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Conjoint axiomatization of Min, DiscriMin and LexiMin

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Abstract

In many multi-criteria decision-making applications, the preferential information is of an ordinal nature and appropriate aggregation procedures should be used. In this paper, we build a common axiomatic framework to characterize the "Minimum" procedure as well as two of its refinements, the "DiscriMin" and the "LexiMin". In practical situations, this axiomatic framework could help to select the best-suited procedure.

Keywords: Multi-criteria; Decision making; Ordinal data; Maxmin

1. Introduction

More often than not, data in decision problems—constraint satisfaction problems (CSP), multi-criteria decision aiding (MCDA), social choice—are of an ordinal nature and it may be so even when data are encoded as numbers. Obviously, when data are ordinal, they should be "treated" accordingly: determining whether $a \succcurlyeq b$ ("a is at least as good as b" or "a is preferred or indifferent to b") should not involve performing arithmetic operations on ordinal data unless a meaningful quantitative recoding can be obtained from the decision maker (as is current practice when building, for instance, an additive utility model, see [33]).

In a flexible constraint satisfaction problem (FCSP) [19,21], a solution a is assessed for each constraint C_i by the degree $\mu_{C_i}(a) \in [0,1]$ up to which a satisfies C_i. Indeed, C_i is allowed to be partially satisfied, i.e. up to a certain degree between 0 and 1; "0" (respectively "1") means the complete violation (resp. satisfaction) of the constraint C_i. It is generally recognized that satisfaction degrees, although expressed as numbers, are a matter of feeling and thus hardly measurable; it is thus advisable to treat these numbers as ordinal data. The comparison of two solutions a and b in the FCSP approach has to be consistent with this ordinal character. In view of the meaning of a constraint, it makes sense to associate a solution a with a global satisfaction degree [19] that is often

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defined as:

$$\mu_{\mathcal{C}}(a) = \min_{i=1,\ldots,n} \mu_{C_i}(a), \tag{1}$$

$\mathcal{C}$ being the set of constraints $\{C_1, \ldots, C_n\}$. Therefore, a is preferred or indifferent to b, if its global satisfaction degree is larger than that of b:

$$a \succcurlyeq b \;\Leftrightarrow\; \mu_{\mathcal{C}}(a) \geqslant \mu_{\mathcal{C}}(b) \;\Leftrightarrow\; \min_{i=1,\ldots,n} \mu_{C_i}(a) \geqslant \min_{i=1,\ldots,n} \mu_{C_i}(b). \tag{2}$$

More generally, we consider the problem of comparing and ranking a set $\mathcal{A}$ of alternatives $\{a, b, \ldots\}$, where each alternative is evaluated with respect to several viewpoints. In other words, we have to compare and rankorder *vectors of performances* (also called *profiles*): $a \equiv (a_1, \ldots, a_n)$. In this paper, we identify alternatives with their profiles, which supposes that the selected viewpoints describe all relevant aspects for the comparison of the alternatives by the decision maker in the considered context.

A common way to proceed in such a case is to use an *aggregation function* u, that summarizes the profile $(a_1, \ldots, a_n)$ of a in a single numerical value $u(a)$. This *global score* or *utility function* induces a weak order $\succcurlyeq$ on $\mathcal{A}$:

$$a \succcurlyeq b \;\Leftrightarrow\; u(a) \geqslant u(b). \tag{3}$$

A weak order is a complete ($\forall\, a, b \in \mathcal{A}, a \succcurlyeq b$ or $b \succcurlyeq a$) and transitive ($\forall\, a, b, c \in \mathcal{A}, a \succcurlyeq b$ and $b \succcurlyeq c$ implies $a \succcurlyeq c$) relation. Aggregation functions and their properties (such as monotonicity, continuity, ...) have been investigated in various contexts (see e.g. [1–4,24,27]); there are few axiomatizations of the "min" function as an operator acting on vectors of numbers and returning a number; it is often the case that a family of aggregation functions (e.g. Sugeno integrals) are characterized, of which "min" is a special case (e.g. [27] or [28]; see also [16]).

In the present paper, we follow another path. We do not aim at characterizing a function or an operator, but instead, we directly characterize the global preference relation on the set of alternatives. Of course, it may happen (and it will) that this preference is determined via some score u like in Eq. (3) but we do not restrict ourselves a priori to relations that can be obtained in that way. This has important consequences since all relations that can be described by Eq. (3), using some function u, are weak orders, which will not be true in general in our more general framework. This approach is not newer than the characterization of functional operators; it is more in the tradition of utility theory and conjoint measurement theory [26,31] as well as social choice [32] and game theory [29]. In all these frameworks (but especially the latter two), characterizations of preference relations on sets of alternatives described by vectors of ordinal data have been obtained, including the Min procedure (that produces the relation according to rule (2)) and the refinements of the Min that we consider in this paper (see Section 3).

What then distinguishes the present work? First the "style" of the axioms. And a common axiomatic framework for three procedures. Our axioms are not intended to be "normative" in the sense that they would point to one of the procedures and designate it as the best one (w.r.t. some criterion such as rationality for instance). They mainly refer to the behaviour of the procedure in response to changes in the evaluations of the alternatives. For instance, monotonicity (i.e., roughly, "if a is preferred to b and the performance of a improves on some criterion, this should not result in b being preferred to improved a") is an axiom of that type, but monotonicity is also a rationality

axiom: one does not expect (except perhaps in very special situations) that any reasonable procedure would not satisfy monotonicity. Our axioms relate to the behaviour of the procedures, but are not supposed to be universally accepted: a property could make sense in a preference modelling context and seem unnatural in another context. Since for characterizing the Min and its two variants, we use various relaxations and strengthenings of the same basic axioms, we hope to provide a kind of "common language" that could help to question about the opportunity of using a procedure or another in a given context. Another advantage of this conjoint axiomatization is to enhance the understanding of what may distinguish the various procedures studied. This work is in line with previous papers [12,30].

The paper is organized as follows. In the next section, we present a problem, namely, the bottleneck assignment problem, in which the Min procedure appears as insufficient; this motivates the introduction of the two refinements of the Min that are described in Section 3; some of their main properties are recalled. In that section, we discuss in some detail several procedures that have been proposed in various fields of research; we show that those procedures all correspond to what we call "DiscriMin". In Section 4, we present the axioms and prove the characterizations of Min, DiscriMin and LexiMin. That of the DiscriMin procedure is new and bridges the gap between those of Min and LexiMin that were already in [30].

2. A bottleneck assignment problem

This section exemplifies difficulties emerging in a classical bottleneck assignment problem. The same difficulties are observed in many other decision problems where the maxmin (or the minmax) rule seems a reasonable criterion for ranking the alternatives [20,21].

The bottleneck assignment problem (BAP) is a variant of the linear assignment problem (LAP). In the latter, the optimization criterion is linear, while in the former it is based on the min operator.

Consider a set of n tasks to be assigned to n operators. One and only one task has to be committed to each operator; for each operator $i \in \{1, \ldots, n\}$, let $x_i \in \{1, \ldots, n\}$ be the task allocated to him. Each operator has been asked to express his preferences for the various tasks; let $\mu_i(j) \in [0, 1]$ be the degree to which operator i "likes" task j.

