On positive solutions to the Lane-Emden problem with Neumann boundary conditions

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The Lane-Emden problem

Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded, $N \geqslant 2$, and $2 < p$. We consider

$$(\mathcal{P}_p) \begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \Omega \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

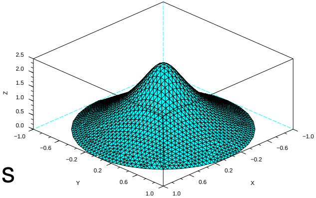
If $p \leqslant 2^* := \frac{2N}{N-2}$, solutions are critical points of the functional

$$\mathcal{E}_p : H^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 - \frac{1}{p}\int_\Omega |u|^p$$

- *Remark:* 0 and ± 1 are always (trivial) solutions.
- In this talk $\Omega = B_R = B(0,R)$ (mostly).
- *Notation:* $0 = \lambda_1 < \lambda_2 < \cdots$ denote the eigenvalues of $-\Delta$ with NBC, E_i denote the corresponding eigenspaces

Dirichlet boundary conditions

$$\begin{cases} -\Delta u + u = |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$



- The ground state solution is positive and is even w.r.t. any hyperplane leaving Ω invariant (when Ω is convex). In particular, it is radially symmetric on a ball.
- Uniqueness of the positive solution when Ω is a ball.
- If Ω is strictly starshaped and $p \geqslant 2^*$, no solution exist.

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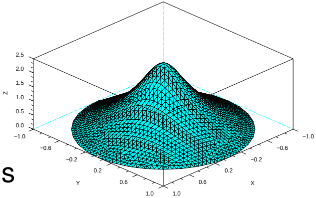

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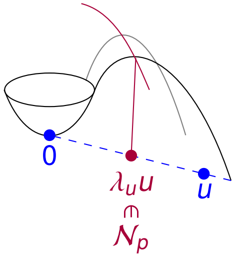
- Uniqueness of the positive solution when Ω is a ball.
- If Ω is strictly starshaped and $p \geqslant 2^*$, no solution exist.

All this is false for Neumann boundary conditions!

Well known facts...

$$(\mathcal{P}_p) \begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \Omega \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$



Theorem (Z. Nehari, A. Ambrosetti, P.H. Rabinowitz)

For any $p \in\,]2, 2^*[$, $(\mathcal{P}_p)$ possesses

1. *a ground state solution to $(\mathcal{P}_p)$;*
2. *it is a one-signed function;*
3. *its Morse index is 1.*

(I'll shed some light on $p = 2^*$ and $\Omega = B_R$ with numerical experiments.)

Outline

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$p \approx 2$: symmetry of ground state solutions

Theorem (D. Bonheure, V. Bouchez, C. Grumiau, C. T., J. Van Schaftingen, '08)

For p close to 2 and any $R \in O(N)$ s.t. $R(\Omega) = \Omega$, ground state solutions to $(\mathcal{P}_p)$ are symmetric w.r.t. R.

E.g. if Ω is radially symmetric, so must the the ground state solution be.

Remark that the seminal method of moving planes is not easily applicable.

Outline

Uniqueness of the positive solution

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Let $v := P_{E_1} u$ (constant function) and $w := P_{E_1^\perp} u$ (zero mean).

$$\int_\Omega |\nabla w|^2 + w^2 = \int_\Omega |u|^{p-1} w$$

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$$
\begin{aligned}
(1+\lambda_2)\int_\Omega w^2 &\leqslant \int_\Omega |\nabla w|^2 + w^2 = \int_\Omega |u|^{p-1} w = \int_\Omega \Bigl((v+w)^{p-1} - v^{p-1}\Bigr) w \\
&= \int_\Omega (p-1)(v+\vartheta_p w)^{p-2} w^2 \qquad (\vartheta_p \in \,]0,1[) \\
&\leqslant (p-1)(|v| + \|w\|_\infty)^{p-2} \int_\Omega w^2 \leqslant (p-1) K^{p-2} \int_\Omega w^2.
\end{aligned}
$$

As $\lambda_1 = 0 < \lambda_2$, for $p \approx 2$, $w = 0$ and then $u = v = 1$.

A priori bounds for positive solutions (1/3)

Lemma

Positive solutions (u_p) are bounded in L^∞ as $p \approx 2$.

- Integrating the equation & Hölder: $\int_\Omega u_p^{p-1} = \int_\Omega u_p \leqslant |\Omega|$ (recall that $u_p > 0$).
- Brezis-Strauss: from the bound on $\int_\Omega u_p^{p-1}$, we deduce a bound on $\|u_p\|_{W^{1,q}(\Omega)}$, $1 \leqslant q < N/(N-1)$.
- Sobolev embedding: (u_p) bounded in $L^r(\Omega)$, $1 < r < N/(N-2)$.
- Bootstrap: $\|u_p\|_{W^{2,r}(\Omega)}$ is bounded for some $r > N/2$ when $p \approx 2$.

A priori bounds for positive solutions (2/3)

Proposition

Let $2 < \bar{p} < 2^*$. There exists $C_{\bar{p}} > 0$ such that any positive solution to $(\mathcal{P}_p)$ with $2 < p \leqslant \bar{p}$ satisfies $\max\big\{\|u\|_{H^1}, \|u\|_{L^\infty}\big\} \leqslant C_{\bar{p}}$.

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It remains to obtain a bound for $2 < \underline{p} < \bar{p} < 2^*$ in L^∞. Blow up argument (Gidas-Spruck). Suppose on the contrary that there is a sequence $(p_n) \subseteq [\underline{p}, \bar{p}]$ and (u_{p_n}) s.t.

$$u_{p_n}(x_{p_n}) := \|u_{p_n}\|_{L^\infty} \to +\infty \qquad \text{and} \qquad p_n \to p^* \in [\underline{p}, \bar{p}].$$

(Drop index n.) Define

$$v_p(y) := \mu_p u_p\Big(\mu_p^{(p-2)/2} y + x_p\Big) \qquad \text{where } \mu_p := 1/\|u_p\|_{L^\infty} \to 0.$$

Note: $v_p(0) = \|v_p\|_{L^\infty} = 1$.

A priori bounds for positive solutions (3/3)

The rescaled function v_p satisfies

$$-\Delta v_p + \mu_p^{p-2} v_p = v_p^{p-1} \quad \text{on } \Omega_p := (\Omega - x_p)/\mu_p^{(p-2)/2}$$

with NBC. By elliptic regularity, (v_p) is bounded in $W^{2,r}$ and $C^{1,\alpha}$, $0 < \alpha < 1$ on any compact set. Thus, taking if necessary a subsequence,

$$v_n \to v^* \quad \text{in } W^{2,r} \text{ and } C^{1,\alpha} \text{ on compact sets of } \Omega^* = \mathbb{R}^N \text{ or } \mathbb{R}^{N-1} \times \mathbb{R}_{>a}.$$

One has $v^* \geqslant 0$, $v^*(0) = 1 = \|v\|_{L^\infty}$ and v^* satisfies

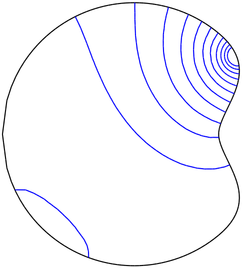
$$-\Delta v^* = (v^*)^{p^*-1} \quad \text{in } \mathbb{R}^N \qquad \text{or} \qquad \begin{cases} -\Delta v^* = (v^*)^{p^*-1} & \text{in } \mathbb{R}^{N-1} \times \mathbb{R}_{>a} \\ \partial_N v^* = 0 & \text{when } x_N = a \end{cases}$$

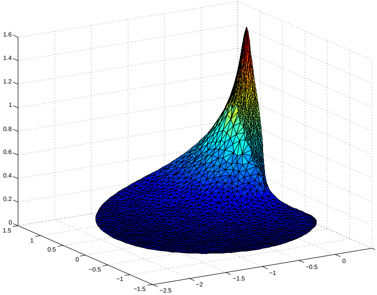
Liouville theorems imply $v^* = 0$. □

Symmetry breaking of the ground state

Theorem (W.-M. Ni, I. Takagi, '93; Adimurthi, F. Pacella, S.L. Yadava '93)

When R is sufficiently large, ground state solutions possess a unique maximum point $P_R \in \partial(R\Omega)$. Moreover, $u_R \to 0$ outside a small neighborhood of P_R. P_R is situated at the "most curved" part of $\partial(R\Omega)$.





p large: symmetry breaking of the ground state

Corollary

1 *cannot remain the ground state on "large" domains.*

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Proposition

1 *cannot be the ground state solution when* $p - 2 > \lambda_2(B_R) = \lambda_2(B_1)/R^2$.

Proof. The Morse index of 1 is the sum of the dimension of the eigenspaces corresponding to negative eigenvalues λ of

$$\begin{cases} -\Delta v + v = (p-1)v + \lambda v, & \text{in } \Omega, \\ \partial_\nu v = 0, & \text{on } \partial\Omega. \end{cases}$$

Clearly $p - 2 + \lambda = \lambda_i(B_R)$. When $p - 2 > \lambda_2$, the Morse index of the solution 1 is > 1.

