# Partially-massless spin-2 fields: twisted duality and interactions in $(A)dS_n$

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Work in collaboration with Andrea Campoleoni, Nacho Cortese and Lucas Traina

based on 1804.05588 and 1807.04524 for the original parts on twisted duality

and in collaboration with Sebastian Garcia-Saenz and Lucas Traina for interactions

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# Plan of the talk

1. Generalities

2. Review of duality for spin-2 in flat spacetime

* 3. Parent action in Stückelberg formulation

4. Twisted (self-) duality in $(A)dS_n$ : massless and PM cases

5. A theory for multiple PM spin-2 fields

# 1. Generalities

- Electric-magnetic duality, perhaps as fundamental as Lorentz symmetry.

  In non-Abelian theory, relates strong and weak coupling regimes. [Long story: Heavyside, Dirac, ...]

- For spin-2 (linearized), studied by P. West, Hull (2001). Previous attempts in the massive case by Curtright & Freund in 80's. Further studied in 2002, on-shell, by X. Bekaert & N.B., all these studies in flat spacetime.

Quid of massless spin-2 in $(A)dS_n$: duality property?

↳ same question for partially-massless spin-2, only defined in $(A)dS_n$

## 2. Review of duality for spin-2 in flat spacetime

Use condensed notation (X. Bekaert & N.B., 2002)

EI : $\mathrm{Tr}\, K \approx 0 \iff K^{\mu}{}_{\alpha\mu\nu} \approx 0$ , where $K = d^{(1)} d^{(2)} h$ ,

BI : $\mathrm{Tr}_{12} *_1 K \equiv 0 \iff K_{[\mu\nu|\varrho]\sigma} \equiv 0$ ,

EII : $d^{\dagger} K \approx 0 \iff \partial^{\mu} K_{\mu\nu\varrho\sigma} \approx 0$ ,

BII : $d K \equiv 0 \iff \partial_{[\mu} K_{\nu\varrho]|\alpha\beta} \equiv 0$ .

- $K \equiv K_{[2,2]} = \frac{1}{4}\, d^{(1)}x^{\mu}\, d^{(1)}x^{\nu}\, d^{(2)}x^{\alpha}\, d^{(2)}x^{\beta}\, K_{\mu\nu|\alpha\beta}$
- dual $d^{\dagger}_{(i)} x^{\mu}$ s.t. $\boxed{\{ d^{(i)}x^{\mu} , d^{\dagger}_{(i)} x^{\nu} \} = \eta^{\mu\nu}}$ ,
- $d^{(i)} := d^{(i)}x^{\mu} \dfrac{\partial}{\partial x^{\mu}}$ , $\quad d^{\dagger}_{(i)} := d^{\dagger}_{(i)} x^{\mu} \dfrac{\partial}{\partial x^{\mu}}$ ,
- $\mathrm{Tr}_{ij} = \eta_{\mu\nu}\, d^{\dagger}_{(i)} x^{\mu}\, d^{\dagger}_{(j)} x^{\nu}$

As operators: $\quad | K_{[2,2]} \rangle = \frac{1}{4}\, d^{(1)}x^\mu\, d^{(1)}x^\nu\, d^{(2)}x^\alpha\, d^{(2)}x^\beta\, K_{\mu\nu|\alpha\beta}\, |0\rangle$

$$d^\dagger_{(i)} x^\mu |0\rangle \overset{!}{=} 0 \qquad \text{destruction.}$$

$$\left[ d^{(i)}x^\mu , d^{(j)}x^\nu \right]_{\mathbb{Z}_2} = 0 \, , \qquad \left[ d^{(i)}x^\mu , d^\dagger_{(j)}x^\nu \right]_{\mathbb{Z}_2} = \delta^i_j\, \eta^{\mu\nu} .$$

Twisted-duality relations $\quad K \longmapsto *_1 K \, , \quad *_1 K \longmapsto -K$

$$\vec{K} := \begin{pmatrix} K \\ *_1 K \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} K \\ *_1 K \end{pmatrix} = J \vec{K} \, , \quad J = \pi/2 \text{ rotation.}$$

$$\rightsquigarrow \begin{pmatrix} E\,I \\ E\,II \end{pmatrix} \longleftrightarrow \begin{pmatrix} B\,I \\ B\,II \end{pmatrix} \quad \text{under duality.}$$

Example, $n = 5$:

$\tilde{K} := * K$

$$\left.\begin{array}{ll} \underline{BI} : & \mathrm{Tr}_{12} *_1 K \equiv 0 \\ \underline{BII} : & d^\dagger K \equiv 0 \end{array}\right\} \Longrightarrow K_{[2,2]} = d^{(1)} d^{(2)} h_{[1,1]} \, , \quad h \sim \ydiagram{2}$$

$\underline{EI}$ : $\mathrm{Tr}\, K = 0 \iff \mathrm{Tr} * \tilde{K} = 0 \iff \tilde{K} \sim \ydiagram{2,2,1}$

$\underline{EII}$ : $d^\dagger K = 0 \iff d\tilde{K} = 0 \iff \tilde{K} \sim \ytableaushort{{}{},{}\partial,\partial}$ , $C \sim \ydiagram{2,1} \equiv \tilde{h}$

- So far: on-shell. One can find an action $S[C]$ s.t. $\mathrm{Tr}\,\tilde{K} = 0$ is the e.o.m.

