A refined rectangle covering number as lower bound for extended formulations.

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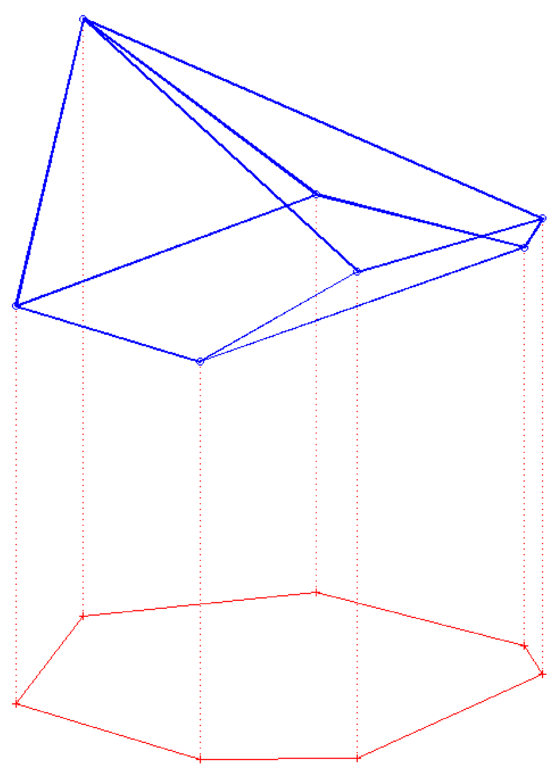
Introduction

An extended formulation of a polytope P is a polytope Q of higher dimension which can be projected onto P with a linear projection.

If P is the convex hull of the feasible domain of a combinatorial optimization problem (usually with exponentially many facets), it is especially interesting to find an extension Q with much fewer facets (in the best case, polynomial in the dimension) as less inequalities must be used to describe the polytope.

In this poster, we consider the problem of determining the *extension complexity* $xc(P)$ which is the smallest number of facets that an extended formulation of P can have.



Let P be a polytope with m facets and n vertices. The slack matrix of P is the matrix $S(m \times n)$ where S_{ij} is defined as the distance between the i^{th} facet and the j^{th} vertex of P.

The smallest number r such that S can be written as the product of two non-negative matrices $W(m \times r)$ and $H(r \times n)$ is called the non-negative rank $r_+(S)$.

Computing the non-negative rank is NP-hard (Vavasis, 2009).

Theorem (Yannakakis, 1991). *Let P be a polytope and S be the slack matrix of P. Then we have:*

$$xc(P) = r_+(S).$$

This result is the main reason many bounds for $xc(P)$ are based on $r_+(S)$.

The rectangle covering number

The rectangle covering bound is one of the most famous lower bound on $r_+(S)$.

One can define a rectangle as a set $(I \times J)$ where I and J are subsets of rows and columns of S. Moreover a covering rectangle can only contain entries (i, j) where $S_{ij} > 0$.

A rectangle covering of a matrix S is a set of rectangles covering all positive entries of S.

The rectangle covering number denoted by $rc(S)$ is defined as the smallest number of rectangles in any rectangle covering of S.

In summary, with the slack matrix S of a polytope P, we have:

$$rc(S) \le r_+(S) = xc(P).$$

$$\begin{matrix} 0 & 0 & 2 & 0 & 2 & 1 & 0 & 3 & 2 & 2 & 1 \\ 0 & 2 & 0 & 0 & 2 & 3 & 2 & 1 & 0 & 2 & 1 \\ 2 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 & 2 & 1 & 2 & 2 & 3 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 2 & 2 & 0 & 1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{matrix}$$

For this slack matrix, we have $rc(S) = 9$

Let us focus on the following submatrix:

$$\begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix}$$

- Entries (1,1), (2,1) and (2,2) are covered once by the same rectangle.
- Entry (1,2) is covered twice by two different rectangles.

Actually, this covering could not be extended to non-negative rank-1 matrices. The current covering is impossible since the entry (1,2) is smaller than 2.

A rectangle covering only relies on the support of the matrix. The challenge is to take into account the values of the entries of the matrices in order to avoid such situations. This is what our refined bound tries to do.

