $m \in M$) to this system. In particular, this proves that each open neighbourhood of $(m_0, \ldots, m_n)$ in M^{n+1} has non-empty intersection with $J_n(M)$.

The co-density is trivial since, given any $n \geq 1$, the projection $\pi_1 : J_n(M) \to M$ is a bijection. $\square$

We now introduce some notation that will be used in the subsequent developments of this text.

Definition 2.3. Let M be a model of *CODF*, $A_\varphi := \{\overline{x} \in M^k \mid \varphi(\overline{x})\}$ be a *quantifier free*[4] L'-definable subset of M^k and assume that the highest derivative of X_i appearing non-trivially in φ is $X_i^{(n_i)}$. Hence the L'-formula φ can be interpreted as an L-formula φ^L in the differential variables $X_1, {X_1}', \ldots, {X_1}^{(n_1)}; \ldots; X_k, {X_k}', \ldots, {X_k}^{(n_k)}$ such that:

$$\forall X_1, \ldots, X_k(\varphi(X_1, \ldots, X_k) \Leftrightarrow \varphi^L(X_1, {X_1}', \ldots, {X_1}^{(n_1)}; \ldots; X_k, {X_k}', \ldots, {X_k}^{(n_k)})).$$

(i) The tuple $(n_1; \ldots; n_k)$ is called the **order** of φ (this generalizes the usual notion of order for differential polynomials).

2 We use the notation ";" to clearly distinguish the roles of the different differential variables.

3 In fact the same result holds for other examples of model complete theories of differential fields equipped with a definable topology (see [1]).

4 By this we mean that the formula φ considered in the definition of A is quantifier free. This not really restrictive since *CODF* admits quantifier elimination, but it is technically needed in the subsequent developments.

(ii) ${A_\varphi}^L := \{(x_{10}, \dots, x_{1n_1}; \dots; x_{k0}, \dots, x_{kn_k}) \in M^N \mid M \models \varphi^L(\overline{x_1}; \dots; \overline{x_k})\}$ where $\overline{x_i} = (x_{i0}, \dots, x_{in_i})$ and $N = (n_1 + 1) + \cdots + (n_k + 1)$.

(iii) ${A_\varphi}^* := \{(x_1, \dots, {x_1}^{(n_1)}; \dots; x_k, \dots, {x_k}^{(n_k)}) \in M^N \mid M \models \varphi(x_1; \dots; x_k)\}$.

In other words, *since φ is quantifier free*, ${A_\varphi}^*$ is the intersection between the L-definable set ${A_\varphi}^L$ and the $(n_1; \dots; n_k)$-jet-space of M^k. In the same way, for any element $\overline{a} = (a_1; \dots; a_k)$ of A, we introduce the notation a^* for the tuple $(a_1, a_1', \dots, a_1^{(n_1)}; \dots; a_k, a_k', \dots, a_k^{(n_k)}) \in {A_\varphi}^L$. Remark that the singleton $\{a^*\}$ is equal to $J_{(n_1;\dots;n_k)}(\{\overline{a}\})$.

(iv) Since φ is quantifier free, A_φ is the projection of ${A_\varphi}^*$ onto some appropriate coordinates (namely $X_{10}, \dots, X_{k0}$) and this projection is a bijection. We call this projection the **canonical projection** of A^* (or of ${A_\varphi}^L$ when the context is clear). We will also say that the L-definable set ${A_\varphi}^L$ **gives rise to** (or is **a source for**) the L'-definable set A_φ.

Remark 2.4. In what follows and in order to simplify the notation, we will drop the subscript $_\varphi$ in the sets A_φ, ${A_\varphi}^L$ and ${A_\varphi}^*$ defined above and simply denote them by A, A^L and A^* respectively. In other words, given any L'-definable set A, we arbitrarily chose a quantifier free L'-formula φ defining A (such a formula always exists by quantifier elimination) and define the sets A^L and A^* via the L-formula φ^L.

Let us illustrate these definitions by some examples.

Examples. . Let $A := \{x \in M \mid x' > 0\}$, then A^L is the half-plane $\{(x_o, x_1) \in M^2 \mid x_1 > 0\}$ and A^* is the intersection of this half-plane with $J_1(M) = \{(x, x') \mid x \in M\}$. It is clear that A is the projection of A^* onto the first coordinate.

. Let $f(X) = X^{(3)}X + 4XX'$ and denote by $f^L \in M[X_0, X_1, X_2]$ the ordinary polynomial obtained by replacing $X, X', X'', X^{(3)}$ by new variables X_0, X_1, X_2, X_3. More precisely,

$$f^L(X_0, X_1, X_2, X_3) := X_3X_0 + 4X_0X_1.$$

Then if $\varphi(X)$ is the L'-formula "$f(X) = 0$", the L-formula φ^L defined above is "$f^L(X_0, X_1, X_2, X_3) = 0$" and it defines a subset A^L of M^4. The set A^* is then the subset of A^L consisting in all the 4-tuples of the form $(x, x', x'', x^{(3)})$ and, projecting it onto the first coordinate, we recover the subset A of M defined by $\varphi(X)$.

Remark 2.5. (i) The operation $^L : \varphi \mapsto \varphi^L$ introduced at the beginning of Definition 2.3 commutes with the usual connection operators on the set $\mathcal{F}$ (resp. $\mathcal{F}'$) of quantifier free L-formulas (resp. L'-formulas). More precisely, for any quantifier free L'-formulas φ and ψ,

$$(\varphi \wedge \psi)^L \equiv_L \varphi^L \wedge \psi^L, \quad (\varphi \vee \psi)^L \equiv_L \varphi^L \vee \psi^L \quad \text{and} \quad (\neg\varphi)^L \equiv_L \neg(\varphi^L)$$

where $\equiv_L$ denotes the equivalence w.r.t. the L-theory *RCF* (recall that any model of *CODF* is a real closed field).

(ii) In the rest of this section we will often assume that the order of the formula defining A is equal to $(n; \dots; n)$ (i.e. $n_1 = \cdots = n_k = n$). In terms of formulas this is equivalent to consider φ as a formula in the variables $X_1, \dots, X_1^{(n)}; \dots; X_k, \dots, X_k^{(n)}$ where n is at least equal to the maximum of the n_i's. This assumption is harmless since we can **"swell"** the set A^L by taking appropriate direct products with some powers of M and obtain, after intersection with the jet-space and canonical projection, the same L'-definable set A.

More precisely, we can consider the following L-definable subset of M^{nk+k}:

$$\hat{A}^L := \{(x_{10}, \dots, x_{1n_1}, \overline{y_1}; \dots; x_{k0}, \dots, x_{kn_k}, \overline{y_k}) \mid (\overline{x_1}, \dots, \overline{x_k}) \in A^L \wedge \overline{y_i} \in M^{n-n_i}\}$$

where for any $i \in \{1, \dots, k\}$, $\overline{x_i} = (x_{i0}, \dots, x_{in_i})$.

It is easy to check that $\hat{A}^L$ also gives rise to A after intersection with the $(n; \dots; n)$-jet-space of M^k and projection onto the coordinates $X_{10}, \dots, X_{k0}$ (see also the example).

Example. Let $A = \{(x_1; x_2) \in M^2 \mid M \models x_1' = 0 \wedge x_2 > 0\}$. Then

$$A^L = \{(x_{10}, x_{11}; x_{20}) \in M^3 \mid M \models x_{11} = 0 \wedge x_{20} > 0\}.$$

But one could also consider the L-definable set

$$\hat{A}^L = A^L \times M = \{(x_{10}, x_{11}; x_{20}, x_{21}) \in M^4 \mid M \models x_{11} = 0 \wedge x_{20} > 0\}.$$

If we intersect $\hat{A}^L$ with the (1; 1)-jet-space of M^2 and project canonically (i.e. onto the variables X_{10} and X_{20}) then we recover the set A.

Unfortunately, the construction described in the previous remark is not the only way to produce examples of distinct L-definable sets giving rise to the same L'-definable set. In terms of formulas, this means that two L'-formulas φ, ψ can be equivalent in the L'-theory *CODF* even if the corresponding L-formulas φ^L and ψ^L are not equivalent in the L-theory *RCF*.

The following example illustrates this phenomenon.

Example. Let A be the subset of M defined by the L'-formula "$1 < X < 3 \wedge 1 < X' < 2$". If we add to $A^L = \{(x_0, x_1) \in M^2 \mid 1 < x_0 < 3 \wedge 1 < x_1 < 2\}$ the point with coordinates $(2, 3)$, we obtain a new L-definable set which also gives rise to A (since $2' = 0 \neq 3$ and hence $(2, 3) \notin J_1(M)$, see Fig. 2).

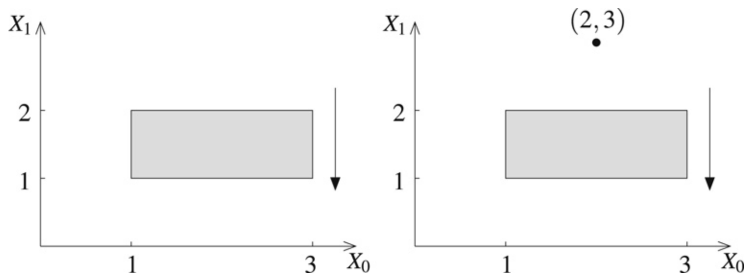

Fig. 2. δ-equivalence.

Nevertheless the following easy lemma shows that the different "sources" of an L'-definable set must be quite like-looking.

Lemma 2.6. *Let*

$$\begin{cases} A_1{}^L = \{(x_{10},\ldots,x_{1m_1};\ldots;x_{k0},\ldots,x_{km_k}) \mid \varphi_1{}^L(x_{10},\ldots,x_{1m_1};\ldots;x_{k0},\ldots,x_{km_k})\} \\ A_2{}^L = \{(y_{10},\ldots,y_{1n_1};\ldots;y_{k0},\ldots,y_{kn_k}) \mid \varphi_2{}^L(y_{10},\ldots,y_{1n1};\ldots;y_{k0},\ldots,y_{kn_k})\} \end{cases}$$

be two quantifier free L-definable sets giving rise to the same L'-definable set $A = A_1 = A_2$ (that is the L'-formulas φ_1 and φ_2 are equivalent in CODF).

