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Using Graph Theory to Derive Inequalities for the Bell Numbers

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Abstract

The Bell numbers count the number of different ways to partition a set of n elements, while the graphical Bell numbers count the number of non-equivalent partitions of the vertex set of a graph into stable sets. This relation between graph theory and integer sequences has motivated us to study properties on the average number of colors in the non-equivalent colorings of a graph to discover new nontrivial inequalities for the Bell numbers. Examples are given to illustrate our approach.

1 Introduction

The Bell numbers $(B_n)_{n\geq 0}$ count the number of different ways to partition a set that has exactly n elements. Starting with $B_0 = B_1 = 1$, the first few Bell numbers are 1, 1, 2, 5, 15, 52, 203 (sequence A141390 in the *On-Line Encyclopedia of Integer Sequences*). The integer B_n can be defined to be the sum

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\},$$

where $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind, with parameters n and k (i.e., the number of partitions of a set of n elements into k blocks). Dobiński's formula [4] gives

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

The 2-Bell numbers $(T_n)_{n\geq 0}$ count the total number of blocks in all partitions of a set of $n+1$ elements. Starting with $T_0 = 1$ and $T_1 = 3$, the first few 2-Bell numbers are 1, 3, 10, 37, 151, 674 (sequence A005493). More formally, the integer T_n is defined by

$$T_n = \sum_{k=0}^{n+1} k \left\{ {n+1 \atop k} \right\} = B_{n+2} - B_{n+1}.$$

Odlyzko and Richmond [12] have studied the average number A_n of blocks in a partition of a set of n elements, which can be defined as

$$A_n = \frac{T_{n-1}}{B_n}.$$

A concept very close to the Bell numbers is also defined in graph theory. More precisely, a coloring of a graph G is an assignment of colors to its vertices such that adjacent vertices have different colors. The chromatic number $\chi(G)$ is the minimum number of colors in a coloring of G. Two colorings are equivalent if they induce the same partition of the vertex set into color classes. For an integer $k > 0$, we define $S(G,k)$ as the number of proper non-equivalent colorings of a graph G that use exactly k colors. Since $S(G,k) = 0$ for $k < \chi(G)$ or $k > n$, the total number $\mathcal{B}(G)$ of non-equivalent colorings of a graph G is defined to be

$$\mathcal{B}(G) = \sum_{k=0}^{n} S(G,k) = \sum_{k=\chi(G)}^{n} S(G,k).$$

In other words, $\mathcal{B}(G)$ is the number of partitions of the vertex set of G whose blocks are stable sets (i.e., sets of pairwise non-adjacent vertices). This invariant has been studied by several authors in the last few years [1, 6, 7, 9, 10, 11] under the name of (graphical) Bell number of G.

Let $\mathcal{T}(G)$ be the total number of stable sets in the set of non-equivalent colorings of a graph G. More precisely, we define

$$\mathcal{T}(G) = \sum_{k=\chi(G)}^{n} kS(G,k).$$

We are interested in computing the average number $\mathcal{A}(G)$ of colors in the non-equivalent colorings of G, that is,

$$\mathcal{A}(G) = \frac{\mathcal{T}(G)}{\mathcal{B}(G)}.$$

Clearly, $\mathcal{B}(\overline{K}_n) = B_n, \mathcal{T}(\overline{K}_n) = T_{n-1} = B_{n+1} - B_n$, and $\mathcal{A}(\overline{K}_n) = \frac{B_{n+1}-B_n}{B_n}$, where $\overline{K}_n$ is the empty graph with n vertices. As another example, consider the cycle C_5 on 5 vertices. As shown in Figure 1, there are five colorings of C_5 with 3 colors, five with 4 colors, and one with 5 colors, which gives $\mathcal{B}(C_5) = 11$, $\mathcal{T}(C_5) = 40$ and $\mathcal{A}(G) = \frac{40}{11}$.

