

**Holographic superconductors in 3 + 1 dimensions away from the probe limit**Yves Brihaye<sup>1,\*</sup> and Betti Hartmann<sup>2,†</sup><sup>1</sup>*Physique-Mathématique, Université de Mons-Hainaut, 7000 Mons, Belgium*<sup>2</sup>*School of Engineering and Science, Jacobs University Bremen, 28759 Bremen, Germany*

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We study holographic superconductors in 3 + 1 dimensions away from the probe limit, i.e. taking backreaction of the space-time into account. We consider the case of pure Einstein and Gauss-Bonnet gravity, respectively. Similar to the probe limit we observe that the critical temperature at which condensation sets in decreases with increasing Gauss-Bonnet coupling. The decrease is however stronger when taking backreaction of the space-time into account. We observe that the critical temperature becomes very small, but stays positive for all values of the Gauss-Bonnet coupling no matter how strong the backreaction of the space-time is.

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**I. INTRODUCTION**

The gravity-gauge theory duality [1] has attracted a lot of attention in the past years. The most famous example is the AdS/CFT correspondence [2] which states that a gravity theory in a  $d$ -dimensional anti-de Sitter (AdS) space-time is equivalent to a conformal field theory on the  $(d - 1)$ -dimensional boundary of AdS.

Recently, this theory has been used to describe so-called holographic superconductors with the help of black holes in higher dimensional space-time [3–5] and many aspects have been discussed such as holographic superconductors in Horava-Lifshitz gravity [6] and in Born-Infeld electrodynamics [7], fermions [8], the behavior of holographic superconductors in external magnetic fields [9] and at zero temperature [10], hydrodynamical aspects of holographic superconductors [11] as well as rotating superconductors [12]. Holographic superconductors in extended models that allow for a first order phase transition [13] as well as holographic superconductors in  $M$  theory [14] have also been studied. Non-Abelian (or  $p$ -wave) holographic superconductors have been studied in [15–23]. In [19,20] a string theory realization of  $p$ -wave holographic superconductors in the probe limit has been discussed and the Meissner effect has been studied in detail [20]. (For a related analytical study see [21].) Sound modes for  $p$ -wave superconductors have been considered in [22], while fermions in these superconductors have been discussed in [23]. Various other aspects have also been studied [24].

The general idea behind holographic superconductors comes from the observation that below a critical temperature electrically charged black holes become unstable to form scalar hair, i.e. they possess nonvanishing scalar fields on the horizon [3]. The reason for this is that close to the

horizon of the black hole the effective mass of the scalar field can become negative with masses below the Breitenlohner-Freedman bound [25] such that the scalar field becomes unstable and possesses a nonvanishing value on and close to the horizon of the black hole. The value of the scalar field on the AdS boundary is then associated with the corresponding condensate in the dual theory.

In most cases, holographic superconductors have been studied in the “probe limit” neglecting backreaction of the space-time. This limit corresponds to letting the electric charge  $e$  tend to infinity or equivalently Newton’s constant  $G$  tend to zero. Backreaction of the space-time was considered in [4] for a  $(2 + 1)$ -dimensional holographic superconductor. It was found that the qualitative results are similar for small charges, but that surprisingly the scalar field can even form a condensate when being uncharged.

In [26]  $(3 + 1)$ -dimensional superconductors were studied by investigating scalar hair formation on black holes in Gauss-Bonnet gravity. This has been extended to higher dimensions in [27]. The motivation for this is the apparent contradiction between the Mermin-Wagner theorem that forbids spontaneous symmetry breaking in  $2 + 1$  dimensions at finite temperatures and the fact that  $(2 + 1)$ -dimensional holographic superconductors do exist. Consequently, it has been suggested that higher curvature corrections should suppress condensation, where higher curvature corrections can, of course, only be studied for  $(3 + 1)$ -dimensional superconductors (or higher dimensional ones). [26,27] were concerned with the “probe limit” and it was found that condensation cannot be suppressed in Gauss-Bonnet gravity.

In this paper, we are interested in the model studied in [26] but away from the probe limit, i.e. taking backreaction of the space-time into account. We study  $(3 + 1)$ -dimensional superconductors in pure Einstein and Gauss-Bonnet gravity, respectively. While for large temperatures, i.e. when the scalar field vanishes identically analytic solutions to the equations of motion are known,

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this is different for a black hole with scalar hair that forms below the condensation temperature. These solutions have to be constructed numerically.

In Sec. II, we present the model, the equations of motion and the boundary conditions. In Sec. III, we discuss our numerical results, while Sec. IV contains our conclusions.

## II. THE MODEL

In this paper, we are studying the formation of scalar hair on an electrically charged black hole in (4 + 1)-dimensional anti-de Sitter space-time. The action reads

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \left( R - 2\Lambda + \frac{\alpha}{4} (R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} - 4R^{\mu\nu} R_{\mu\nu} + R^2) + 16\pi G \mathcal{L}_{\text{matter}} \right), \quad (1)$$

where  $\Lambda = -6/L^2$  is the cosmological constant and  $\alpha$  the Gauss-Bonnet coupling.  $\mathcal{L}_{\text{matter}}$  denotes the matter Lagrangian:

$$\mathcal{L}_{\text{matter}} = -\frac{1}{4} F_{MN} F^{MN} - (D_M \psi)^* D^M \psi - m^2 \psi^* \psi, \quad (2)$$

$M, N = 0, 1, 2, 3, 4,$

where  $F_{MN} = \partial_M A_N - \partial_N A_M$  is the field strength tensor and  $D_M \psi = \partial_M \psi - ie A_M \psi$  is the covariant derivative.  $e$  and  $m^2$  denote the electric charge and mass of the scalar field  $\psi$ , respectively.