Then if we assume that these two sets lie in the same ambient space M^N (i.e. $m_i = n_i = n$ for all $i \in \{1,\ldots,k\}$ and $N = k(n+1)$; see the swelling procedure described in Remark 2.5(ii)), they only differ by a set of empty interior in M^N.

Proof. By the density of the jet-spaces (Lemma 2.2), if $(A_1{}^L \setminus A_2{}^L) \cup (A_2{}^L \setminus A_1{}^L)$ has non-empty interior in M^N then it contains a point $(a_1,\ldots,a^{(n)};\ldots;a_k,\ldots,a_k^{(n)})$ of the associated jet-space $J_{(n;\ldots;n)}(M^k)$. But then $(a_1;\ldots;a_k)$ belongs to $(A_1 \setminus A_2) \cup (A_2 \setminus A_1) = \emptyset$, a contradiction. $\square$

We now formalize this ambiguity between the different sources of a given L'-definable set via the following definition.

Definition 2.7. Two quantifier free L-definable sets are δ**-equivalent** (denoted by $\equiv_\delta$) if they give rise to the same L'-definable set. This is equivalent to say that, considering these two sets as subsets of the same ambient space M^N (where $N = k(n+1)$), they have the same intersection with $J_{(n;\ldots;n)}(M^k)$.

Lemma 2.6 directly implies the following result.

Corollary 2.8. *Let A and B be two disjoint L'-definable subsets of M^k. Then for any sources A^L of A and B^L of B lying in the same ambient space M^N, $A^L \cap B^L$ has empty interior in M^N. In particular, if A^L and B^L are open in M^N then they are disjoint.*

Remark 2.9. We can also define in the same way a notion of δ-equivalence on the set $\mathcal{F}$ of quantifier free L-formulas. With this the operation ^{L} described Remark 2.5(i) becomes a bijection between $\mathcal{F}'/_{\equiv_{L'}}$ and $\mathcal{F}/_{\equiv_\delta}$ (where $\equiv_{L'}$ denotes the equivalence between two L'-formulas in *CODF*).

We end this section with the following technical lemma which ensures that the operation ^{L} does not behave too badly when we take fibers of definable sets.

Lemma 2.10. *Assume that*

- $A = \{\bar{x} \in M^k \mid \varphi(\bar{x})\}$ *is a quantifier free L'-definable subset of M^k;*
- $a = (a_{j_1};\ldots;a_{j_l}) \in \pi_{(j_1;\ldots;j_l)}(A)$ *(where $1 \le j_1 < \cdots < j_l \le k$);*
- $\pi_{(\overline{j_1};\ldots;\overline{j_l})} : M^N \to M^{(n_{j_1}+1)+\cdots+(n_{j_l}+1)}$ *is the* **"blocks" projection** *associated with $\pi_{(j_1;\ldots;j_l)}$:*

$$\pi_{(\overline{j_1};\ldots;\overline{j_l})} : (X_{10},\ldots,X_{1n_1};\ldots;X_{k0},\ldots,X_{kn_k}) \mapsto (X_{j_10},\ldots,X_{j_1n_{j_1}};\ldots;X_{j_l0},\ldots,X_{j_ln_{j_l}});$$

- $a^* = (a_{j_1}, a'_{j_1},\ldots,a_{j_1}^{(n_{j_1})};\ldots;a_{j_l},a'_{j_l},\ldots,a_{j_l}^{(n_{j_l})})$ *is the point of $\pi_{(\overline{j_1};\ldots;\overline{j_l})}(A^L)$ corresponding to a (i.e. $\pi_{(j_10;\ldots;j_l0)}(a^*) = a$).*

Then the fiber $(A_a)^*$ is equal to the intersection of the fiber $(A^L)_{a^*}$ with an "appropriate" jet-space J of M^{k-l} (where A^L is the L-definable set defined by the L-formula φ^L).

Remark 2.11. This lemma seems to be very complicated and its statement could appear totally uninviting. Anyway the main idea of this result (and the only thing one needs to remember) is the following:
The L-definable fiber $(A^L)_{a^*}$ of A^L gives rise to the L'-definable fiber A_a of A. Equivalently $(A_a)^L = (A^L)_{a^*}$.

Proof. The result is an immediate consequence of Definition 2.3 and of the fact that the fibers of a definable set are definable by the same quantifier free formula as the set, just by adding parameters.

Assume that $\{1,\ldots,k\} = \{i_1,\ldots,i_{k-l}\}\dot{\cup}\{j_1,\ldots,j_l\}$ and let J be the $(n_{i_1};\ldots;n_{i_{k-l}})$-jet-space of M^{k-l}. Then

$$\begin{aligned} (A^L)_{a^*} \cap J &= \{(a_{i_1},\ldots,a_{i_1}^{(n_{i_1})};\ldots;a_{i_{k-l}},\ldots,a_{i_{k-l}}^{(n_{i_{k-l}})}) \mid (a_1;\ldots;a_k) \in A\} \\ &= (A_a)^*. \quad \square \end{aligned}$$

3. δ-topology

When we work with real closed fields, the order topology is really convenient since, for example, all polynomials are continuous with respect to this topology. In fact this topology can be seen as the "natural" topology associated with the language L of ordered rings. The word natural means here that the relation symbols of the language define open sets for this topology and that the interpretation of each function symbol is continuous w.r.t. this topology. This corresponds to the notion of topological system introduced by van den Dries in [16] and studied intensively by L. Mathews in his thesis [6,5].

Unfortunately the result of continuity does not hold anymore when we consider differential polynomials. For example one can deduce from the axiomatization of *CODF* that the differential polynomial $p(X) = X'$ is not continuous on M (w.r.t. the order topology). Indeed, the preimage by p of the set $\{x \in M \mid x > 0\}$ is a dense an co-dense subset of M.

This observation leads us to consider another topology on M.

Definition 3.1. An L'-definable subset A of M is a basic open set for the δ-**topology** (we say that A is a **basic δ-open set**) if $A^L \subseteq M^n$ is δ-equivalent to a basic open L-definable set for the product topology in M^n.

Example. Let $a_0, b_0, a_1, b_1 \in M$ be such that $a_0 < b_0$ and $a_1 < b_1$. Then the L'-definable set

$$O := \{x \in M \mid a_0 < x < b_0 \wedge a_1 < x' < b_1\}$$

is a basic δ-open subset of M (since O^L is a basic open box of M^2).

Remark 3.2. (i) The δ-topology can be seen as the topology induced on the infinite jet-space $J_\omega(M) := \{(x, x', \dots, x^{(n)}, \dots) \mid x \in M\}$ by the usual product topology on $M^\omega = M \times M \times \cdots$. The basic δ-open subsets of M are canonical projections (cf. Definition 2.3) of basic open sets of M^ω. We recall that these latter are of the form $I_0 \times \cdots \times I_n \times \cdots$ where each I_j is an open interval in M and only finitely many of these I_j's are not equal to M. In particular each basic open subset of M^ω is definable by a quantifier free L-formula.

In the same way each basic δ-open subset of M is definable by a quantifier free L'-formula even if this δ-topology itself cannot be uniformly defined by such an L'-formula.

(ii) Proposition 3.3(i) below shows that Definition 3.1 naturally extends to the product topology induced by the δ-topology on any Cartesian power of M.

(iii) The δ-topology can also be considered as the *natural* topology on M associated to the language L' (as the order topology is the natural topology associated to L). In particular, since ordinary polynomials are continuous (w.r.t. the order topology) and the derivative of a differential polynomial is still a differential polynomial, one can deduce that each differential polynomial in $M\{X_1; \dots, X_k\}$ is continuous w.r.t. the δ-topology (we say that it is δ-continuous). This will be developed more explicitly in the forthcoming note [10].

(iv) In what follows we will use the prefix "δ-" before any topological object to specify that we consider it in the δ-topology (e.g.: δ-open, δ-closed, δ-interior, δ-continuous, etc.).

Here are some elementary properties of the δ-topology.

Proposition 3.3. *Let $A, A_1, \dots, A_l$ be quantifier free[5] L'-definable subsets of M^k.*

(i) *A is a basic δ-open set of M^k iff A^L is δ-equivalent to a basic open subset of M^N (where $N = (n_1 + 1) + \cdots + (n_k + 1)$ and $(n_1; \dots; n_k)$ is the order of a quantifier free formula defining A).*

(ii) *A has non-empty δ-interior in M^k iff any L-definable set B^L which is δ-equivalent to A^L has non-empty interior in M^N. In particular A has non-empty δ-interior iff A^L has non-empty interior in M^N.*

(iii) *A is δ-open (resp. δ-closed) in M^k iff A^L is δ-equivalent to an open (resp. closed) subset of M^N.*

(iv) *If $A = \bigcup_{i=1}^{l} A_i$ and each A_i has empty δ-interior in M^k then A has empty δ-interior in M^k.*

Proof. (i) Assume first that A is the product $A_1 \times \cdots \times A_k$ where each A_i is a basic δ-open set of M. For each i in $\{1, \dots, k\}$, Definition 3.1 implies that ${A_i}^L$ is δ-equivalent to a L-definable basic open subset ${O_i}^L$ of M^{n_i+1}. The Cartesian product $O^L = {O_1}^L \times \cdots \times {O_k}^L$ is a basic open subset of M^N and, since $J_{(n_1;\dots;n_k)}(M^k) = J_{n_1}(M) \times \cdots \times J_{n_k}(M)$, O^L is δ-equivalent to A^L proving that A is a basic δ-open subset of M^k.