Solving the problem is determining the n variables x_i such that the satisfaction of the least satisfied operator, $\min_i \mu_i(x_i)$ is maximized, while respecting the natural constraints of the problems (in particular, $i \neq j \Rightarrow x_i \neq x_j$). In other words, if a and b represent two different complete assignments, these solutions are compared taking into account their performance profiles ($a_i = \mu_i(x_i^{(a)})$ and similarly for b_i) and,

$$a \succcurlyeq b \quad \Leftrightarrow \quad \min_i a_i \geqslant \min_i b_i. \tag{4}$$

The following numerical example is intended to clarify our point. Let

$$M = \begin{pmatrix} 0.4 & 0.0 & 0.1 & 0.2 & 0.4 \\ 0.2 & 0.5 & 0.1 & 0.5 & 0.0 \\ 0.8 & 0.1 & 0.6 & 0.0 & 0.2 \\ 0.0 & 0.7 & 0.2 & 0.6 & 0.1 \\ 0.7 & 0.2 & 0.6 & 0.1 & 0.6 \end{pmatrix}.$$

The matrix M contains the preference degrees for an assignment problem with five tasks and five operators. Each line (resp. row) corresponds to an operator (resp. a task).

The solution $x^{(a)} = (1, 2, 3, 4, 5)$ assigns the first task to the first operator, the second task to the second operator and so on; the corresponding performance profile is $a = (0.4, 0.5, 0.6, 0.6, 0.6)$. This solution is "optimal", since there does not exist a solution $x^{(b)}$ such that $b \succ a$ according to relation (4). Indeed, the least satisfied operator, i.e. the one corresponding to the first row and satisfied only to degree 0.4, cannot be more satisfied in any other assignment. However, there exists a solution that one would probably prefer to solution $x^{(a)}$. As a matter of fact, if we consider solution $x^{(b)} = (5, 4, 3, 2, 1)$ with profile $b = (0.4, 0.5, 0.6, 0.7, 0.7)$, we observe that the first three operators enjoy the same degree of satisfaction as in the previous solution, while the satisfaction for both the fourth and the fifth ones have increased. This situation claims for a refined comparison procedure, which would enforce not only min-optimality but also Pareto-optimality (see Section 3.2).

Before going into further details, let us examine a third solution $x^{(c)} = (5, 4, 1, 2, 3)$ with performance profile $c = (0.4, 0.5, 0.8, 0.7, 0.6)$. The latter solution is also to be preferred to solution $x^{(a)}$, since each individual operator is at least as satisfied with solution $x^{(c)}$ as with solution $x^{(a)}$. But should it be preferred to solution $x^{(b)}$, some operators being more satisfied in $x^{(b)}$ than in $x^{(c)}$, and vice versa? Which is/are the best solution(s) and with respect to which ordering procedure?

3. Min and its refinements

We formalize in this section the Min procedure to build a global preference relation on vectors of performances. As shown in the previous section, the Min shows some weaknesses. Various procedures that refine the Min have been proposed in the literature. We recall them and draw some links between apparently different definitions that turn out to yield the same relation.

3.1. Notation

Consider a set of objects. We assume that each object is described in all its relevant aspects by a n-component vector—a profile—belonging to the set $X = [0, 1]^n$. The problem is to determine a relevant procedure to rank the objects, on the basis of their profile. This is a suitable framework for representing and dealing with, for instance,

(a) Multiple-criteria decision problems: objects are alternatives and each alternative is evaluated along n criteria; evaluations may be normalized in order to fit in the $[0, 1]$ interval or they may be supposed to be recoded into "utilities";
(b) Flexible constraint satisfaction problems (FCSP): objects are solutions and each solution is characterized by the degree to which it fulfills each constraint (in a set of n);
(c) In voting theory, each candidate is characterized by a profile indicating the rank of the candidate in the preference ordering of each voter; normalizing the profile by dividing the ranks by the number of voters yields a vector of performances in the $[0, 1]$ interval.

The set of profiles of the objects under consideration will be denoted by $\mathcal{A}$ and is thus a subset of $X = [0, 1]^n$.

A ranking procedure associates to any set $\mathcal{A}$, a relation, i.e. a subset of $\mathcal{A} \times \mathcal{A}$. Often, but not always, this relation may have special properties such as transitivity or completeness. In the sequel, procedures will be denoted by $\succcurlyeq$, possibly with a superscript like *min*, *disc* or *lex* that identifies it. The procedures we consider can be applied to any subset $\mathcal{A}$ of $X = [0,1]^n$ including X itself; we assume that the relation induced on each particular subset $\mathcal{A}$ is the restriction of the relation defined by the procedure on X. We shall therefore make no distinction between a procedure and the relation it induces on X.

A binary relation $\succcurlyeq$ on A is said to be:

- *reflexive* if $[a \succcurlyeq a]$,
- *irreflexive* if $[\text{Not}(a \succcurlyeq a)]$,
- *complete* if $[a \succcurlyeq b$ or $b \succcurlyeq a]$,
- *symmetric* if $[a \succcurlyeq b] \Rightarrow [b \succcurlyeq a]$,
- *asymmetric* if $[a \succcurlyeq b] \Rightarrow [\text{Not}[b \succcurlyeq a]]$,
- *transitive* if $[a \succcurlyeq b$ and $b \succcurlyeq c] \Rightarrow [a \succcurlyeq c]$,

for all $a, b, c \in \mathcal{A}$. The asymmetric (resp. symmetric) part of relation $\succcurlyeq$ is denoted by $\succ$ (resp. $\sim$).

3.2. Min

In the field of fuzzy set theory, the Min and the Max operators have been first proposed by Zadeh [34] to define the intersection and union of fuzzy sets (see also [9]). Bellman and Zadeh [10] have argued in favour of using those operators in a multi-objective programming framework (that could be described as FCSP).

The *Min-procedure* $\succcurlyeq^{\min}$ compares alternatives on the basis of their worst performance:

$$a \succcurlyeq^{\min} b \;\Leftrightarrow\; \min_{i \in I} a_i \geqslant \min_{i \in I} b_i. \tag{5}$$

The definitions of the asymmetric and the symmetric parts $\succ^{\min}$ and $\sim^{\min}$ of $\succcurlyeq^{\min}$ are obvious. The lack of discriminating power exhibited by the min-based procedure in the assignment problem is also known as the "drowning effect" [20,21]. The "Min" has another drawback. A *min-optimal* solution in $\mathcal{A}$ (i.e. a solution b such that there is no $a \in A$ for which $a \succ^{\min} b$) is not necessarily Pareto-optimal; there may exist a solution a that dominates b in the Pareto ordering:

$$a \succ^{\text{Pareto}} b \;\Leftrightarrow\; \begin{cases} \forall i \in I\colon\; a_i \geqslant b_i, \\ \exists i \in I\colon\; a_i > b_i. \end{cases} \tag{6}$$

This may not occur however if there is only one min-optimal solution; a unique min-optimal solution is Pareto-optimal.

Despite its weaknesses, the Min enjoys particular features that command attention; it is well suited for aggregating *ordinal* evaluations when meaningful recodings into cardinal scales seem out of reach or when preferences are non-compensatory [11,13,23] but commensurate (i.e. all criteria are evaluated on the same ordinal scale). Therefore, it makes sense to keep considering this procedure or to start from it and build more discriminating ones. In the next subsections we recall the definitions of some refinements of the Min.

3.3. DiscriMin

The DiscriMin procedure compares solutions according to the inclusion of the α-cuts of their performance profiles. This procedure has been proposed independently by Brewka [14] in a logical setting and by Barbera and Jackson [6] in a welfare economics context (where it is called "Protective Criterion").