  ↳ [P. West 2001, 2002], [N.B., S. Cnockaert, M. Henneaux 2003: contact with Curtright's action for $n=5$]

Strategy: Use a Parent action $\longrightarrow$ $S[h, Y^{ab|}{}_c]$

$$
\begin{tikzcd}
 & & S[h, Y^{ab|}{}_c] \arrow[dl, "Y \text{ auxilliary}"'] \arrow[dr, "h \text{ Lagrange multipl. for } \partial_a Y^{ab|}{}_c = 0"] & \\
 & S^{FP}[h] & & \phantom{X} \arrow[dll, "\text{Solve } Y^{ab|}{}_d = \partial_c \tilde{Y}^{abc|}{}_d"] \\
\text{Dual actions} \arrow[ur] \arrow[r] & S[\tilde{Y}^{\cdots|}{}_{\cdot}] & &
\end{tikzcd}
$$

- Therefore: in flat spacetime

$h \sim$  $\xleftrightarrow{\text{dual}}$ $C \sim$  $:= *_1 \tilde{Y}^{\cdots|}{}_{\cdot}$

$[1,1]$ $\qquad$ $[n-3,1]$

- Question: What about $(A)dS_n$? $\longrightarrow$ Early investigations by B. Julia, Yu. Zinoviev
  still no identification of what field
  is the dual graviton in $(A)dS$.

Idea : [Y. Zinoviev, Th. Basile – X. Bekaert – N. B.] : Parent action = linearised action from Chamseddine – West / MacDowell – Mansouri

- $S_m[\underline{h}, \underline{\omega}] = \int_{AdS_n} \Big[ -\frac{\epsilon_{abcp[n-3]}}{2(n-3)!} \Big( \nabla h^a \wedge \omega^{bc} - \frac{1}{n-2} \omega^{a}{}_{\cdot} \, \omega^{\cdot b} \, \bar{e}^c \Big) \bar{e}^{p_1} \ldots \bar{e}^{p_{n-3}}$
$\qquad - \frac{\epsilon_{abc[n-2]}}{(n-1)!} \mu^2 \frac{n-2}{4} h^a h^b \bar{e}^c \ldots \bar{e}^c \Big]$,

$\mu^2 := \frac{2(n-1)}{n-2} \left( 2m^2 + \sigma (n-2) \lambda^2 \right)$, $\nabla^2 V^a = -\sigma \lambda^2 \bar{e}^a \bar{e}_b V^b$, $\sigma = \begin{cases} 1 & AdS_n \\ -1 & dS_n \end{cases}$

- Set $m = 0$ and integrate out $h^a = dx^\mu h_{\mu|}{}^a$, auxiliary for $\lambda \neq 0$ :

$$\hookrightarrow \boxed{S[\omega_{\mu|}{}^{ab}] = \frac{1}{\lambda^2} \int_{(A)dS_n} \overset{(1)}{C}{}^{ab} \wedge * \overset{(1)}{C}{}^{cd}}$$

- $\overset{(1)}{R}{}_{\mu\nu|}{}^{ab} := 2 \nabla_{[\mu} \omega_{\nu]|}{}^{ab}$

for
- $\overset{(1)}{C}{}^{ab} := \overset{(1)}{R}{}^{ab} - 2 \bar{e}^{[a} \overset{(1)}{P}{}^{b]}$
- $\overset{(1)}{P}{}_{\mu|}{}^a := \frac{1}{n-2} \left( \overset{(1)}{R}{}_{\mu|}{}^a - \frac{\bar{e}_\mu{}^a}{2(n-1)} R \right)$ Schouten – like

- $S[\omega_{\mu|}{}^{ab}]$ invariant under $\delta_\epsilon \omega_{\mu|}{}^{ab} = \nabla_\mu \epsilon^{ab} + \frac{\mu^2}{n-1} \bar{e}_\mu{}^{[a} \epsilon^{b]}$

- Hodge-dualise: $T_{[n-2,1]} := *_1 \hat{\omega}_{[2,1]}$ ( $\hat{\omega}$ = traceless part of $\omega$)

$\hookrightarrow$ Relation with $C_{[n-3,1]}$ via Stückelberg shift:

$$\hat{\omega}_{a_1}{}^{bc} \longmapsto \hat{\omega}_{a_1}{}^{bc} + \frac{1}{\lambda} \nabla_d \hat{W}^{bcd_1}{}_a$$

- Hodge dualise $T_{[n-2,1]} := *_2 \hat{\omega}_{\cdot 1}{}^{\cdot\cdot}$ & $C_{[n-3,1]} := *_1 \hat{W}^{\cdots 1}$.

and observe that, in the flat limit $\lambda \to 0$ of $\lambda^2 \mathcal{L}_0(C,T)$,

$T_{[n-2,1]}$ becomes a topological field:

$$\frac{\delta \mathcal{L}_0}{\delta T} \equiv \mathrm{Tr}\, K_{[n-1,2]}(T) \approx 0 \quad \underset{n+1>n}{\Longleftrightarrow} \quad K(T) \approx 0$$

$T$ pure gauge.

- Hence, only $\mathcal{L}(C, \partial C)$ remains for local d.o.f.