The right-to-left implication is similarly proved. Assume that A^L is δ-equivalent to the basic open subset $O^L = {O_1}^L \times \cdots \times {O_k}^L$ of M^k. Then $O_1, \dots, O_k$ are basic δ-open subsets of M and $A = O_1 \times \cdots \times O_k$ is a basic δ-open subset of M^k.

(ii) Suppose that A^L has non-empty interior in M^N. Then it contains a basic open set O^L. Since real closed fields admit quantifier elimination in the language L, we can assume that O^L is quantifier free L-definable. Hence $O^* \subset A^*$ and the δ-open set O is contained in A, implying that this latter has non-empty δ-interior in M^k.

Suppose now that A contains a basic δ-open subset O and let B^L be any L-definable set δ-equivalent to A^L. Using the swelling procedure introduced in Remark 2.5(ii), we can assume that O^L and B^L lie in the same ambient space M^N. Remark that we cannot assure that $O^L \subseteq B^L$. Nevertheless

$$B^L \cap O^L \equiv_\delta A^L \cap O^L \equiv_\delta O^L$$

since all these sets give rise to the L'-definable set O. But O^L is open and then, by Lemma 2.6, $B^L \cap O^L$ has non-empty interior in M^N. Hence B^L has non-empty interior in M^N.

[5] As in Definition 2.3 this means that we can assume that each L-definable set $A^L, {A_1}^L, \dots, {A_l}^L$ gives rise (after canonical projection) to $A, A_1, \dots, A_l$ respectively.

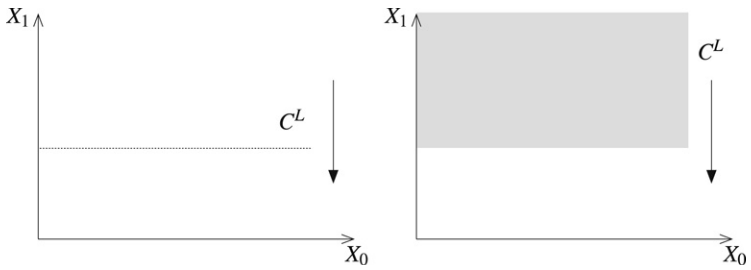

Fig. 3. Examples of δ-cells.

(iii) Suppose first that A^L is δ-equivalent to the open set $O^L = \cup_i {O_i}^L$ where the ${O_i}^L$'s are basic open subsets of M^N. Then A is equal to the union of the basic δ-open subsets O_i of M^k and hence it is δ-open in M^k.
Assume now that $A = \cup_i A_i$ where each A_i is a basic δ-open subset of M^k. Then each ${A_i}^L$ is δ-equivalent to a basic open subset ${O_i}^L$ of M^{n_i}. Since we can again assume that these open L-definable sets belong to the same ambient space M^ω, A^L is δ-equivalent to the open subset $\cup_i {O_i}^L$ of M^ω. To finish the proof we just remark that the definability of A and A^L (in their respective language) implies that $(A^L)^c = (A^c)^L$ (where ^{c} denotes the complement of a set) and that two sets are δ-equivalent iff their complements are (cf. Remark 2.5(i)).

(iv) As before we assume that the ${A_i}^L$'s lie in the same ambient space. By (ii) each ${A_i}^L$ has empty interior in this space and, since the order topology in real closed fields satisfies (iv) (see for example [6, Lemma 5.4]), $\bigcup_{i=1}^l {A_i}^L$ has empty interior. But this union gives rise, after canonical projection, to A and hence this set has empty δ-interior in M^k (again by (ii)).

□

4. A theorem of δ-decomposition for definable sets in *CODF*

We begin with the definition of δ-cells which generalizes the usual definition of cells in o-minimal structures. Let us first recall that any model of *CODF* is a real closed field and then is an o-minimal L-structure. We will continually use this fact in what follows.

Definition 4.1. An L'-definable set $C \subseteq M$ is a (**1**)-δ-**cell** if C^L is δ-equivalent to an (o-minimal) open cell D^L of M^{n+1} for some $n \in \mathbb{N}$. If C^L is δ-equivalent to a non-open cell (i.e. a cell containing a 0 in its type[6]) then C is a (**0**)-δ-**cell**.

Examples. . Let $C^L \subseteq M^2$ be the $(1, 0)$-cell $C^L = \{(x_0, x_1) \mid x_1 = 1\}$. Then C^l gives rise to the (0)-δ-cell $C = \{x \in M \mid x' = 1\}$.

. If we replace the symbol $=$ by $>$ in the definition of C^L above, the latter becomes the $(1, 1)$-cell $\{(x_0, x_1) \mid x_1 > 1\}$ and then C is the (1)-δ-cell whose elements have derivative strictly greater than 1 (see Fig. 3).

We proceed similarly to define δ-cells in higher dimension.

Definition 4.2. $C \subseteq M^k$ is an $(\mathbf{i_1}; \dots; \mathbf{i_k})$-$\delta$-**cell** if C^L is δ-equivalent to an $(i_{10}, \dots, i_{1n_1}; \dots; i_{k0}, \dots, i_{kn_k})$-cell D^L such that: for any $j \in \{1, \dots, k\}$,

$$\begin{cases} i_j = 1 & \text{if } i_{jl} = 1 \text{ for each } l \in \{0, \dots n_j\}, \\ i_j = 0 & \text{otherwise.} \end{cases}$$

The idea of this definition is the following: the digit i_j in the δ-type of C is equal to 0 iff the tuple $(i_{j0}, \dots, i_{jn_j})$ in the o-minimal type of C^L contains a 0.

Definition 4.3. (i) As in Definition 2.3, the o-minimal cell D^L appearing in Definitions 4.1 and 4.2 will be called a **source cell** of C. Obviously this cell is not unique but the definition of a δ-cell ensures the existence of at least one source cell.
Hence, in the rest of this work, we will always use the notation C^L to denote a source cell giving rise to C even if this source cell does not correspond exactly to the L-definable set C^L appearing in Definition 2.3.

(ii) The tuple $(i_1; \dots; i_k)$ appearing in Definitions 4.1 and 4.2 is called a δ-**type** of C. Furthermore, for each l in $\{1, \dots, k\}$, i_l is called a δ-type of C in the variable X_l.

Example. Assume C is the L'-definable subset of M^2 given by the formula "$X' = 0 \wedge Y'' > 0$". Then C^L is the $(1, 0; 1, 1, 1)$-cell equal to

$$\{(x_0, x_1; y_0, y_1, y_2) \in M^5 \mid x_1 = 0 \wedge y_2 > 0\}$$

and, taking the intersection of C^L with $J_{(1,2)}(M^2)$ and projecting onto the coordinates X_0 and Y_0, we see that C is a $(0; 1)$-δ-cell.

[6] For the purposes of this work, we do not need to distinguish between different kinds of (0)-δ-cell. More exactly, we do not care about the positions of the 0's in the type of C^L since these positions do not play any roles in the dimension theory developed in the subsequent sections.

Remark 4.4. As in the previous section, we could replace C^L by any *appropriate* direct product with powers of M and obtain the same δ-cell C. In this case we introduce the following notation: if $\bar{s} = (s_1, \ldots, s_k)$ is a tuple of positive integers then

$$C^L{}_{\bar{s}} := \{(x_{10}, \ldots, x_{1n_1}, M^{s_1}; \ldots; x_{k0}, \ldots, x_{kn_k}, M^{s_k}) \mid (x_{10}, \ldots, x_{1n_1}; \ldots; x_{k0}, \ldots, x_{kn_k}) \in C^L\}$$

is an $(i_{10}, \ldots, i_{1n_1}, \underbrace{1, \ldots, 1}_{s_1 \text{ times}}; \ldots; i_{k0}, \ldots, i_{kn_k}, \underbrace{1, \ldots, 1}_{s_k \text{ times}})$-cell δ-equivalent to C^L.

Before proceeding further we have to be careful and verify that these definitions make sense (this is not clear a priori since a δ-cell can be obtained from many different o-minimal source cells). In other words: *are we sure that to each δ-cell C corresponds a unique δ-type?*

Next lemma gives a positive answer to this question, so that we can now speak of the δ-type of a δ-cell.

Lemma 4.5. *Let C be an $(i_1; \ldots; i_k)$-δ-cell and B^L be an L-definable set which is δ-equivalent to C^L. Then each (o-minimal) cell decomposition of B^L contains a cell which gives rise to a δ-cell with δ-type $(i_1; \ldots; i_k)$. In particular, each source cell of C gives rise to this δ-type $(i_1; \ldots; i_k)$.*

We first prove the following lemma which makes explicit the behavior of δ-cells under coordinates projections and fibrations.

Lemma 4.6. *If C is an $(i_1; \ldots; i_k)$-δ-cell then $\pi_{k-1}(C)$ is an $(i_1; \ldots; i_{k-1})$-δ-cell and for each $a \in \pi_{k-1}(C)$ the fiber $C_a = \{y \in M \mid (a; y) \in C\}$ is an (i_k)-δ-cell.*

Proof. Assume that C is obtained via the $(i_{10}, \ldots, i_{1n_1}; \ldots; i_{k0}, \ldots, i_{kn_k})$-cell C^L. By a classical result on o-minimal cells [17, Proposition 3.5, p. 60], the projection of C^L onto the $(n_1 + 1) + \cdots + (n_{k-1} + 1)$ first coordinates (let us denote this projection by $\pi_{\overline{k-1}}$) and the fiber $C^L{}_{a^*}$ (with $\{a^*\} = J_{(n_1;\ldots;n_{k-1})}(\{a\})$) are still cells. Furthermore a quick look at the proof of this property shows that these two cells have types $(i_{10}, \ldots, i_{1n_1}; \ldots; i_{k-1,0}, \ldots, i_{k-1,n_{k-1}})$ and $(i_{k0}, \ldots, i_{kn_k})$ respectively. But one can easily see that $\pi_{\overline{k-1}}(C^L)$ is a source cell of $\pi_{k-1}(C)$ and, by Lemma 2.10, $C^L{}_{a^*}$ is a source cell of C_a. Hence $\pi_{k-1}(C)$ is an $(i_1; \ldots; i_{k-1})$-δ-cell and C_a is an (i_k)-δ-cell. □

Proof of Lemma 4.5. Let $C \subseteq M^k$ be a δ-cell and B^L be an L-definable set which is δ-equivalent to the source cell C^L of C. Let $\mathfrak{C}^L = \{C_1^L, \ldots, C_l^L\}$ be a cell decomposition of B^L.
We proceed by induction on k:

$k = 1$: Assume that C is a (1)-δ-cell. Then, by Lemma 2.6, B^L has non-empty interior in its ambient space. By [6, Lemma 5.4], there exists j in $\{1, \ldots, l\}$ such that C_j^L has non-empty interior, i.e. is an open cell. Hence C_j^L gives rise to a δ-cell C_j with δ-type (1).