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Proposition (Lopez, '96)

On radial domains, the ground state is either constant or (e.g. when $p > 2 + \lambda_2$) not radially symmetric.

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Proposition

When Ω is a ball or an annulus, the Morse index of a non-constant positive radial solution is at least $N+1$. In particular, when $p > 2+\lambda_2$, ground state solutions cannot be radial.

Based on: A. Aftalion, F. Pacella, Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains, CRAS, 339(5), '04.

Let u be non-constant positive radial solution of $(\mathcal{P}_p)$. We have to show that

$$L v := -\Delta v + v - (p-1)|u|^{p-2} v$$

with NBC possesses $N+1$ negative eigenvalues.

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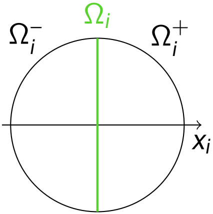
u radial $\Rightarrow \partial_{x_i} u = 0$ on $\partial\Omega$ and on Ω_i.



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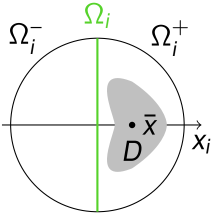
Let $\bar{x} \in \Omega_i^+$ s.t. $\partial_{x_i} u(\bar{x}) \neq 0$. Let D be the connected component of $\{\partial_{x_i} u(\bar{x}) \neq 0\}$ containing $\bar{x}$. $D \subseteq \Omega_i^+$.



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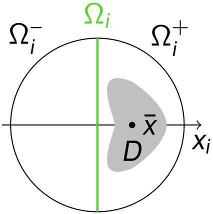

$$L(\partial_{x_i} u) = 0, \quad \text{on } D; \qquad \partial_{x_i} u = 0, \quad \text{on } \partial D.$$

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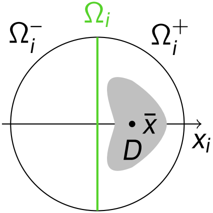

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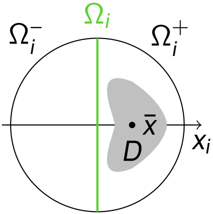

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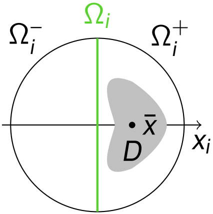

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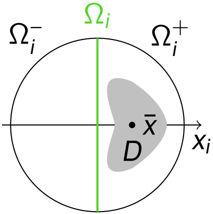

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All ψ_j^*, $j \neq i$ vanish on the axis $x_i \Rightarrow$ the family $(\psi_j^*)_{j=1}^N$ is lin. indep.

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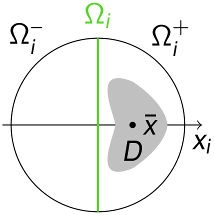

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All ψ_j^*, $j \neq i$ vanish on the axis $x_i \Rightarrow$ the family $(\psi_j^*)_{j=1}^N$ is lin. indep.
None of the $(\psi_j^*)_{j=1}^N$ is a first eigenfunction.

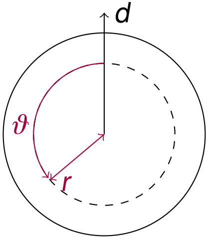
p large: symmetry breaking of the ground state

Theorem (Lopes, '96)

On radial domains, ground state solutions are symmetric w.r.t. any hyperplane containing a line L passing through the origin.

Theorem (J. Van Schaftingen, '04)

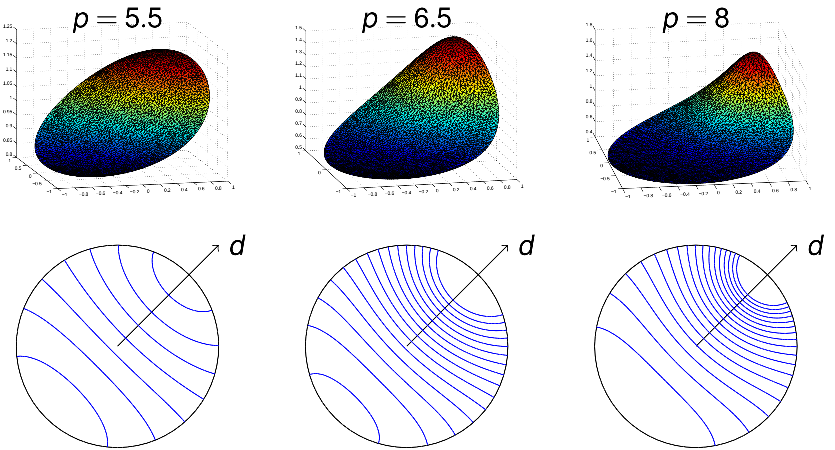
On radial domains, ground state solutions are foliated Schwarz symmetric.



There exists a unit vector d s.t. u depends only on $r = |x|$ and $\vartheta = \arccos(\frac{x}{|x|} \cdot d)$ and is non-increasing in ϑ.

p large: non-radially symmetric ground state

$$\Omega = B_1 \subseteq \mathbb{R}^2 \ \Rightarrow\ 2 + \lambda_2 \approx 5.39$$



Ground states — summary

- When $p \approx 2$, 1 is the sole positive solution (hence the GS are ± 1).
- When $p > 2 + \lambda_2$,
  - 1 is not the GS anymore;
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Theorem (Lin, Ni, Takagi '88)

Let $\Omega_1 \subseteq \mathbb{R}^N$ be a bounded smooth domain and $p \in \,]2, 2^*[$. There exists $0 < R_0 \leqslant R_1$ such that the equation $-\Delta u + u = |u|^{p-2} u$ with NBC on $\Omega = R\Omega_1$ possesses

1. *only constant positive solutions for $R < R_0$;*
2. *a non-constant positive solution for $R > R_1$.*

We showed that one can quantify $R > R_1$ as $p - 2 > \lambda_2(B_R) = \lambda_2(B_1)/R^2$.

Outline

Symmetry breaking at exactly $p = 2+\lambda_2$?

Conjecture

± 1 are the ground states of $-\Delta u + u = |u|^{p-2}u$ with NBC for all $p \leqslant 2+\lambda_2$.

- *If $2+\lambda_2 \geqslant 2^*$, no concentration therefore occurs when $p \to 2^*$.*
- *If $2+\lambda_2 < 2^*$, the GS solutions for $p \in\,]2+\lambda_2, 2^*[$ lie on the branch emanating from $(p,u) = (2+\lambda_2, 1)$.*

Evidence for this conjecture: examine the bifurcation at $p = 2+\lambda_2$ on a ball.

Symmetry breaking at exactly $p = 2 + \lambda_2$?

Recall: the linearisation of the equation around $u = 1$,

$$Lv := -\Delta v + v - (p-1)v$$

is not invertible iff $p = 2 + \lambda_i$, $i \geqslant 2$.

A basis of E_2 is

$$x \mapsto r^{-\frac{N-2}{2}} J_{N/2}(\sqrt{\lambda_i} r) \frac{x_j}{|x|}, \qquad j = 1, \dots, N.$$

There is single function (up to a multiple) that is invariant under rotation in $(x_2, \dots, x_N)$.

Symmetry breaking at exactly $p = 2 + \lambda_2$?

Theorem (Crandall-Rabinowitz '71)

Let X and Y two Banach spaces, $u^* \in X$, and a function $F : \mathbb{R} \times X \to Y : (p,u) \mapsto F(p,u)$ such that $\forall p \in \mathbb{R},\ F(p,u^*) = 0$. Let $p^* \in \mathbb{R}$ be such that $\ker(\partial_u F(p^*,u^*)) = \operatorname{span}\{\varphi^*\}$ has a dimension 1 and $\operatorname{codim}\big(\operatorname{Im}(\partial_u F(p^*,u^*))\big) = 1$. Let $\psi : Y \to \mathbb{R}$ be a continuous linear map such that $\operatorname{Im}(\partial_u F(p^*,u^*)) = \{y \in Y : \langle \psi, y\rangle = 0\}$.

In our case

- $F(p,u) = -\Delta u + u - |u|^{p-2}u$,
- $p^* = 2 + \lambda_2$, $u^* = 1$,
- $\varphi^* = \varphi_2$,
- $\langle \psi, f\rangle = \int_\Omega f\varphi_2$.

Symmetry breaking at exactly $p = 2 + \lambda_2$?

Theorem (Crandall-Rabinowitz (cont'd))

If $a := \langle \psi, \partial_{pu} F(p^*, u^*)[\varphi^*] \rangle \neq 0$, then (p^*, u^*) is a bifurcation point for F. In addition, the set of non-trivial solutions of $F = 0$ around (p^*, u^*) is given by a unique $\mathcal{C}^1$ curve $t \mapsto (p(t), u(t))$. The local behavior of the branch for p close to p^* is as follows.

$$a = -\int_\Omega \varphi_2^2 = -1$$

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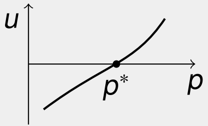
- *If* $b := -\frac{1}{2a}\Big\langle \psi, \partial_u^2 F(p^*, u^*)[\varphi^*, \varphi^*] \Big\rangle \neq 0$ *then the branch is* *transcritical* *and*

$$u_p = u^* + \frac{p - p^*}{b}\varphi^* + o(p - p^*).$$



$$a = -\int_\Omega \varphi_2^2 = -1 \qquad \text{and} \qquad b = -\tfrac{1}{2}\lambda_2(\lambda_2 - 1)\int_\Omega \varphi_2^3 = 0.$$

Symmetry breaking at exactly $p = 1 + \lambda_2$?