5 bis

## 3. Parent action in Stückelberg formulation

The best way is to use a Stückelberg reformulation $\longrightarrow$ [N.B., A. Campoleoni, I. Cortes] 2018

↳ Continue the analysis of Yurii Zinoviev [2008] and use frame-like,

1st-order action for $m \neq 0$:

$$\underset{\text{spin: }}{S}\,[\underbrace{h^a, \omega^{ab}}_{2}, \underbrace{A, F^{ab}}_{1}, \underbrace{\varphi, \pi^a}_{0}] = \int_{(A)dS_n} \left( \mathcal{L}^{(2)} + \mathcal{L}^{(1)} + \mathcal{L}^{(0)} + \mathcal{L}^{cross} \right)$$

$$\bullet\ \mathcal{L}^{(2)} \sim -\frac{\epsilon_{abc\,p[n-3]}}{2\,(n-3)!}\left(\nabla h^{a} \wedge \omega^{bc} - \frac{1}{n-2}\,\omega^{a}{}_{\cdot}\,\omega^{\cdot b}\,\bar{e}^{c}\right)\bar{e}^{p_1}\ldots\bar{e}^{p_{n-3}}$$

$$\bullet\ \mathcal{L}^{(1)} \sim F^{ab}\left(\nabla A - \tfrac{1}{4}\,F_{cd}\,\bar{e}^{c}\,\bar{e}^{d}\right) \qquad \bullet\ \mathcal{L}^{(0)} \sim \pi^{a}\left(\nabla\varphi - \tfrac{1}{2}\,\pi_{b}\,\bar{e}^{b}\right),$$

$$\bullet\ \mathcal{L}^{cross} \sim \Big[m\,(n-1)\,\omega^{ab}A + m\,F^{a}{}_{d}\,h^{pd}\,\bar{e}^{b} + \mu\,\pi^{a}A\,\bar{e}^{b} - \frac{n-2}{4}\,\mu^{2}\,h^{a}h^{b} - m\mu\,\varphi\,h^{a}\bar{e}^{b} - \frac{m^{2}}{n-2}\,\varphi^{2}\,\bar{e}^{a}\bar{e}^{b}\Big]$$

$$\times\ \epsilon_{abc[n-2]}\,\bar{e}^{c}\ldots\bar{e}^{c}.$$

$S[h,\omega,A,F,\varphi,\pi]$ is *invariant* under the gauge transformations :

$$\begin{cases} \delta h^{a} = \nabla\xi^{a} - \lambda^{a}{}_{b}\,\bar{e}^{b} + \dfrac{2m}{n-2}\,\epsilon\,\bar{e}^{a} \\[2ex] \delta\omega^{ab} = \nabla\Lambda^{ab} + \dfrac{\mu^{2}}{n-1}\,\bar{e}^{[a}\,\xi^{b]} \end{cases} \qquad \begin{cases} \delta A = \nabla\epsilon - m\,\xi^{a}\,\bar{e}_{a} \\[2ex] \delta F^{ab} = 2m\,\Lambda^{ab} \end{cases}$$

$$\begin{cases} \delta\varphi = -\mu\,\epsilon \\[2ex] \delta\pi^{a} = -m\mu\,\xi^{a} \end{cases}$$

where

$$\mu^{2} := \frac{2(n-1)}{n-2}\left(2m^{2} + \sigma\,(n-2)\,\lambda^{2}\right)$$

- When $m = 0$ : Spin-1 & 0 sectors decouple $\to$ recovers 1st order formalism of line. gravity in $(A)dS_n$ ;

- When $\mu = 0$ : spin-0 sector decouples $\longrightarrow$ 1st order formulation of PM spin-2.

  $\longrightarrow$ smooth flat limit : $\pm 2$ & $\pm 1$ helicities.

Rem 1 : $\mu = 0$ only if $\sigma = -1$ apparently, but if $\mathcal{L}^{(1)} \longmapsto -\sigma \mathcal{L}^{(1)}$, PM limit also in $AdS_n$

$$\mu^2 \longmapsto \tilde{\mu}^2 := \frac{2(n-1)}{n-2}\left(2m^2 - (n-2)\lambda^2\right), \text{ in both } \sigma = \pm 1 .$$

Rem 2 : $Y^{ab|}{}_c = \omega_{c|}{}^{ab} + 2\,\delta_c^{[a}\,\omega_{\circ|}{}^{b]\circ}$

Two things can be done

Ⓐ Electric reduction : Eliminate auxil. $Y$, $F$ & $\pi$ $\longrightarrow S[h_{ab}, A_a, \varphi]$

$\hookrightarrow$ Stückelberg formulation for massive spin-2 in $(A)dS_n$ [Yu. Zinoviev 2001, 2006, 2008]

- Same limits $m \to 0$ and $\mu \to 0$ as above.

## Ⓑ Magnetic reduction

### B.1) Massless $m=0$ case : Spin 1 & 0 decouple :

$$\mathcal{L} \longrightarrow \mathcal{L}_0 = \nabla_b h_{c|}{}^{a}\, Y^{bc|}{}_{a} - \tfrac{1}{2}\left( Y^{bc|a} Y_{ab|c} + \frac{1}{n-2} Y^{a\cdot|}{}_{\cdot}\, Y_{a\cdot|}{}^{\cdot} \right) - \frac{n-2}{2}\,\sigma\lambda^2 \left( h_{a|b}\, h^{b|a} - h'^2 \right)$$

- $\delta h^a = \nabla \xi^a - \Lambda^{ab}\,\bar{e}_b$
- $\delta Y^{bc|}{}_{a} = \nabla_a \Lambda^{bc} + 2\,\bar{e}_a{}^{[b}\,\nabla_{\cdot}\Lambda^{c]\cdot} - 2\sigma(n-2)\,\bar{e}_a{}^{[b}\,\xi^{c]}$

For $\lambda \neq 0$, $h$ auxilliary $\longrightarrow$ $\mathcal{L}(\nabla Y, Y) = \dfrac{\sigma}{2(n-2)\lambda^2}\left[ \nabla_a \hat{Y}^{cd|}{}_{b}\, \nabla_c \hat{Y}^{ab|}{}_{d} + \sigma\lambda^2\, \hat{Y}_{bc|a}\, \hat{Y}^{ba|c} \right]$

where $\hat{Y}$ = traceless part of $Y^{ab|}{}_{c}$.