The Ansatz for the metric reads

$$ds^2 = -f(r)a^2(r)dt^2 + \frac{1}{f(r)}dr^2 + \frac{r^2}{L^2} d\Sigma_{k,3}^2, \quad (3)$$

where  $f$  and  $a$  are functions of  $r$  only. The 3-dimensional metric is

$$d\Sigma_{k,3}^2 = \begin{cases} d\Omega_3^2 & \text{for } k = 1 \\ dx^2 + dy^2 + dz^2 & \text{for } k = 0 \\ d\Xi_3^2 & \text{for } k = -1 \end{cases} \quad (4)$$

where  $k$  denotes the curvature of the 3-dimensional space. We are only interested in plane-symmetric black holes in this paper, so we will set  $k = 0$ . However, we will keep the  $k$  in the equations for completeness.

For the electromagnetic field and the scalar field we choose [4]

$$A_M dx^M = \phi(r)dt, \quad \psi = \psi(r), \quad (5)$$

such that the black hole possesses only electric charge.

The coupled Einstein and Euler-Lagrange equations are obtained from the variation of the action with respect to the matter and metric fields, respectively. They read:

$$f' = 2r \frac{k - f + 2r^2/L^2}{r^2 + 2\alpha(k - f)} - \gamma \frac{r^3}{2fa^2} \times \left( \frac{2e^2 \phi^2 \psi^2 + f(2m^2 a^2 \psi^2 + \phi'^2) + 2f^2 a^2 \psi'^2}{r^2 + 2\alpha(k - f)} \right), \quad (6)$$

$$a' = \gamma \frac{r^3 (e^2 \phi^2 \psi^2 + a^2 f^2 \psi'^2)}{af^2 (r^2 + 2\alpha(k - f))}, \quad (7)$$

$$\phi'' = -\left(\frac{3}{r} - \frac{a'}{a}\right) \phi' + 2 \frac{e^2 \psi^2}{f} \phi, \quad (8)$$

$$\psi'' = -\left(\frac{3}{r} + \frac{f'}{f} + \frac{a'}{a}\right) \psi' - \left(\frac{e^2 \phi^2}{f^2 a^2} - \frac{m^2}{f}\right) \psi, \quad (9)$$

where  $\gamma = 16\pi G$ . Here and in the following the prime denotes the derivative with respect to  $r$ . In [26], (3 + 1)-dimensional holographic superconductors have been studied in the probe limit corresponding to  $\gamma = 0$ . For  $\gamma \neq 0$  we take backreactions of the space-time into account. Note that this limit is equivalent to letting  $e \rightarrow \infty$  since we can perform the rescalings  $\psi \rightarrow \psi/e$ ,  $\phi \rightarrow \phi/e$ , and  $\gamma \rightarrow e^2 \gamma$ . Hence without losing generality we can set  $e \equiv 1$ .

In order to find an explicit solution of the equations of motion, we have to fix appropriate boundary conditions. In the following, we are interested in the formation of scalar hair on electrically charged black holes with the horizon at  $r = r_h$  such that

$$f(r_h) = 0, \quad (10)$$

with  $a(r_h)$  finite. In order for the matter fields to be regular at the horizon we need to impose

$$\begin{aligned} \phi(r_h) &= 0, \\ \psi'(r_h) &= \frac{m^2 \psi(r_h^2 + 2\alpha k)}{2rk + 4r/L^2 - \gamma r^3 (m^2 \psi^2 + \phi'^2 / (2a^2))} \Big|_{r=r_h}. \end{aligned} \quad (11)$$

Asymptotically, we want the space-time to be that of a Reissner-Nordström-anti-de Sitter black hole, i.e. we can choose  $a(r \rightarrow \infty) \rightarrow 1$ . Other choices of the asymptotic value of  $a(r)$  would simply correspond to a rescaling of the time coordinate. The matter fields on the other hand obey [26]

$$\phi(r \gg 1) = \mu - \rho/r^2, \quad \psi(r \gg 1) = \frac{\psi_-}{r^{\lambda_-}} + \frac{\psi_+}{r^{\lambda_+}}, \quad (12)$$

with

$$\begin{aligned}\lambda_- &= 2 - \sqrt{4 - 3(L_{\text{eff}}/L)^2}, \\ \lambda_+ &= 2 + \sqrt{4 - 3(L_{\text{eff}}/L)^2}, \\ L_{\text{eff}}^2 &\equiv \frac{2\alpha}{1 - \sqrt{1 - 4\alpha/L^2}} \sim L^2(1 - \alpha/L^2 + O(\alpha^2)).\end{aligned}\quad (13)$$

Note that the value of the Gauss-Bonnet coupling  $\alpha$  is bounded from above:  $\alpha \leq L^2/4$  where  $\alpha = L^2/4$  is the Chern-Simons limit. For larger values of  $\alpha$  the solution would possess a naked singularity.

The parameters  $\mu$ ,  $\rho$  are the chemical potential and density of electric charge, respectively. Along with [26] we choose  $\psi_- = 0$ .  $\psi_+$  will correspond to the expectation value  $\langle \mathcal{O} \rangle$  of the operator  $\mathcal{O}$  which, in the context of the gauge theory-gravity duality, is dual to the scalar field and as such represents the value of the condensate.

