Observer Design for Chemical Tubular Reactors Based on System Linearization

Ivan Francisco Yupanqui Tello1, Daniel Coutinho2, Joseph Winkin3, Alain Vande Wouver4

Abstract—This paper presents an observer design technique for a class or nonlinear coupled parabolic PDEs using the infinite dimensional approach in Hilbert spaces. In particular, the linear approximate model incorporating the spatially varying coefficients of the set of PDEs is used for designing a Luenberger-like observer in order to achieve exponential convergence of the estimation error dynamics. The proposed observer is applied to a chemical tubular reactor considering only boundary measurements. The observer performance is assessed via numerical simulations.

Key words: distributed parameter systems, boundary observation, parabolic PDEs, tubular reactor.

I. INTRODUCTION

Distributed parameter systems (DPSs) are a class of important processes in which process variables depend not only on time but also on spatial coordinates. Examples of DPSs can be found in (bio)process monitoring and control, robotics, glass feeders, biomedical engineering and flexible structures [1], [2], [3], [4]. The description of DPSs often takes the form of hyperbolic, parabolic or elliptic partial differential equations (PDEs). Parabolic PDEs represent the dynamics of industrial processes involving convection and diffusion effects. One of the most important examples of such class of systems is the chemical tubular reactor (CTR) with axial dispersion. In order to capture the effects of reactions, diffusion and convection in the reactor, the reactor model is often described by a set of coupled parabolic PDEs.

For the operation and monitoring of processes described by PDEs, the knowledge of system states is of fundamental importance. In most cases it is not possible to have full information of the system states because of economical reasons and/or the fact that not all variables can be measured, that is, installing all the necessary sensors may not be physically possible or their costs may become prohibitive. In such a case, the internal states can be estimated from the measurement of process inputs and outputs. State observers are usually based on the system model with an additional output injection term to improve the convergence of the observation error and they are digitally implemented by a computer for monitoring and/or feedback purposes. Due to the fact that many processes in real world applications can be modeled in terms of PDEs, the synthesis of observers for DPSs has received a growing interest in the last decades. Among the available survey papers, we can cite those by [5], [6], or a more recent textbook [7] where a broad class of estimation techniques for DPSs are presented.

In this work, we are interested in boundary observation of a system described by a set of nonlinear parabolic PDEs. By linearizing the nonlinear equations in the vicinity of the steady state profile, a set of linear parabolic PDEs with spatially varying coefficients is derived. Then, the state estimation problem can be transformed into a well-posed abstract boundary control problem by applying an exact transformation. The spectrum of the resulting linear operator is computed by solving an eigenvalue problem using techniques for the solution of the heat equation for composite media. Finally, the output injection operator is obtained using the decomposition of the state space induced by the spectral projection operator.

The majority of the previous work on boundary observation of infinite-dimensional systems concentrated on cases that are described by a single spatially invariant parabolic PDE [8], [9], [10], [11]. This work investigates an approach for systems that are described by a set of PDEs with spatially varying coefficients, which would model many chemical engineering processes. A case study involving a tubular reactor, wherein the catalytic cracking of gas oil takes place, is used to illustrate our method. The reaction scheme results in a triangular operator, which simplifies the computation of the spectrum of the system.

The paper is organized as follows. Section II focuses on the mathematical description of the system of interest. In particular, the nonlinear system is linearized around a steady state profile and appropriate state transformations are used to cast the linearized system as a well-posed infinite-dimensional system. In Section III, the dynamics of the tubular reactor (wherein the catalytic cracking of gas oil takes place) is presented. This reaction consists of two series parallel reactions. The mass balance for the reactor results in a set of coupled nonlinear parabolic PDEs. Section IV considers the design of the observer system based on the spectrum decomposition where the triangular structure of the state operator simplifies the computation of its spectrum. Throughout this work, the mathematical notation is standard.
for infinite dimensional systems; see, for instance, [8].