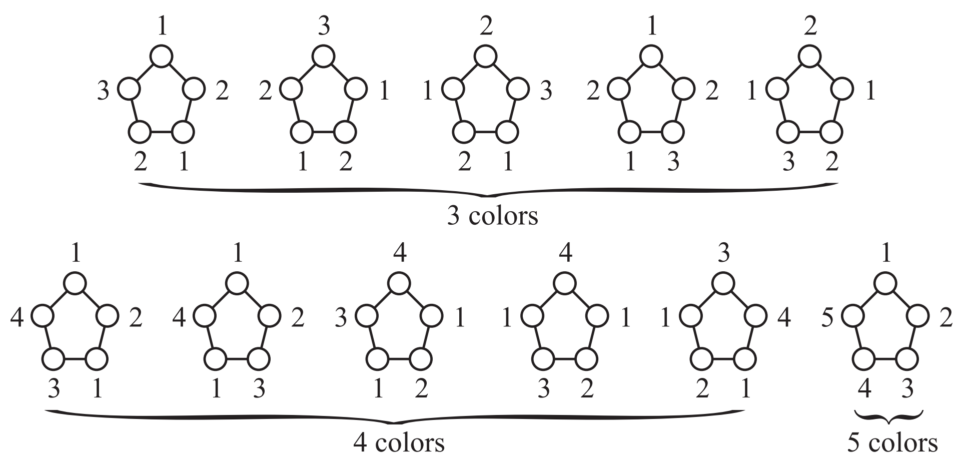

Figure 1: The non-equivalent colorings of C_5.

This close link between Bell numbers and graph colorings indicates that it is possible to use graph theory to derive inequalities for the Bell numbers. This is the aim of this article. The next section gives values of $\mathcal{A}(G)$ for some families of graphs and basic properties involving $\mathcal{A}(G)$. In Section 3, we give several examples of inequalities for the Bell numbers that can be deduced from relations involving $\mathcal{A}(G)$.

Let u and v be two vertices in a graph G. We let $G_{|uv}$ denote the graph obtained by identifying (merging) the vertices u and v and, if u and v are adjacent vertices, by removing the edge that links u and v. If parallel edges are created, we keep only one. Also, if u is adjacent to v, we let $G - uv$ denote the graph obtained from G by removing the edge that links u with v, while if u is not adjacent to v, we let $G + uv$ denote the graph obtained by linking u with v. In what follows, we let K_n, P_n and C_n be the complete graph of order n, the path of order n, and the cycle of order n, respectively. We denote the disjoint union of two graphs G_1 and G_2 by $G_1 \cup G_2$. We refer to Diestel [3] for basic notions of graph theory that are not defined here.

2 Some values and properties of $\mathcal{A}(G)$

The *deletion-contraction* rule (also often called the *fundamental reduction theorem* [5]) is a well known method to compute $\mathcal{B}(G)$ [7, 11]. More precisely, let u and v be any pair of distinct vertices of G. We have

$$S(G,k) = S(G-uv,k) - S(G_{|uv},k) \text{ for every pair } u,v \text{ of adjacent vertices in } G, \quad (1)$$

$$S(G,k) = S(G+uv,k) + S(G_{|uv},k) \text{ for every pair } u,v \text{ of non-adjacent vertices in } G. \quad (2)$$

It follows that

$$\left.\begin{array}{l} \mathcal{B}(G) = \mathcal{B}(G-uv) - \mathcal{B}(G_{|uv}) \\ \mathcal{T}(G) = \mathcal{T}(G-uv) - \mathcal{T}(G_{|uv}) \end{array}\right\} \text{for every pair } u,v \text{ of adjacent vertices in } G, \quad (3)$$

$$\left.\begin{array}{l} \mathcal{B}(G) = \mathcal{B}(G+uv) + \mathcal{B}(G_{|uv}) \\ \mathcal{T}(G) = \mathcal{T}(G+uv) + \mathcal{T}(G_{|uv}) \end{array}\right\} \text{for every pair } u,v \text{ of non-adjacent vertices in } G. \quad (4)$$

Let v be a vertex in a graph G. We let $G-v$ denote the graph obtained from G by removing v and all its incident edges. A vertex of a graph G is *dominating* if it is adjacent to all other vertices of G, and it is *simplicial* if its neighbors are pairwise adjacent.