- Perform the shift $\hat{Y}^{bc|}{}_{a} \longmapsto \hat{Y}^{bc|}{}_{a} + \frac{1}{\lambda}\nabla_d \hat{W}^{bcd|}{}_{a}$ to find

$$\mathcal{L}_0(\hat{Y},\hat{W}) = \frac{1}{\lambda^2}\left[ \frac{1}{2}\nabla_c \hat{W}^{abc|d}\,\nabla^e W_{dbe|a} + \lambda\,\hat{Y}^{ab|c}\,\nabla^e \hat{W}_{cbe|a} + \frac{\sigma}{2(n-2)}\nabla_b \hat{Y}^{ab|c}\,\nabla^d \hat{Y}_{cd|a} + \frac{\lambda^2}{2}\hat{Y}^{ab|c}\hat{Y}_{ab|c} \right]$$

- Hodge dualise $T_{[n-2,1]} := *_1 \hat{Y}_{[2,1]}$

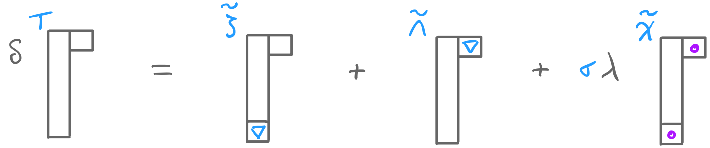
  and $C_{[n-3,1]} := *_1 \hat{W}_{[3,1]}$

$\Big\} \longrightarrow \mathcal{L}_0 = -\dfrac{1}{2\lambda^2 (n-3)!}\left[ \mathcal{L}(\nabla C) + 2\lambda\, \text{“}T\cdot\nabla C\text{”} + \dfrac{\sigma}{(n-2)^2}\,\mathcal{I} \right]$

where $\mathcal{I} := \mathcal{L}(\nabla T) + \sigma(n-2)\lambda^2 \left( T^2 - (n-2)\,T'^2 \right)$

- $\mathcal{L}_0(T,C)$ invariant under

  - $\delta\, T_{[n-2,1]}$ :




  - $\delta\, C_{[n-3,1]}$ :


- In the flat limit $\lambda \longrightarrow 0$ , the cross terms vanish and

$$\boxed{\lambda^2\, \mathcal{L}_0(T,C) \longrightarrow \mathcal{L}^{\text{Curt}}(C_{[n-3,1]}) + \frac{\sigma}{(n-2)^2}\, \mathcal{L}^{\text{Curt.}}(T_{[n-2,1]})}$$

Note :

$\mathcal{L}^{\text{Curt.}}(T_{[n-2,1]})$ is topological : curvature $K^T_{[n-1,2]} \simeq 0$ vanishes on-shell

where $K^T_{[n-1,2]} := d^{(1)} d^{(2)}\, T_{[n-2,1]}$ [X. Bekaert & N.B. 2002]

Hence there remains only $C_{[n-3,1]} \equiv \tilde{h}_{[n-3,1]}$ propagating.

## B.2) Partially-Massless $\mu = 0$ case : Spin 0 decouples :

- In order to accommodate both $AdS_n$ and $dS_n$, rescale $\mathcal{L}^{(1)} \to (-\sigma)\, \mathcal{L}^{(1)}$ so that

$$\mu^2 \mapsto \tilde{\mu}^2 := \frac{2(n-1)}{n-2}\left(2m^2 - (n-2)\lambda^2\right) .$$

- Take $\tilde{\mu} \to 0$, get

$$\mathcal{L}_{PM} = h_{a|b}\, C^{a|b} + \frac{\sigma}{\tilde{m}} A_a \nabla_b C^{a|b} - \frac{1}{2}\left( Y^{bc|a}\, Y_{ab|c} + \frac{1}{n-2}\, Y^{a\bullet|}{}_{\bullet}\, Y_{a\blacktriangle|}{}^{\blacktriangle} \right) - \frac{\sigma}{4} F_{ab} F^{ab}$$

where $C^{a|b} := \nabla_c Y^{ac|b} - \tilde{m}\, F^{ab}$ and $\tilde{m}^2 := \frac{n-2}{2}\lambda^2$ .

- As in flat spacetime, $h_{a|b}$ is Lagrange multiplier. Constraint $C^{a|b} = 0$ solved identically by $Y^{bc|}{}_a = \frac{1}{\lambda} \nabla_d \hat{W}^{bcd|}{}_a - \frac{\sigma}{2\tilde{m}}\left(\nabla_a F^{bc} + 2\, \bar{e}_a{}^{[b} \nabla_\bullet F^{c]\bullet}\right)$ .

- Substituting in $\mathcal{L}_{PM}(h, Y, A, F)$ gives $\boxed{\mathcal{L}_{PM}(\nabla \hat{W}) = -\frac{1}{2\lambda^2} \nabla_\bullet \hat{W}^{ab\bullet|c}\, \nabla^{\blacktriangle} \hat{W}_{ab\blacktriangle|c} + \text{T.D.}}$

  $\hookrightarrow$ $F^{ab}$ enters through total derivative "T.D."

  $\hookrightarrow$ $\mathcal{L}_{PM}$ invariant under $\delta \hat{W}^{bcd|}{}_a = \nabla_e \hat{v}^{bcde|}{}_a$ .