On another hand if C has δ-type (0), the same argument implies that B^L has empty interior in its ambient space and then each cell included in B^L is non-open in this space (i.e. its o-minimal type contains a zero). Hence each cell included in B^L gives rise to a δ-cell with δ-type (0).

$k > 1$: Let C have δ-type $(i_1; \ldots; i_k)$. Remark first that $\pi_{\overline{k-1}}(B^L)$ is a source set for $\pi_{k-1}(C)$ and hence, by Lemma 4.6 and the induction hypothesis, $\pi_{\overline{k-1}}(\mathfrak{C}^L)$ (which is a cell decomposition of $\pi_{\overline{k-1}}(C^L)$) contains a cell $\tilde{C}^L$ which gives rise to a δ-cell $\tilde{C} \subseteq \pi_{k-1}(C)$ with δ-type $(i_1; \ldots; i_{k-1})$. For the following, it is worth noting that

$$\tilde{C}^L = \pi_{\overline{k-1}}(C_{j_1}^L) = \cdots = \pi_{\overline{k-1}}(C_{j_s}^L)$$

for some subset $\{j_1, \ldots, j_s\}$ of $\{1, \ldots, l\}$ (see Fig. 4). Assume that $i_k = 1$ and let

$$C_a = C_{(a_1;\ldots;a_{k-1})} = \{a_k \mid (a_1; \ldots; a_k) \in C\}$$

where $a = (a_1; \ldots; a_{k-1})$ belongs to $\tilde{C}$. By Lemma 4.6, C_a is a (1)-δ-cell and hence it has non-empty δ-interior in M. But

$$\tilde{C}^L_{a_1{}^*;\ldots;a_{k-1}{}^*} = \tilde{C}^L_{a^*}$$

is also a source cell of C_a (cf. Lemma 2.10) and then it has non-empty interior in its ambient space. Furthermore

$$\tilde{C}^L_{a^*} = (C_{j_1}^L)_{a^*} \cup \cdots \cup (C_{j_s}^L)_{a^*}$$

and then (again by [6, Lemma 5.4]) there exists t in $\{j_1, \ldots, j_s\}$ such that $(C_{j_t}^L)_{a^*}$ is an open cell (i.e. has type $(1, \ldots, 1)$). This implies that $C_{j_t}^L$ has a type equal to (type of $\pi_{\overline{k-1}}(C_{j_t}^L)$; $1, \ldots, 1$) and so it gives rise to a δ-cell with δ-type $(i_1; \ldots; i_{k-1}; 1)$ which is the δ-type of C.

The case where $i_k = 0$ can be proved in a similar way using, as in the case where $k = 1$, the fact that any source set of C_a has empty interior in its ambient space.

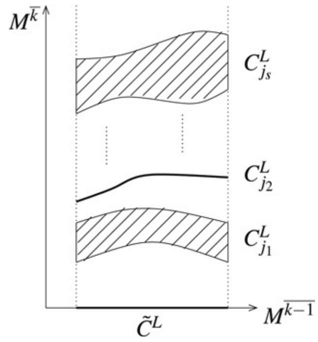

Fig. 4.

Remark 4.7. . As in o-minimal structures, $(1;\ldots;1)$-δ-cells will be called **δ-open δ-cells** and, by Proposition 3.3(ii) and Definition 4.2, they are exactly the δ-cells which are δ-open in their ambient space. Furthermore any L'-definable subset of M^k with non-empty δ-interior contains a δ-open δ-cell. This follows from Proposition 3.3(ii) and the analogous result for o-minimal structures.

. In the same way, $(0;\ldots;0)$-δ-cells will be called **trivial** δ-cells. Remark that, contrary to the o-minimal case and many other examples of structures admitting a cell decomposition (e.g.: weakly o-minimal structures, p-adically closed fields, etc.), the trivial δ-cells are not necessarily finite (d-minimal theories provide other examples of structures where the basic "cells" are not finite, see [7]).

Now we have proved that the notion of δ-cells is well-defined, we can make our way to the statement of a "differential cell decomposition theorem" for CODF.
Before that we generalize the notion of decomposition introduced for o-minimal structures.

Definition 4.8. A **δ-decomposition of** M is a partition of M into finitely many δ-cells. A **δ-decomposition of** M^k $(k > 1)$ is a partition $\mathcal{C}$ of M^k into finitely many δ-cells such that the projection $\pi_{k-1}(\mathcal{C})$ is still a δ-decomposition of M^{k-1}.

We are now able to state the main result of this section.

Theorem 4.9 (δ-decomposition Theorem). *Let M be a closed ordered differential field. For any finite collection $\mathcal{A} = \{A_1,\ldots,A_l\}$ of L'-definable (over $P \subseteq M$) subsets of M^k there exists a finite δ-decomposition $\mathcal{C}$ of M^k (definable over P) compatible with $\mathcal{A}$ (i.e. partitioning each of the A_i's).*

Proof. Let $\mathcal{A} = \{A_1,\ldots,A_l\}$ be a finite collection of L'-definable subsets of M^k and suppose that the order of each L'-formula φ_j defining A_j is equal to the tuple $(n_{j1};\ldots;n_{jk})$. For each $s \in \{1,\ldots k\}$, let $N_s = max\{n_{1s},\ldots,n_{ls}\}$ and consider the sets $A_j{}^L$ as subsets of the space $M^{(N_1+1)+\cdots+(N_k+1)} = M^N$.

Since real closed fields are o-minimal there exists a finite cell decomposition $\mathcal{C}^L$ of M^N compatible with the L-definable collection $\mathcal{A}^L = \{A_1{}^L,\ldots,A_l{}^L\}$. Take the intersection between $\mathcal{C}^L$ and the $(N_1;\ldots;N_k)$-jet-space of M^k and then the projection onto the coordinates $X_{10},\ldots,X_{k0}$ to obtain a finite partition $\mathcal{C}$ of M^k into δ-cells compatible with $\mathcal{A}$. By Lemma 4.6, $\mathcal{C}$ is a δ-decomposition of M^k.

The definability of the δ-cells over the set of parameters P follows from the analogous fact in the o-minimal cell decomposition theorem (Theorem 1.2), since a δ-cell is definable from the same set of parameters as its source cell (cf. Definitions 4.1 and 4.2). □

Remark 4.10. In the proof of Theorem 4.9, it is possible that the cell decomposition $\mathcal{C}^L$ of M^N contains several cells which do not intersect the jet-space $J_{(N_1,\ldots,N_k)}(M^k)$. Without lost of generality we can forget these cells and simply consider the subset of $\mathcal{C}^L$ consisting of all the cells having a non-empty intersection with $J_{(N_1,\ldots,N_k)}(M^k)$. Actually, this subset of $\mathcal{C}^L$ is δ-equivalent to $\mathcal{C}^L$ so that it gives rise to the same δ-decomposition of M^k. In the rest of the paper we will always assume that the cells we consider have non-empty intersection with the jet-space.

Example. Let A^L be the line

$$\{(x_0,x_1,x_2) \in M^3 \mid x_1 = 0 \wedge x_0 = x_2\}.$$

Then $A^L = C^L_1 \cup C^L_2 \cup C^L_3$ where $C^L_1 = \{(0,0,0)\}$ is a $(0,0,0)$-cell and C^L_2 (resp. C^L_3) is the $(1,0,0)$-cell containing the elements of A^L which have a strictly positive (resp. negative) first coordinate. One can see that C^L_2 and C^L_3 have empty intersection with $J_{(2)}(M)$ and that the L'-definable set $A = \{x \in M \mid x' = 0 \wedge x'' = x\}$ resumes to the singleton $\{0\}$. In fact, in this example, it is easy to see that the L'-formula defining A is equivalent to the formula $x = 0$.

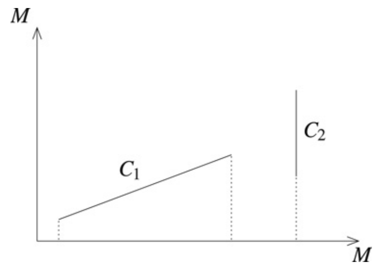

Fig. 5.

5. δ-dimension in *CODF*

In this section we develop a dimension theory for the theory of closed ordered differential fields based on the δ-decomposition theorem.

We prove that this notion of dimension is a **dimension function** on the class of definable sets in *CODF* in the sense of the axioms introduced by van den Dries in [16].

We also show that it is strongly related with two other notions of dimension (or rank) on definable sets in *CODF*, namely the differential transcendence degree and the topological dimension induced by the δ-topology.

5.1. Definition and first properties

We first define the δ**-dimension** of a δ-cell.

Definition 5.1. Let C be an $(i_1; \ldots ; i_k)$-δ-cell, then δ-$\mathbf{dim}(\mathbf{C}) = \sum_{j=1}^{k} i_j$.

Lemma 4.5 ensures that this definition makes sense and furthermore it trivially implies the following result.