Proposition 1. *If G has a dominating vertex v, then $\mathcal{A}(G) = 1 + \mathcal{A}(G-v)$.*

Proof. Clearly, $S(G,k) = S(G-v,k-1)$ for all k, which implies

$$\begin{aligned} \mathcal{B}(G) &= \sum_{k=\chi(G)}^{n} S(G,k) \\ &= \sum_{k=\chi(G)}^{n} S(G-v,k-1) = \sum_{k=\chi(G-v)}^{n-1} S(G-v,k) = \mathcal{B}(G-v) \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}(G) &= \sum_{k=\chi(G)}^{n} kS(G,k) \\ &= \sum_{k=\chi(G)}^{n} kS(G-v,k-1) = \sum_{k=\chi(G-v)}^{n-1} (k+1)S(G-v,k) = \mathcal{T}(G-v) + \mathcal{B}(G-v). \end{aligned}$$

Hence $\mathcal{A}(G) = \dfrac{\mathcal{T}(G-v)+\mathcal{B}(G-v)}{\mathcal{B}(G-v)} = 1 + \dfrac{\mathcal{T}(G-v)}{\mathcal{B}(G-v)} = 1 + \mathcal{A}(G-v)$. □

Duncan [7] proved that if G is a tree, then $S(G,k) = \left\{ {n-1 \atop k} \right\}$ for $k = 1, \ldots, n$. This leads to our second proposition.

Proposition 2. *Let G be a tree of order n. Then $\mathcal{B}(G) = B_{n-1}$ and $\mathcal{T}(G) = B_n$.*

Proof. Since $S(G,k) = \left\{ {n-1 \atop k-1} \right\}$, we immediately get

$$\mathcal{B}(G) = \sum_{k=1}^{n} \left\{ {n-1 \atop k-1} \right\} = \sum_{k=0}^{n-1} \left\{ {n-1 \atop k} \right\} = B_{n-1}$$

and

$$\begin{aligned}\mathcal{T}(G) &= \sum_{k=1}^{n} k \left\{ {n-1 \atop k-1} \right\} = \sum_{k=0}^{n-1} (k+1) \left\{ {n-1 \atop k} \right\} \\ &= \sum_{k=0}^{n-1} k \left\{ {n-1 \atop k} \right\} + \sum_{k=0}^{n-1} \left\{ {n-1 \atop k} \right\} = (B_n - B_{n-1}) + B_{n-1} = B_n.\end{aligned}$$

□

Proposition 3. *Let $T \cup pK_1$ be the graph obtained from a tree T of order $n \geq 1$ by adding p isolated verices. Then $\mathcal{B}(T \cup pK_1) = \sum_{i=0}^{p} \binom{p}{i} B_{n+i-1}$ and $\mathcal{T}(T \cup pK_1) = \sum_{i=0}^{p} \binom{p}{i} B_{n+i}$.*

Proof. For $p = 0$, the result follows from Proposition 2. For larger values of p, we proceed by induction. Let T' be the tree obtained from T by adding a new vertex and linking it to exactly one vertex in T. Equations (4) give the following:

$$\begin{aligned}\mathcal{B}(T \cup pK_1) &= \mathcal{B}(T' \cup (p-1)K_1) + \mathcal{B}(T \cup (p-1)K_1) \\ &= \sum_{i=0}^{p-1} \binom{p-1}{i} B_{n+i} + \sum_{i=0}^{p-1} \binom{p-1}{i} B_{n+i-1} \\ &= \sum_{i=1}^{p} \binom{p-1}{i-1} B_{n+i-1} + \sum_{i=0}^{p-1} \binom{p-1}{i} B_{n+i-1} \\ &= B_{n+p-1} + \sum_{i=1}^{p-1} \left(\binom{p-1}{i-1} + \binom{p-1}{i} \right) B_{n+i-1} + B_{n-1} \\ &= \sum_{i=0}^{p} \binom{p}{i} B_{n+i-1}.\end{aligned}$$