Let $\mathcal{I}(a)_\alpha = \{i \in I | a_i > \alpha\}$ be the strong α-cut of the performance profile of a solution a. The *DiscriMin* ordering is defined as follows:

$$a \succcurlyeq^{\text{disc}} b \quad \Leftrightarrow \quad \begin{cases} a = b \quad \text{or} \\ \begin{bmatrix} \exists \alpha' \in [0,1] \text{ such that } \mathcal{I}(a)_{\alpha'} \backslash \mathcal{I}(b)_{\alpha'} \neq \emptyset \\ \quad \text{and} \\ \forall \alpha \in [0,1] \text{ with } \alpha < \alpha',\ \mathcal{I}(a)_\alpha = \mathcal{I}(b)_\alpha \end{bmatrix}. \end{cases} \tag{7}$$

Note that only the α-cuts for values not larger than α' are taken into account; above that level, there is no constraint imposed on the profiles. a is better than b in the sense of the DiscriMin ordering if although both solutions are indifferent for low levels of aspiration, raising this level of aspiration breaks the ties in favour of a.

For ease of reference, we explicitly define the asymmetric and symmetric parts of $\succcurlyeq^{\text{disc}}$:

$$a \succ^{\text{disc}} b \quad \Leftrightarrow \quad \exists \alpha' \in [0,1] \text{ such that: } \begin{cases} \mathcal{I}(a)_{\alpha'} \supsetneq \mathcal{I}(b)_{\alpha'} \quad \text{and} \\ \forall \alpha \in [0,1] \quad \text{with } \alpha < \alpha',\ \mathcal{I}(a)_\alpha = \mathcal{I}(b)_\alpha; \end{cases} \tag{8}$$

$$a \sim^{\text{disc}} b \quad \Leftrightarrow \quad \begin{cases} a = b \quad \text{or} \\ \begin{bmatrix} \exists \alpha' \in [0,1] \text{ such that } \mathcal{I}(a)_{\alpha'} \backslash \mathcal{I}(b)_{\alpha'} \neq \emptyset \neq \mathcal{I}(b)_{\alpha'} \backslash \mathcal{I}(a)_{\alpha'} \\ \quad \text{and} \\ \forall \alpha \in [0,1] \text{ with } \alpha < \alpha', \mathcal{I}(a)_\alpha = \mathcal{I}(b)_\alpha \end{bmatrix}. \end{cases} \tag{9}$$

An equivalent definition [19] relies on the *difference set*, i.e. the set of points of view $\mathcal{D}(a,b)$ on which the evaluations of alternatives a and b differ:

$$\mathcal{D}(a,b) = \{i \in I\colon\ a_i \neq b_i\}, \tag{10}$$

$$a \succcurlyeq^{\text{disc}} b \quad \Leftrightarrow \quad \min_{i \in \mathcal{D}(a,b)} a_i \geqslant \min_{i \in \mathcal{D}(a,b)} b_i. \tag{11}$$

Note that if $a = b$, $\min_{i \in \mathcal{D}(a,b)} a_i = \min_{i \in \mathcal{D}(a,b)} b_i = 1$, since the minimum of an empty subset of $[0,1]$ is the maximal element in $[0,1]$. The above definitions are proven to be equivalent in [20]. The DiscriMin ordering compares alternatives on the basis of their worst performance-like the *Min*—but only taking into account the points of view on which they differ. It is clear from (10–11) that the DiscriMin ordering refines the min-ordering in the sense that $a \succ^{\min} b \Rightarrow a \succ^{\text{disc}} b$ and $a \succ^{\text{disc}} b \Rightarrow a \succcurlyeq^{\min} b$. A solution a is said *DiscriMin-optimal* in a set $\mathcal{A}$, if there is no solution $b \in \mathcal{A}$ such that $b \succ^{\text{disc}} a$. DiscriMin-optimal solutions are Pareto-optimal but the converse is not true in general (see [19]).

$\succ^{\text{disc}}$ is a partial order (an asymmetric and transitive relation). The symmetric part $\sim^{\text{disc}}$ is not necessarily transitive as shown by the following example. Let $a=(0.8,0.4)$, $b=(0.4,0.5)$ and $c=(0.5,0.4)$; we have $a\sim^{\text{disc}} b\sim^{\text{disc}} c$ but $a\succ^{\text{disc}} c$. Hence $\succcurlyeq^{\text{disc}}$—contrary to $\succcurlyeq^{\min}$—is not a weak order.

3.3.1. No reason for regret

Behringer [7] proposed a notion called "No Reason for Regret (NR)" as a refinement of both min- and Pareto-optimality (see also [8]). This author showed that NR-optimal solutions are min- and Pareto-optimal while the converse is not true in general. As stated in [8], a is NR-better ($\succ^{\text{NR}}$) than b if the following holds:

$$a\succ^{\text{NR}} b \;\Leftrightarrow\; \exists j:(b_j<a_j \text{ and } [\forall i,(b_j<a_i) \text{ or } (b_i\leqslant a_i)]). \tag{12}$$

This definition can be motivated as follows. Assume there is a bundle of "goods" to be distributed among m persons. Each person assigns a degree of satisfaction to each subset of goods; different persons may feel different degrees of satisfaction for the same set of goods. Let $a=(a_1,\ldots,a_m)$ and $b=(b_1,\ldots,b_m)$ be two admissible distributions, where a_i is the satisfaction degree of person i for the set of goods he/she receives in distribution a. Each person i wants to maximize his/her individual satisfaction. In human societies, there are collective mechanisms (unanimity, veto power) that generate a feeling for equity or fairness. The reasoning described below could be a basis for generating such a feeling.

Distribution a will be (socially) preferred to b if

1. at least one individual feels better with a than with b:

   $$\exists j: a_j>b_j$$

   and, for each other person i,

2. (a) either i is not worse off:

   $$\forall i: a_i\geqslant b_i$$

   or

   (b) if i receives less satisfaction ($a_i<b_i$), he/she has "no reason for regretting" the choice of distribution a instead of b, because he/she is still better off than member j was with distribution b:

   $$a_i>b_j.$$

   With distribution b, j would have reasons for complaining and might put his/her veto, disabling the society agreement.

The NR ordering, proposed by Behringer in the domain of multi-objective optimization, is also known in the field of cooperative games theory as Osborne and Rubinstein's ordering [18,29]. It is usually defined in that context as a complete relation:

$$a\succcurlyeq^{\text{NR}} b \;\Leftrightarrow\; a=b \quad\text{or}\quad \exists j:[a_j>b_j \text{ and } \forall i,\ a_i\geqslant \min(b_j,b_i)]. \tag{13}$$

It is not immediately evident that the relation defined by (12) is the asymmetric part of the relation defined by (13), i.e. $a \succ^{\mathrm{NR}} b$ iff Not$[b \succcurlyeq^{\mathrm{NR}} a]$. The easiest way of showing this might be indirectly, as a consequence of Proposition 1.

Proposition 1.

$$a \succ^{\mathrm{disc}} b \quad \Leftrightarrow \quad a \succ^{\mathrm{NR}} b$$

and

$$a \succcurlyeq^{\mathrm{disc}} b \quad \Leftrightarrow \quad a \succcurlyeq^{\mathrm{NR}} b.$$

Proof. $[\succ^{\mathrm{disc}} \Rightarrow \succ^{\mathrm{NR}}]$ Suppose that $a \succ^{\mathrm{disc}} b$; using (11), we have $\min_{i \in \mathcal{D}(a,b)} a_i > \min_{i \in \mathcal{D}(a,b)} b_i$ where $\mathcal{D}(a,b) = \{i \in I\colon\ a_i \neq b_i\}$. Let a_k (resp. b_j) be the minimum of a_i (resp. b_i) for i in $\mathcal{D}(a,b)$. We have

$$b_j < a_k \leqslant a_j,$$

$$b_i = a_i, \quad \forall i \notin \mathcal{D}(a,b),$$

$$b_j < a_k \leqslant a_i, \quad \forall i \in \mathcal{D}(a,b).$$

Therefore, $a \succ^{\mathrm{NR}} b$.