There are analytic solutions of the equations of motion for  $\psi(r) \equiv 0$ :

$$\begin{aligned}f(r) &= k + \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 - \frac{4\alpha}{L^2} + \frac{4\alpha M}{r^4} - \frac{4\alpha\gamma\rho^2}{r^6}} \right), \\ a(r) &= 1, \quad \phi(r) = \frac{\rho}{r_h^2} - \frac{\rho}{r^2},\end{aligned}\quad (14)$$

where  $M$  and  $\rho$  are arbitrary integration constants that can be interpreted as the mass and the charge density of the solution, respectively. In the limit  $\alpha \rightarrow 0$ , the metric function  $f(r)$  becomes  $f(r) = k + \frac{r^2}{L^2} - \frac{M}{r^2} + \frac{\gamma\rho^2}{r^4}$ . These solutions are electrically charged black holes which are the only solutions for temperatures larger than the critical temperature  $T_c$ . For  $T < T_c$  these solutions will be unstable to form scalar hair, i.e. develop a nonvanishing value of  $\psi$  on the horizon. The aim of this paper is to study the formation of scalar hair black holes in dependence on  $\alpha$  and  $\gamma$ . The temperature mentioned here corresponds to the Hawking temperature of the black hole and reads

$$T = \frac{1}{4\pi} \sqrt{-g^{tt} g^{MN} \partial_M g_{tt} \partial_N g_{tt}} \Big|_{r=r_h} = \frac{1}{4\pi} f'(r_h) a(r_h),$$

$$M, N = 1, 2, 3, 4. \quad (15)$$

In the gauge theory-gravity duality  $T_c$  is the temperature below which superconductivity appears.

### III. NUMERICAL RESULTS

In the following we are only interested in the plane-symmetric black holes with  $k = 0$ . The equations of motion (6)–(9) depend in principle on a number of constants but due to the scale invariances noted in [4] two of them can be scaled out and hence be fixed to particular values without losing generality. In the following we fix  $r_h = 0.5$  and  $L = 1$ . Along with [26] we set  $m^2 = -3/L^2 \equiv -3$  which guarantees the stability of AdS<sub>5</sub> since  $m^2 < m_{\text{BF}}^2 =$

$-4/L^2$  with  $m_{\text{BF}}^2$  the Breitenlohner-Freedman (BF) mass [25].

To find a unique solution to the equations of motions we fix the boundary conditions (10) and (11) at the horizon, and choose  $a(\infty) = 1$ ,  $\psi_- = 0$ . In addition we fix  $\psi_+$  to a particular value. In this way we are able to construct branches of solutions labeled by the parameter  $\psi_+$ . Note that  $\mu$  and  $\rho$  are uniquely fixed by the choice of  $\psi_+$  and are not free parameters. However, in the literature on holographic superconductors, the solutions are typically presented for fixed electric charge density  $\rho$ , while the horizon value  $r_h$  is treated as a free parameter. These two approaches are connected to each other. Indeed, it is easy to convert a branch of solutions with fixed  $r_h$  and varying  $\psi_+$ —where  $\rho = \rho(\psi_+)$ —into a branch of solutions with a constant charge density. In the following we denote quantities corresponding to a fixed charge density by a hat. Setting  $\hat{\rho} = 1$ , the relevant Hawking temperature  $\hat{T}$  and condensate  $\hat{\psi}_+$  are, respectively, given by

$$\hat{T} = \frac{T}{\rho^{1/3}}, \quad \hat{\psi}_+ = \frac{\psi_+}{\rho^{(\lambda_+/3)}}, \quad (16)$$

where  $\lambda_+$  is defined in (13).

#### A. Effect of backreaction in Einstein gravity

This corresponds to the case  $\gamma \neq 0$  and  $\alpha = 0$ . We solved the equations for several values of  $\gamma$  and  $\psi_+$  and find that solutions exist for generic values of these parameters.

When studying solutions for  $\gamma$  fixed and varying  $\psi_+$  we find that in the limit  $\psi_+ \rightarrow 0$  the solutions tend to (14) for very specific values of  $\mu$  and  $\rho$  which depend on the choice of  $\gamma$  and can only be determined numerically. Correspondingly, the critical temperature  $\hat{T}_c$  at which  $\hat{\psi}_+ = 0$  can also only be determined numerically. We find the values given in the table below:

$\gamma$	0.0	0.025	0.05	0.1	0.15	0.2	0.3	0.35
$4\pi\hat{T}_c$	2.48	2.02	1.61	0.99	0.57	0.33	0.10	0.06

For large  $\gamma$  the construction of the solutions becomes increasingly difficult. In principle we would want to know what happens for very large  $\gamma$ . In order to understand this, we fitted the numerical data and found that

$$T_c \approx 0.198 \cdot \exp(-10.6 \cdot \gamma) \rho^{1/3} \quad (17)$$

fits the data for  $\gamma \geq 0.2$  very well. This, on the other hand, means that no matter how large we choose  $\gamma$ , we will always have  $T_c > 0$ . This has already been observed for superconductors in 2 + 1 dimensions [4], where it was shown that  $T_c > 0$  in the limit  $e \rightarrow 0$  which corresponds to  $\gamma \rightarrow \infty$  here. Apparently, this phenomenon persists for (3 + 1)-dimensional superconductors.