Corollary 5.2. *If a* δ-cell C *is equal to an o-minimal cell then its* δ*-dimension is equal to its o-minimal dimension.*

We now generalize the δ-dimension to any L'-definable subset of M^k.

Definition 5.3. Let A be a non-empty L'-definable subset of M^k, the δ-dimension of A (δ-$\mathbf{dim}(\mathbf{A})$) is the maximal δ-dimension of a δ-cell C included in A. By convention we assign to the empty set a δ-*dim* equal to $-\infty$.

The following result directly follows from Definition 5.3.

Corollary 5.4. *If* $A \subseteq B$ *are two* L'*-definable subsets of* M^k *then* δ-$dim(A) \le \delta$-$dim(B)$.

Definition 5.5. Let A be a non-empty definable subset of M^k. A δ-cell $C \subseteq A$ of maximal δ-dimension is called a **witness** δ-cell of A.

Remark 5.6. (i) Similarly to what happens in o-minimal structures (where we can also define a notion of witness cell), the witness δ-cell of A is not necessarily unique. Furthermore, a given δ-decomposition of A can contain several witness cells with different δ-types (although they have the same δ-dimension). E.g.: if A is the disjoint union of two δ-cells C_1 and C_2 with δ-type $(1, 0)$ and $(0, 1)$ respectively then each of C_1, C_2 is a witness δ-cell of A (see Fig. 5).

(ii) It is worth noting that the canonical projection of a witness cell of an L-definable set A^L must not be a witness δ-cell of the L'-definable set A. For example, let

$$A^L = \{(x_0, x_1, x_2) \in M^3 \mid x_1 = 0 \wedge x_0 = x_2\} = C_1^L \cup C_2^L \cup C_3^L$$

where $C_1^L = \{(0, 0, 0)\}$ and C_2^L, C_3^L are $(1, 0, 0)$-cells. Remark that C_2^L and C_3^L are witness cells for A^L while the corresponding δ-cells C_1, C_2 are empty and then are not witnessing the δ-dimension of A (which is actually witnessed by the δ-cell $C_1 = \{0\}$).

The following theorem proves that the δ-dimension of an L'-definable set A is detectable in any partition of A.

Theorem 5.7. *Any partition* $\mathcal{C} = \{C_1, \ldots , C_s\}$ *of* $A \subset M^k$ *into* δ-cells *contains a witness* δ-cell *of* A.

We begin with the proof of a slightly stronger o-minimal analogue of this theorem.

Lemma 5.8. *Let* M *be an o-minimal* L*-structure and* $A^L \subseteq M^N$ *be* L*-definable. If* $C^L \subseteq A^L$ *is an* $(i_1, \ldots , i_N)$*-cell then any finite (not necessarily disjoint) family* $\mathcal{C}^L$ *of cells covering* A^L *contains a* $(j_1, \ldots , j_N)$*-cell such that* $i_l \le j_l$ *for any* $l \in \{1, \ldots , N\}$.

In particular, if C^L *is a witness cell for* A^L *then any cell decomposition of* A^L *contains a witness cell of* A^L *which has the same type as* C^L.

Proof. Suppose that A^L has dimension d and let C^L be an $(i_1, \ldots, i_N)$-cell contained in A^L. We first recall that, given two cells ${C_1}^L \subset {C_2}^L \subset M^N$ of types $(i_1, \ldots, i_N)$ and $(j_1, \ldots, j_N)$ respectively, the equality $j_l = i_l$ holds for each $l \in \{1, \ldots, N\}$ iff ${C_1}^L$ and ${C_2}^L$ have the same dimension [17, Lemma 1.14, Ch. 4]. Moreover, a straightforward induction on N shows that, in the case where $dim({C_1}^L) \le dim({C_2}^L)$, $j_l \ge i_l$ for each $l \in \{1, \ldots, N\}$.

Consider a cell decomposition $\mathcal{D}^L$ of A^L which is compatible with $\mathcal{C}^L$ and which partitions C^L (such a cell decomposition exists by Theorem 1.2). Since C^L is a definable set, $\mathcal{D}^L$ contains a witness cell D^L for C^L and then D^L has the same type as C^L. Furthermore, $\mathcal{D}^L$ is compatible with $\mathcal{C}^L$ and then there exists $\tilde{C}^L \in \mathcal{C}$ containing D^L. Hence $\tilde{C}^L$ has type $(j_1, \ldots, j_N)$ with $j_l \ge i_l$ for each l in $\{1, \ldots, N\}$.

In particular, if C^L is a witness cell of A^L then $\tilde{C}^L$ is also a witness cell of A^L and $j_l = i_l$ for any $l \in \{1, \ldots, N\}$. □

We are now able to prove Theorem 5.7.

Proof. If δ-$dim(A) = 0$ then, by Definition 5.3, each δ-decomposition of A contains only trivial δ-cells. Hence we can assume that the δ-dimension of A is equal to $m \ge 1$.

We proceed by induction on k:

$k = 1$: In this case δ-$dim(A) = 1$ and A contains a δ-open δ-cell C of M. Hence A has non-empty δ-interior in M and, Lemma 3.3(iv), each finite δ-decomposition of A must contain a δ-cell with non-empty δ-interior, i.e. a δ-open δ-cell.

$k > 1$: Assume that the theorem is proved for any $k' < k$ and let π denote the projection onto the $(k-1)$ first coordinates.

Let $A \subseteq M$ be L'-definable and $\mathcal{C} = \{C_1, \ldots, C_s\}$ be a partition of A into δ-cells. Remark that, by Definition 4.8 and Lemma 4.6, δ-$dim(\pi(A)) \ge m - 1$. On another hand, if $\pi(A)$ contains a δ-cell C of δ-dimension at least $m + 1$, the induction hypothesis implies that any δ-decomposition $\mathcal{D}$ of $(C \times M) \cap A$ contains a δ-cell D such that $\pi(D)$ is a witness cell for C, i.e. δ-$dim(\pi(D)) \ge m + 1$ (since any such δ-decomposition $\mathcal{D}$ projects onto a δ-decomposition of C). This implies that δ-$dim(D) \ge m + 1$, contradicting the fact that δ-$dim(A) = m$. It follows that *the δ-dimension of $\pi(A)$ is either m or $m - 1$*.

Assume first that δ-$dim(\pi(A)) = m$.

The induction hypothesis implies that $\{\pi(C_1), \ldots, \pi(C_s)\}$ contains an element $\pi(C_j)$ of δ-dimension m. Hence C_j has δ-dimension at least m and since it is included in A, this δ-dimension is exactly m.

Suppose now that $\pi(A)$ has δ-dimension $m - 1$.

By Lemma 4.6 and the inductive hypothesis, A contains a witness δ-cell C of δ-type $(i_1; \ldots; i_{k-1}; 1)$ with $\sum_{h=1}^{k-1} i_h = m - 1$. Otherwise each witness δ-cell of A has δ-type $(i_1; \ldots; i_{k-1}; 0)$ with $\sum_{h=1}^{k-1} i_h = m$. Taking the projection of one of these δ-cell onto the $k - 1$ first coordinate, we get a $(i_1; \ldots; i_{k-1})$-δ-cell included in $\pi(A)$. This contradicts the fact that δ-$dim(\pi(A)) = m - 1$.

Remark now that, since $\mathcal{C}$ is a partition of A, $(C_1^L \cup \cdots \cup C_s^L) \cap C^L$ is δ-equivalent to C^L. Hence, by Lemma 4.5, $(C_1^L \cup \cdots \cup C_s^L)$ contains a cell $\tilde{C}^L$ which gives rise to the δ-type $(i_1, \ldots, i_{k-1}, 1)$. Let $(\overline{i_1}; \ldots; \overline{i_{k-1}}; 1, \ldots, 1)$ be the (o-minimal) type of $\tilde{C}^L$. By Lemma 5.8, there exists $r \in \{1, \ldots, s\}$ such that C_r^L has type $(\overline{j_1}; \ldots; \overline{j_{k-1}}; 1, \ldots, 1)$ with

$$\overline{j_1} \ge \overline{i_1}, \ldots, \overline{j_{k-1}} \ge \overline{i_{k-1}} \text{ (where } \overline{j_l} \ge \overline{i_l} \text{ means } j_{l0} \ge i_{l0} \wedge \cdots \wedge j_{ln_l} \ge i_{ln_l}).$$

Hence C_r has type $(j_1; \ldots; j_{k-1}; 1)$ with $j_1 \ge i_1, \ldots, j_{k-1} \ge i_{k-1}$ and its δ-dimension is at least m. But $C_r \subset A$ and then δ-$dim(C_r) = m$. □

Remark that in the last part of the proof above, C^L may not be a witness cell for A^L and it is why we needed to introduce Lemma 5.8.

We end this section by the following easy corollary of Theorem 5.7.

Corollary 5.9. *If A is an L'-definable subset of M^k which is also definable in the language L then its δ-dimension is equal to its o-minimal dimension.*

Proof. By Corollary 5.2 and Theorem 5.7. □

5.2. Homeomorphism and δ-cells

We first recall a classical definition from topology.

Definition 5.10. Let X, Y be two topological spaces and $f : X \to Y$ be a bijection. Then f is a **homeomorphism** if both f and the inverse function $f^{-1} : Y \to X$ are continuous (we say that f is bi-continuous).

In o-minimal structures cells have the following nice property.

Proposition 5.11. *Let M be an o-minimal structure and C^L be an $(i_1, \ldots, i_N)$-cell. Let $1 \le j_1, \ldots, \le j_l \le N$ be such that $i_v = 1$ iff $v \in \{j_1, \ldots, j_l\}$. Then the projection $\pi_{(j_1,\ldots,j_l)}$ is an homeomorphism between C^L and the open cell $\pi_{(j_1,\ldots,j_l)}(C^L) \subseteq M^l$.*

Proof. By induction, using the definition of cells. □

Remark 5.12. This property remains true in many other variants of o-minimality and was taken as a basis for the definition of topological cell by L. Mathews in his thesis [6,5].