The proof for $\mathcal{T}(T \cup pK_1)$ is similar. □

Proposition 4. *Let C_n be a cycle of order $n \geq 3$. Then,*

$$\mathcal{B}(C_n) = \sum_{j=1}^{n-1} (-1)^{j+1} B_{n-j} \quad \textit{and} \quad \mathcal{T}(C_n) = \sum_{j=1}^{n-1} (-1)^{j+1} B_{n-j+1}.$$

Proof. Duncan [7] proved that $\mathcal{B}(C_n) = \sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j}$. It is therefore sufficient to prove that $\mathcal{T}(C_n) = \sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j+1}$.

If $n = 3$, then $\mathcal{T}(C_3) = 3 = B_3 - B_2$. If $n > 3$, Equations (3) together with the fact that P_n is a tree give $\mathcal{T}(C_n) = \mathcal{T}(P_n) - \mathcal{T}(C_{n-1}) = B_n - \mathcal{T}(C_{n-1})$, and the result follows by induction. □

Proposition 5. *Let $C_n \cup pK_1$ be the graph obtained from a cycle of order $n \geq 3$ by adding p isolated verices. Then*

$$\mathcal{B}(C_n \cup pK_1) = \sum_{j=1}^{n-1}(-1)^{j+1}\sum_{i=0}^{p}\binom{p}{i}B_{n+i-j} \quad \text{and} \quad \mathcal{T}(C_n \cup pK_1) = \sum_{j=1}^{n-1}(-1)^{j+1}\sum_{i=0}^{p}\binom{p}{i}B_{n+i-j+1}.$$

Proof. For $p = 0$, the result follows from Proposition 4. For larger values of p, we proceed by induction. If $n = 3$ then Equations (3) and Proposition 3 give

$$\begin{aligned}\mathcal{B}(C_3 \cup pK_1) &= \mathcal{B}(P_3 \cup pK_1) - \mathcal{B}(P_2 \cup pK_1)\\ &= \sum_{i=0}^{p}\binom{p}{i}B_{3+i-1} - \sum_{i=0}^{p}\binom{p}{i}B_{2+i-1}\\ &= \sum_{j=1}^{2}(-1)^{j+1}\sum_{i=0}^{p}\binom{p}{i}B_{3+i-j}.\end{aligned}$$

Hence, the result is valid for $n = 3$. So assume $n > 3$ and that the statement holds for smaller values of n:

$$\begin{aligned}\mathcal{B}(C_n \cup pK_1) &= \mathcal{B}(P_n \cup pK_1) - \mathcal{B}(C_{n-1} \cup pK_1)\\ &= \sum_{i=0}^{p}\binom{p}{i}B_{n+i-1} - \sum_{j=1}^{n-2}(-1)^{j+1}\sum_{i=0}^{p}\binom{p}{i}B_{n+i-j-1}\\ &= \sum_{i=0}^{p}\binom{p}{i}B_{n+i-1} + \sum_{j=2}^{n-1}(-1)^{j+1}\sum_{i=0}^{p}\binom{p}{i}B_{n+i-j}\\ &= \sum_{j=1}^{n-1}(-1)^{j+1}\sum_{i=0}^{p}\binom{p}{i}B_{n+i-j}.\end{aligned}$$

The proof for $\mathcal{T}(C_n \cup pK_1)$ is similar. □

Proposition 6. *Let G be a graph with a simplicial vertex v. Then $\mathcal{A}(G) > \mathcal{A}(G - v)$.*

Proof. Let r be the number of neighbors of v in G. We have $S(G, k) = (k - r)S(G - v, k) +$