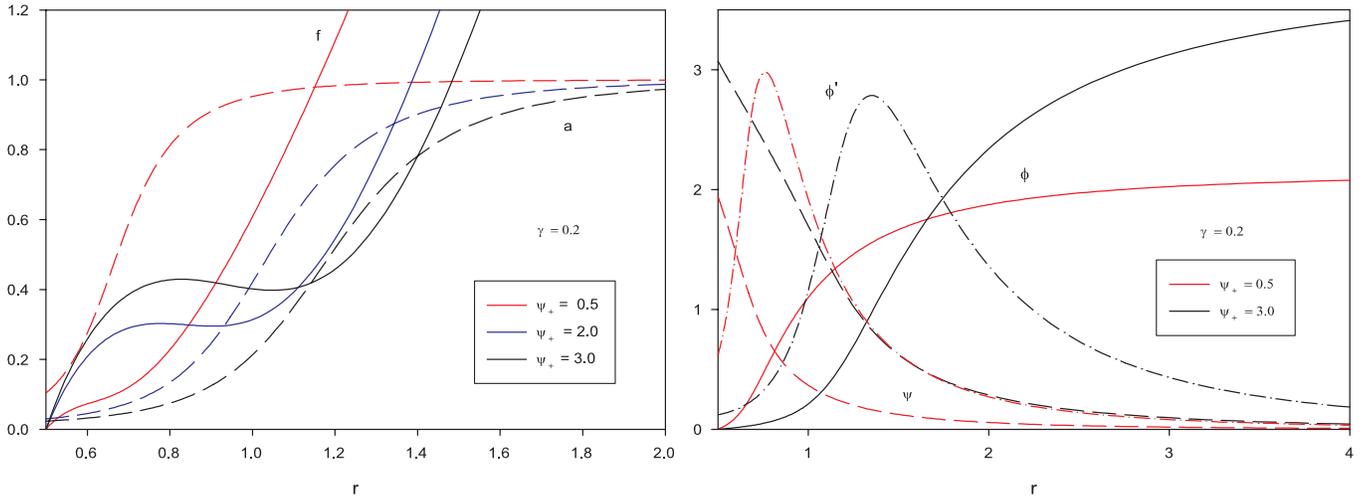


FIG. 1 (color online). The metric functions  $f(r)$  and  $a(r)$  (left) and the matter functions  $\phi(r)$ ,  $\psi(r)$ , and  $\phi'(r)$  (right) for  $\gamma = 0.2$  and for three values of  $\psi_+$ .

Fixing  $\gamma$  and increasing the value of the condensate  $\psi_+$  we find that the values  $a(r_h)$  and  $\phi'(r_h)$  slowly approach zero. At the same time, the function  $f(r)$  develops a local maximum and a local minimum at values  $r_M, r_m$  such that  $r_h < r_M < r_m < \infty$ . This is illustrated for the metric functions  $f(r)$  and  $a(r)$  in Fig. 1 (left) for  $\gamma = 0.2$  and three different values of  $\psi_+$ .

This would suggest that for sufficiently large  $\psi_+$  the function  $f(r)$  develops a double zero at  $r = r_m$  which would correspond to the formation of an extremal black hole with vanishing Hawking temperature. A detailed analysis, however, shows that the value  $f(r_m)$  remains strictly positive, while the value of  $a(r_h)$  decreases with  $\psi_+$  increasing according to an exponential behavior  $a(r_h) \sim \exp(-c\psi_+)$  with  $c$  some constant. This result suggests that the black hole solutions are not limited by a maximal value of the condensate  $\psi_+$  and that the temperature stays positive for all values of  $\psi_+$ .

In Fig. 1 (right) we show the matter field functions  $\phi(r)$ ,  $\psi(r)$ , and  $\phi'(r)$ . We observe that for fixed  $\gamma$  and increasing  $\psi_+$  the maximum of  $\phi'(r)$  is pushed further away from the horizon of the black hole. Indeed, for small values of  $\gamma$  and  $\psi_+$ , the maximum of  $\phi'(r)$  is on the horizon of the black hole. Note that when fixing  $\psi_+$  and increasing  $\gamma$  we observe a similar phenomenon.

To understand how the scalar field  $\psi$  becomes unstable close to the horizon, we plot the effective mass

$$m_{\text{eff}}^2 = m^2 + e^2 A_t^2 g^{tt} = -\frac{3}{L^2} - \frac{\phi^2}{fa^2} \quad (18)$$

in Fig. 2 for  $L = 1$ . Indeed, the effective mass drops below the Breitenlohner-Freedman bound of  $-4/L^2 = -4$  close to the horizon. For fixed  $\psi_+$  and increasing  $\gamma$  the quantity  $-\frac{3}{L^2} - \frac{\phi^2}{fa^2}$  becomes more narrow and smaller in absolute

value. For fixed  $\gamma$  and increasing  $\psi_+$  it becomes broader and larger in absolute value.

To understand this in more detail note that close to the horizon the functions can be expanded as follows:

$$f(r) = f'(r_h)(r - r_h) + \frac{f''(r_h)}{2}(r - r_h)^2 + \dots, \quad (19)$$

$$a(r) = a(r_h) + a'(r_h)(r - r_h) + \frac{a''(r_h)}{2}(r - r_h)^2 + \dots, \quad (20)$$

$$\phi(r) = \phi'(r_h)(r - r_h) + \frac{\phi''(r_h)}{2}(r - r_h)^2 + \dots, \quad (21)$$