$\bullet\ \mathcal{L}_{PM}(\nabla W) = -\frac{1}{2\lambda^2}\,\nabla_{\bullet}\hat{W}^{bc\bullet|a}\,\nabla^{\bullet}\hat{W}_{bc\bullet|a}$

**Stückelberg shift** $\quad \hat{W}^{bcd|}{}_{a} \longmapsto \hat{W}^{bcd|}{}_{a} + \frac{1}{(n-3)\,\tilde{m}}\left(\nabla_a U^{bcd} - \text{Trace}\right)$

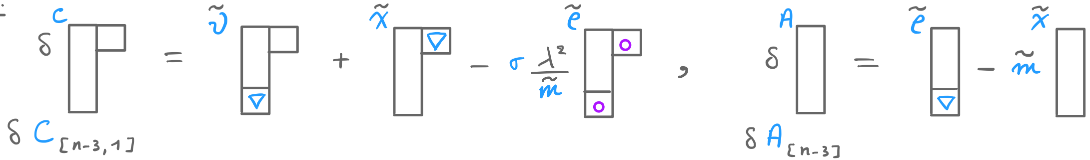
**Hodge dualise** $\quad \begin{cases} A_{[n-3]} := *_1 U_{[3]} \\ C_{[n-3,1]} := *_1 \hat{W}_{[3,1]} \end{cases}$

$$\tilde{\mathcal{L}}_{PM}(C,A) = \frac{-1}{2(n-3)!\,\lambda^2}\left[\mathcal{L}(C_{[n-3,1]}) - \frac{2\sigma\lambda^2}{(n-2)\,\tilde{m}^2}\,\mathcal{L}(A,\nabla A) + \frac{4\sigma\lambda^2}{\tilde{m}}\,\tilde{\mathcal{L}}^{cross.}\right]$$

where $\bullet\ \mathcal{L}(A,\nabla A) = (\nabla_a A_{b[n-3]})^2 - (n-3)\,(\nabla\cdot A)^2 + 3\sigma\lambda^2 A^2$

$\bullet\ \mathcal{L}^{cross} \sim A^{a[n-3]}\left(\nabla_b C_{a[n-3]|}{}^{b} + (-)^{n-1}(n-3)\,\nabla_a C'_{a[n-4]}\right)$

Gauge transfo :




Smooth flat limit $\quad \lambda^2\,\tilde{\mathcal{L}}_{PM} \longrightarrow \mathcal{L}^{Curt.}(\partial C_{[n-3,1]}) - \sigma\,\mathcal{L}(A_{[n-3]}) \quad$ (unitary in $dS_n$)

$\bullet$ Rem. : Unitarity at classical level from $\sigma$ in helicity 1 : consistent with rules in BMV ($\sigma = 1$) and [Th. Basile, X. Bekaert, N.B. 2017] $\sigma = -1$.

## 4. Twisted (self-) duality in $(A)dS_n$ : massless and PM cases

### 4.1) Massless case

→ In the standard formulation from $\mathcal{L}^{FP}_{\lambda}(\nabla h, h)$ :

$$K^{ab|cd} := -\frac{1}{2}\left(\nabla^a \nabla^{[c} h^{d]b} - \nabla^b \nabla^{[c} h^{d]a} + \nabla^c \nabla^{[a} h^{b]d} - \nabla^d \nabla^{[a} h^{b]c}\right) + \sigma \lambda^2 \left(\bar{g}^{a[c} h^{d]b} - \bar{g}^{b[c} h^{d]a}\right)$$

primary gauge-invariant quantity s.t.

$$\boxed{\begin{array}{ll} \mathrm{Tr} * K \equiv 0 \quad (\mathrm{BI}) , & \nabla^{[a} K^{bc]|ef} \equiv 0 \quad (\mathrm{BII}) \\ \mathrm{Tr}\, K \approx 0 \quad (\mathrm{EI}) , & \nabla^a K_{ab|cd} \approx 0 \quad (\mathrm{EII}) \end{array}}$$

→ In the dual formulation $\mathcal{L}_0(\hat{Y}^{\cdot\cdot|\cdot}, \hat{W}^{\cdot\cdot|\cdot}) \sim \nabla\hat{W}\nabla\hat{W} + \lambda\, \hat{Y}\, \nabla\cdot\hat{W} + \nabla\hat{Y}\nabla\hat{Y} + \lambda^2\, \hat{Y}^2$

$$R_{ab|}{}^{cd} := 2\, \nabla_a \left(\nabla\cdot \hat{W}^{cd\cdot|}{}_b + \lambda\, \hat{Y}^{cd|}{}_b\right) , \quad R_{a|}{}^{c} := \mathrm{Tr}\, R$$

$$K_{ab|}{}^{d} := 2\, \nabla_a \nabla\cdot \hat{Y}^{\cdot d|}{}_b + 2\sigma(n-2)\lambda \left(\nabla\cdot \hat{W}^{cd|}{}_{[a|b]} + \lambda\, \hat{Y}^{d}{}_{[a|b]}\right) , \quad K_a := K_{a\cdot|}{}^{\cdot} .$$

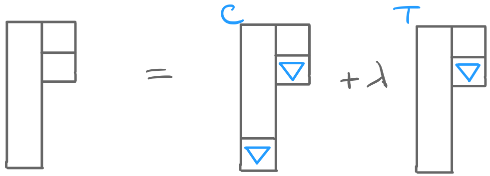
s.t.
- $V_{ab|}{}^{cd} := R_{ab|}{}^{cd}$ − Traces is gauge-invariant
- $X_{ab|}{}^{c} := K_{ab|}{}^{c} + \frac{2}{n-1}\, \delta^c_{[a} K_{b]}$ is gauge-invariant