$S(G-v,k-1)$. Assuming that G is of order n, we have

$$\begin{aligned}\mathcal{B}(G) &= \sum_{k=0}^{n} S(G,k) = \sum_{k=0}^{n-1} kS(G-v,k) - r\sum_{k=0}^{n-1} S(G-v,k) + \sum_{k=0}^{n-1} S(G-v,k)\\ &= \sum_{k=0}^{n-1}(k-r+1)S(G-v,k)\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}(G) &= \sum_{k=0}^{n} kS(G,k) = \sum_{k=0}^{n}(k^2-kr)S(G-v,k) + \sum_{k=0}^{n} kS(G-v,k-1)\\ &= \sum_{k=0}^{n-1}(k^2-kr)S(G-v,k) + \sum_{k=0}^{n-1}(k+1)S(G-v,k)\\ &= \sum_{k=0}^{n}(k^2-k(r-1)+1)S(G-v,k).\end{aligned}$$

We therefore have

$$\begin{aligned}&\mathcal{B}(G-v)\mathcal{T}(G) - \mathcal{T}(G-v)\mathcal{B}(G)\\ &= \sum_{k=0}^{n-1} S(G-v,k)\sum_{k'=0}^{n}(k'^2-k'(r-1)+1)S(G-v,k') - \sum_{k=0}^{n-1} kS(G-v,k)\sum_{k'=0}^{n-1}(k'-r+1)S(G-v,k')\\ &= \sum_{k=0}^{n-1}\left(S(G-v,k)\right)^2\left(k^2-k(r-1)+1-k(k-r+1)\right)\\ &+ \sum_{k=0}^{n-2}\sum_{k'=k+1}^{n-1} S(G-v,k)S(G-v,k')(k'^2-k'(r-1)+1+k^2-k(r-1)+1-k(k'-r+1)-k'(k-r+1))\\ &= \sum_{k=0}^{n-1}\left(S(G-v,k)\right)^2 + \sum_{k=0}^{n-2}\sum_{k'=k+1}^{n-1} S(G-v,k)S(G-v,k')\left((k-k')^2+2\right) > 0\end{aligned}$$

which implies $\mathcal{A}(G)-\mathcal{A}(G-v) = \dfrac{\mathcal{T}(G)}{\mathcal{B}(G)} - \dfrac{\mathcal{T}(G-v)}{\mathcal{B}(G-v)} = \dfrac{\mathcal{B}(G-v)\mathcal{T}(G)-\mathcal{T}(G-v)\mathcal{B}(G)}{\mathcal{B}(G)\mathcal{B}(G-v)} > 0.$ □

Proposition 7. *Let G, H and $F_1,\ldots,F_r$ be $r+2$ graphs, and let $\alpha_1,\ldots,\alpha_r$ be r positive numbers such that*

- $\mathcal{B}(G) = \mathcal{B}(H) + \sum_{i=1}^{r}\alpha_i\mathcal{B}(F_i)$
- $\mathcal{T}(G) = \mathcal{T}(H) + \sum_{i=1}^{r}\alpha_i\mathcal{T}(F_i)$
- $\mathcal{A}(F_i) < \mathcal{A}(H)$ *for $i = 1,\ldots,r$.*

Then $\mathcal{A}(G) < \mathcal{A}(H)$.

Proof. Since $\mathcal{A}(F_i) < \mathcal{A}(H)$, we have $\mathcal{T}(F_i) < \frac{\mathcal{T}(H)\mathcal{B}(F_i)}{\mathcal{B}(H)}$ for $i = 1, \ldots, r$. Hence

$$\begin{aligned}
\mathcal{A}(G) &= \frac{\mathcal{T}(G)}{\mathcal{B}(G)} = \frac{\mathcal{T}(H) + \sum_{i=1}^{r} \alpha_i \mathcal{T}(F_i)}{\mathcal{B}(H) + \sum_{i=1}^{r} \alpha_i \mathcal{B}(F_i)} \\
&< \frac{\mathcal{T}(H) + \sum_{i=1}^{r} \alpha_i \frac{\mathcal{T}(H)\mathcal{B}(F_i)}{\mathcal{B}(H)}}{\mathcal{B}(H) + \sum_{i=1}^{r} \alpha_i \mathcal{B}(F_i)} = \frac{\mathcal{T}(H)\left(\mathcal{B}(H) + \sum_{i=1}^{r} \alpha_i \mathcal{B}(F_i)\right)}{\mathcal{B}(H)\left(\mathcal{B}(H) + \sum_{i=1}^{r} \alpha_i \mathcal{B}(F_i)\right)} \\
&= \frac{\mathcal{T}(H)}{\mathcal{B}(H)} = \mathcal{A}(H).
\end{aligned}$$