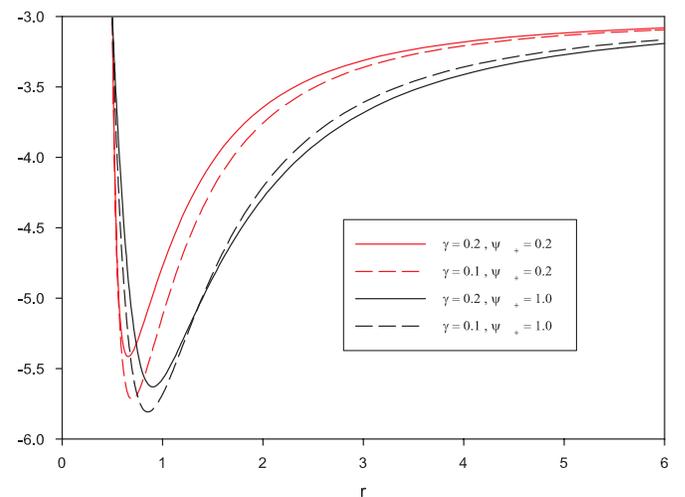


FIG. 2 (color online). The value of the effective mass  $m_{\text{eff}}^2 = m^2 + e^2 A_t^2 g^{tt} = -\frac{3}{L^2} - \frac{\phi^2}{fa^2}$  close to the horizon of the black hole for several values of  $\gamma$  and  $\psi_+$ .

$$\begin{aligned} \psi(r) &= \psi(r_h) + \psi'(r_h)(r - r_h) + \frac{\psi''(r_h)}{2}(r - r_h)^2 \\ &+ \dots \end{aligned} \quad (22)$$

Moreover note that there are the following relations between the values of the functions at  $r = r_h$ :

$$\phi''(r_h) = \phi'(r_h) \left( \frac{a'(r_h)}{a(r_h)} + 2e^2 \frac{\psi(r_h)^2}{f'(r_h)} - \frac{3}{r_h} \right), \quad (23)$$

$$\psi'(r_h) f'(r_h) = m^2 \psi(r_h) = -\frac{3}{L^2} \psi(r_h), \quad (24)$$

$$\begin{aligned} f'(r_h) &= 4 \frac{r_h}{L^2} - r_h \gamma \left( m^2 \psi(r_h)^2 + \frac{\phi'(r_h)^2}{2a(r_h)^2} \right) \\ &= 4 \frac{r_h}{L^2} + r_h \gamma \left( \frac{3}{L^2} \psi(r_h)^2 - \frac{\phi'(r_h)^2}{2a(r_h)^2} \right), \end{aligned} \quad (25)$$

$$a'(r_h) = r_h \gamma \left( a(r_h) \psi'(r_h)^2 + e^2 \frac{\phi'(r_h)^2 \psi(r_h)^2}{a(r_h) f'(r_h)^2} \right). \quad (26)$$

Several quantities characterizing the solutions are given in Fig. 3 for  $\gamma$  varying and  $\psi_+ = 0.2$  fixed (left) and for  $\psi_+$  varying and  $\gamma = 0.2$  fixed (right), respectively. First, note that  $\psi(r_h)$  increases with  $\gamma$  and  $\psi_+$ , respectively. That  $\psi(r_h)$  is an increasing function of the condensate  $\psi_+$  was already noticed in [26] for the probe limit. Here, we find in addition that the stronger the backreaction the higher the value  $\psi(r_h)$  for a given condensate  $\psi_+$ . Since we would like  $\psi$  to have its maximal positive value on the horizon we have  $\psi(r_h) > 0$ ,  $\psi'(r_h) < 0$  and then from (24) obviously  $f'(r_h) > 0$ . For  $\gamma = 0$ , the value of  $f'(r_h) = 4 \frac{r_h}{L^2}$  which for  $r_h = 0.5$  and  $L = 1$  is just  $f'(r_h) = 2.0$ . For increasing  $\gamma$  the value  $f'(r_h)$  is first decreasing due to the decrease of the electric field  $\phi'(r_h)/a(r_h)$  on the horizon [the negative term in (25)] and then for sufficiently strong

backreaction  $\psi(r_h)$  becomes larger and larger such that  $f'(r_h)$  starts increasing again. This is similar for  $\gamma$  fixed and varying  $\psi_+$ . For  $\psi_+ = 0$ , the solution is given by (14). For small  $\psi_+ > 0$  the electric field on the horizon  $\phi'(r_h)/a(r_h)$  decreases and leads to a slight decrease in  $f'(r_h)$ . For increasing  $\psi_+$  the value  $\psi(r_h)$  becomes larger and  $f'(r_h)$  increases. For both  $\gamma$  and  $\psi_+$  increasing, respectively, the value of  $a(r_h)$  decreases from  $a(r_h) = 1$ .

Finally, the charge density  $\rho$  decreases for small  $\psi_+$  and increases for larger  $\psi_+$  when  $\gamma$  is fixed. For fixed  $\psi_+$  and varying  $\gamma$  the behavior is qualitatively similar.

When studying the dependence of the condensate  $\psi_+$  on the temperature  $T$  one can take two different viewpoints. First, we consider the system for fixed horizon ( $r_h = 0.5$  here). The dependence of  $\langle \mathcal{O} \rangle = \psi_+$  as a function of the temperature  $T$  is given in Fig. 4 (left) for several values of  $\gamma$ .

The solid line represents the condensate  $\psi_+$  and the dashed line represents the charge density  $\rho$ , respectively. For completeness we give the corresponding lines for  $\gamma = 0$  (dashed line), where the bullet indicates the minimal value of  $\rho$ . Apparently, the behavior is quite different when comparing large and small  $\gamma$ . For small  $\gamma$  the temperature of the condensate is larger than the temperature of the critical limit, for large  $\gamma$  it is vice versa.