Theorem (Crandall-Rabinowitz — extended)

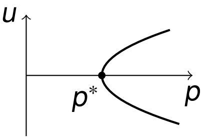
- *If $b = 0$, let us define*

$$c := -\frac{1}{6a}\Big(\big\langle \psi, \partial_u^3 F(p^*, u^*)[\varphi^*, \varphi^*, \varphi^*]\big\rangle + 3\big\langle \psi, \partial_u^2 F(p^*, u^*)[\varphi^*, w]\big\rangle\Big)$$

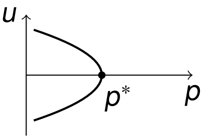
where $w \in X$ is any solution of the equation $\partial_u F(p^*, u^*)[w] = -\partial_u^2 F(p^*, u^*)[\varphi^*, \varphi^*]$. If $c \neq 0$ then

$$u_p = u^* \pm \Big(\frac{p - p^*}{c}\Big)^{1/2} \varphi^* + o\big(|p - p^*|^{1/2}\big).$$

In particular, the branch is supercritical if $c > 0$ and subcritical if $c < 0$.



Supercritical



Subcritical

Symmetry breaking at exactly $p = 2 + \lambda_2$?

In our case,

$$c = \frac{1}{6}\lambda_2(\lambda_2 - 1)\Big(-(\lambda_2 - 2)\int_{B_R}\varphi_2^4 - 3\lambda_2(\lambda_2 - 1)\int_{B_R}\varphi_2^2 w\Big)$$

where $(-\Delta + 1 - \lambda_2)w = \varphi_2^2$ with NBC on B_R.

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where $(-\Delta + 1 - \lambda_2)w = \varphi_2^2$ with NBC on B_R.

$$= \frac{1}{6}\bar\mu_2 R^{-(N+2)}\Big(2 + \frac{\bar\mu_2}{R^2}\Big)\Big((\beta - \alpha)\frac{\bar\mu_2}{R^2} + \beta + \alpha\Big)$$

where $\alpha := \int_{B_1} \bar\varphi_2^4, \quad \beta := -3\bar\mu_2 \int_{B_1} \bar\varphi_2^2 \bar w,$

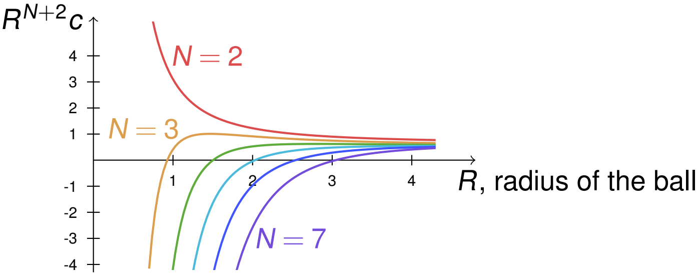
$(-\Delta - \bar\mu_2)\bar w = \bar\varphi_2^2$ with NBC on B_1,

$\bar\varphi_2$ and $\bar\mu_2 > 0$ are "the" second eigenfunction and eigenvalue of $-\Delta$ with NBC on B_1 s.t. $|\bar\varphi_2|_{L^2} = 1$.

Symmetry breaking at exactly $p = 2 + \lambda_2$?

We numerically have

N	α	β	$\beta - \alpha$	$\beta + \alpha$
2	0.5577	0.5884	0.0306	1.1461
3	0.4632	0.3096	−0.1536	0.7728
4	0.4222	0.1694	−0.2528	0.5916
5	0.4171	0.0858	−0.3313	0.5029
6	0.4421	0.0250	−0.4171	0.4671



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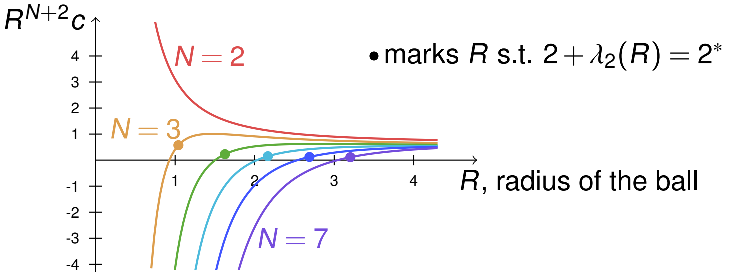

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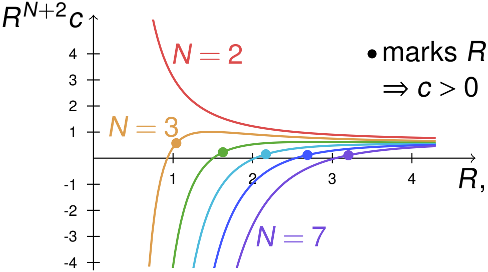


• marks R s.t. $2+\lambda_2(R) = 2^*$
$\Rightarrow c > 0$ in the subcritical range

R, radius of the ball

Symmetry breaking at exactly $p = 2 + \lambda_2$?

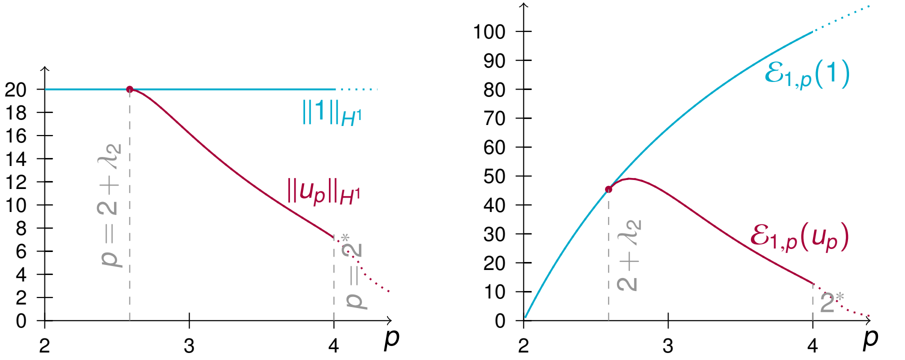
Theorem (Crandall-Rabinowitz — extended)

Assume $F(p,u) = \partial_u \mathcal{E}(p,u)$. If (p, u_p) is the branch of nontrivial solutions emanating from (p^*, u^*), $b = 0$ and $c \neq 0$,

$$\mathcal{E}(p,u_p) - \mathcal{E}(p,u^*) = \frac{a}{6c}(p-p^*)^2 + o\big((p-p^*)^2\big) \qquad \textit{when } \frac{p-p^*}{c} > 0.$$

In our case, $a = -1 < 0$ and $c > 0$. Consequence: the energy along the super-critical branch emanating from $(2+\lambda_2, 1)$ has lower energy than the trivial solution 1.

Symmetry breaking at exactly $p = 2 + \lambda_2$?



Norm and energy of the ground state for $N = 4$, $R = 3$.

Outline

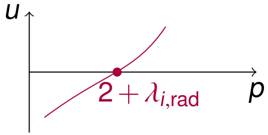
Transcritical *radial* bifurcations

Proposition

On balls, two branches radial solutions in $\mathcal{C}^{2,\alpha}(\Omega)$ of

$$(\mathcal{P}_p)\begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \Omega \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

start from each $(p,u) = (2+\lambda_{i,rad}, 1)$, $i > 1$. Locally, these branches form a unique $\mathcal{C}^1$-curve. Moreover, for all $i \geqslant 2$, the bifurcation is transcritical.



Spectrum of $-\Delta$ with NBC

Eigenfunctions of $-\Delta$ with NBC have the form:

$$\varphi(x) = r^{-\frac{N-2}{2}} J_\nu(\sqrt{\lambda} r)\, P_k\left(\frac{x}{|x|}\right), \qquad \text{where } \nu = k + \frac{N-2}{2},$$

$r = |x|$, and $P_k : \mathbb{R}^N \to \mathbb{R}$ is an harmonic homogenous polynomial of degree k for some $k \in \mathbb{N}$. To satisfy the boundary conditions:

$$\sqrt{\lambda} R \text{ is a root of } z \mapsto (k-\nu) J_\nu(z) + z \partial J_\nu(z) = k J_\nu(z) - z J_{\nu+1}(z).$$

where $\lambda \geqslant 0$ is the corresponding eigenvalue.

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where $\lambda \geqslant 0$ is the corresponding eigenvalue.

Radial eigenfunctions correspond to $k = 0$ (thus $P_k = 1$).

Let us denote $\lambda_{i,\mathrm{rad}}$ the eigenvalues that possess a radial eigenfunction (simple in H^1_{rad} and $C^{2,\alpha}_{\mathrm{rad}}(\Omega)$).