$[\succ^{\mathrm{NR}} \Rightarrow \succ^{\mathrm{disc}}]$ Suppose that $a \succ^{\mathrm{NR}} b$; in view of (12), for all $i \in \mathcal{D}(a,b)$, we have either $b_j < a_i < b_i$ or $b_i < a_i$. Let $\mathcal{D}_1 = \{i \in \mathcal{D}(a,b), b_j < a_i < b_i\}$ and $\mathcal{D}_2 = \{i \in \mathcal{D}(a,b), b_i < a_i\}$. $(\mathcal{D}_1, \mathcal{D}_2)$ is a partition of $\mathcal{D}(a,b)$. $\mathcal{D}(a,b)$ is not empty since $j \in \mathcal{D}_2$. We have:

$$\min_{i \in \mathcal{D}_1} b_i > \min_{i \in \mathcal{D}_1} a_i > b_j$$

and

$$\min_{i \in \mathcal{D}_2} b_i \leqslant b_j \quad \text{since } j \in \mathcal{D}_2;$$

hence

$$\min_{i \in \mathcal{D}(a,b)} b_i = \min_{i \in \mathcal{D}_2} b_i \leqslant b_j.$$

We also have

$$\min_{i \in \mathcal{D}_1} a_i > b_j \geqslant \min_{i \in \mathcal{D}(a,b)} b_i$$

and

$$\min_{i \in \mathcal{D}_2} a_i > \min_{i \in \mathcal{D}_2} b_i \geqslant \min_{i \in \mathcal{D}(a,b)} b_i;$$

hence

$$\min_{i \in \mathcal{D}(a,b)} a_i > \min_{i \in \mathcal{D}(a,b)} b_i.$$

Therefore, $a \succ^{\mathrm{disc}} b$.

A similar argument proves the equivalence of $\succcurlyeq^{\text{disc}}$ and $\succcurlyeq^{\text{NR}}$. $\square$

3.3.2. Remark on Justmann order

A variant of $\succcurlyeq^{\text{NR}}$ has been proposed in [25] and was compared with $\succcurlyeq^{\text{NR}}$ in [18]. The definition of the *Justmann order* $\succcurlyeq^{\text{J}}$ is as follows:

$$a \succcurlyeq^{\text{J}} b \;\Leftrightarrow\; a = b \quad \text{or} \quad \exists j : [a_j > b_j \text{ and } \forall i,\ a_i \geqslant \min(a_j, b_i)]. \tag{14}$$

The definition of this ordering can be motivated by considerations in game theory similar to those for $\succcurlyeq^{\text{NR}}$ presented in the example above. Derks and Peters [18] have shown that $a \succcurlyeq^{\text{J}} b$ implies $a \succcurlyeq^{\text{NR}} b$ (it is obvious from the comparison of (13) and (14)). Although we have $\succcurlyeq^{\text{J}} \subseteq \succcurlyeq^{\text{NR}}$, we may not infer that the converse inclusion holds between the asymmetric parts of these relations, since $\succcurlyeq^{\text{J}}$ is not a complete relation. The following simple example makes the point. Let $a = (0.8, 0.4)$ and $b = (0.2, 0.6)$; it is readily seen that neither $a \succcurlyeq^{\text{J}} b$ nor $b \succcurlyeq^{\text{J}} a$, while $a \succ^{\min} b$ and hence $a \succ^{\text{NR}} b$. We thus do not have $\succ^{\text{NR}} \subset \succ^{\text{J}}$. The same example also shows that $\succ^{\min}$ is not included in $\succ^{\text{J}}$ and it is not hard to prove that $\succ^{\text{J}}$ is not included in $\succ^{\min}$ either.

Let us mention an alternative definition of $\succcurlyeq^{\text{J}}$ in terms of α-cuts, that allows for a comparison with (7). We leave it to the reader to establish the equivalence of (15) and (14).

$$a \succcurlyeq^{\text{J}} b \;\Leftrightarrow\; \begin{cases} a = b \quad \text{or} \\ \begin{bmatrix} \exists i \in \{1, \ldots, n\} \text{ such that } \forall \alpha < a_i,\ \mathcal{I}(a)_\alpha \supseteq \mathcal{I}(b)_\alpha \\ \quad \text{and} \\ \exists \alpha' < a_i \text{ with } \mathcal{I}(a)_{\alpha'} \supsetneq \mathcal{I}(b)_{\alpha'} \end{bmatrix} \end{cases}. \tag{15}$$

We do not investigate relation $\succcurlyeq^{\text{J}}$ further in the sequel.

3.4. LexiMin

Well-known in the theory of social choice and welfare, the LexiMin order [32] relies on the cardinality of the α-cuts of the performance profiles [20], in contrast with their inclusion like in the previous procedure.

Solution a is preferred to solution b, according to the LexiMin order, if there is a threshold α such that for all $\beta < \alpha$, the number of points of view that pass level β is the same for both a and b, while the number of points of view that reach or pass α is strictly larger for a than for b. In other words, this definition is similar to (8–9) where cardinalities of cutsets are compared, instead of the cutsets themselves.

Let us show the differences between the three procedures on an example. Let $a = (0.8, 0.3, 0.9)$, $b = (0.3, 0.3, 0.9)$, and $c = (0.7, 0.3, 0.4)$. Since the minimal value of the three profiles is 0.3, they are all indifferent with respect to $\succcurlyeq^{\min}$. The DiscriMin makes a distinction between a and c, and between b and c, but not between a and b (because the value 0.3 on the second criterion is common to both alternatives; when this common value has been neutralized, the minimum of the remaining ones is 0.3 both in a and in b): $c \succ^{\text{disc}} b$; $c \succ^{\text{disc}} a$ and $a \sim^{\text{disc}} b$. The LexiMin discriminates between the three alternatives; we have $c \succ^{\text{lex}} a \succ^{\text{lex}} b$; indeed, when comparing a and c, one sees that 0.3 is the smallest value of the threshold for which there are more criteria above the threshold in c than

in a; the same occurs for the value 0.8 of the threshold when comparing a and b, and for $\alpha = 0.7$, when comparing b and c.

When comparing two vectors, this procedure, like DiscriMin, may take into account their values on several coordinates, contrary to the Min which bases the comparison on the sole smallest value. In contrast with DiscriMin, the different performance values of vectors a and b are ranked in increasing order before the vectors are compared.

Let $\underline{a}$ denote the lowest value in the profile of a. $([1], \ldots, [n])$ is a permutation of the indices from I such that the corresponding profile is arranged in nondecreasing order. In other words, $a_{[i]}$, $i \in I$, is the ith smallest value of a:

$$\underline{a} = a_{[1]} \leqslant a_{[2]} \leqslant \cdots \leqslant a_{[m]}.$$

Let b another profile; we have

$$\underline{b} = b_{[1]} \leqslant b_{[2]} \leqslant \cdots \leqslant b_{[m]};$$

clearly, the permutations corresponding to the profiles a and b may be different.