Following the literature, we also present our results for fixed charge density  $\hat{\rho} = 1$ . The dimensionless quantity  $\langle \hat{\mathcal{O}} \rangle^{1/\lambda_+} / \hat{T}_c = (\hat{\psi}_+)^{1/\lambda_+} / \hat{T}_c$  (with  $\lambda_+ = 3$  in the limit  $\alpha = 0$ ) as a function of the rescaled temperature  $\hat{T} / \hat{T}_c$  is given in Fig. 4 (right). Qualitatively, the behavior for large  $\gamma$  is similar to that for small  $\gamma$ . However, the condensate can become quite large when increasing  $\gamma$ . Moreover, the critical temperature  $\hat{T}_c$  at which  $\hat{\psi}_+ = 0$  decreases with increasing  $\gamma$ .

Note that we are in fact plotting  $(e \hat{\psi}_+)^{1/\lambda_+} / \hat{T}_c$  but that  $e$  does not appear here due to our choice  $e \equiv 1$ . Moreover,

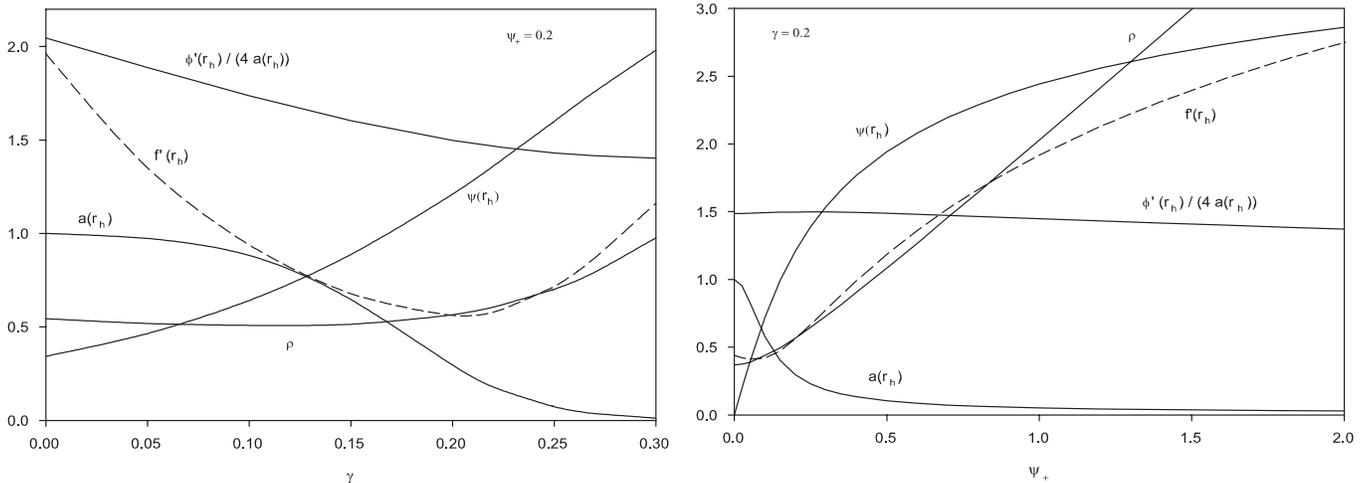


FIG. 3. Several quantities characterizing the black holes in pure Einstein gravity ( $\alpha = 0$ ) with  $\psi_+ = 0.2$  for  $\gamma$  varying (left) and for  $\gamma = 0.2$  and  $\psi_+$  varying (right).

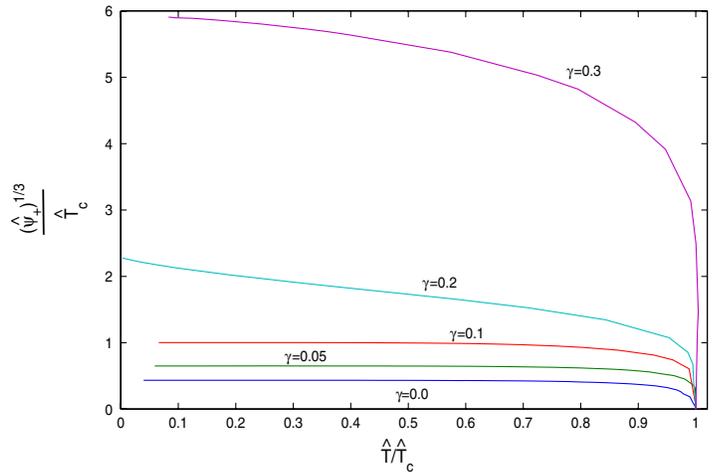
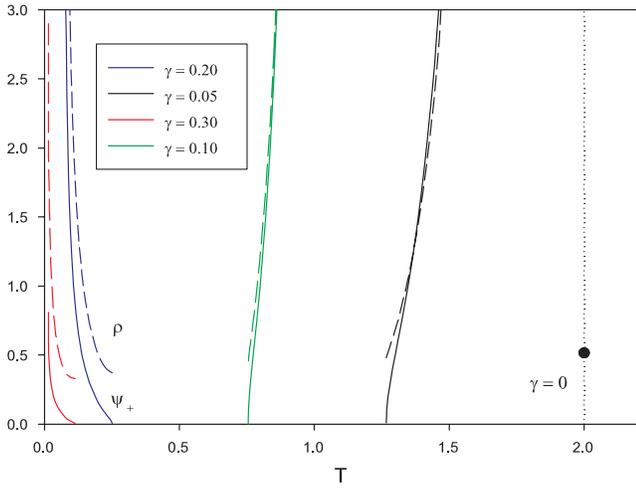


FIG. 4 (color online). The values  $\psi_+$  and  $\rho$  as functions of the temperature  $T$  for several values of  $\gamma$  with  $r_h = 0.5$  (left). The values  $(\hat{\psi}_+)^{1/3}/\hat{T}_c$  as a function of  $\hat{T}/\hat{T}_c$  for several values of  $\gamma$  with  $\hat{\rho} = 1$  (right).