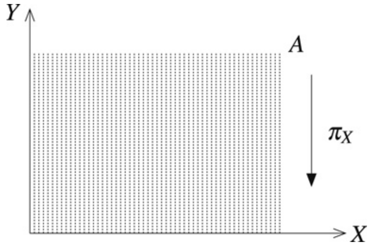

Fig. 6.

Unfortunately the example below shows that we cannot expect such a powerful result in our context.

Example. Let $A = \{(x; y) \mid x > 0 \wedge y' = 0\}$, it is easy to see that A is a $(1, 0)$-δ-cell but, since the subfield of constants of M is infinite, $\pi_X : M^2 \to M$ is not injective on A (see Fig. 6).

Hence in order to express an analogous result as in Proposition 5.11 we have to weaken the condition of injectivity.

Definition 5.13. An L'-definable function $f : A \to B$ is **almost injective** if for each b in B, the set $f^{-1}(b) = \{a \in A \mid f(a) = b\}$ has δ-dimension 0 (i.e. $f^{-1}(b)$ is a finite union of trivial δ-cells, see Theorems 4.9 and 5.7).

We then have the following theorem.

Theorem 5.14. *Let C be an $(i_1; \ldots; i_k)$-δ-cell and suppose that there exist $l > 0$ and $1 \leq j_1, \ldots, \leq j_l \leq k$ such that $i_v = 1$ iff $v \in \{j_1, \ldots, j_l\}$. Then $\pi_{(j_1;\ldots;j_l)}(C)$ is δ-open in M^l and the projection $\pi_{(j_1,\ldots,j_l)}$ is almost injective and δ-continuous on C.*

Proof. We prove the theorem by induction on k.

If $k = 1$ then $l = 1$ and there is nothing to prove. Assume then that $k > 1$ and that the result is true for all $k' < k$. Let C_{k-1} be the projection of C onto the $k - 1$ first coordinates. We consider two cases (in both cases the δ-continuity is immediate since in any topological space, coordinate projections are continuous).

$\underline{i_k = 0}$: By the inductive hypothesis, there exists an almost injective projection $\pi_{(j_1;\ldots;j_l)}$ which sends C_{k-1} onto a δ-open subset in M^l. It remains to prove that this projection applied to C (let us denote it π) is still almost injective.

Let u belong to $\pi(C) = \pi_{(j_1;\ldots;j_l)}(C_{k-1})$. Then

$$\pi^{-1}(u) = \pi_{k-1}^{-1} \circ \pi_{(j_1;\ldots;j_l)}^{-1}(u).$$

Hence, since $\pi_{(j_1;\ldots;j_l)}^{-1}(u)$ is a finite union of trivial δ-cells and $i_k = 0$, $\pi^{-1}(u)$ has also δ-dimension zero (Lemma 4.6).

$\underline{i_k = 1}$: Let $\pi_{(j_1;\ldots;j_{l-1})}$ be an almost injective projection which sends C_{k-1} to a δ-open subset of M^{l-1}. Define a projection π on C by

$$\pi(a_1; \ldots; a_k) := (a_{j_1}; \ldots; a_{j_{l-1}}; a_k)$$

so that

$$\pi(C) = \bigcup_{a \in C_{k-1}} \pi_{(j_1;\ldots;j_{l-1})}(a) \times C_a.$$

Recall that $\pi_{(j_1;\ldots;j_{l-1})}(C_{k-1})$ is δ-open in M^{l-1} and remark that, since $i_k = 1$, a source cell ${C^L}_{a^*}$ of C_a has type $(1, \ldots, 1)$ (i.e. is open in its ambient space). Hence, for each a in C_{k-1}, C_a is δ-open in M.

Suppose now that $\pi(C)$ is not δ-open in M^l. Then there exists $a = (a_1; \ldots; a_k) \in C$ such that each δ-open ball of M^l centered on $(a_{j_1}; \ldots; a_{j_{l-1}}; a_k) = \pi(a)$ contains a point which does not belong to $\pi(C)$. Furthermore, since C_a is a δ-open δ-cell, this point can be chosen outside the fiber $(a_{j_1}; \ldots; a_{j_{l-1}}) \times C_a$. Hence each δ-open ball in M^{l-1} centered on $(a_1; \ldots; a_{l-1})$ contains a point which does not belong to $\pi_{(j_1;\ldots;j_{l-1})}(C_{k-1})$, contradicting the fact that this set is δ-open in M^{l-1}. This implies that $\pi(C)$ is δ-open.

It remains to prove the almost injectivity of π. This follows from the fact that, if $u \in \pi(C)$, then u is of the form $(a_{j_1}; \ldots, a_{j_l}; a_u)$ with $(a_{j_1}; \ldots, a_{j_l}) \in \pi_{(j_1,\ldots,j_{l-1})}(C_{k-1})$. Hence $\pi^{-1}(u)$ is equal to the set

$$(\pi_{(j_1,\ldots,j_{l-1})}^{-1}(a_{j_1}; \ldots, a_{j_l})) \times a_u.$$

By the induction hypothesis, the left member of this direct product has δ-dimension 0 and then $\pi^{-1}(u)$ has also δ-dimension 0. $\square$

$$\begin{tikzcd} C \arrow[r, "\tau"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowright"] & C^\tau \arrow[d, "p^\tau"] \\ p(C) \arrow[r, dashed, "\tau^*"'] & p^\tau(C^\tau) \end{tikzcd}$$

Fig. 7. The application τ^*.

5.3. The δ-dimension as a dimension function

We now prove some properties of the δ-dimension. In particular we show that it satisfies the axioms of a **definable dimension function** as appearing in [16].

Lemma 5.15. *Let A, B be two L'-definable subsets of M^k,*

(i) δ-$dim(A) = -\infty$ *iff* $A = \emptyset$;
(ii) *if* $A = \{\overline{a}\}$ *with* $\overline{a} \in M^k$ *then* δ-$dim(A) = 0$;
(iii) *if* $A = M^k$ *then* δ-$dim(A) = k$.

Proof. This is an immediate consequence of Theorem 5.7 or Corollary 5.9 and of the fact that the o-minimal dimension satisfies the same properties. $\square$

Lemma 5.16. *Let A, B be two L'-definable subsets of M^k,*

$$\delta\text{-}dim(A \cup B) = max\{\delta\text{-}dim(A), \delta\text{-}dim(B)\}.$$

Proof. The case where $A \cap B = \phi$ directly follows from Definition 5.3 and Theorem 5.7. If this intersection is non-empty we write

$$B = (B \setminus A)\dot{\cup}(A \cap B)$$

and then

$$\begin{aligned} max\{\delta\text{-}dim(A), \delta\text{-}dim(B)\} &= max\{\delta\text{-}dim(A), \delta\text{-}dim(B \setminus A), \delta\text{-}dim(A \cap B)\} \\ &= max\{\delta\text{-}dim(A), \delta\text{-}dim(B \setminus A)\} \quad (\text{since } A \cap B \subseteq A) \\ &= \delta\text{-}dim(A \dot{\cup} (B \setminus A)) \\ &= \delta\text{-}dim(A \cup B). \quad \square \end{aligned}$$

Lemma 5.17. *Let A be an L'-definable subsets of M^k. If σ is any permutation of $\{1, \ldots, k\}$ and*

$$A^\sigma := \{(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \mid (x_1, \ldots, x_k) \in A\},$$

then A and A^σ have the same δ-dimension.

Proof. Since any permutation can be expressed as a product of transpositions $(i, i+1)$ it suffices to prove the lemma in the case where σ is equal to such a transposition τ. Furthermore, if we prove that δ-$dim(A^\tau) \geq \delta$-$dim(A)$ then the reverse equality is obtained by the same method using the inverse transposition $\tau^{-1} = \tau$.

Suppose that the result holds for δ-cells and that A contains a witness δ-cell C of δ-*dim* l. Then $A^\tau \supseteq C^\tau$ (remark that C^τ is not necessarily a δ-cell) and Corollary 5.4 implies

$$\delta\text{-}dim(A^\tau) \geq \delta\text{-}dim(C^\tau) \geq \delta\text{-}dim(C) = \delta\text{-}dim(A).$$

We now prove the result for a δ-cell C with δ-$dim(C) = l$. By Theorem 5.14, there exists an almost injective coordinate projection $p = \pi_{(j_1,\ldots,j_l)}$ on C such that $p(C)$ is δ-open in M^l. Let p^τ be the "twisted" projection

$$\pi_{(\tau(j_1),\ldots,\tau(j_l))} : M^k \to M^l : (X_1, \ldots, X_k) \mapsto (X_{\tau(j_1)}, \ldots, X_{\tau(j_l)})$$

and define the application τ^* such that the diagram 7 below commutes.

Remark that, for any $(a_{j_1}, \ldots, a_{j_l})$ in $p(C)$,

$$\tau^*(a_{j_1}, \ldots, a_{j_l}) = \begin{cases} (a_{\tau(j_1)}, \ldots, a_{\tau(j_l)}) & \text{if } i \text{ and } i+1 \text{ belong to } \{j_1, \ldots, j_l\} \\ (a_{j_1}, \ldots, a_{j_l}) & \text{if } i \text{ or } i+1 \text{ does not belong to } \{j_1, \ldots, j_l\}. \end{cases}$$

It is easy to see that τ^* is 1-1 and δ-bi-continuous (i.e. τ^* is a δ-homeomorphism). Hence $\tau^*(C) = p^\tau(C^\tau)$ is δ-open in M^l implying that its δ-dimension is equal to l.

It follows that δ-$dim(C^\tau) \geq l = \delta$-$dim(C)$. $\square$

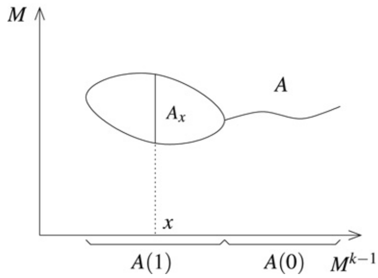

Fig. 8.