□

3 Inequalities for the Bell numbers

In this section, we show how to derive inequalities for the Bell numbers, using properties related to the average number $\mathcal{A}(G)$ of colors in non-equivalent colorings of G. We start by analyzing paths. As already mentioned, $P_n \cup pK_1$ is the graph obtained by adding p isolated vertices to a path on n vertices.

Theorem 8. $\mathcal{A}(P_n \cup (p+1)K_1) < \mathcal{A}(P_{n+1} \cup pK_1)$ *for all* $n \geq 1$ *and* $p \geq 0$.

Proof. It follows from Equations (4) that

$$\mathcal{B}(P_n \cup (p+1)K_1) = \mathcal{B}(P_{n+1} \cup pK_1) + \mathcal{B}(P_n \cup pK_1)$$

and

$$\mathcal{T}(P_n \cup (p+1)K_1) = \mathcal{T}(P_{n+1} \cup pK_1) + \mathcal{T}(P_n \cup pK_1).$$

Also, we know from Proposition 6 that $\mathcal{A}(P_n \cup pK_1) < \mathcal{A}(P_{n+1} \cup pK_1)$. Hence, it follows from Proposition 7 that $\mathcal{A}(P_n \cup (p+1)K_1) < \mathcal{A}(P_{n+1} \cup pK_1)$. □

Proposition 3 immediately gives the following corollary.

Corollary 9. *If* $n \geq 1$ *and* $p \geq 0$ *then*

$$\left(\sum_{i=0}^{p+1} \binom{p+1}{i} B_{n+i}\right) \left(\sum_{i=0}^{p} \binom{p}{i} B_{n+i}\right) < \left(\sum_{i=0}^{p+1} \binom{p+1}{i} B_{n+i-1}\right) \left(\sum_{i=0}^{p} \binom{p}{i} B_{n+i+1}\right).$$

Examples 10. For $p = 0$ and $n \geq 1$, Corollary 9 provides the following inequality:

$$(B_n + B_{n+1})B_n < (B_{n-1} + B_n)B_{n+1} \iff B_n^2 < B_{n-1}B_{n+1}.$$

This inequality for the Bell numbers also follows from Proposition 6. Indeed, P_n is obtained from P_{n+1} by removing a vertex of degree 1, which implies

$$\mathcal{A}(P_n) < \mathcal{A}(P_{n+1}) \iff \frac{B_n}{B_{n-1}} < \frac{B_{n+1}}{B_n} \iff B_n^2 < B_{n-1}B_{n+1}.$$

Note that Engel [8] has shown that the sequence $(B_n)_{n\geq 0}$ is log-convex, which implies $B_n^2 \leq B_{n-1}B_{n+1}$ (with a non-strict inequality) for $n \geq 1$. Recently, Alzer [2] proved that the sequence $(B_n)_{n\geq 0}$ is strictly log-convex by showing that

$$B_{n-1}B_{n+1} - B_n^2 = \frac{1}{2e^2}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\frac{j^{n-1}(k-j)^{n-1}}{j!(k-j)!}(k-2j)^2$$

for all $n \geq 2$. Since $B_1^2 = 1 < 2 = B_0B_2$, this also implies $B_n^2 < B_{n-1}B_{n+1}$ for all $n \geq 1$.