comparing our results to the  $(2 + 1)$ -dimensional case [4] our choices of  $\gamma = 0.0, 0.05, 0.1, 0.2,$  and  $0.3,$  respectively, would correspond to  $e = \infty, 4.47, 3.16, 2.24,$  and  $1.83$  when setting  $\gamma = 1$  instead of  $e = 1.$

**B. Effect of backreaction in Gauss–Bonnet gravity**

This corresponds to the case  $\gamma \neq 0$  and  $0 < \alpha \leq 0.25.$  The  $\gamma = 0$  limit was studied in [26]. Our numerical results indicate that, also for  $\gamma \neq 0,$  the presence of the Gauss–Bonnet term leads to a decrease in the critical temperature. This is shown in Fig. 5 where we give the dependence of  $\hat{T}_c$  on  $\alpha$  for different values of  $\gamma.$  Apparently backreaction on the space-time also cannot suppress condensation, i.e. the critical temperature stays positive for all values of  $\gamma$  and  $\alpha$  that we have studied in this paper. We find, e.g., for  $\alpha =$

$0.1$  that  $T_c = 0.185\rho^{1/3}$  for  $\gamma = 0$  (in agreement with [26]),  $T_c = 0.051\rho^{1/3}$  for  $\gamma = 0.1,$  and  $T_c = 0.008\rho^{1/3}$  for  $\gamma = 0.2.$  For  $\alpha = 0.25$  we find  $T_c = 0.158\rho^{1/3}$  for  $\gamma = 0$  (again in agreement with [26]),  $T_c = 0.024\rho^{1/3}$  for  $\gamma = 0.1,$  and  $T_c = 0.001\rho^{1/3}$  for  $\gamma = 0.2.$  Hence, the critical temperature can become arbitrarily close to zero for  $\alpha$  and  $\gamma$  large enough, however within our numerical accuracy, we never find  $T_c = 0$  for finite values of  $\gamma$  and  $\alpha \leq 0.25.$

**IV. CONCLUSIONS**

In this paper, we have studied holographic superconductors in  $3 + 1$  dimensions away from the probe limit. We considered the case of pure Einstein and Gauss–Bonnet gravity, respectively, and have constructed numerically electrically charged black holes that carry scalar hair. For pure Einstein gravity we find that, in agreement with the results for holographic superconductors in  $2 + 1$  dimensions [4], the critical temperature at which condensation sets in is strictly positive for all values of the gravitational coupling. Considering Gauss–Bonnet corrections further decreases the critical temperature, but all our numerical results indicate that it stays positive when taking backreaction into account. Hence, even when taking the gravitational coupling to infinity—which corresponds to letting the electric charge  $e$  of the condensate tend to zero—there would still be condensation. Similar to  $2 + 1$  dimensions this signals the existence of an additional instability of the scalar field. The explanation is similar to that in  $2 + 1$  dimensions [4,5]: since the scalar field is uncharged the instability cannot be caused by the spontaneous symmetry breaking. Rather it is caused by the fact that for large  $\gamma$  the black hole is close to the extremal limit in which its horizon geometry would correspond to  $\text{AdS}_2 \times \mathbb{R}_3.$  In  $\text{AdS}_2$  the Breitenlohner–Freedman bound [25] is  $m_{\text{BF}}^2 = -1/(4L^2).$

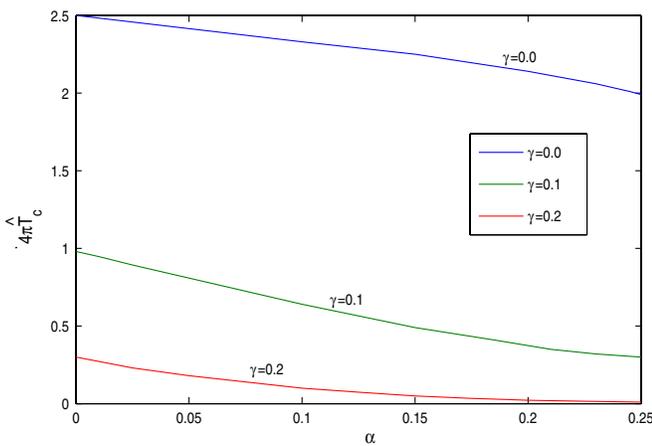


FIG. 5 (color online). The critical temperature  $\hat{T}_c$  at which superconductivity sets in as a function of the Gauss–Bonnet coupling constant  $\alpha$  for several values of the gravitational coupling  $\gamma.$

Hence a scalar field with mass  $m^2 = -3/L^2$  that is stable in  $\text{AdS}_5$  is certainly unstable in  $\text{AdS}_2$ .

When considering non-Abelian holographic superconductors it has been observed that the phase transition that leads to the formation of vector hair becomes first order if the gravitational coupling is large enough [17]. It would be

interesting to see how the Gauss-Bonnet term influences this result.