• Recall Hodge-dual $\hat{W}^{\cdots|}{}_{\cdot} = *_1 C_{[n-3,1]}$ , $\hat{Y}^{\cdot\cdot|}{}_{\cdot} = *_1 T_{[n-2,1]}$

and define $K^C_{[n-2,2]} := *_2 V_{\cdot\cdot|}{}^{\cdot\cdot}(\hat{W},\hat{Y})$ , $K^T_{[n-1,2]} := *_2 X_{\cdot\cdot|}{}^{\cdot}(\hat{W},\hat{Y})$

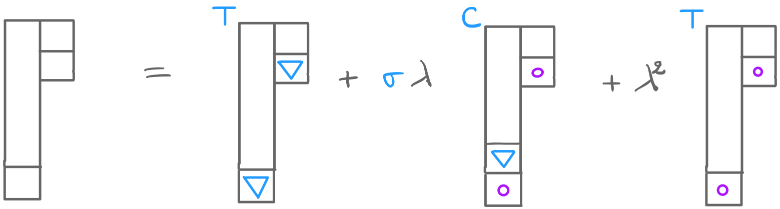
where $\hat{W}$ & $\hat{Y}$ are expressed in terms of $C$ and $T$ .

$K^C_{[n-2,2]} \sim$




s.t. $\boxed{\mathrm{Tr} *_1 K^C \equiv 0}$ $(\widetilde{BI}_1)$

$K^T_{[n-1,2]} \sim$




$\mathrm{Tr} *_1 K^T \equiv 0$ $(\widetilde{BI}_2)$

• $\nabla^{(1)} K^C_{[n-2,2]} \equiv \lambda\, K^T_{[n-1,2]}$ $(\widetilde{BII}_1)$ • $\nabla^{(2)} K^C_{[n-2,2]} \equiv \lambda\, \sigma^2{}_1 (K^T_{[n-1,2]})$ $(\widetilde{BII}_2)$

From the action, finds $\boxed{\mathrm{Tr}\, K^C \approx 0}$ $(\widetilde{EI}_1)$ and $\mathrm{Tr}\, K^T_{[n-1,2]} \approx 0$ $(\widetilde{EI}_2)$

implying that $\boxed{K^T \approx 0}$ , whence $\boxed{\nabla_{(1)} K^C \approx 0 \quad \& \quad \nabla_{(2)} K^C \approx 0}$ $(\widetilde{EII})$

Therefore twisted-duality in (A)dS$_n$ : $\boxed{K^{c}_{[n-2,2]} \simeq *_1 K_{[2,2]}}$

as in flat spacetime, relates $\begin{pmatrix} BI \\ BII \end{pmatrix} \longleftrightarrow \begin{pmatrix} \tilde{E}I \\ \tilde{E}II \end{pmatrix}$ and $\begin{pmatrix} EI \\ EII \end{pmatrix} \longleftrightarrow \begin{pmatrix} \tilde{B}I \\ \tilde{B}II \end{pmatrix}$

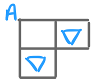
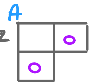
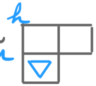
In the flat limit $\lambda \to 0$, reproduces the twisted-duality of Hull.

4.2) Partially-massless case

• Standard (electric) Stückelberg formulation with $\mathcal{L}_{PM}(h_{ab}, A_a)$ invariant under

$$\delta h_{ab} = 2\nabla_{(a}\xi_{b)} + \frac{2\tilde{m}}{n-2} g_{ab}\,\epsilon \quad , \quad \delta A_a = \nabla_a \epsilon + 2\sigma \tilde{m}\, \xi_a$$

• $H_{ab} := h_{ab} - \frac{\sigma}{\tilde{m}} \nabla_{(a} A_{b)}$ invariant under $\xi$, NOT under $\epsilon$

$\Rightarrow$ • $\boxed{K_{ab|c} := -4\sigma\tilde{m}\, \nabla_{[a} H_{b]c}}$ $\sim$  $-\,\lambda^2$  $-\,\tilde{m}$  fully invariant.

• $\boxed{Q^{ab|cd} := -\frac{1}{2}\left(\nabla^a \nabla^{[c} H^{d]b} - \dots - \nabla^d \nabla^{[a} H^{b]c}\right)}$ s.t. $\dfrac{\delta \mathcal{L}_{PM}}{\delta h^{ab}} \equiv -2\underbrace{\left(Q_{a\cdot|b}{}^{\cdot} - \frac{1}{2}\bar{g}_{ab}\, Q''\right)}_{G^{PM}_{ab}}$

and $\dfrac{\delta \mathcal{L}_{PM}}{\delta A^a} \equiv -\dfrac{2\sigma}{\tilde{m}} \nabla^b G^{PM}_{ab} \equiv \sigma K_{a\cdot|}{}^{\cdot}$

$\boxed{\mathrm{Tr}\, K_{\cdot\cdot|\cdot} \simeq 0 \simeq \mathrm{Tr}\, Q}$ $(\tilde{E}I)$

$$\boxed{\mathrm{Tr}\, K_{\cdot\cdot|\cdot} \approx 0 \approx \mathrm{Tr}\, Q} \qquad (\tilde{EI})$$

One also derives that

$$\boxed{\nabla^{[a} Q^{bc]|}{}_{mn} \equiv -\frac{\tilde m}{n-2}\, \delta^{[a}_{[m}\, K^{bc]|}{}_{n]}} \qquad (\tilde{BII})$$