II. MATHEMATICAL MODEL DESCRIPTION

The dynamics of a one-dimensional tubular reactor with axial convection, dispersion and reaction is given by

$$\theta_t(z,t) = \Gamma_0 \theta_{zz}(z,t) - \nabla \theta_z(z,t) + F(\theta(z,t))$$

subject to the following boundary and initial conditions

$$\begin{align*}
\Gamma_0 \theta_z(0,t) &= \nabla (\theta(0,t) - u_{in}(t)) \\
\theta_z(t,0) &= 0 \\
\theta(z,0) &= \theta_0(z)
\end{align*}$$

where \(\theta (. , t) = [\theta_1(. , t), \ldots, \theta_n(. , t)]^T \in H = L_2^2(0,l)\) denotes the vector of state variables that represents the components concentration of the process, \(z \in [0,l]\) (where \(l\) is the reactor length) and \(t \in [0,\infty)\) denote the spatial and time domains, respectively. \(\Gamma_0 \) and \(\nabla\) are \(n\)-dimensional diagonal matrices of constant entries that represent the diffusion coefficients and the constant advective velocity respectively. \(\Gamma_0\) is assumed to be nonsingular and \(F\) is a locally Lipschitz continuous nonlinear function from a specific subset of \(H\) into \(H\) describing the kinetic part of the process (reaction rate equations). The associated state estimation problem for system (1)-(2) consists in designing a dynamical observer on the steady state profile and the input signal such that \(\lim_{t \to \infty} \theta(z,t) - \hat{\theta}(z,t) = 0\).

The nonlinear system (1)-(2) can be linearized around the steady state profile \((\theta_{ss}(z),u_{ss})\) and the resulting linear system is given by:

$$\begin{align*}
\hat{\theta}_t(z,t) &= \Gamma_0 \hat{\theta}_{zz}(z,t) - \nabla \hat{\theta}_z(z,t) + K_0(z) \hat{\theta}(z,t) \\
\hat{\theta}_z(t,0) &= 0 \\
\hat{\theta}(z,0) &= \hat{\theta}_0(z)
\end{align*}$$

subject to the following boundary and initial conditions

$$\begin{align*}
\Gamma_0 \hat{\theta}_z(0,t) &= \nabla (\hat{\theta}(0,t) - \tilde{u}_{in}(t)) \\
\hat{\theta}_z(t,0) &= 0 \\
\hat{\theta}(z,0) &= \hat{\theta}_0(z)
\end{align*}$$

where \(K_0(z) = \frac{\partial F}{\partial \theta}(\theta_{ss}(z)), \) and \(\hat{\theta}(z,t) = \theta(z,t) - \theta_{ss}(z)\) and \(\tilde{u}_{in}(t) = u_{in}(t) - u_{ss}\) are the vector state and control input deviations with respect to their steady state profiles, respectively.

Remark 1: It should be emphasized that the formulation developed in this section can be extended to the case where the matrices \(\Gamma_0 \) and \(\nabla\) are diagonalizable. Indeed, the state transformation can be used to return to the case where the matrices are diagonal. Moreover, in most chemical engineering processes, \(\Gamma_0 \) and \(\nabla\) are symmetric and then diagonalizable.

The equation in (3) is of type diffusion-convection-reaction PDE. In view of solving the eigenvalue problem, it is much easier to convert the equation to a diffusion-reaction type. To this end, consider the following transformation:

$$x(z,t) = T \hat{\theta}(z,t) = \exp \left( -\frac{1}{2} \Gamma_0^{-1} \nabla \right) \hat{\theta}(z,t).$$

By using the above transformation, the PDE system (3)-(4) can be described in terms of a new state vector \(x(z,t)\) leading to the following linear diffusion-reaction coupled parabolic PDE:

$$x_t(z,t) = \Gamma x_{zz}(z,t) + K(z)x(z,t)$$

subject to the boundary and initial conditions given by

$$\begin{align*}
\Gamma x_z(0,t) &= \frac{1}{2} \nabla x(0,t) - \nabla \tilde{u}_{in}(t) \\
\Gamma x_z(t,0) &= -\frac{1}{2} \nabla x(t,0) \\
x(z,0) &= T^{-1} \hat{\theta}_0(z)
\end{align*}$$

where the matrices \(K(z)\) and \(\Gamma\) are given by

$$\begin{align*}
K(z) &= T[K_0(z) - \frac{1}{4} \nabla \Gamma_0^{-1} \nabla] T^{-1} \\
\Gamma &= TT\Gamma_0 T^{-1}.
\end{align*}$$