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- [1] See, e.g., O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Phys. Rep.* **323**, 183 (2000); E. D'Hoker and D. Z. Freedman, [arXiv:hep-th/0201253](#); M. Benna and I. Klebanov, [arXiv:0803.1315](#).
- [2] J. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998); *Int. J. Theor. Phys.* **38**, 1113 (1999).
- [3] S. S. Gubser, *Phys. Rev. D* **78**, 065034 (2008).
- [4] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, *Phys. Rev. Lett.* **101**, 031601 (2008); *J. High Energy Phys.* **12** (2008) 015; G. T. Horowitz and M. M. Roberts, *Phys. Rev. D* **78**, 126008 (2008).
- [5] For recent reviews see C. P. Herzog, *J. Phys. A* **42**, 343001 (2009); S. A. Hartnoll, *Classical Quantum Gravity* **26**, 224002 (2009); G. Horowitz, [arXiv:1002.1722](#).
- [6] R. G. Cai and H. Q. Zhang, *Phys. Rev. D* **81**, 066003 (2010); M. R. Setare and D. Momeni, [arXiv:1003.0376](#).
- [7] J. Jing and S. Chen, *Phys. Lett. B* **686**, 68 (2010).
- [8] S. S. Gubser, F. D. Rocha, and A. Yarom, [arXiv:1002.4416](#); S. S. Gubser, F. D. Rocha, and P. Talavera, [arXiv:0911.3632](#); T. Faulkner, G. T. Horowitz, J. McGreevy, M. M. Roberts, and D. Vegh, *J. High Energy Phys.* **03** (2010) 121.
- [9] X. H. Ge, B. Wang, S. F. Wu, and G. H. Yang, [arXiv:1002.4901](#); K. Maeda, M. Natsuume, and T. Okamura, *Phys. Rev. D* **81**, 026002 (2010); M. Montull, A. Pomarol, and P. J. Silva, *Phys. Rev. Lett.* **103**, 091601 (2009); T. Albash and C. V. Johnson, *Phys. Rev. D* **80**, 126009 (2009); [arXiv:0906.0519](#); *J. High Energy Phys.* **09** (2008) 121; E. Nakano and W. Y. Wen, *Phys. Rev. D* **78**, 046004 (2008).
- [10] R. A. Konoplya and A. Zhidenko, *Phys. Lett. B* **686**, 199 (2010); G. T. Horowitz and M. M. Roberts, *J. High Energy Phys.* **11** (2009) 015; S. S. Gubser and A. Nellore, *Phys. Rev. D* **80**, 105007 (2009).
- [11] C. P. Herzog and A. Yarom, *Phys. Rev. D* **80**, 106002 (2009); I. Amado, M. Kaminski, and K. Landsteiner, *J. High Energy Phys.* **05** (2009) 021; A. Yarom, *J. High Energy Phys.* **07** (2009) 070.
- [12] J. Sonner, *Phys. Rev. D* **80**, 084031 (2009).
- [13] F. Aprile and J. G. Russo, *Phys. Rev. D* **81**, 026009 (2010); S. Franco, A. Garcia-Garcia, and D. Rodriguez-Gomez, *J. High Energy Phys.* **04** (2010) 92.
- [14] J. P. Gauntlett, J. Sonner, and T. Wiseman, *J. High Energy Phys.* **02** (2010) 60; J. P. Gauntlett, J. Sonner, and T. Wiseman, *Phys. Rev. Lett.* **103**, 151601 (2009).
- [15] S. S. Gubser, *Phys. Rev. Lett.* **101**, 191601 (2008).
- [16] S. S. Gubser and S. S. Pufu, *J. High Energy Phys.* **11** (2008) 033.
- [17] M. Ammon, J. Erdmenger, V. Grass, P. Kerner, and A. O'Bannon, *Phys. Lett. B* **686**, 192 (2010).
- [18] A. R. Lugo, E. F. Moreno, and F. A. Schaposnik, *J. High Energy Phys.* **03** (2010) 13; P. Basu, J. He, A. Mukherjee, and H. H. Shieh, [arXiv:0911.4999](#); H. b. Zeng, Z. y. Fan, and Z. z. Ren, *Phys. Rev. D* **80**, 066001 (2009); R. Manvelyan, E. Radu, and D. H. Tchrakian, *Phys. Lett. B* **677**, 79 (2009); M. M. Roberts and S. A. Hartnoll, *J. High Energy Phys.* **08** (2008) 035.
- [19] M. Ammon, J. Erdmenger, M. Kaminski, and P. Kerner, *Phys. Lett. B* **680**, 516 (2009).
- [20] M. Ammon, J. Erdmenger, M. Kaminski, and P. Kerner, *J. High Energy Phys.* **10** (2009) 067.
- [21] P. Basu, J. He, A. Mukherjee, and H. H. Shieh, *J. High Energy Phys.* **11** (2009) 070.
- [22] C. P. Herzog and S. S. Pufu, *J. High Energy Phys.* **04** (2009) 126.
- [23] M. Ammon, J. Erdmenger, M. Kaminski, and A. O'Bannon, [arXiv:1003.1134](#).
- [24] F. Aprile, S. Franco, D. Rodriguez-Gomez, and J. G. Russo, [arXiv:1003.4487](#); C. P. Herzog, [arXiv:1003.3278](#); J. W. Chen, Y. J. Kao, D. Maity, W. Y. Wen, and C. P. Yeh, *Phys. Rev. D* **81**, 106008 (2010); O. C. Umeh, *J. High Energy Phys.* **08** (2009) 062; K. Maeda, M. Natsuume, and T. Okamura, *Phys. Rev. D* **79**, 126004 (2009).
- [25] P. Breitenlohner and D. Z. Freedman, *Ann. Phys. (N.Y.)* **144**, 249 (1982).
- [26] R. Gregory, S. Kanno, and J. Soda, *J. High Energy Phys.* **10** (2009) 010.
- [27] Q. Pan, B. Wang, E. Papantonopoulos, J. Oliveira, and A. B. Pavan, *Phys. Rev. D* **81**, 106007 (2010).