Transcritical radial bifurcations

Proof. $\Omega = B_R$. Using Crandall-Rabinowitz' theorem, one has to show

$$b = -\tfrac{1}{2}(1 + \lambda_{i,\mathrm{rad}})\lambda_{i,\mathrm{rad}} \int_{B_R} \varphi_{i,\mathrm{rad}}^3 \neq 0.$$

Given that radial eigenfunctions are given by constant spherical harmonics ($k = 0$, $\nu = (N-2)/2$), this amounts to

$$\int_0^R \Bigl(r^{-\frac{N-2}{2}} J_\nu\bigl(r\sqrt{\bar\lambda_{i,\mathrm{rad}}}/R\bigr)\Bigr)^3 r^{N-1}\,\mathrm{d}r \neq 0 \quad \text{i.e.} \quad \int_0^{\sqrt{\bar\lambda_{i,\mathrm{rad}}}} t^{1-\nu} J_\nu^3(t)\,\mathrm{d}t \neq 0$$

where $\lambda_{i,\mathrm{rad}} = \bar\lambda_{i,\mathrm{rad}}/R^2$. This is true for large i because

$$\int_0^\infty t^{1-\nu} J_\nu^3(t)\,\mathrm{d}t = \frac{2^{\nu-1}(3/16)^{\nu-1/2}}{\pi^{1/2}\Gamma(\nu+1/2)} > 0.$$

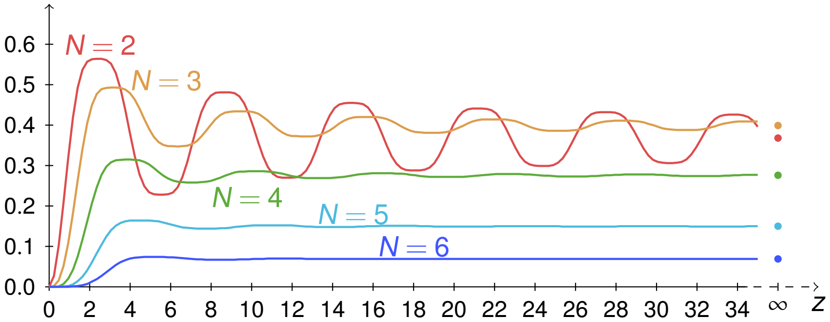
For any i, the proof is harder. Thus $b < 0$.

Transcritical radial bifurcations

Here are the graphs of the functions

$$]0,+\infty[\to \mathbb{R} : z \mapsto \int_0^z t^{1-\nu} J_\nu^3(t)\,\mathrm{d}t, \qquad \nu = (N-2)/2,$$

indicating that radial bifurcations are transcritical for all i.

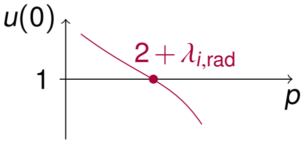


Shape of transcritical radial bifurcations

$$u_p = 1 + \frac{p - (2 + \lambda_{i,\text{rad}})}{b} \varphi_{i,\text{rad}} + o\big(p - (2 + \lambda_{i,\text{rad}})\big)$$

where $\varphi_{i,\text{rad}}(x) = |x|^{-\nu} J_\nu(\sqrt{\lambda_{i,\text{rad}}}\,|x|)$. Thus

- $u_p(0) > 1$ if $p < 2 + \lambda_{i,\text{rad}}$
- $u_p(0) < 1$ if $p > 2 + \lambda_{i,\text{rad}}$



These facts remain true along the whole banches.

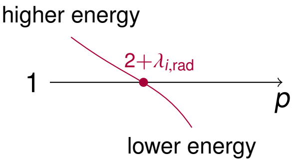
Energy of transcritical radial bifurcations

Theorem (Crandall-Rabinowitz — extended)

Assume $F(p,u) = \partial_u \mathcal{E}(p,u)$. If (p,u_p) is the branch of nontrivial solutions emanating from (p^*,u^*) and $b \neq 0$,

$$\mathcal{E}(p,u_p) - \mathcal{E}(p,u^*) = \frac{a}{6b^2}(p-p^*)^3 + o\big((p-p^*)^3\big).$$

In our case $a = -1$. Consequence: the energy along the right (resp. left) branch is lower (resp. higher) than the one of the trivial solution.



Positive transcritical radial bifurcations

Corollary

The branches consist of positive functions.

SKETCH: If it was not the case, there would be a point solution along the branch with a double root, hence $= 0$. There is no bifurcation from 0. □

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Theorem

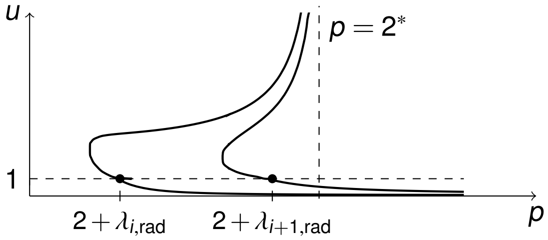
Radial bifurcations obtained for the $C^{2,\alpha}(\Omega)$-norm are unbounded and do not intersect each other. Moreover, along bifurcations starting from $(2 + \lambda_{i,rad}, 1)$, the solutions always possess the same number of intersections with 1.

SKETCH: The number of crossings with 1 stays constant because otherwise a non-constant radial solution u s.t. $u - 1$ has a double root would exists. Since the branches do not intersect each other, Rabinowitz's principle says they must be undounded.

Multiplicity results (radial domains)

Theorem (p subcritical)

Assume $\Omega = B_R \subseteq \mathbb{R}^N$ with $N \geqslant 3$. For any $p > 2 + \lambda_{n+1,rad}$, $(\mathcal{P}_p)$ has $2n$ distinct non-constant positive radial solutions, among which there is an increasing one.



Degeneracy results (radial domains)

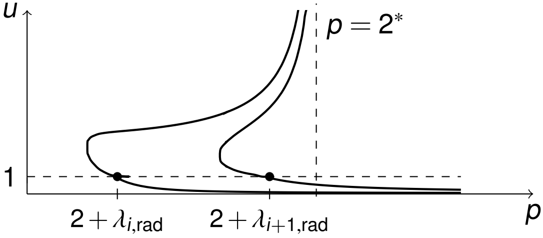
Theorem

On balls, there exists a degenerate positive radial solution for some p provided that the measure of Ω is large enough.

Multiplicity results (radial domains, supercritical)

Theorem ($p \geqslant 2^*$)

Assume $\Omega = B_R \subseteq \mathbb{R}^N$ with $N \geqslant 3$. For any $p > 2 + \lambda_{n+1,rad}$, $(\mathcal{P}_p)$ has n distinct non-constant positive radial solutions, among which there are an increasing and a decreasing one. These solutions are bounded in L^∞.



$p \geqslant 2^*$

Theorem (Adimurthi, Yadava '91)

Let $p = 2^*$ and $\Omega = B_R$. One consider the problem

$$(\mathcal{P}_p) \begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \Omega \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

1. *If $N \geqslant 3$ and $2 + \lambda_{2,rad}(R) < p$, then $(\mathcal{P}_p)$ admits a positive solution which is radially increasing.*
2. *If $N \in \{4, 5, 6\}$ and $p < 2 + \lambda_{2,rad}(R)$, then $(\mathcal{P}_p)$ admits a positive solution which is radially decreasing.*
3. *If $N = 3$, there exists an $R^* > 0$ such that for $R \in]0, R^*[$, $(\mathcal{P}_p)$ only admits constant positive solutions.*

$p \geqslant 2^*$

Theorem (X-J. Wang, '91)

When $p = 2^*$ and $\Omega = R\Omega_1$ with R large enough, $(\mathcal{P}_p)$ possesses at least one non-constant positive solution.

$p \geqslant 2^*$

Theorem (X-J. Wang, '91)

When $p = 2^*$ *and* $\Omega = R\Omega_1$ *with* R *large enough,* $(\mathcal{P}_p)$ *possesses at least one non-constant positive solution.*

Theorem (E. Serra & P. Tilli, '11)

Assume $a \in L^1(]0,R[)$ *is increasing, not constant and satisfies* $a > 0$ *in* $]0,R[$, *then for any* $p \in]2,+\infty[$, $-\Delta u + u = a(|x|)|u|^{p-2}u$ *with NBC possesses a positive radially increasing solution.*

Trick: work on the space of radially increasing functions.

$p \geqslant 2^*$

Proposition

Assume Ω is a ball of radius R. If u is a radial solution of $(\mathcal{P}_p)$ such that $u(0) < 1$, then $\|u\|_{L^\infty} \leqslant \exp(1/2)$ and $\|\partial_r u\|_{L^\infty} \leqslant 1$.

$p \geqslant 2^*$

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Proof. In radial coordinates, the equation writes

$$-u'' - \frac{N-1}{r} u' + u = u^{p-1}.$$

Multiplying by u', we get

$$\frac{\mathrm{d}}{\mathrm{d}r} h(r) = -\frac{N-1}{r} u'^2(r) \leqslant 0,$$

where

$$h(r) := \frac{u'^2(r)}{2} + \frac{u^p(r)}{p} - \frac{u^2(r)}{2}.$$

In particular, this means that $h(r) \leqslant h(0)$ for any r.