The LexiMin relation $\succcurlyeq^{\text{lex}}$ is defined as follows:

$$a \succcurlyeq^{\text{lex}} b \quad \Leftrightarrow \quad \begin{cases} a_{[\,]} = b_{[\,]} \quad \text{or} \\ \exists i \leqslant m\text{:}\ \ [a_{[i]} > b_{[i]} \text{ and } \forall j < i\text{:}\ \ a_{[j]} = b_{[j]}]. \end{cases} \tag{16}$$

Note that $a \sim^{\text{lex}} b$ iff $a_{[\,]} = b_{[\,]}$. It is easy to convince oneself that $\succcurlyeq^{\text{lex}}$ is transitive and complete, hence a weak order. The procedure compares the cardinality of the α-cuts for the fuzzy sets a and b on the universe I, letting the value of α increase. As soon as a difference occurs, the vector whose cut has the highest cardinality is chosen. Vector corresponding to the cut of highest cardinality is chosen.

It can be shown [20,22] that such a ranking leads to the solutions violating the smallest number of fuzzy constraints in FCSP, in the sense of a fuzzy-valued cardinality.

The orderings described above have been presented in increasing order of their discriminating power: solutions being indifferent for an order could be distinguished by the next one:

$$a \succcurlyeq^{\text{lex}} b \Rightarrow a \succcurlyeq^{\text{disc}} b \Rightarrow a \succcurlyeq^{\min} b. \tag{17}$$

In particular, any LexiMin-optimal solution is DiscriMin-optimal.

Before proceeding to an axiomatic characterization of these orderings, we mention a few results about the resolution of problems using them. Dubois and Fortemps [21] showed that the optimal solutions for DiscriMin and for LexiMin can be obtained through an iterative version of the min-based algorithm. A paper by Barbadym [5] provides with a very elegant "vectorial" solution of the lexicographic bottleneck problem. The reader is also referred to [15,17].

4. Characterization of the Min, DiscriMin and LexiMin

Our characterization of the above ordering procedures uses axioms that refer to the expected behaviour of the procedure when the values describing the alternatives are varied in some particular

ways. These properties are related to specific interpretations of the "nature of the data"; in particular, they are consistent with an ordinal interpretation of the values describing the alternatives. We emphasize that our axioms are not supposed to be verified in all circumstances; they do not implement what could be interpreted as rationality principles that any ordering procedure should obey: they are not normative axioms. Our axioms may be described as answers to "What if?" questions of the type: "What if the preferred alternative in a pair is deteriorated (or improved) in such and such a way? What if a pair of alternatives are both altered in such and such a way?". This approach provides axioms that may be relevant and argued for in the described context of FCSP and MCDA.

Consider an alternative a with its performance profile; the profile (x_i, a_{-i}) denotes the profile of another alternative or solution obtained by substituting the value of a on the ith criterion, a_i, by the value x_i; a_{-i} is a $(n-1)$-tuple that contains all values in the profile a except that corresponding to criterion i. In other words, alternative a' described by the profile (x_i, a_{-i}) enjoys the same valuations than a on all criteria except possibly on the ith one, on which the valuation of a' is x_i.

4.1. The Min procedure

We present the axioms for the Min and briefly comment on them. Since they have been introduced in [30], we give the results without proofs; they can be found in the mentioned paper.

4.1.1. Axioms

The main characteristic of the Min procedure is that it is well-suited to deal with *ordinal* data. The usual ordinality axiom consists in requiring that the ordering of two alternatives a, b is not modified when a common increasing transformation of the $[0,1]$ interval is applied to all coordinates of their profiles. Our axiom is related to ordinality but neither weaker nor stronger. We impose that the ordering of a and b is not altered when the values in their profiles undergo a common translation (the relationship between translation invariance and ordinality is discussed in some detail in Section 4.1.3). Let us be more precise.

A translation t_K of value $K \in [-1,1]$ transforms the performance profiles a and b into profiles a' and b', such that $\forall i \in I$

$$a'_i = a_i + K,$$

$$b'_i = b_i + K.$$

The translation t_K is called *feasible* w.r.t. the profiles a and b if both transformed profiles still belong to $\mathcal{A}$. Note that the "feasibility" of a translation is only checked with respect to a and b. The invariance under feasible translation is formulated as follows:

Axiom 1 (Translation Invariance—TI). *For all pairs of profiles $(a,b) \in \mathcal{A}^2$ and for all K such that t_K is a feasible translation, we have*

$$a \succcurlyeq b \Rightarrow a' \succcurlyeq b',$$

where a' and b' are the profiles resulting from the translation.

The second axiom makes sense, in particular, in the context of constraint satisfaction. It requires that the preference of one alternative a over another b can be reverted through appropriate change on any single dimension. More precisely, if alternative b does not violate completely any constraint, it is possible to deteriorate the degree to which a satisfies a single particular constraint in such a way that b is strictly preferred to the transformed a. For example, setting the performance of a on a constraint to 0 does it. We call the corresponding property "Strong Reversal":

Axiom 2 (Strong Reversal—SR). *If $a \succcurlyeq b$ and for every criterion $j \in I$, $b_j > 0$ then for each criterion $i \in I$, there exists $a'_i < a_i$ such that $b \succ (a'_i, a_{-i})$.*

A similar property, called "Weak Reversal", deals with the case where it cannot be excluded that b fully violates a constraint. In such a case, the preference of a over b can be reverted but indifference of transformed a and b can result and it may be impossible to obtain strict preference of b over transformed a.

Axiom 3 (Weak Reversal—WR). *If $a \succcurlyeq b$ then for each criterion $i \in I$, there exists $a'_i \leqslant a_i$ such that $b \succcurlyeq (a'_i, a_{-i})$.*

In [30], examples are provided that show the independence of these three axioms. A procedure considering all the elements of $\mathcal{A}$ as equivalent verifies TI and WR but not SR; LexiMin satisfies TI and SR but not WR; procedures deciding on the basis of the ratio $\underline{a}/\underline{b}$ of the smallest values in both profiles, for instance, do not satisfy TI whilst WR and SR may be true.

4.1.2. Characterization

The first lemma states direct consequences of the reversal axioms. If alternative a completely dissatisfies a criterion, it is impossible for a to be strictly preferred to any other alternative. In particular, if b has only strictly positive performances, then it is strictly preferred to a.

Lemma 2. (a) *For any procedure satisfying WR, for all pairs of alternatives $(a, b) \in \mathcal{A}^2$,*

$$\underline{a} = 0 \Rightarrow b \succcurlyeq a. \tag{18}$$

(b) *For any procedure satisfying SR, for all pairs of alternatives $(a, b) \in \mathcal{A}^2$,*

$$\underline{a} = 0 < \underline{b} \Rightarrow b \succ a. \tag{19}$$

For procedures that enjoy property TI, translation invariance enables to establish the previous results for levels other than 0.