As a second example, assume $p = 1$ and $n \geq 1$. Corollary 9 provides the following inequality for the Bell numbers, which also follows from the strict log-convexity of the sequence $(B_n)_{n\geq 0}$:

$$\begin{aligned} &(B_n + B_{n+1} + B_{n+2})(B_n + B_{n+1}) < (B_{n-1} + B_n + B_{n+1})(B_{n+1} + B_{n+2}) \\ \Longleftrightarrow\ & B_n(B_n + B_{n+1}) < B_{n-1}(B_{n+1} + B_{n+2}). \end{aligned}$$

For $n \geq 3$ and $r \geq 0$, we denote $H_{n,r}$ the graph obtained by linking one extremity of P_r to one vertex of C_n (see Figure 2). For $r = 0$, $H_{n,0}$ is equal to C_n. Also, $H_{n,r} \cup pK_1$ is the graph obtained from $H_{n,r}$ by adding p isolated vertices. We now compare $\mathcal{A}(H_{3,n-3} \cup pK_1)$ with $\mathcal{A}(P_{n+1} \cup pK_1)$ to derive new inequalities involving the Bell numbers.

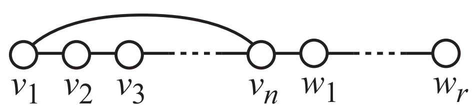

Figure 2: The graph $H_{n,r}$

Theorem 11. *$\mathcal{A}(H_{3,n-3} \cup pK_1) < \mathcal{A}(P_{n+1} \cup pK_1)$ for all $n \geq 4$ and $p \geq 0$.*

Proof. Note first that Equations (3) give $\mathcal{B}(H_{3,n-3} \cup pK_1) = \mathcal{B}(P_n \cup pK_1) - \mathcal{B}(P_{n-1} \cup pK_1)$ and $\mathcal{T}(H_{3,n-3} \cup pK_1) = \mathcal{T}(P_n \cup pK_1) - \mathcal{T}(P_{n-1} \cup pK_1)$. Hence

$$\begin{aligned} &\mathcal{A}(P_{n+1} \cup pK_1) - \mathcal{A}(H_{3,n-3} \cup pK_1) = \frac{\mathcal{T}(P_{n+1} \cup pK_1)}{\mathcal{B}(P_{n+1} \cup pK_1)} - \frac{\mathcal{T}(H_{3,n-3} \cup pK_1)}{\mathcal{B}(H_{3,n-3} \cup pK_1)} \\ &= \frac{\mathcal{T}(P_{n+1} \cup pK_1)}{\mathcal{B}(P_{n+1} \cup pK_1)} - \frac{\mathcal{T}(P_n \cup pK_1) - \mathcal{T}(P_{n-1} \cup pK_1)}{\mathcal{B}(P_n \cup pK_1) - \mathcal{B}(P_{n-1} \cup pK_1)} \\ &= \frac{\mathcal{T}(P_{n+1} \cup pK_1)\left(\mathcal{B}(P_n \cup pK_1) - \mathcal{B}(P_{n-1} \cup pK_1)\right) - \mathcal{B}(P_{n+1} \cup pK_1)\left(\mathcal{T}(P_n \cup pK_1) - \mathcal{T}(P_{n-1} \cup pK_1)\right)}{\mathcal{B}(P_{n+1} \cup pK_1)\mathcal{B}(H_{3,n-3} \cup pK_1)}. \end{aligned}$$

Let $f(n,p)$ be the numerator of the above fraction. It follows from Proposition 3 that

$$f(n,p) = \sum_{i=0}^{p}\binom{p}{i}B_{n+i+1}\left(\sum_{\ell=0}^{p}\binom{p}{\ell}(B_{n+\ell-1} - B_{n+\ell-2})\right) - \sum_{i=0}^{p}\binom{p}{i}B_{n+i}\left(\sum_{\ell=0}^{p}\binom{p}{\ell}(B_{n+\ell} - B_{n+\ell-1})\right).$$

It remains to prove that $f(n,p) > 0$ for all $n \geq 4$ and $p > 0$. Since $B_n = \frac{1}{e}\sum_{k=1}^{\infty} \frac{k^n}{k!}$, we have

$$B_n - B_{n-1} = \frac{1}{e}\sum_{k=1}^{\infty}\left(\frac{k^n}{k!} - \frac{k^{n-1}}{k!}\right) = \frac{1}{e}\sum_{k=2}^{\infty}\frac{k^{n-1}}{k!}(k-1).$$