A. Infinite-dimensional formulation

We can formulate the system as an abstract boundary system on the infinite-dimensional space \(H\) [8] by considering that \(u(t) = \nabla \tilde{u}_{in}(t)\) yielding the following state space representation:

$$\begin{align*}
\dot{x}(t) &= \mathcal{U}x(t) \\
\mathcal{B} x(t) &= u(t) \\
y(t) &= \mathcal{C} x(t)
\end{align*}$$

where \(x(. , t) \in H\) and the operators \(\mathcal{U} : D(\mathcal{U}) \to H, \mathcal{B} : D(\mathcal{B}) \to \mathbb{R}^{n_u}, \mathcal{C} : D(\mathcal{C}) \to \mathbb{R}^{n_y}\) are defined as

$$D(\mathcal{U}) = \left\{ x \in H : x, \frac{dx}{dz} \text{ are a.c., } \frac{d^2x}{dz^2} \in H \right\}$$

and

$$D(\mathcal{B}) = \left\{ x \in H : -\frac{dx}{dz}(0) + \frac{1}{2} \nabla x(0) = 0 \right\}$$

$$D(\mathcal{C}) = \left\{ x \in H : \nabla x = [x(0)] \right\}.$$

B. Reformulation on the extended space

Boundary problems occur frequently in applications, but unfortunately they do not fit into the standard formulation \(\Sigma(A, B, C)\) with bounded observation and control operators \(C\) and \(B\), respectively. However, for sufficiently smooth control inputs \(u(t)\) it is possible to reformulate such problem on an extended state space so that they lead to an associated system in the standard form \(\Sigma(A^e, B^e, C^e)\) as described in [8]. Firstly, a new operator \(A\) is defined by

$$A x = \mathcal{U} x,$$

$$D(A) = D(\mathcal{U}) \cap \ker \mathcal{B}$$

$$= \left\{ x \in H : x, \frac{dx}{dz} \text{ are a.c., } \frac{d^2x}{dz^2} \in H \right\}$$

and

$$\Gamma \frac{dx}{dz}(0) - \nabla x(0) = 0, \Gamma \frac{dx}{dz}(l) + \nabla x(l) = 0.$$
The latter are bounded, then $A$ is the infinitesimal generator of a $C_0$-semigroup on $H$. Assume that there exists an operator $B \in \mathcal{L}(U,H)$ such that for all $u \in U$, $Bu \in D(\mathcal{U})$ and the following holds:

$$\Box Bu = u. \quad (12)$$

By defining the new input as $\hat{u}(t)$ and a new state

$$x^e(t) = \begin{bmatrix} x_1^e(t) \\ x_2^e(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ x(t) - Bu(t) \end{bmatrix} \quad (13)$$

the system representation can be reformulated on the extended state space $H^e = U \oplus H$ as a well-posed system with bounded input operator and unbounded output operator:

$$\dot{x}^e(t) = A^e x^e(t) + B^e \hat{u}(t) \quad y(t) = C^e x^e(t) \quad (14)$$

where

$$A^e = \begin{bmatrix} 0 & 0 \\ \Omega B & A \end{bmatrix}, \quad B^e = \begin{bmatrix} -B \\ I \end{bmatrix}, \quad C^e = \mathcal{C} \begin{bmatrix} B & I \end{bmatrix}. \quad (15)$$

Observe that the operator $C^e$ is well-defined on $U \oplus D(\mathcal{C})$, provided that $D(B) \subset D(\mathcal{C})$. Hence, the operator $A^e$ generates a $C_0$-semigroup on $H^e$ given by

$$T^e(t) = \begin{bmatrix} I & 0 \\ \int_0^t T(s) \Omega Bu ds & T(t) \end{bmatrix} \quad (16)$$

where $T(t)$ is the $C_0$-semigroup generated by $A$. Since the performed transformation is an exact transformation, the resulting system (14) is equivalent to the original system in the sense of [8]. As we need to estimate $x(t)$ we must focus on the second state equation of (14) and the measured output $y(t)$ which can be rewritten as

$$\dot{x}_2^e(t) = \hat{A} x_2^e(t) + \Omega Bu(t) - Bu(t) \quad y(t) = \mathcal{C} x_2^e(t) + \mathcal{C} Bu(t) \quad (17)$$

Therefore, the proposed observer is formally given by

$$\dot{\tilde{x}}_2^e(t) = \hat{A} \tilde{x}_2^e(t) + \Omega Bu(t) - Bu(t) + L(\tilde{y}(t) - y(t)) \quad \tilde{y}(t) = \mathcal{C} \tilde{x}_2^e(t) + \mathcal{C} Bu(t). \quad (18)$$