• In the dual formulation for PM spin-2, $\mathcal{L}(\hat W^{\cdots|}{}_{\cdot}, U^{\cdot\cdot})$ Stückelberg

$$\begin{cases} \bullet\ \mathcal{R}_{ab|}{}^{cd} := 2\,\nabla_{a}\big(\underbrace{\nabla_{\cdot} \hat W^{cd\cdot|}{}_{b}} - \frac{\sigma\lambda^2}{\tilde m}\, \underbrace{U_b{}^{cd}}\big) & \bullet\ \mathcal{V}_{ab|}{}^{cd} = \mathcal{R}_{ab|}{}^{cd} - \text{Traces} \\ \bullet\ \mathcal{K}^{U}{}_{ab|}{}^{c} := 2\,\nabla_{[a}\nabla^{\cdot} U_{b]}{}^{c}{}_{\cdot} + 2(n-2)\,\tilde m\, \big(\nabla_{\cdot} \hat W^{\cdot c}{}_{[a|b]} - \frac{\sigma}{\tilde m}\lambda^2\, U_{ab}{}^{c}\big) \end{cases}$$

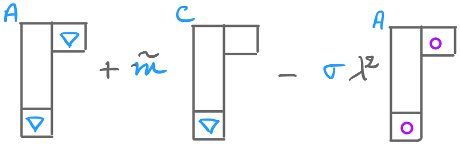
$\rightarrow$ gauge invariant

• Define dual curvatures • $\mathcal{K}^{c}{}_{[n-2,2]} := *_2\, \mathcal{V}_{\cdot\cdot|}{}^{\cdot\cdot}$ $\qquad (\mathrm{Tr}\, \mathcal{K}^U \equiv 0)$

$$\bullet\ \tilde{\mathcal{K}}^{a[n-2]|}{}_{b} := \frac{(-)^n}{2}\, \epsilon^{a[n-3]cde}\Big(\delta^a_e\, \mathcal{K}^{U}{}_{cd|b} - \frac{n-2}{2}\, \delta^a_b\, \mathcal{K}^{U}{}_{cd|e}\Big)$$

i.e. $\mathcal{K}^{U}{}_{ab|}{}^{c} = (-)^{n-1} \dfrac{2}{(n-2)!}\, \epsilon_{d[n-2][a}{}^{c}\, \tilde{\mathcal{K}}^{d[n-2]|}{}_{b]}$

i.e. $\tilde{\mathcal{K}}_{[n-2,1]} \sim$




and $\mathcal{K}^{C}{}_{[n-2,2]}(\nabla\nabla C,\ \bar g_{\cdot\cdot}\, \nabla A_{[n-3]})$

similar to the massless case

$$\boxed{\mathrm{Tr}\, *_1 \mathcal{K}^{C} \equiv 0 \equiv \mathrm{Tr}\, *_1 \tilde{\mathcal{K}}} \qquad (\tilde{BI})$$

- $\mathrm{Tr}\ \widetilde{K}_{[n-2,1]} \simeq 0$     $\mathrm{Tr}\ K^C_{[n-2,2]} \simeq 0$     $(\widetilde{EI})$

- $\nabla^{(1)} K^C_{[n-2,2]} \equiv \dfrac{\lambda^2}{\widetilde{m}}\, T^{12}\left(\widetilde{K}_{[n-2,1]}\right)$     $(\widetilde{BII}_1)$

- $\nabla^{(2)} K^C_{[n-2,2]} \equiv \dfrac{\lambda^2}{\widetilde{m}}\, T^{12} \sigma^2{}_1\left(\widetilde{K}_{[n-2,1]}\right)$     $(\widetilde{BII}_2)$

- $\nabla_{(2)} K^C_{[n-2,2]} \simeq \dfrac{\sigma\lambda^2}{\widetilde{m}}\, \widetilde{K}_{[n-2,1]}$     $(\widetilde{EII})$     • $\nabla_{(1)} K^C_{[n-2,2]} \simeq \dfrac{\lambda^2}{\widetilde{m}}\, \sigma^2{}_1\, \widetilde{K}_{[n-2,1]}$

PM Twisted-duality : $\boxed{K^C_{[n-2,2]} \simeq *_1\, Q_{[2,2]}}$     $(TD_1)_\lambda$

However, $(TD_1)$ not enough for smoothness of flat limit !

$\longrightarrow$ act $(*_1 \nabla^{(1)})\left[(TD_1)_{[n-2,2]}\right]$, use $(\widetilde{BII}_1)$, $\mathrm{Tr}_{12}(BII)$, $(EI)$ & $(\widetilde{EI})$ to get

$$\boxed{\widetilde{K}_{[n-2,1]} \simeq (-)^{n-1}\,\frac{\sigma\,\widetilde{m}^2}{2\lambda^2}\, *_1 K_{[2,1]}} \quad (TD_2)_\lambda$$

so that

$$(TD_1)_\lambda \underset{\lambda\neq 0}{\iff} (TD_2)_\lambda$$

When $\lambda \to 0$ : $(TD_1)_\lambda \longrightarrow$ Hull's spin-2 TD

$(TD_2)_\lambda \longrightarrow d^{(2)}\left(\widetilde{F}_{[n-2]} \simeq *_1 F_{[2]}\right) \iff \widetilde{F}_{[n-2]} \simeq *_1 F_{[2]}$

- In flat limit, $(TD_1)_\lambda \longrightarrow$ Pair of usual twisted duality relations
$(TD_2)_\lambda \longrightarrow$ for spin 2 $(h_{ab} \sim C_{[n-3,1]})$
and spin 1 $(A_a \sim A_{[n-3]})$.