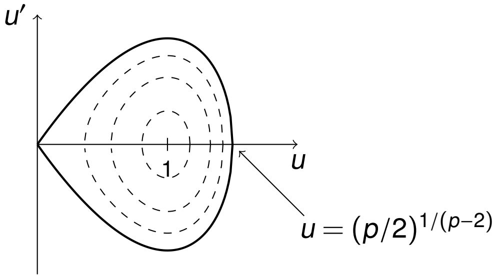
$p \geqslant 2^*$

Proof (cont'd). The assumption $u(0) < 1$ implies

$$h(0) = \frac{u^p(0)}{p} - \frac{u^2(0)}{2} = u^2(0)\Big(\frac{u^{p-2}(0)}{p} - \frac{1}{2}\Big) \leqslant 0.$$

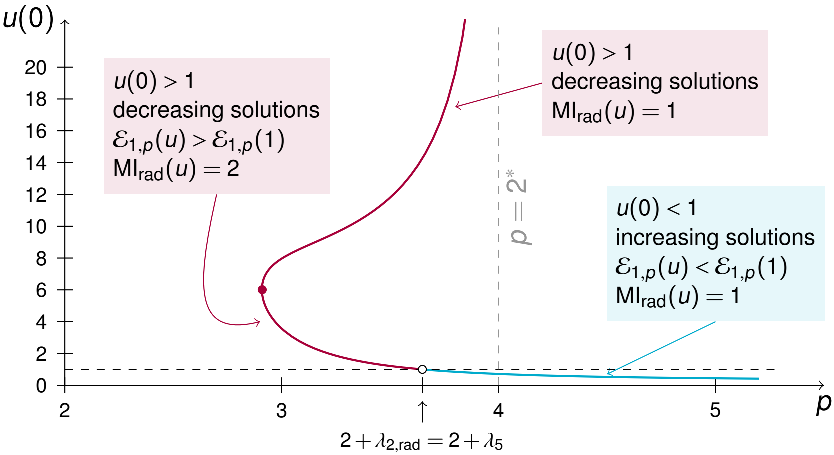
Thus

$$\|u\|_{L^\infty} \leqslant \Big(\frac{p}{2}\Big)^{1/(p-2)} \leqslant \exp(1/2). \qquad \square$$



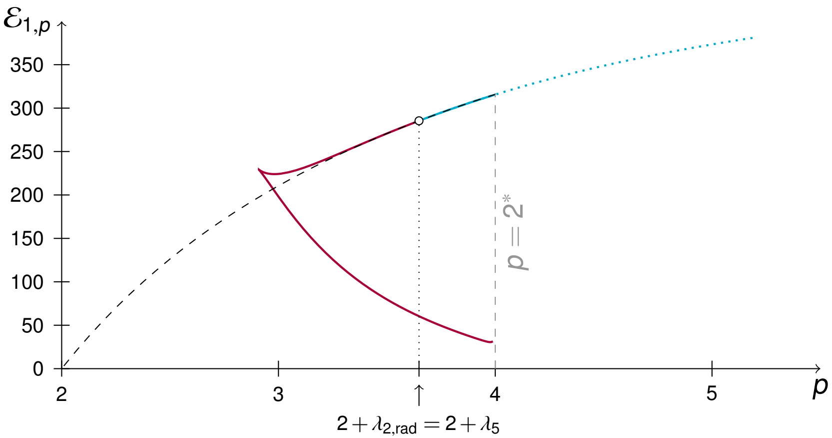
Outline

Radial branch from $2 + \lambda_{2,\mathrm{rad}}$



$N = 4$, $R = 4$.

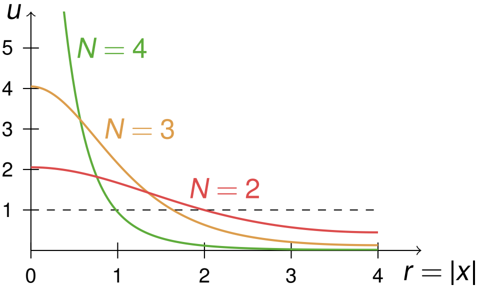
Radial branch from $2 + \lambda_{2,\text{rad}}$



Energy along the first radial branch ($N = 4$, $R = 4$).

Radial ground state for $p = 1.95 + \lambda_{2,\mathrm{rad}} < 2^*$ on B_4

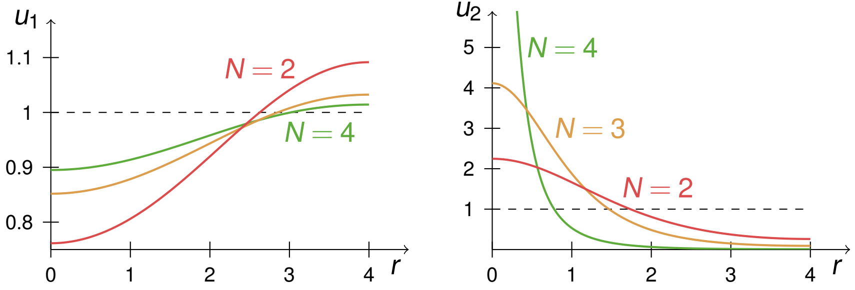
Using the Mountain Pass Algorithm in the space of radial functions:



N	2^*	$2 + \lambda_{2,\mathrm{rad}}$	$\mathcal{E}(1)$	$\min u$	$\max u$	$\mathcal{E}(u)$
2	∞	2.92	7.60	0.447	2.05	7.45
3	6	3.26	50.58	0.130	4.05	34.85
4	4	3.65	280.58	0.016	13.31	66.39

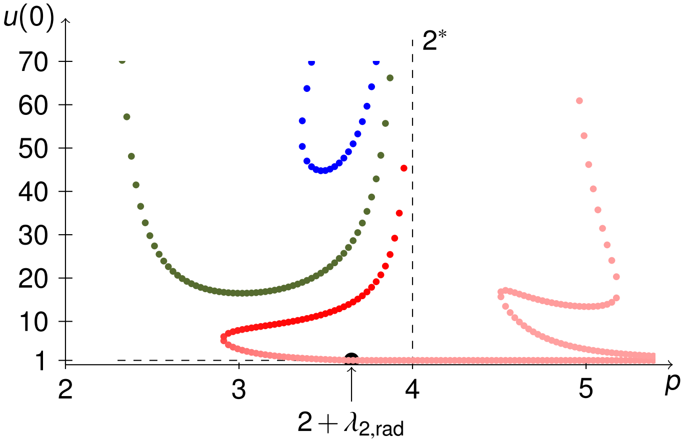
Radial ground state for $p = 2.1 + \lambda_{2,\mathrm{rad}} < 2^*$ on B_4

Using the Mountain Pass Algorithm in the space of radial functions with initial functions $x \mapsto 1 \pm 0.2|x|$.



N	$2+\lambda_{2,\mathrm{rad}}$	$\mathcal{E}(1)$	$\min u_1$	$\max u_1$	$\mathcal{E}(u_1)$	$\min u_2$	$\max u_2$	$\mathcal{E}(u_2)$
2	2.92	8.48	0.76	1.09	8.47	0.261	2.25	7.39
3	3.26	54.30	0.85	1.03	54.29	0.092	4.12	30.74
4	3.65	294.63	0.90	1.01	294.62	0.008	17.25	49.61

Bifurcation diagram $N = 4$, $R = 4$



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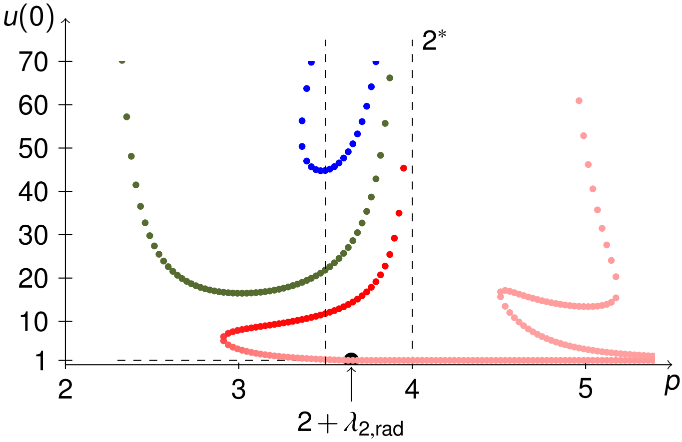
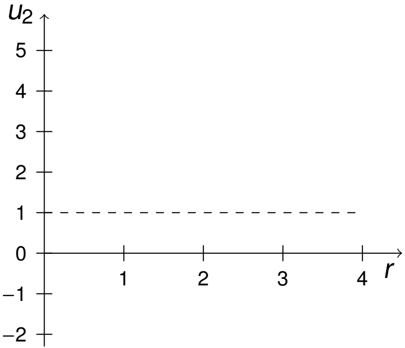


Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

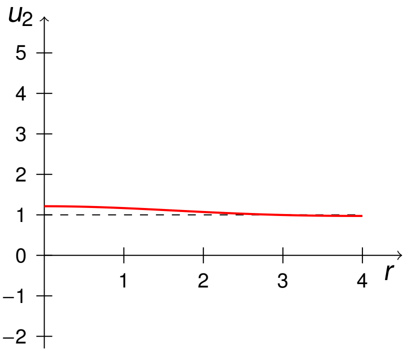
Shape of the solutions for $p = 3.5 < 2 + \lambda_{2,\text{rad}}$.