The convergence of the observer (18) is achieved by finding a bounded operator $L$ which ensures the exponential stability of the corresponding error dynamics. Finally, we can recover the estimation of the original states by using

$$\dot{x}(t) = \tilde{x}_2^e(t) + Bu(t). \quad (19)$$

**III. APPLICATION TO A TUBULAR REACTOR**

The process considered in this section is the tubular catalytic cracking reactor presented in [13]. This process involves axial dispersion, convection and the following reactions taking place in it:

$$\Theta_1 \xrightarrow{k_1} \Theta_2 \xrightarrow{k_2} \Theta_3 \quad \Theta_1 \xrightarrow{k_3} \Theta_3 \quad (20)$$

where $\Theta_1$ represents gas oil, $\Theta_2$ gasoline and $\Theta_3$ other products (e.g. butanes, coke, etc.). Putting aside the reactions related to $\Theta_3$, the dynamics of the system are described by the following parabolic PDEs representing the component balances of $\Theta_1$ and $\Theta_2$ within the reactor:

$$\theta_1(z,t) = \gamma \theta_1(z,t) - \nu \dot{\theta}_1(z,t) - k_0 \dot{\theta}_1^2(z,t) \quad (21)$$

$$\theta_2(z,t) = \gamma \theta_2(z,t) - \nu \dot{\theta}_2(z,t) + k_1 \dot{\theta}_1^2(z,t) - k_2 \dot{\theta}_2(z,t) \quad (22)$$

subject to the boundary conditions given by

$$\gamma \theta_1(z,0,t) = v(\theta_1(0,t) - u_1(t)) \quad \theta_1(z,l,t) = 0 \quad (23)$$

$$\gamma \theta_2(z,0,t) = v(\theta_2(0,t) - u_2(t)) \quad \theta_2(z,l,t) = 0$$

for all $t \geq 0$ and all $z \in [0,l]$, where $l$ is the reactor length, and $\theta_1(z,t)$, $\theta_2(z,t)$, $\gamma$, $v$, $u_1(t)$ and $u_2(t)$ denote the weight fractions of reactant $\Theta_1$ and $\Theta_2$, the axial dispersion coefficient, the superficial velocity, the inlet weight fraction of component $A$ and the inlet weight fraction of component $B$, respectively. Such model takes the form of (1) by considering

$$\theta(z,t) = \begin{bmatrix} \theta_1(z,t) \\ \theta_2(z,t) \end{bmatrix}, \quad \Gamma_0 = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix} \quad (24)$$

$$\Psi = \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} \quad F(\theta) = \begin{bmatrix} -k_0 \theta_1^2 \theta_2 \\ k_1 \theta_1^2 - k_2 \theta_2 \end{bmatrix}. \quad (25)$$

The corresponding steady-state equations of the PDEs model defined by (21)-(22) are given by the following ODEs:

$$\gamma \frac{d^2 \theta_{1ss}}{dz^2} - v \frac{d \theta_{1ss}}{dz} - k_0 \theta_{1ss}^2 = 0 \quad (24)$$

$$\gamma \frac{d^2 \theta_{2ss}}{dz^2} - v \frac{d \theta_{2ss}}{dz} + k_1 \theta_{1ss}^2 - k_2 \theta_{2ss} = 0 \quad (25)$$

subject to the boundary conditions

$$\gamma \frac{d \theta_{1ss}}{dz}(0) = v(\theta_{1ss}(0) - u_{1ss}) \quad \frac{d \theta_{1ss}}{dz}(l) = 0 \quad (26)$$

$$\gamma \frac{d \theta_{2ss}}{dz}(0) = v(\theta_{2ss}(0) - u_{2ss}) \quad \frac{d \theta_{2ss}}{dz}(l) = 0. \quad (27)$$

The adopted numerical values for the process parameters are taken from Table I (see [13]).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Numerical Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l$</td>
<td>1 m</td>
</tr>
<tr>
<td>$k_0$</td>
<td>22.9 (l x weight fraction)</td>
</tr>
<tr>
<td>$k_1$</td>
<td>18.1 (l x weight fraction)</td>
</tr>
<tr>
<td>$k_2$</td>
<td>1.7 (h)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.5 m$^2$ x h$^{-1}$</td>
</tr>
<tr>
<td>$v$</td>
<td>2 m x h$^{-1}$</td>
</tr>
<tr>
<td>$u_1(t)$</td>
<td>0.7 weight fraction</td>
</tr>
<tr>
<td>$u_2(t)$</td>
<td>0 weight fraction</td>
</tr>
</tbody>
</table>

**TABLE I: Parameters of the process.**

The steady state profiles of $\theta_1(z,t)$ and $\theta_2(z,t)$ are shown in Figure 1. They were obtained by solving numerically (24)-(27) considering the parameters given in Table I.