Proposition 3. *If a procedure satisfies properties TI and WR, then for all pairs of alternatives $(a, b) \in \mathcal{A}^2$,*

$$\underline{a} = \underline{b} \Rightarrow a \sim b \tag{20}$$

and

$$\underline{a} < \underline{b} \Rightarrow b \succcurlyeq a. \tag{21}$$

Proposition 4. *If a procedure satisfies properties TI and SR, then for all pairs of alternatives* $(a,b) \in \mathcal{A}^2$,

$$\underline{a} < \underline{b} \Rightarrow b \succ a. \tag{22}$$

Combining the above two propositions clearly yields the following characterization of the Min:

Theorem 5. *The Min procedure is the only one that satisfies TI, WR and SR.*

4.1.3. Relationship between translation invariance and ordinality

We discuss here the somewhat mysterious translation invariance axiom and its connections with ordinality. Let us distinguish two types of feasible translations for a given pair of profiles a and b. A feasible translation of type I avoids the border of the $[0,1]$ interval, i.e., if all a_i and b_i are different of 0 and 1, $t_K(a_i)$ and $t_K(b_i)$ are equal neither to 0 nor 1. A feasible translation is said of type II, if some $a_i \neq 0,1$ or some $b_i \neq 0,1$ is mapped onto either 0 or 1. It is not difficult to realize that, for any feasible translation of type I w.r.t. a and b, one can build an increasing transformation of the $[0,1]$ interval that modifies a, b just like the translation does. This implies that ordinal invariance entails translation invariance for translations of type I. Translations of type II cannot be embedded into an increasing (one-to-one) transformation of the $[0,1]$ interval but they can be embedded in a non-decreasing one. Suppose for instance that x and y are respectively the minimum and the maximum values in the profiles a and b and that $K = 1 - y$; t_K maps the interval $[x,y] \subset]0,1[$onto$[x+K,1]$. A non-decreasing transformation f that produces the same effect on the $[x,y]$ interval than t_K is the following:

$$f(u) = \begin{cases} ((x+K)/x)u & \text{if } u \in [0,x] \\ u+K & \text{if } u \in [x,y] \\ 1 & \text{if } u \in [y,1] \end{cases}.$$

Although the Min satisfies *strong ordinality* (i.e. $a \succcurlyeq b$ implies $f(a) \succcurlyeq f(b)$, where f is any non-decreasing transformation of the $[0,1]$ interval that fixes 0 and 1), this is not the case of DiscriMin and LexiMin. One can thus conceive of translation invariance as to a property that is intermediary between ordinality and strong ordinality; it is well-suited for describing the behaviour of the refinements of the Min we are dealing with.

Translation invariance of type II could be roughly described as "ordinality + continuity near the borders". It is interesting to note that Bouyssou and Pirlot [12] have proved the equivalence of strong ordinality with ordinality + continuity. This might help to understand intuitively the position of the translation invariance property w.r.t. ordinality and strong ordinality.

4.2. The DiscriMin procedure

The above characterization of Min suggests that it might be interesting to pay attention to two families of procedures: those satisfying TI and WR on the one hand and, on the other hand, those satisfying TI and SR. Those families, respectively denoted TIWR and TISR have been investigated in some detail in [30]. The intersection of the families TIWR and TISR contains a single procedure,

the Min. LexiMin and DiscriMin belong to the TISR family, which we shall thus further explore in the sequel.

4.2.1. Axioms

To keep in line with our axiomatization of the Min, we need to weaken WR into WWR. WWR limits the validity of the WR property to those criteria on which alternative b is not completely unsatisfactory, i.e. the criteria i for which $b_i > 0$.

Axiom 4 (Weakened Weak Reversal—WWR). *If $a \succcurlyeq b$ then for each criterion $i \in I$ such that $b_i \neq 0$, there exists $a'_i \leqslant a_i$ such that $b \succcurlyeq (a'_i, a_{-i})$.*

At the same time, in order to reject the Min procedure that also satisfies the WWR axiom, we add an independence condition (ICE) stating that the degree of satisfaction of a constraint does not matter if the compared profiles achieve the same degree of satisfaction on this constraint.

Axiom 5 (Independence w.r.t. Criterion–Equality—ICE). *If $a \succcurlyeq b$ and for some i, $a_i = b_i$, then $\forall x_i \in [0,1], (x_i, a_{-i}) \succcurlyeq (x_i, b_{-i})$.*

This is the classical condition of "mutual preference independence". A consequence of ICE is that, instead of comparing the profiles a and b, we can compare the profiles $(1_{\mathcal{E}}, a_{-\mathcal{E}})$ and $(1_{\mathcal{E}}, b_{-\mathcal{E}})$, where $\mathcal{E}$ denotes the set of criteria on which a and b reach the same performance, namely, $\mathcal{E} = \{i \in I : a_i = b_i\} = I \backslash \mathcal{D}(a,b)$; $1_{\mathcal{E}}$ denotes a sub-profile on the set $\mathcal{E}$ with all values equal to 1, and $a_{-\mathcal{E}}$ (resp. $b_{-\mathcal{E}}$) sets the evaluations on the coordinates in the complement of $\mathcal{E}$ to the corresponding values in the profile a (resp. b). The following corollary is obvious.

Corollary 6. *For any procedure satisfying ICE, for all pairs of alternatives $(a,b) \in \mathcal{A}^2$,*

$$a \succcurlyeq b \quad \Leftrightarrow \quad a^* \succcurlyeq b^*, \tag{23}$$

where the profile $a^* = (1_{\mathcal{E}}, a_{-\mathcal{E}})$ obtains by setting to 1 the degrees of satisfaction for the criteria in $\mathcal{E}$.

4.2.2. Characterization

Using a^* and b^* defined above, let $\underline{a}^*$ (resp. $\underline{b}^*$) denote the minimum value in the profile a^* (resp. b^*); we have $\underline{a}^* = \min_{i \in \mathcal{D}(a,b)} a_i$; the usual convention applies in the limit case where $\mathcal{D}(a,b) = \emptyset$, i.e. $\min_{i \in \emptyset} a_i = 1$. The result of Lemma 2(b) related to SR is still relevant for DiscriMin. As a consequence of the weak reversal property, if a completely violates a criterion which b does not, then a cannot be strictly preferred to b. Lemma 2(a) is modified into the following.

Lemma 7. *For any procedure satisfying WWR, for all pairs of alternatives $(a,b) \in \mathcal{A}^2$, we have*

$$\exists i : a_i = 0 \neq b_i \Rightarrow b \succcurlyeq a. \tag{24}$$

Proof. Assume that $\exists i : a_i = 0 \neq b_i$ together with $a \succ b$. In view of WWR, it should be possible to find $a'_i \leqslant a_i$ such that $b \succcurlyeq (a'_i, a_{-i})$. But, this is impossible, since $a_i = 0$. □

Taking into account the ICE property, we obtain the following modifications of Proposition 4 and Proposition 3.

Proposition 8. *If a procedure enjoys properties TI, ICE and WWR, then for all pairs of alternatives* $(a,b)\in\mathcal{A}^2$,

$$\underline{a}^* = \underline{b}^* \Rightarrow a \sim b \tag{25}$$

and

$$\underline{a}^* < \underline{b}^* \Rightarrow b \succcurlyeq a. \tag{26}$$

Proof. Because of Corollary 6, we can compare the modified profile $a^* = (1_{\mathcal{E}}, a_{-\mathcal{E}})$ and $b^* = (1_{\mathcal{E}}, b_{-\mathcal{E}})$.

For establishing the first part of the proposition, we use the TI property with $K = -\underline{a}^* = -\underline{b}^*$. Applying the translation t_K to a^* and b^* yields the vectors $a^{*\prime}$ and $b^{*\prime}$. There exist two different indices $i \neq j$, such that $a_i^{*\prime} = 0 \neq b_i^{*\prime}$ and $b_j^{*\prime} = 0 \neq a_j^{*\prime}$. Applying Lemma 7 for both indices i and j yields $b \succcurlyeq a$ and $a \succcurlyeq b$.