$\mathcal{E}(1) \approx 270.709$

Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

Shape of the solutions for $p = 3.5 < 2 + \lambda_{2,\mathrm{rad}}$.

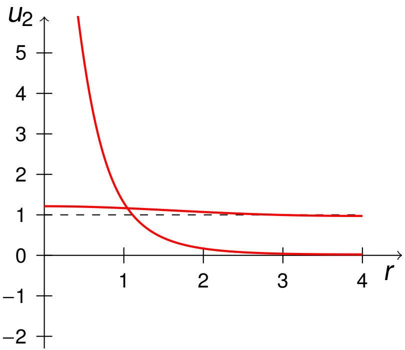


$$\mathcal{E}(1) \approx 270.709$$

$$u(0) \approx 1.213 \quad \Rightarrow \mathcal{E} \approx 270.753$$

Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

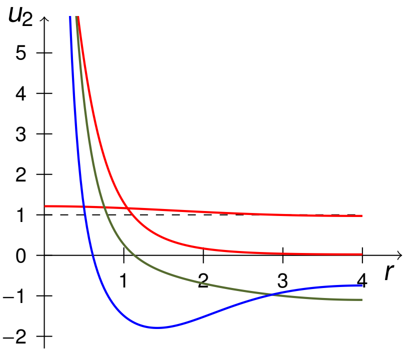
Shape of the solutions for $p = 3.5 < 2 + \lambda_{2,\text{rad}}$.



$$\mathcal{E}(1) \approx 270.709$$
$$u(0) \approx 1.213 \Rightarrow \mathcal{E} \approx 270.753$$
$$u(0) \approx 11.803 \Rightarrow \mathcal{E} \approx 79.730$$

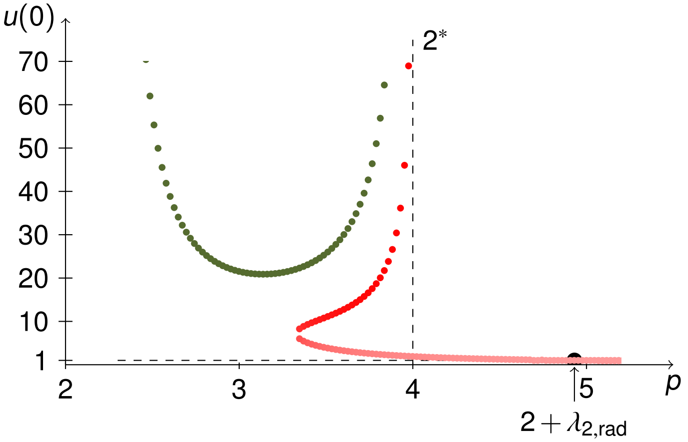
Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

Shape of the solutions for $p = 3.5 < 2 + \lambda_{2,\text{rad}}$.

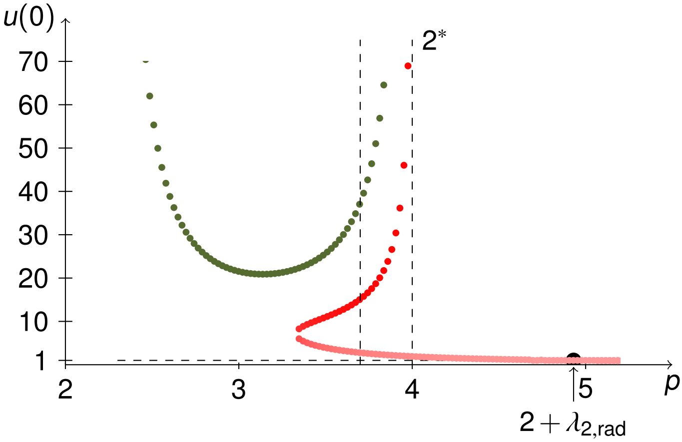


$$\mathcal{E}(1) \approx 270.709$$
$$u(0) \approx 1.213 \Rightarrow \mathcal{E} \approx 270.753$$
$$u(0) \approx 11.803 \Rightarrow \mathcal{E} \approx 79.730$$
$$u(0) \approx 21.887 \Rightarrow \mathcal{E} \approx 390.387$$
$$u(0) \approx 44.830 \Rightarrow \mathcal{E} \approx 436.267$$

Bifurcation diagram $N = 4$, $R = 3$

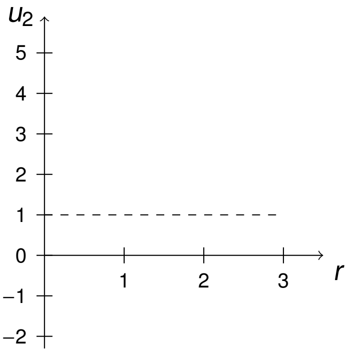


Bifurcation diagram $N = 4$, $R = 3$



Bifurcation diagram $N = 4$, $R = 3$ (cont'd)

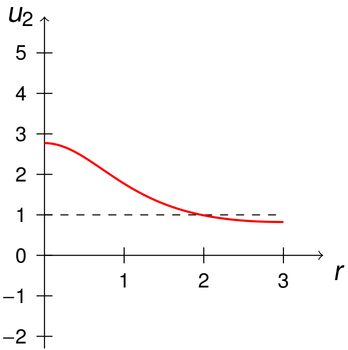
Shape of the solutions for $p = 3.7 < 2^* < 2 + \lambda_{2,\mathrm{rad}}$.



$\mathcal{E}(1) \approx 91.8273$

Bifurcation diagram $N = 4$, $R = 3$ (cont'd)

Shape of the solutions for $p = 3.7 < 2^* < 2 + \lambda_{2,\text{rad}}$.

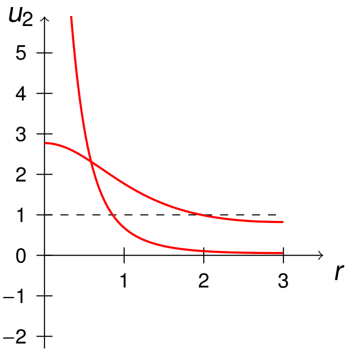


$$\mathcal{E}(1) \approx 91.8273$$

$$u(0) \approx 2.77189 \Rightarrow \mathcal{E} \approx 95.7796$$

Bifurcation diagram $N = 4$, $R = 3$ (cont'd)

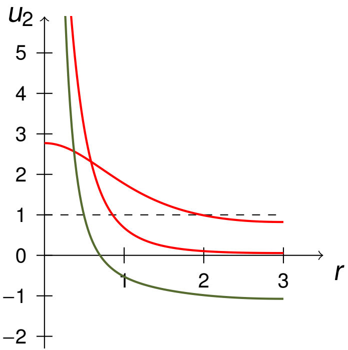
Shape of the solutions for $p = 3.7 < 2^* < 2 + \lambda_{2,\text{rad}}$.



$$\mathcal{E}(1) \approx 91.8273$$
$$u(0) \approx 2.77189 \Rightarrow \mathcal{E} \approx 95.7796$$
$$u(0) \approx 15.1307 \Rightarrow \mathcal{E} \approx 54.283$$

Bifurcation diagram $N = 4$, $R = 3$ (cont'd)

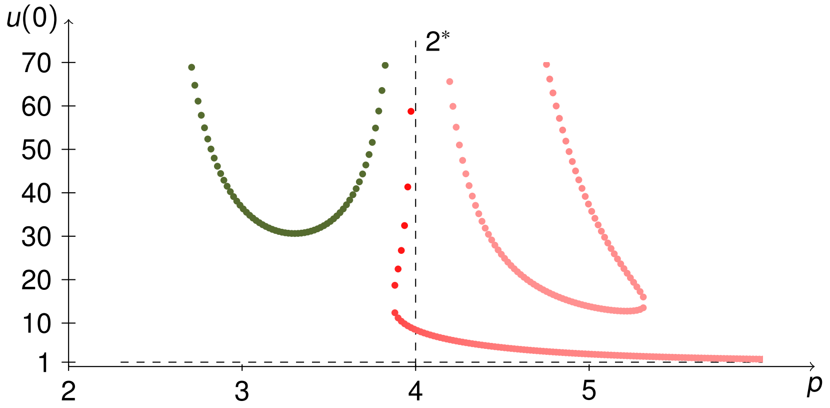
Shape of the solutions for $p = 3.7 < 2^* < 2 + \lambda_{2,\text{rad}}$.



$$\begin{aligned}
\mathcal{E}(1) &\approx 91.8273\\
u(0) \approx 2.77189 \Rightarrow \mathcal{E} &\approx 95.7796\\
u(0) \approx 15.1307 \Rightarrow \mathcal{E} &\approx 54.283\\
u(0) \approx 37.412 \quad \Rightarrow \mathcal{E} &\approx 168.972
\end{aligned}$$