The deviation vector $\theta(z,t) = \theta(z,t) - \theta_{ss}(z)$ is substituted by the new vector variable $x(z,t)$ according to the transformation defined in (5) leading to

$$\begin{bmatrix} x_1(z,t) \\ x_2(z,t) \end{bmatrix} = \begin{bmatrix} e^{-\frac{z}{\gamma}} & 0 \\ 0 & e^{-\frac{z}{\gamma}} \end{bmatrix} \begin{bmatrix} \theta_1(z,t) \\ \theta_2(z,t) \end{bmatrix}. \quad (28)$$
By applying the procedure described in Section II and assuming that there exists a single input \( u_1(t) \) (setting \( u_2(t) = 0 \)), we can formulate the system as an abstract boundary system on the Hilbert space \( H = L^2(0,1) \oplus L^2(0,1) \) according to

\[
\begin{align*}
\dot{x}(t) &= \mathcal{U}x(t) \quad x(0) = x_0 \\
\mathcal{B}x(t) &= u(t) \\
y(t) &= \mathcal{C}x(t)
\end{align*}
\]

where

\[ x(\cdot, t) = \left\{ \begin{array}{c} x_1(\cdot, t) \\ x_2(\cdot, t) \end{array} \right\}, \quad 0 \leq z \leq l \in H \]

and the operators \( \mathcal{U} : D(\mathcal{U}) \to H, \mathcal{B} : D(\mathcal{B}) \to \mathbb{R}, \mathcal{C} : D(\mathcal{C}) \to \mathbb{R}^2 \) are defined as

\[
\mathcal{U} = \begin{bmatrix} \gamma \frac{d^2}{dz^2} - \hat{k}_1(z) & 0 \\ 2k_1x_{1ss}(z) & \gamma \frac{d^2}{dz^2} - \hat{k}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{U}_{11} & 0 \\ \mathcal{U}_{21} & \mathcal{U}_{22} \end{bmatrix}
\]

\[
D(\mathcal{U}) = \left\{ x \in H : x, \frac{dx}{dz}, \frac{d^2x}{dz^2} \in H \right\}
\]

\[
\mathcal{U}_{11} x_1 (l) + \frac{\nu}{2} x_1 (l) = 0,
\]

\[
\mathcal{U}_{21} x_2 (0) - \frac{\nu}{2} x_2 (0) = 0,
\]

\[
\mathcal{U}_{22} x_2 (l) + \frac{\nu}{2} x_2 (l) = 0
\]

\[
\mathcal{B}x = -\gamma \frac{dx_1}{dz}(0) + \frac{\nu}{2} x_1 (0)
\]

\[
\mathcal{C}x = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}
\]

with \( \hat{k}_1(z) = \frac{\nu^2}{4} + 2k_1 \delta_{1ss}(z) \) and \( \hat{k}_2 = \frac{\nu^2}{4\gamma} + k_2 \).

### IV. Observer Design

The observer design is based on the procedure described in Section II-B with \( \mathcal{A} \) as defined below

\[
\mathcal{A} = \begin{bmatrix} \gamma \frac{d^2}{dz^2} - \hat{k}_1(z) & 0 \\ 2k_1x_{1ss}(z) & \gamma \frac{d^2}{dz^2} - \hat{k}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}
\]

\[
D(\mathcal{A}) = D(\mathcal{U}) \cap \ker \mathcal{B}
\]

\[
\begin{align*}
\gamma \frac{dx}{dz}(0) - \frac{1}{2} \gamma x(0) &= 0, \\
\gamma \frac{dx}{dz}(l) + \frac{1}{2} \gamma x(l) &= 0
\end{align*}
\]

The study of the spectrum of \( \mathcal{A} \) allows designing an observer with a specified convergence rate. This section summarizes the computation of the eigenvalues and eigenvectors of operator \( \mathcal{A} \).