For proving the second part, K is chosen equal to $-\underline{a}^*$; $a^{*\prime}$ and $b^{*\prime}$ denote the corresponding translation of the vectors a^*, b^*. Since there exist $a_i^{*\prime} = 0 \neq b_i^{*\prime}$, we have by Lemma 7, $b^{*\prime} \succcurlyeq a^{*\prime}$ and, by TI, $b^* \succcurlyeq a^*$, which amounts to $b \succcurlyeq a$. □

Proposition 9. *If a procedure enjoys properties TI, ICE and SR, then, for all pairs of alternatives* $(a,b)\in\mathcal{A}^2$,

$$\underline{a}^* < \underline{b}^* \Rightarrow b \succ a. \tag{27}$$

Proof. Let us again compare the modified profiles a^* and b^*.

Let $\underline{a}^* < \underline{b}^*$ and assume that $a^* \succcurlyeq b^*$. Use the TI property with $K = -\underline{a}^* < 1$. The corresponding translated profiles are such that $a^{*\prime} \succcurlyeq b^{*\prime}$ together with $\underline{a}^{*\prime} = 0 < \underline{b}^{*\prime}$. This contradicts Lemma 2(b). □

Putting together the above two propositions yields the following characterization of the DiscriMin.

Theorem 10. *The "DiscriMin" procedure is the only one that satisfies TI, ICE, WWR and SR.*

4.3. The LexiMin procedure

In view of a characterization of LexiMin, which is more discriminant than DiscriMin, we need relax the weak reversal axiom and modify the independence axiom. W'WR essentially requires that the "zero-profile", $\overline{0}$, is not preferred to any alternative (see Lemma 11). The independence axiom (IRE) consists in a modification of ICE, where rank matters instead of criterion index.

4.3.1. Axiom

Axiom 6 (Weakest Weak Reversal—W'WR). *If* $a \succcurlyeq b$ *then there exists* $a' \in \mathcal{A}$ *with* $a'_i \leqslant a_i$ $\forall i$, *such that* $b \succcurlyeq a'$.

Axiom 7 (Independence w.r.t. Rank–Equality—IRE). *If* $a \succcurlyeq b$ *and for some* $i \in I$, $a_{[i]} = b_{[i]}$, *let* k (*resp.* l) *be such that* $a_{[i]} = a_k$ (*resp.* $b_{[i]} = b_l$). *Then*, $\forall x_k = x_l \in [0,1]$, $(x_k, a_{-k}) \succcurlyeq (x_l, b_{-l})$.

4.3.2. Characterization

Lemma 11. *For any procedure satisfying W'WR, for all pairs of alternatives* $(a,b) \in \mathcal{A}^2$,

$$(\forall i, a_i = 0) \Rightarrow b \succcurlyeq a.$$

The proof of the lemma is straightforward. The next two propositions parallel Propositions 8 and 9.

Proposition 12. *If a procedure enjoys properties IRE and W'WR, then for all pairs of alternatives* $(a,b) \in \mathcal{A}^2$,

$$(\forall i, a_{[i]} = b_{[i]}) \Rightarrow a \sim b. \tag{28}$$

Proof. IRE allows us to substitute $a_{[i]} = b_{[i]}$ by 0 for all i and we conclude using Lemma 11. □

Notice that, contrary to Proposition 8, the TI property is not needed in Proposition 12; it cannot be dispensed of in the next one however.

Proposition 13. *If a procedure enjoys properties TI, IRE and SR, then for all pairs of alternatives* $(a,b) \in \mathcal{A}^2$,

$$\exists j \in I: \left.\begin{array}{l} \forall i < j, a_{[i]} = b_{[i]} \\ \textit{and } a_{[j]} < b_{[j]} \end{array}\right\} \Rightarrow b \succ a. \tag{29}$$

Proof. Suppose that $a \succcurlyeq b$. We denote by k_i (resp. l_i) the index corresponding to rank i in profile a (resp. b). We consider the profiles a' and b' such that

$$\forall i < j, \quad a'_{k_i} = b'_{l_i} = 1 \geqslant b_{[j]} = b_{l_j}$$

$$\forall i \geqslant j, \quad a'_{k_i} = a_{k_i}, \quad b'_{l_i} = b_{l_i}.$$

By IRE, we still have $a' \succcurlyeq b'$. Apply the translation t_K to these profiles with $K = -a_{[j]} = -a_{k_j} = -a'_{k_j}$. We get the profiles a'' and b'', with $\forall i < j$,

$$a''_{k_i} = b''_{l_i} \geqslant b''_{l_j} > 0 \quad \text{and} \quad a''_{k_j} = 0.$$

By TI, we have $a'' \succcurlyeq b''$. The fact that $\underline{a}'' = 0 < \underline{b}'' = b''_{l_j}$ contradicts Lemma 2(b). □

Corollary 14. *A procedure satisfying TI, IRE and SR is either* $\succcurlyeq^{lex}$ *or its asymmetric part* $\succ^{lex}$.

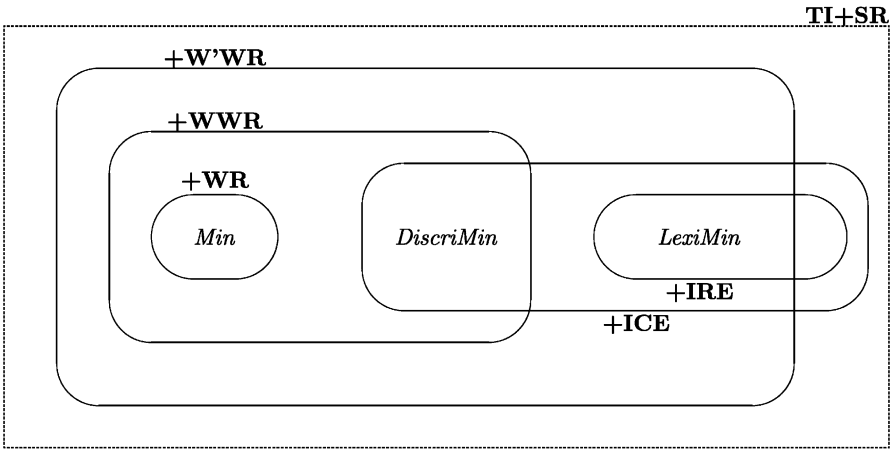

Fig. 1. Axiomatic framework: in the set of TI+SR aggregation procedures, ovals represent subsets enjoying an additional axiom (written in bold); procedures are in italics.

Proof. In view of Proposition 13, it is clear that if $a \neq b$, either $a \succcurlyeq b$ or $b \succcurlyeq a$ but not both and $a \succcurlyeq b$ iff $a \succ^{\text{lex}} b$. Furthermore, any relation that satisfies IRE is either reflexive or irreflexive, i.e. $[\forall a, a \succcurlyeq a]$ or $[\forall a, \text{Not}[a \succcurlyeq a]]$. □

In view of Corollary 14, the sole role played by W'WR is forcing the reflexivity of $\succcurlyeq$. Putting together Propositions 12 and 13 yields the following characterization of LexiMin (that was already in [30]).

Theorem 15. *The LexiMin procedure is the only one which satisfies TI, IRE, W'WR and SR.*

4.4. The common axiomatic framework

The axiomatic framework, depicted in Fig. 1, focusses on the procedures that satisfy TI and SR. Inside the family TISR, WR determines the Min. The difference between the three procedures under investigation is explained by the WR axiom and its variants on the one hand and by the independence axioms on the other hand.