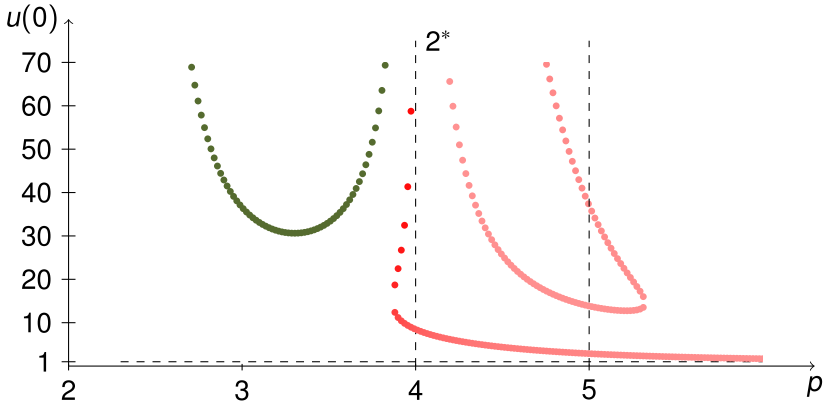
Bifurcation diagram $N = 4$, $R = 2$

$2 + \lambda_2 \approx 8.59365$



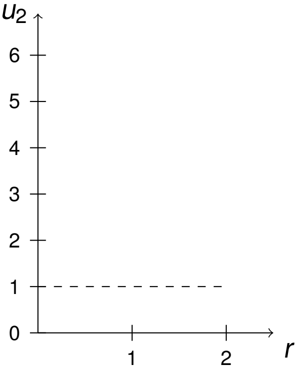
Bifurcation diagram $N = 4$, $R = 2$

$2 + \lambda_2 \approx 8.59365$



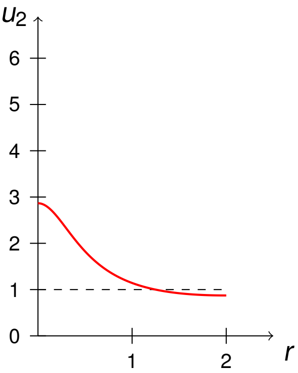
Bifurcation diagram $N = 4$, $R = 2$ (cont'd)

Shape of the solutions for $2^* < p = 5 < 2 + \lambda_{2,\text{rad}}$.



Bifurcation diagram $N = 4$, $R = 2$ (cont'd)

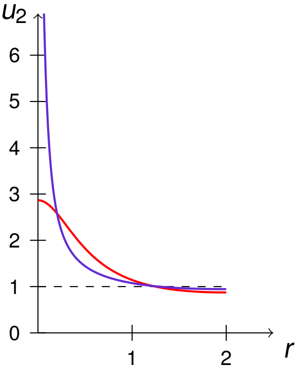
Shape of the solutions for $2^* < p = 5 < 2 + \lambda_{2,\mathrm{rad}}$.



$u(0) \approx 2.86611$

Bifurcation diagram $N = 4$, $R = 2$ (cont'd)

Shape of the solutions for $2^* < p = 5 < 2 + \lambda_{2,\mathrm{rad}}$.

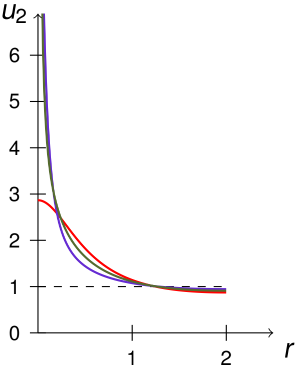


$u(0) \approx 2.86611$

$u(0) \approx 13.8393$

Bifurcation diagram $N = 4$, $R = 2$ (cont'd)

Shape of the solutions for $2^* < p = 5 < 2 + \lambda_{2,\text{rad}}$.