Notice that \( -\mathcal{A}_{11} \) and \( -\mathcal{A}_{22} \) are both Sturm-Liouville operators, which are self-adjoint with respect to an appropriate inner product.

Now, let \( \lambda_n \) and \( \chi_n \) be the eigenvalues and eigenfunctions of the operator \( \mathcal{A}_{11} \), and \( \mu_n \) and \( \psi_n \) be the eigenvalues and eigenfunctions of the operator \( \mathcal{A}_{22} \). Then, it follows that:

- \( \mathcal{A}_{22} \) is a linear operator with constant coefficients and its eigenvalues are given by

\[
\mu_n = -\gamma w_n^2 - \hat{k}_2, \quad \text{with} \ \tan(w_n l) = \frac{4\gamma w_n \nu}{4\gamma^2 w_n^2 - \nu^2}
\]

and the corresponding eigenfunctions are given by

\[
\psi_n = \cos(w_n z) + \frac{\nu}{2\gamma w_n} \sin(w_n z).
\]

- \( \mathcal{A}_{11} \) is a linear operator with reaction coefficient depending on \( z \), consequently the calculation of the spectrum of the operator \( \mathcal{A}_{11} \) is a challenging issue. This problem can be carried out by dividing the length of the reactor into a finite number \( N \) of segments where it is assumed that at each segment the values of coefficients are constant. The mathematical formulation of this approach can be found in [14]. Hence, the eigenvalues \( \lambda_n \) are given by

\[
\lambda_n = -\gamma w_n^2 - \hat{k}_{1n}, \quad \text{with} \ \tan(w_n l) = \frac{4\gamma w_n \nu}{4\gamma^2 w_n^2 - \nu^2}
\]

and the corresponding eigenfunctions are given by

\[
\chi_n(z) = \rho_n \sin(w_n z) + \eta_n \sin(w_n z)
\]

where \( \rho_1 = 1; \ \rho_n = s_{n,n-1} \sin(w_{n-1} z) + \eta_n \cos(w_{n-1} z) \) and

\[
s_{n,n-1} = \frac{\sin(w_{n-1} z) + \eta_n \cos(w_{n-1} z)}{\sin(w_n z) + \eta_n \cos(w_n z)}
\]

\[
s_{N,N-1} = \frac{1}{h_N} \sin(w_{N-1} z) + \eta_N \cos(w_{N-1} z)
\]

\[
\eta_1 = -\nu \sin(w_1 z) - 2h_1 w_1 \cos(w_1 z)
\]

\[
\eta_i = \frac{num_i}{den_i}
\]
is the union of two parts, estimation error pair $(B, Luenberger-observer)$ with the corresponding eigenvectors given by $A$ where $A$ and are given by

\[ x(\tau, z, t) = e^{\tau A} x(0, z, t) + \int_0^\tau e^{(\tau - \tau') A} \beta(\tau', z, t) \, d\tau'. \]

Hence, the first five eigenvalues of the operators $A_{11}$ and $A_{22}$ are as follows

$\lambda = \{ -2.39 \times 10^{-5}, -1.34 \times 10^{-4}, -4.46 \times 10^{-4}, -1.12 \times 10^{-3}, 2.35 \times 10^{-3} \}$

$\mu = \{ -2.04 \times 10^{-6}, 1.096 \times 10^{-5}, -5.68 \times 10^{-5}, -2.08 \times 10^{-4}, -5.78 \times 10^{-4} \}$.

**B. Luenberger-observer**

In order to design a dynamical observer of the form (18), we must guarantee the $\beta$-exponential detectability of the pair $(A, C)$, where $C$ should be interpreted as a bounded linear functional on $H$ which approaches the action of the Dirac delta distribution. Assuming a decay rate for the state estimation error $\beta = -2.5 \times 10^{-6}$, which provides faster convergence than the one of the states, the spectrum of $\sigma(A)$ is the union of two parts, $\sigma_\beta^+(A)$ and $\sigma_\beta^-(A)$, such that a rectifiable, closed, simple curve $\mathcal{C}_\beta$ can be drawn so as to enclose an open set containing $\sigma_\beta^+(A)$ in its interior and $\sigma_\beta^-(A)$ in its exterior. The operator, $P C_\beta$, defined by

\[ P C_\beta x = \frac{1}{2\pi j} \int_{C_\beta} (\lambda I - A)^{-1} x \, d\lambda \]

is the spectral projection operator that induces a decomposition of the state space as well as the operators according to

\[ H = H_\beta^+ \oplus H_\beta^- \]

\[ P C_\beta x = \sum_{\sigma(A) \in \sigma(A)_\beta^+} \langle x, \Psi_n \rangle \phi_n \]

\[ A_\beta^+ x = A P C_\beta x = \sum_{\sigma(A) \in \sigma(A)_\beta^+} \sigma_n(A) \langle x, \Psi_n \rangle \phi_n \]