Min satisfies WR as well as WWR and W'WR, while DiscriMin enjoys the weakened and weakest versions, and LexiMin only satisfies the latter. While the chain of implications between the various versions of WR is quite evident, the relationships between ICE and IRE are less clear. In general, under no additional condition, neither ICE implies IRE nor the contrary. In the subset TISR, IRE obviously implies ICE, due to Corollary 14 since both $\succcurlyeq^{\text{lex}}$ and $\succ^{\text{lex}}$ satisfy ICE. Conversely, in the same subset TISR, ICE does not imply IRE. Indeed, $\succcurlyeq^{\text{disc}}$ does not satisfy IRE (neither does $\succcurlyeq^{\text{min}}$). Take for instance $n = 3$, $a = (0.2, 0.2, 0.3)$ and $b = (0.4, 0.3, 0.2)$; we have $a \succcurlyeq^{\text{disc}} b$; changing $a_{[1]} = b_{[1]} = 0.2$ into 0.4 yields a' and b' with $b' \succ^{\text{disc}} a'$, which proves that IRE does not apply. In the oval "+IRE" representing the set of procedures satisfying TI, SR and IRE, there are only two procedures, the one, $\succcurlyeq^{\text{lex}}$, denoted "LexiMin" in Fig. 1, inside the zone labelled "+W'WR" and the other, $\succ^{\text{lex}}$, outside that zone (and not represented on the picture).

5. Conclusion

We have shown that a common axiomatic framework allows for the characterization of several ordinal procedures, namely the Min procedure as well as two of its refinements: DiscriMin and LexiMin. These aggregation procedures are widely used in practical situations, in Flexible Constraint Satisfaction Programming, in Multi-Criteria Decision-Aiding, in Game Theory or in Social Welfare.

The kind of axioms we propose have an intuitive content that should allow for questioning the Decision Maker in order to select the most appropriate procedure. Since this choice is context-dependent, it has to be reconsidered for each new problem. Our proposal aims also at clarifying the differences between the investigated procedures.

References

[1] J. Aczél, On weighted synthesis of judgements, Aequationes Math. 27 (1984) 288–307.
[2] J. Aczél, C. Alsina, Synthesising judgements: a functional equations approach, Math. Modelling 9 (1987) 311–320.
[3] J. Aczél, F. Roberts, On the possible merging functions, Math. Social Sci. 17 (1989) 205–243.
[4] J. Aczél, F. Roberts, Z. Rosenbaum, On scientific laws without dimensional constants, J. Math. Anal. Appl. 119 (1986) 389–416.
[5] V.A. Barbadym, The modifications of algorithms for general lexicographic bottleneck optimisation problem, in: Proc. Fifth Twente Workshop, 1997.
[6] S. Barberà, M. Jackson, Maximin, leximin, and the protective criterion: characterizations and comparisons, J. Econom. Theory 46 (1988) 34–44.
[7] F.A. Behringer, On optimal decisions under complete ignorance: a new criterion stronger than both pareto and maxmin, Eur. J. Oper. Res. 1 (1977) 295–306.
[8] F.A. Behringer, Lexmaxmin in fuzzy multiobjective decision making, Optimization 21 (1990) 23–49.
[9] R.E. Bellman, M. Giertz, On the analytic formalism of the theory of fuzzy sets, Inform. Sci. 5 (1973) 149–156.
[10] R.E. Bellman, L.A. Zadeh, Decision-making in a fuzzy environment, Management Sci. 17 (4) (1970) 141–164.
[11] D. Bouyssou, Some remarks on the notion of compensation in MCDM, Eur. J. Oper. Res. 26 (1986) 150–160.
[12] D. Bouyssou, M. Pirlot, Choosing and ranking on the basis of fuzzy preference relations with the 'Min in Favor', in: G. Fandel, T. Gal (Eds.), Multiple Criteria Decision Making—Proc. 12th Internat. Conf. Hagen, Germany, Springer, Berlin, 1997, pp. 115–127.
[13] D. Bouyssou, J.-C. Vansnick, Noncompensatory and generalized noncompensatory preference structures, Theory and Decision 21 (1986) 251–266.
[14] G. Brewka, Preferred subtheories: an extended logical framework for default reasoning, in: Proc. 11th Joint Conf. on Artificial Intelligence (IJCAI 89), Detroit, MI, 1989, pp. 1043–1048.
[15] R.E. Burkard, F. Rendl, Lexicographic bottleneck problems, Oper. Res. Lett. 10 (1991) 303–308.
[16] E. Czogala, J. Drewniak, Associative monotonic operations in fuzzy set theory, Fuzzy Sets and Systems 12 (3) (1984) 249–269.
[17] F. Della Croce, V.Th. Paschos, A. Tsoukias, An improved general procedure for lexicographic bottleneck problems, Oper. Res. Lett. 24 (1999) 187–194.
[18] J. Derks, H. Peters, Orderings, excess functions, and the nucleolus, Math. Social Sci. 36 (1998) 175–182.
[19] D. Dubois, H. Fargier, H. Prade, Fuzzy constraints in job-shop scheduling, J. Intell. Manuf. 6 (1995) 215–234.
[20] D. Dubois, H. Fargier, H. Prade, Refinements of the maximin approach to decision-making in a fuzzy environment, Fuzzy Sets and Systems 81 (1996) 103–122.
[21] D. Dubois, P. Fortemps, Computing improved optimal solutions to max-min flexible constraint satisfaction problems, Eur. J. Oper. Res. 118 (1999) 95–126.

[22] H. Fargier, Problèmes de satisfaction de contraintes flexibles: application à l'ordonnancement de production, Ph.D. Thesis, Université Paul Sabatier, Toulouse, 1994.
[23] P.C. Fishburn, Noncompensatory preferences, Synthese 33 (1976) 393–403.
[24] J.C. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer, Dordrecht, 1994.
[25] M. Justmann, Iterative processes with 'nucleolar' restrictions, Internat. J. Game Theory 6 (1977) 62–66.
[26] D.H. Krantz, R.D. Luce, P. Suppes, A. Tversky, Foundations of Measurement, Additive and Polynomial Representations, vol. 1, Academic Press, New York, 1971.
[27] J.-L. Marichal, Aggregation operations for multicriteria decision aid, Ph.D. Thesis, Université de Liège, Belgium, 1998.
[28] J.-L. Marichal, On Sugeno integral as an aggregation function, Fuzzy Sets and Systems 114 (3) (2000) 347–365.
[29] M.J. Osborne, A. Rubinstein, A Course in Game Theory, The MIT Press, Cambridge, MA, 1994.
[30] M. Pirlot, A characterization of 'min' as a procedure for exploiting valued preference relations and related results, J. Multi-Criteria Decision Anal. 4 (1995) 37–56.
[31] F.S. Roberts, Measurement Theory with Applications to Decision Making, Utility and the Social Sciences, Addison-Wesley, Reading, MA, 1979.
[32] A.K. Sen, Social choice theory, in: K.J. Arrow, M.D. Intriligator (Eds.), Handbook of Mathematical Economics, vol. 3, North-Holland, Amsterdam, 1986, pp. 1073–1181.
[33] D. von Winterfeldt, W. Edwards, Decision Analysis and Behavioral Research, Cambridge University Press, Cambridge, 1986.
[34] L.A. Zadeh, Fuzzy Sets, Inform. Control 8 (1965) 338–353.