A refined rectangle covering number

Let A be a (2×2) nonnegative matrix. Suppose A can be written as the sum of nonnegative rank-1 matrices.

Suppose further that all elements are non-zero such that the determinant of that submatrix is positive:

$$a_{11}a_{22} - a_{12}a_{21} > 0.$$

Theorem. *The number of rank-1 matrices whose support touches at least one of the entries a_{11} and a_{22} must be at least two.*

Proof. Suppose there is only one rank-1 matrix R such whose support touches the entries a_{11} and a_{22}. Since the sum of the rank-1 matrices is equal to A, we must have:

$$R_{11} = a_{11},$$
$$R_{22} = a_{22}.$$

Without loss of generality, since R is a nonnegative rank-1 matrix, it can be written with $\alpha, \beta \in \mathbb{R}_0^+$ as:

$$R = \begin{pmatrix} \frac{a_{11}}{\alpha} \\ \frac{a_{22}}{\beta} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^T = \begin{pmatrix} a_{11} & \frac{\beta}{\alpha}a_{11} \\ \frac{\alpha}{\beta}a_{22} & a_{22} \end{pmatrix}.$$

Since the rank-1-matrices are all nonnegative, we must further have that

$$R_{12} = \frac{\beta}{\alpha}a_{11} \le a_{12},$$
$$R_{21} = \frac{\alpha}{\beta}a_{22} \le a_{21}.$$

These two inequalities can be combined to obtain

$$\frac{a_{11}}{a_{12}} \le \frac{\alpha}{\beta} \le \frac{a_{21}}{a_{22}}.$$

This leads us to

$$a_{11}a_{22} - a_{12}a_{21} \le 0,$$

which is a contradiction. □

This additional constraint can be applied to all (2×2) submatrices in a slack matrix. By denoting $rrc(S)$ this new refined bound, we have:

$$rc(S) \le rrc(S) \le r_+(S) = xc(P).$$

For this slack matrix, we have $rrc(S) = 10$.
This allows us to prove that $xc(P) = 10$.

$$\begin{matrix} 0 & 0 & 2 & 0 & 2 & 1 & 0 & 3 & 2 & 2 & 1 \\ 0 & 2 & 0 & 0 & 2 & 3 & 2 & 1 & 0 & 2 & 1 \\ 2 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 & 2 & 1 & 2 & 2 & 3 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 2 & 2 & 0 & 1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{matrix}$$

Applications

For regular n-gons, we have a nonnegative matrix factorization for corresponding slack matrices up to $n = 128$ verifying:

$$r_+(n) \le \begin{cases} 2k-1 & 2^{k-1} < n \le 2^{k-1} + 2^{k-2}, \\ 2k & 2^{k-1} + 2^{k-2} < n \le 2^k. \end{cases}$$

n	rc	rrc	UB
6	5	5	5
7	6	6	6
8	6	6	6
9	6	7	7

n	rc	rrc	UB
10	7	7	7
11	7	7	7
12	7	7	7
13	7	8	8

For all 4-dimensional 0/1 polytopes, it turns out that:

$$rrc(S) = xc(P)$$

More particularly, among the 202 affine equivalent classes:

- For 195 classes, $rc(S) = rrc(S) = xc(P)$.
- For 7 classes, $rc(S) = 9$, $rrc(S) = 10$ and $xc(P) = 10$.

Reference: Michael Oelze, Arnaud Vandaele and Stefan Weltge. Computing the extension complexities of all 4-dimensional 0/1-polytopes. *To appear*, 2014.

A particular case of Linear Euclidean Matrices are the matrices defined by:

$$M_n(i,j) = (i-j)^2 \qquad 1 \le i,j \le n.$$

For these matrices, an upper bound on r_+ is known (Gillis, Glineur, 2012):

$$r_+(M_n) \le 2 + \left\lceil \frac{n}{2} \right\rceil.$$

n	rc	rrc	UB
4	4	4	4
5	4	5	5
6	4	5	5
7	5	6	6

n	rc	rrc	UB
8	5	6	6
9	5	6	7