(36)

\[ \mathcal{C}_\beta^+ x = \mathcal{C} P C_\beta x = \sum_{\sigma(A) \in \sigma(A)_\beta^+} \langle x, \Psi_n \rangle \mathcal{C} \phi_n . \]

The system $\Sigma(A, -, \mathcal{C})$ is $\beta$-exponentially detectable by $L = i_\beta L_0$, where $L_0$ is such that $A_\beta^+ + L_0 \mathcal{C}_\beta^+$ is $\beta$-exponentially stable and $i_\beta$ is the injection operator from $H_\beta^+$ to $H$ as shown in [8]. To guarantee the well-posedness of our formulation with bounded operators, the operator $C$ is redefined as

\[ \mathcal{C} x = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \approx \begin{bmatrix} \int_0^1 1_{[0, \epsilon]} x_1(z,t) \, dz \\ \int_0^1 1_{[0, \epsilon]} x_2(z,t) \, dz \end{bmatrix} \]

where

\[ 1_{[0, \epsilon]} = \begin{cases} 1, & 0 \leq z \leq \epsilon, \\ 0, & \text{elsewhere}, \end{cases} \]

for a sufficiently small $\epsilon$. Considering that $\epsilon = 0.01$, we obtain

\[ (A_\beta^+, -, \mathcal{C}_\beta^+) = \left( \begin{array}{c} \mu_2, -j \int_0^{0.91} 1_{[0, \epsilon]} \psi_1(z) \, dz \\ 0 \end{array} \right) = \left( \begin{array}{c} 2.40 \times 10^{-6}, -j \int_0^{0.91} 1_{[0, \epsilon]} \psi_1(z) \, dz \\ 0 \end{array} \right) \]

(38)

and $L_0 = [0 - 5.05 \times 10^{-7}]$. Notice that $L_0$ is such that $A_\beta^+ + L_0 \mathcal{C}_\beta^+$ is $\beta$-exponentially stable. Thus,

\[ L y = i_\beta L_0 y = \begin{bmatrix} 0 \\ \psi_1(z) \end{bmatrix} [0 - 5.05 \times 10^{-7}] y . \]

(39)

The observer system has been simulated considering the parameters of the system given in Table I with the initial profiles for the observer system as $\hat{\theta}_1 = \hat{\theta}_2 = 0$. The numerical simulation of PDE systems can be achieved using the method of lines and the MATMOL library [15]. Figures 2 and 3 show the evolution of the actual states $\theta_1$, $\theta_2$ (red lines) and the estimated states $\hat{\theta}_1$, $\hat{\theta}_2$ (blue lines) related to the proposed observer.

**V. CONCLUDING REMARKS**

In this paper, a Luenberger-type state observer is presented for a certain class of nonlinear coupled parabolic PDEs considering only boundary measurements. To this end, a linear approximate model of the system around a given steady-state profile is considered, which results in a linear DPS with spatially-varying coefficients. The observer output injection operator is thereby initially determined by making use of the boundary conditions for the stability of a triangular operator having Sturm-Liouville operators in its diagonal.
Subsequently, the output injection operator is designed by using spectrum decomposition conditions ensuring a certain convergence decay rate for the estimation error dynamics. The proposed observer was applied to a tubular catalytic cracking reactor and the observer performance was studied via numerical simulations. It has been observed that the formulated observer has provided an accurate estimation of the states of the original plant.

Moreover, in view of the promising numerical results obtained in the application (see e.g. Figures 2 and 3), an interesting open problem is the extension of the method described in this paper to a more general class of coupled parabolic PDEs with not necessarily triangular structure. This problem is currently under investigation by the authors. Thus, the proposed study is a point of departure to design observers based on the spectrum for more general state operators.

**References**