Lemma 5.18. *Let A be an L'-definable subset of M^k and, for each $x \in M^{k-1}$ and $i \in \{0, 1\}$, let*

$$A_x := \{y \in M \mid (x, y) \in A\} \text{ and } A(i) := \{x \in M^{k-1} \mid \delta\text{-}dim(A_x) = i\}.$$

Then $A(i)$ is L'-definable and δ-$dim(\{(x, y) \in A \mid x \in A(i)\}) = \delta$-$dim(A(i)) + i$ (Fig. 8).

Proof. The proof is similar to the proof of the o-minimal analogous result (see [17, ch. 4, Prop. 1.5]).

Suppose first that A is an $(i_1; \dots; i_k)$-δ-cell. Then Lemma 4.6 implies that $A(i_k) = \pi(A)$ and $A((i_k + 1) \bmod 2) = \emptyset$. The result then follows trivially.

Suppose now that A is any L'-definable subset of M^k and let $\mathcal{C}$ be a δ-decomposition of M^k partitioning A. Let $E \in \mathcal{C}$ and consider $C_1, \dots, C_l$, the elements of $\mathcal{C}$ which partition A and such that

$$\pi(C_1) = \cdots = \pi(C_l) = \pi(E).$$

For each $x \in \pi(E)$,

$$A_x = (C_1)_x \cup \cdots \cup (C_l)_x$$

and hence, by Lemma 5.16,

$$i = \delta\text{-}dim(A_x) = max_{j=1}^{l}\{\delta\text{-}dim((C_j)_x)\}.$$

Furthermore Lemma 4.6 implies that, for any j in $\{1, \dots, l\}$,

$$\delta\text{-}dim(C_j) = \delta\text{-}dim(\pi(C_j)) + \delta\text{-}dim((C_j)_x)$$

for any $x \in \pi(C_j) = \pi(E)$. Hence

$$\begin{aligned} i &= max_j\{\delta\text{-}dim(C_j) - \delta\text{-}dim(\pi(E))\} \\ &= max_j\{\delta\text{-}dim(C_j)\} - \delta\text{-}dim(\pi(E)) \\ &= \delta\text{-}dim\left(\bigcup_j C_j\right) - \delta\text{-}dim(\pi(E)) \\ &= \delta\text{-}dim\left(\bigcup_{x\in\pi(E)} (\{x\} \times A_x)\right) - \delta\text{-}dim(\pi(E)) . \end{aligned}$$

Remark that, in the equality above, i is independent of the choice of x in $\pi(E)$. Hence $\pi(E) \subseteq A(i)$ and $A(i)$ is an union of δ-cells belonging to $\pi(\mathcal{C})$. It is then L'-definable. Taking the union over all $E \in \mathcal{C}$ such that $\pi(E) \subseteq A(i)$ in this equality we get

$$\begin{aligned} &\delta\text{-}dim\left(\bigcup_{x\in A(i)} (\{x\} \times A_x)\right) - \delta\text{-}dim(A(i)) = i \\ &\quad\Rightarrow \delta\text{-}dim(\{(x, y) \in A \mid x \in A(i)\}) = \delta\text{-}dim(A(i)) + i . \quad \square \end{aligned}$$

Lemmas 5.15–5.18 resume into the following theorem (see [16, p.189]).

Theorem 5.19. *δ-dim is a definable dimension function on the class of L'-definable sets in a closed ordered differential field.*

Corollaries (i) and (ii) below present some usual properties of dimension functions and their proofs can be found in [16, ch. 1]. The first one is a coordinate free version of Lemma 5.18 and it is easy to see that it implies (iii).

Corollary 5.20. (i) *Let $A \subset M^k$ and $f : A \to M^l$ be an L'-definable function. For each $j \in \{0, \dots, k\}$, let*

$$B(j) := \{y \in M^l \mid \delta\text{-}dim(f^{-1}(y)) = j\}.$$

Then $B(j)$ is definable and

$$\delta\text{-}dim(B(j)) + j = \delta\text{-}dim(f^{-1}(B(j))).$$

(ii) *For any L'-definable sets A, B,*

$$\delta\text{-}dim(A \times B) = \delta\text{-}dim(A) + \delta\text{-}dim(B).$$

(iii) *If f is an L'-definable almost injective function from A to M^l then*

$$\delta\text{-}dim(A) = \delta\text{-}dim(f(A)).$$

In particular, if there exists a definable bijection between A and B then

$$\delta\text{-}dim(A) = \delta\text{-}dim(B).$$

Proof. As said before the proofs of (i) and (ii) can be found in [16, Corollaries 1.5]. To see that (iii) is true, just remark that if f is almost injective then $B(0) = f(A)$ and $f^{-1}(B(0)) = A$. The result now follows immediately from (i). □

In order to compare the δ-dimension with other notions of dimension on the class of definable sets in *CODF* in the next sub-section, we need the following corollary.

Corollary 5.21. *Assume that A is an L'-definable (with parameters from M) subset of M^k, N is an elementary extension of M and A_N is the subset of N^k defined by the same formula as A. Then $\delta\text{-}dim(A) = \delta\text{-}dim(A_N)$.*

Proof. Since A_N is defined by a formula with parameters from M, there exists a finite partition $\mathcal{C}$ of A into δ-cells such that $\mathcal{C}_N$ is a partition of A_N into δ-cells (Theorem 4.9). By Theorem 5.7, it suffices to show that $\delta\text{-}dim(C) = \delta\text{-}dim(C_N)$ for each δ-cell C definable with parameters in M. Remark that the fact that C^L is a source cell of C is first-order L'-definable by the following formula (with parameters from M):

$$\forall X_1, \dots, X_k((X_1; \dots; X_k) \in C \Leftrightarrow (X_1, \dots, X_1^{(n_1)}; \dots; X_k, \dots, X_k^{(n_k)}) \in C^L).$$

Hence, since N is an elementary extension of M, $(C^L)_N$ is a source cell of C_N. Furthermore the type of $(C^L)_N$ is equal to the type of C^L and, by Lemma 4.5, the δ-type of C_N is equal to the one of C. It follows that $\delta\text{-}dim(C_N) = \delta\text{-}dim(C)$. □

5.4. δ-dimension, differential rank and topological dimension

In this sub-section we consider two models M, N of *CODF* where N is an $|M|^+$-saturated elementary extension of M.

<u>*Differential rank:*</u>

Definition 5.22. Let A be an L'-definable subset of M^k and A_N be the subset of N^k defined by the same formula as A. Then

$$\delta\text{-rk}(A) = max_{a \in A_N}\{\text{differential transcendence degree of } M\langle a\rangle \text{ over } M\}.$$

A point a of A_N such that $M\{a\}$ has maximal differential transcendence degree over M is called a **differentially generic (or L'-generic) point** of A.

Theorem 5.23. *If $C \subseteq M^k$ is a δ-*cell *then $\delta\text{-}dim(C) = \delta\text{-rk}(C)$.*

Before we prove this theorem we introduce some intermediate lemmas.

Lemma 5.24. *If a δ-*cell *$C \subseteq M^k$ is δ-open then C_N contains a point $(a_1, \dots, a_k)$ whose components are differentially independent over M (i.e. a differentially generic point of M^k).*

Proof. Let ${C^L}_N$ be the source cell of C_N defined by the same formula as C^L and recall that the analogous algebraic result holds for real closed fields (see for example [6, Lemma 8.11]).
<u>Assume first that $k = 1$.</u>
For each natural number s, the open cell $({C^L}_N)_s = C^L \times N^s$ contains an algebraic generic[7] point $(a_0, \dots, a_{n+s})$ of $M^{(n+1)+s}$ (where n is the order of the formula defining C). Hence each finite system of polynomial inequations

$$f_1(X_0, \dots, X_{n+s}) \neq 0 \wedge \dots \wedge f_t(X_0, \dots, X_{n+s}) \neq 0 \quad (*)$$

7 An algebraic generic point in an L-definable (with parameters from M) set $A_N \subseteq N^k$ is a point of A_N such that the algebraic transcendence degree of $M(a)$ over M is maximal amongst the elements of A_N. In the particular case where $A_N = N^k$ this means (since N is $|M|^+$-saturated) that the components of a are algebraically independent over M.

with $f_1, \ldots, f_t \in M[X_0, \ldots, X_{n+s}]$ defines an open subset of M^{n+1+s}. This set contains $(a_0, \ldots, a_{n+s})$ and so has a non-empty intersection with the open cell $({C^L}_N)_s$. Since this intersection is also an open subset of M^{n+1+s}, it contains an element of $J_{n+s}(M)$ (Lemma 2.2) that we denote by $(a, a', \ldots, a^{(n+s)})$. Hence $(a, a', \ldots, a^{(n+s)}) \in ({C^L}_N)_s$ is a solution to the system $(*)$. It follows that C_N contains a point a which is a solution to the system of differential inequations

$$f_1(X) \neq 0 \wedge \cdots f_t(X) \neq 0$$

with $f_1, \ldots, f_t \in M\{X\}$.

This proves that each finite system of differential inequations has non-empty intersection with C_N. By the saturation of N, C_N contains a differentially transcendental point.

We act similarly in the case where $k > 1$.

We use appropriate direct products of ${C^L}_N$ and N (see Remark 4.4) to prove that any finite system of differential inequations in k variables (with parameters from M) has a solution in C_N and then we conclude again by the saturation of N. □

Since each L'-definable set with non-empty δ-interior contains an open δ-cell, the following result is a trivial consequence of Lemma 5.24:

Corollary 5.25. *If $A \subseteq M^k$ is L'-definable and has non-empty δ-interior in M^k then A_N contains a L'-generic point of M^k.*

The next lemma gives a more precise characterization of the links between the differential transcendence degree and the δ-type of a δ-cell.