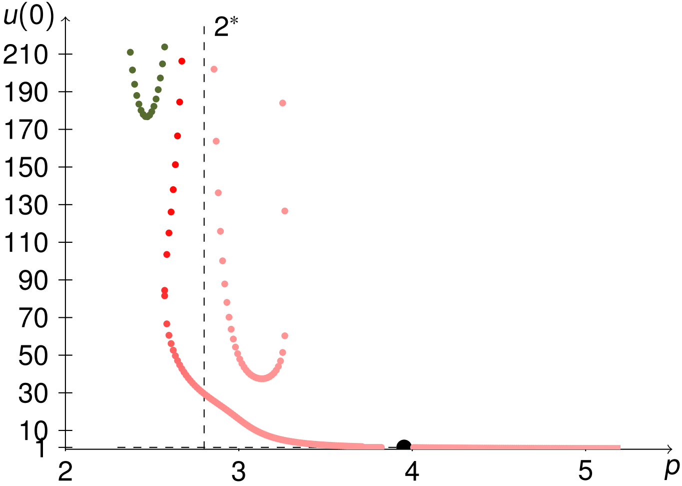
$u(0) \approx 2.86611$

$u(0) \approx 13.8393$

$u(0) \approx 37.0332$

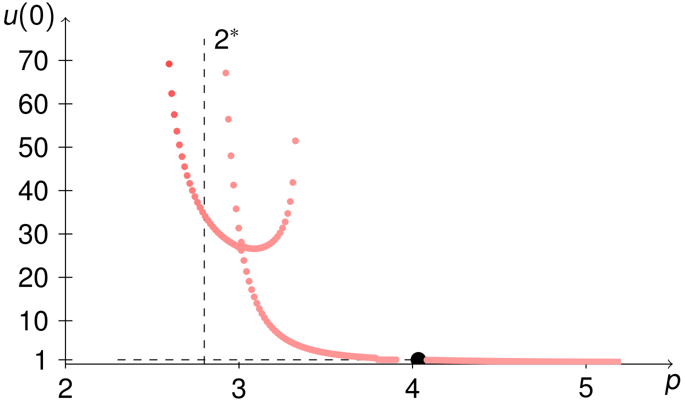
Bifurcation diagram $N = 7$, $R = 5$

$2 + \lambda_{2,\mathrm{rad}} \approx 3.95325$



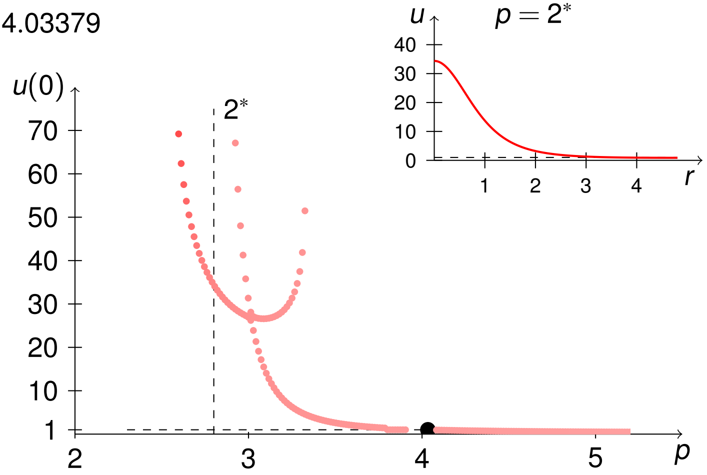
Bifurcation diagram $N = 7$, $R = 4.9$

$2 + \lambda_2 \approx 4.03379$



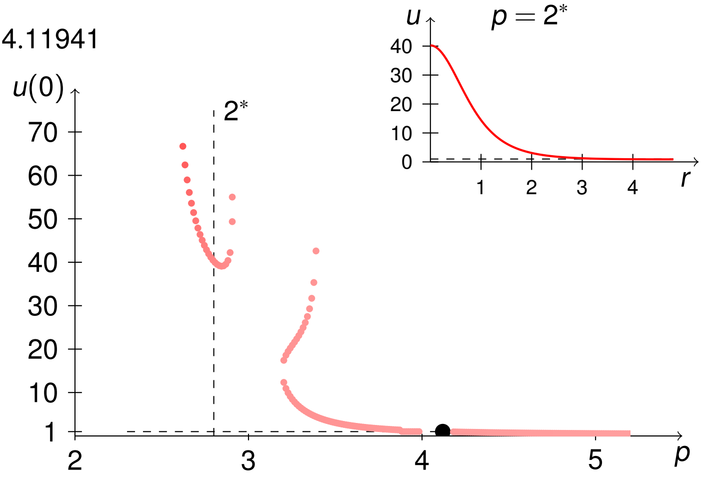
Bifurcation diagram $N = 7$, $R = 4.9$

$2 + \lambda_2 \approx 4.03379$



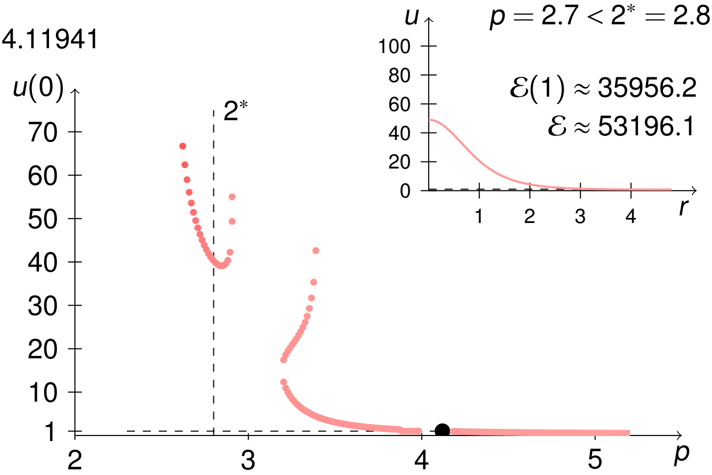
Bifurcation diagram $N = 7$, $R = 4.8$

$2 + \lambda_2 \approx 4.11941$



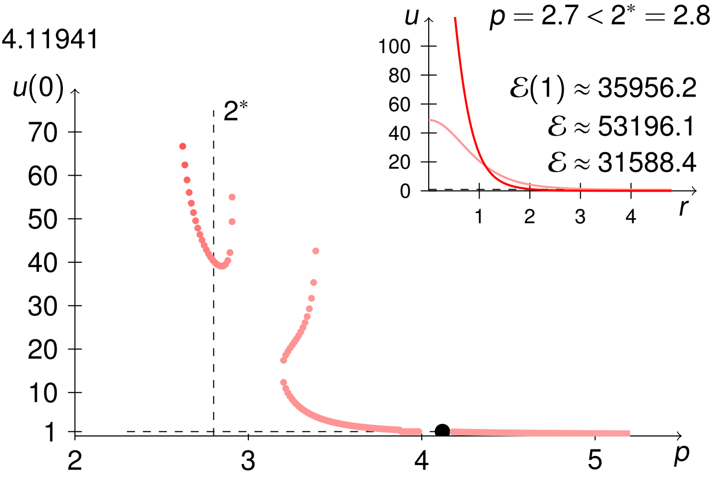
Bifurcation diagram $N = 7$, $R = 4.8$

$2 + \lambda_2 \approx 4.11941$



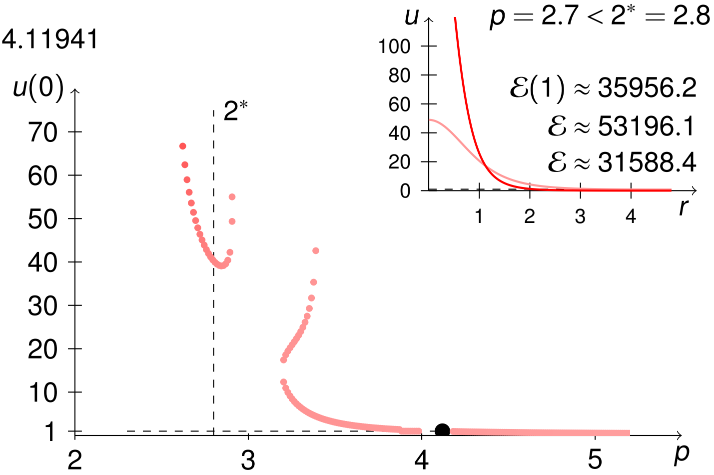
Bifurcation diagram $N = 7$, $R = 4.8$

$2 + \lambda_2 \approx 4.11941$



Bifurcation diagram $N = 7$, $R = 4.8$

$2 + \lambda_2 \approx 4.11941$



It is known [Adimurthi & S. L. Yadava '97] that if $N \geqslant 7$ and R is small enough, positive solutions for $p = 2^*$ must be constant.

Small diffusion

$$\begin{cases} -\varepsilon\Delta u + u = f(u), & \text{in } B_R, \\ u > 0, & \text{in } B_R, \\ \partial_\nu u = 0, & \text{on } \partial B_R \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

Small diffusion

$$\begin{cases} -\varepsilon\Delta u + u = f(u), & \text{in } B_R, \\ u > 0, & \text{in } B_R, \\ \partial_\nu u = 0, & \text{on } \partial B_R \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

Assumptions: f is of class $\mathcal{C}^1$ and satisfies, for some $u_0 > 0$,

$$f(0) = f'(0) = 0; \tag{F_0}$$

$$f(u_0) = u_0 \quad \text{and} \quad f'(u_0) > 1; \tag{F_1}$$

$$F(s) - \frac{s^2}{2} < \varliminf_{s\to+\infty}\left(F(s) - \frac{s^2}{2}\right) \text{ for } 0 \leqslant s \leqslant u_0, \tag{F_2}$$

where $F(s) := \int_0^s f(t)\,\mathrm{d}t$.

Small diffusion

Theorem

Assume $f \in \mathcal{C}^1$ satisfies (F_0), (F_1), (F_2), and $N \geqslant 2$. Then for any $n \in \mathbb{N}_0$ and any $\varepsilon > 0$ such that

$$\varepsilon < \varepsilon_{n+1} := \frac{f'(u_0) - 1}{\lambda_{n+1,rad}(B_R)},$$

Problem $(\mathcal{P}_\varepsilon)$ has at least n distinct non-constant radial solutions.

Small diffusion — a priori bounds

Proposition

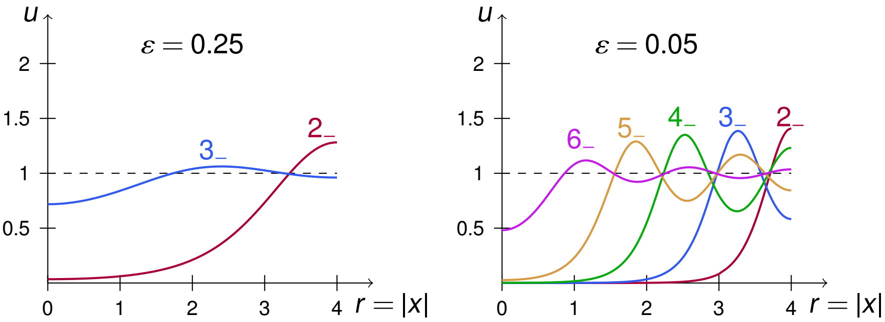
Assume f is of class $\mathcal{C}^k$, $k \geqslant 0$, and (F_2) holds. For any $q \geqslant 1$ and any $\varepsilon_0 > 0$, there exists $C > 0$ such that if u is a classical radial solution of Problem $(\mathcal{P}_\varepsilon)$ with $u(0) \leqslant u_0$ and $\varepsilon \leqslant \varepsilon_0$, then

$$\|u\|_{W^{k+2,q}} \leqslant \varepsilon^{-1} C.$$

Lemma

Assume f is continuous, (F_0), and (F_2) holds. Then there exists $\bar{\varepsilon} > 0$ such that if u is a non constant nonnegative classical radial solution of Problem $(\mathcal{P}_\varepsilon)$ with $\varepsilon \geqslant \bar{\varepsilon}$, then $u(0) > u_0$.

Small diffusion — pictures



Non constant radial solutions for $N = 3$, $p = 3$, $R = 4$, $\varepsilon \to 0$.

Thank you for your attention.

Krasnoselskii-Boehme-Marino theorem (1/2)

Theorem (Krasnoselskii-Boehme-Marino)

Let $F : I \times H \to K : (t,u) \mapsto F(t,u)$ be a continuous function, where $I \subseteq \mathbb{R}$ is an interval, and H and K are Banach spaces, such that $F(\lambda,0) = 0$ for any $\lambda \in I$.

- *If F is of class $\mathcal{C}^1$ in a neighborhood of $(\lambda,0)$ and $(\lambda,0)$ is a bifurcation point of F then $\partial_u F(\lambda,0)$ is not invertible.*
- *Let assume that for each $(\lambda,u) \in I \times H$,*

  $$F(\lambda,u) = L(\lambda,u) - N(\lambda,u), \quad L(\lambda,\cdot) = \lambda \mathbb{1} - T \quad \text{and} \quad N(\lambda,u) = o(\|u\|),$$

  with T linear, T and N compact, and the last equality being uniform on each compact set of λ.
  If λ_* is an eigenvalue of T with odd multiplicity, then $(\lambda_*,0)$ is a global bifurcation point for $F(t,u) = 0$.

Krasnoselskii-Boehme-Marino theorem (2/2)

Theorem (Krasnoselskii-Boehme-Marino (cont'd))

- *Let assume that H is a Hilbert space and that for each* $(\lambda, u) \in I \times \mathbb{R}$, $F(\lambda, u) = \nabla_u h(\lambda, u)$ *where*

$$h(\lambda, u) = \tfrac{1}{2}\langle L(\lambda, u), u\rangle - g(\lambda, u),$$
$$L(\lambda, \cdot) = \lambda \mathbb{1} - T, \quad \text{and} \quad \nabla g(\lambda, u) = o(\|u\|),$$

with T *linear and symmetric,* $g(\lambda, \cdot) \in \mathcal{C}^2$ *for all* λ*, and the last equality being uniform on each compact set of* λ*.*
If λ_* *is an eigenvalue of* T *with finite multiplicity and* $h(\lambda, \cdot)$ *verifies the Palais-Smale condition for each* λ*, then* $(\lambda_*, 0)$ *is a bifurcation point for* $F(t, u) = 0$.

▸ Return