- Considering $(TD_2)_\lambda$ and gauge-fixing $A_{[1]} \overset{(*)}{=} 0 \overset{(*)}{=} A_{[n-3]}$
which is allowed for $\lambda \neq 0$, gets

$$(TD_2)_\lambda \xrightarrow[(*)]{} \boxed{(n-2)\, \nabla^a C^{a[n-3]|b} \approx \frac{(-)^n}{2} \epsilon^{a[n-2]cd} \nabla_c h_d{}^b} \quad (TD2)^*_\lambda$$

while $(TD1)_\lambda \xrightarrow[(*)]{} \nabla^{(2)}(TD2^*_\lambda)$ curl.

(self-duality in $n=4$)

- In the $n=4$ case, $(TD2)^*_\lambda$ reproduces Hinterbichler's duality relation.

- Warning: once the Stückelberg fields $A_{[1]}$ & $A_{[n-3]}$ are fixed to zero $\overset{(*)}{=} 0$,
the flat limit is no-longer smooth for the counting of d.o.f.!

Instead $\begin{pmatrix} (TD1)_\lambda \\ (TD2)_\lambda \end{pmatrix} \xrightarrow[\lambda \to 0]{} \begin{pmatrix} K_{[n-2,2]} \approx *_1 K_{[2,2]} \\ F_{[n-2]} \approx *_1 F_{[2]} \end{pmatrix}$ is smooth.

## 5. A theory for multiple PM spin-2 fields

- Several endeavours to find a consistent theory of non-linear PM spin-2 fields [Y. Zinoviev 2006, C. de Rham - S. Renaux-Petel 2012, S.F. Hassan, A. Schmidt-May, M. von Strauss 2012, E. Joung, K. Mkrtchyan and G. Poghosyan 2019]

$\Rightarrow$ no consistent 2-derivative (cubic) vertex for a single PM field.

- As for gauge algebra, for a set of PM spin-2 [S. Garcia-Saenz, K. Hinterbichler, A. Joyce, E. Mirtsou & R.A. Rosen 2015]

↳ no non-abelian deformation to first order in fields, with assumptions on # derivatives.

• Revisiting these analyses in the BV BRST-cohomological formulation

Start from $S_0[h^a_{\mu\nu}] = -\frac{1}{4}\int d^n x \sqrt{-g}\; k_{ab}\left[F^{a\mu\nu|\rho} F^b{}_{\mu\nu|\rho} - 2 F^{a\mu} F^b{}_\mu\right]$

$$F^a_{\mu\nu|\rho} := 2\nabla_{[\mu} h^a_{\nu]\rho}$$

$$\overset{(0)}{\delta}_\epsilon S_0 = 0 \quad \text{under} \quad \overset{(0)}{\delta}_\epsilon h^a_{\mu\nu} = \nabla_\mu \nabla_\nu \epsilon^a - \frac{\sigma}{L^2} g_{\mu\nu}\epsilon^a$$

1) Most general deformation of gauge algebra :

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] h^a_{\mu\nu} = \overset{(0)}{\delta}_\chi h_{\mu\nu} \qquad \text{where}$$

$\chi = \alpha\,( m^a{}_{bc}\, \epsilon_1^b \epsilon_2^c + n^a{}_{bc}\, \nabla^\mu \epsilon_1^b \nabla_\mu \epsilon_2^c ) \;\longrightarrow$ no field dependence

↳ Consistency requires $m^a{}_{bc} = 0 = n^a{}_{bc} \implies$ Abelian

⟶ no higher-order corrections ?

2) Deformation of gauge symmetry, if 2 $\partial$'s :

Consistency gives only (out of 6 candidates)

$$\overset{(1)}{\delta}_{\epsilon}\, h^{a}_{\mu\nu} = \alpha\, f^{a}{}_{b,c}\, F^{b}_{e(\mu|\nu)}\, \nabla^{e} \epsilon^{c} \quad , \quad \text{only in } n=4 .$$

3) Cubic vertex with 2 $\partial$'s : $S_1 = \int d^4x \sqrt{-g}\, h^{a}_{\mu\nu}\, J^{\mu\nu}_{a}$

$$J^{\mu\nu}_{a} = f_{bc,a}\left[ F^{b\,\mu}{}_{e|\sigma}\, F^{c\,\nu e|\sigma} - \frac{1}{4} g^{\mu\nu} F^{b\,e\sigma|\lambda} F^{c}_{e\sigma|\lambda} + \text{improvements}\right]$$

$\Rightarrow$ # independent deformation at order $\alpha$ : $\frac{1}{2} N^2 (N+1)$

$$f_{ab,c} \sim \boxed{a \,|\, b} \otimes \boxed{c}$$

$\rightarrow$ Uniqueness result (since existence not new)

• Conservation :

Obviously $\nabla_\mu \nabla_\nu J^{\mu\nu}_a - \frac{\sigma}{L^2} g_{\mu\nu} J^{\mu\nu}_a \approx 0$

but also, $n=4$,

$\nabla_\mu J^{\mu\nu}_a \approx 0 \Longrightarrow \mathcal{J}^\mu_{ab} := \sqrt{-g}\, J^{\mu\nu}_a \nabla_\nu \bar{\epsilon}_b$ Noether current

rigid symmetry $\delta h^a_{\mu\nu} = f^a{}_{bc} F^b{}_{e(\mu|\nu)} \nabla^e \bar{\epsilon}^c$

Killing

4) Higher-order consistency :

Provided $f_{ae,b} f^e{}_{c,d} = 0$ (1)

$f_{ab,e} f^e{}_{c,d} = 0$ (2)

Fully consistent to all orders (!) .

But (1) & (2) non-trivial solution only if $k_{ab} \not\succ 0$

i.e. "wrong" relative signs.

THANKS!