Lemma 5.26. *Let C be an $(i_1; \ldots; i_k)$-δ-cell such that $i_j = 1$ iff $j \in \{j_1 \ldots, j_l\}$. Then C_N contains a point whose components $j_1 \ldots, j_l$ are differentially independent over M.*

Proof. If $k = 1$ then we can suppose that C is a (1)-δ-cell and the result follows from Lemma 5.24 (there is nothing to prove in the case where C has δ-dimension 0).

If $k > 1$, we assume that $l < k$ (otherwise the result follows from Lemma 5.24). By Theorem 5.14, $\pi_{(j_1;\ldots;j_l)}(C)$ has non-empty δ-interior in M^l and then $\pi_{(j_1;\ldots;j_l)}(C_N)$ contains an L'-generic point of M^l (Corollary 5.25). Hence C_N contains a point whose components $j_1 \ldots, j_l$ are differentially independent over M. □

We now prove Theorem 5.23.

Proof.

. Lemma 5.26 clearly implies that δ-rk$(C) \geq \delta$-$dim(C)$.

. Let δ-rk$(C) = l$ and $a = (a_1; \ldots; a_k) \in C_N$ be an L'-generic point of C with components $a_{j_1}, \ldots, a_{j_l}$ differentially independent. Assume that there is $j \in \{j_1, \ldots, j_l\}$ such that $i_j = 0$. Then, by Lemma 4.5, if ${C^L}_N$ is a source cell of C_N with o-minimal type $(\ldots; i_{j0}, \ldots, i_{jn_j}; \ldots)$, there exists $s \in \{0, \ldots, n_j\}$ such that $i_{js} = 0$. But

$$(a_1, \ldots, a_1^{(n_1)}; \ldots; a_k, \ldots, a_k^{(n_k)}) \in {C^L}_N$$

and so, using the equivalence between algebraic transcendence degree and dimension in o-minimal structures [6, Lemma 8.11], we deduce that $a_j^{(s)}$ is algebraic over the others components of $(a_1, \ldots, a_1^{(n_1)}; \ldots; a_k, \ldots, a_k^{(n_k)})$. This contradicts the assumption on a and prove that, for each $j \in \{j_1, \ldots, j_l\}$, $i_j = 1$. Hence δ-$dim(C)$ is a least l. □

Corollary 5.27. *For any L'-definable set $A \subseteq M^k$, δ-rk$(A) = \delta$-$dim(A)$.*

Proof. If δ-$dim(A) = l$ then there exists a witness δ-cell C of A with δ-dim l and, by Lemma 5.23, C_N contains a point $a = (a_1; \ldots; a_k)$ such that $M\langle a\rangle$ has differential transcendence degree l. Since C is a witness δ-cell of A, a is an L'-generic point of A and δ-rk$(A) = l$. □

Topological dimension

The notion of topological dimension can be defined in any topological space but here we only consider it in the case where this space is a first-order structure (in a language L) equipped with a topology τ.

Definition 5.28. Let M be any L-structure equipped with a topology τ and let A be an L-definable subset of M^k. The **topological dimension** associated to τ is defined as follow:

$$\begin{aligned} tdim(A) \ = max\{l \in \{1, \ldots, k\} \mid \ & \text{there exist } 1 \leq j_1 < \cdots < j_l \leq k \text{ s.t.} \\ & \pi_{j_1,\ldots,j_k}(A) \text{ has non-empty interior in } M^l \text{ w.r.t. } \tau\}, \end{aligned}$$

and $tdim(A) = 0$ iff for every $j \in \{1, \ldots, k\}$, $\pi_{(j)}(A)$ has empty interior (w.r.t. τ) in M.

Remark that if $A \subseteq M^k$ then $tdim(A) = k$ iff A has non-empty interior in M^k. Moreover, in most of classical examples of first-order topological structures (e.g. *RCF*, *RCR* and *pCF* where the respective topologies are the natural ones associated with the language of the structure), the equivalence "$tdim(A) = 0 \iff A$ *is finite*" holds (see [6]).
This is not true anymore in closed ordered differential fields if we consider the topological dimension associated with the δ-topology. Indeed, there exist infinite definable sets with topological dimension 0 (e.g. the subfield of constants of M is an infinite subset of M which has empty δ-interior in M). This has to be related with the fact that there exist infinite trivial δ-cells (cf. Remark 4.7 and Theorem 5.29).

The next theorem proves the equivalence between the δ-dimension and the topological dimension associated with the δ-topology in *CODF*.

Theorem 5.29. *Let $M \models CODF$ and $A \subset M^k$ be L'-definable, then*

$$tdim(A) = \delta\text{-}dim(A)$$

where tdim is the topological dimension associated with the δ-topology on M.

Proof.

. Suppose first that $tdim(A) = 0$. If $\delta\text{-}dim(A) > 0$ then A contains a non-trivial δ-cell C and Theorem 5.14 implies that a coordinate projection $\pi(C) \subset \pi(A)$ has non-empty δ-interior. Hence $\pi(A)$ has non-empty δ-interior and $tdim(A) > 0$ (Definition 5.28), a contradiction.

On another hand, if $\delta\text{-}dim(A) = 0$ and $\pi(A)$ is a coordinate projection of A which has non-empty δ-interior, then $\pi(A)$ contains a δ-open δ-cell. Hence

$$0 < \delta\text{-}dim(\pi(A)) \leq \delta\text{-}dim(A)$$

and this contradicts the assumption $\delta\text{-}dim(A) = 0$. So $tdim(A) = 0$.

. Suppose now that $\delta\text{-}dim(A) = l > 0$. Then A contains a δ-cell C of δ-dimension l and, by Theorem 5.14, there exists a coordinate projection $\pi_{(j_1;\ldots;j_l)}$ such that $\pi_{(j_1;\ldots;j_l)}(C)$ has non-empty δ-interior in M^l. By Definition 5.28, $tdim(A) \geq l$.

. Finally let $tdim(A) = l > 0$ and $\pi(A) \subset M^l$ be a coordinate projection of A with non-empty δ-interior. By Lemma 5.25, $\pi(A)_N$ contains a differentially generic point $a = (a_1; \ldots; a_l)$ of M^l. Hence A_N contains a point which is projected on a and $\delta\text{-rk}(A) \geq l$. By Theorem 5.23, $\delta\text{-}dim(A) \geq l$, finishing the proof. $\square$

6. A final remark

All the work developed in this paper only depends on three basic results: the quantifier elimination in *CODF*, the existence of a cell decomposition for *RCF* and the density of the jet-spaces w.r.t. the order topology. This is why we think that all these methods should apply to other examples of theories of differential fields (eventually equipped with finitely many commuting derivations) as soon as the three conditions above hold. It is worth noticing that the third author obtained in his thesis the same results as the ones in this paper in the case of existentially closed ordered fields equipped with finitely many commuting derivations [11].

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References

[1] N. Guzy, C. Rivière, Principle of differential lifting for theories of differential fields and Pierce–Pillay axiomatization, Notre-Dame J. Formal Logic 47 (2006) 331–341.
[2] W. Hodges, Model Theory, in: Encyclopedia of Mathematics and Its Applications, vol. 42, Cambridge University Press, Cambridge, 1993, xiii, 772 p.
[3] I. Kaplansky, An Introduction to Differential Algebra, Hermann, 1955.
[4] J. F. Knight, A. Pillay, C. Steinhorn, Definable sets in ordered structures. II, Trans. Amer. Math. Soc. 295 (1986) 593–605.
[5] L. Mathews, Topological analogues of model theoretic stability, Ph. D. Thesis, Oxford University, 1992.
[6] L. Mathews, Cell decomposition and dimension functions in first-order topological structures, Proc. London Math. Soc., Ser. III. 70 (1) (1995) 1–32.
[7] C. Miller, Tameness in expansions of the real field, in: Logic Colloquium 2001, in: Lect. Notes Log. 20, Assoc. Symb. Logic, 2005, pp. 281–316.
[8] A. Pillay, C. Steinhorn, Definable sets in ordered structures. I, Trans. Amer. Math. Soc. 295 (1986) 565–592.
[9] A. Pillay, C. Steinhorn, Definable sets in ordered structures. III, Trans. Amer. Math. Soc. 309 (2) (1988) 469–476.
[10] C. Rivière, Further notes on cell decomposition in closed ordered differential fields, Ann. Pure Appl. Logic. 159 (1–2) (2009) 100–110.
[11] C. Rivière, Model companion of theories of differential fields, Differential cell decomposition in closed ordered differential fields, Ph.D. Thesis, University of Mons-Hainaut, 2005.
[12] A. Robinson, Ordered differential fields, J. Combin. Theory, Ser. A 14 (1973) 324–333.
[13] J.-P. Rolin, P. Speissegger, A.J. Wilkie, Quasianalytic Denjoy–Carleman classes and o-minimality, J. Amer. Math. Soc. 16 (4) (2003) 751–777.
[14] M. F. Singer, The model theory of ordered differential fields, J. Symbolic Logic 43 (1978) 82–91.
[15] P. Speissegger, The Pfaffian closure of an o-minimal structure, J. Reine Angew. Math. 508 (1999) 189–211.
[16] L. van den Dries, Dimension of definable sets, algebraic boundedness and Henselian fields, Ann. Pure Appl. Logic 45 (2) (1989) 189–209.
[17] L. van den Dries, Tame Topology and o-minimal Structures, in: London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, 1998.
[18] L. van den Dries, P. Speissegger, The real field with convergent generalized power series, Trans. Amer. Math. Soc. 350 (11) (1998) 4377–4421.
[19] L. van den Dries, P. Speissegger, The field of reals with multisummable series and the exponential function, Proc. London Math. Soc., Ser. III. 81 (3) (2000) 513–565.
[20] A. Wilkie, Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function, J. Amer. Math. Soc. 9 (4) (1996) 1051–1094.