Hence

$$\begin{aligned}
&e^2 f(n,p)\\
&= \sum_{i=0}^{p}\sum_{\ell=0}^{p}\sum_{j=1}^{\infty}\sum_{k=2}^{\infty}\binom{p}{i}\binom{p}{\ell}\frac{j^{n+i+1}}{j!}\frac{k^{n+\ell-2}}{k!}(k-1) - \sum_{i=0}^{p}\sum_{\ell=0}^{p}\sum_{j=1}^{\infty}\sum_{k=2}^{\infty}\binom{p}{i}\binom{p}{\ell}\frac{j^{n+i}}{j!}\frac{k^{n+\ell-1}}{k!}(k-1)\\
&= \sum_{i=0}^{p}\sum_{\ell=0}^{p}\sum_{j=1}^{\infty}\sum_{k=2}^{\infty}\binom{p}{i}\binom{p}{\ell}\frac{j^{n+i}}{j!}\frac{k^{n+\ell-2}}{k!}(k-1)(j-k)\\
&= \sum_{i=0}^{p}\sum_{\ell=0}^{p}\sum_{j>k\geq 1}\binom{p}{i}\binom{p}{\ell}\left(\frac{j^{n+i}k^{n+\ell-2}}{j!k!}(k-1)(j-k) - \frac{j^{n+i-2}k^{n+\ell}}{j!k!}(j-1)(j-k)\right)\\
&= \sum_{i=0}^{p}\sum_{\ell=0}^{p}\sum_{j>k\geq 1}\binom{p}{i}\binom{p}{\ell}\frac{j^{n-2}k^{n-2}}{j!k!}(j-k)\left(j^{i+2}k^{\ell}(k-1) - j^{i}k^{\ell+2}(j-1)\right)\\
&= \sum_{j>k\geq 1}\frac{j^{n-2}k^{n-2}}{j!k!}(j-k)\left((k-1)j^2\sum_{i=0}^{p}\binom{p}{i}j^i\sum_{\ell=0}^{p}\binom{p}{\ell}k^{\ell} - (j-1)k^2\sum_{i=0}^{p}\binom{p}{i}j^i\sum_{\ell=0}^{p}\binom{p}{\ell}k^{\ell}\right)\\
&= \sum_{j>k\geq 1}\frac{j^{n-2}k^{n-2}}{j!k!}(j-k)(j+1)^p(k+1)^p\left((k-1)j^2 - (j-1)k^2\right).
\end{aligned}$$

Let $g(j,k) = (k-1)j^2 - (j-1)k^2$. We have proved that

$$e^2 f(n,p) = \sum_{j>k\geq 1}\frac{j^{n-2}k^{n-2}}{j!k!}(j-k)(j+1)^p(k+1)^p g(j,k).$$

Note that $g(j,1) = 1-j$, $g(j,2) = (j-2)^2$, and

$$g(j,k) = j^2k - j^2 - jk^2 + k^2 = (j-k)(jk-j-k) = (j-k)\left(k(\frac{j}{2}-1) + j(\frac{k}{2}-1)\right).$$

Hence, $g(j,k) > 0$ for $j > k \geq 3$, and it remains to prove that

$$\sum_{k=1}^{2}\sum_{j=k+1}^{\infty}\frac{j^{n-2}k^{n-2}}{j!k!}(j-k)(j+1)^p(k+1)^p g(j,k) > 0.$$

We have

$$\sum_{k=1}^{2}\sum_{j=k+1}^{\infty}\frac{j^{n-2}k^{n-2}}{j!k!}(j-k)(j+1)^p(k+1)^p g(j,k)$$

$$= \sum_{j=3}^{\infty} 2^{n-3}\frac{j^{n-2}}{j!}3^p(j+1)^p(j-2)^3 - \sum_{j=2}^{\infty}\frac{j^{n-2}}{j!}2^p(j+1)^p(j-1)^2$$

$$\geq 12^p\left(\sum_{j=3}^{5}2^{n-3}\frac{j^{n-2}}{j!}(j-2)^3 - \sum_{j=2}^{5}\frac{j^{n-2}}{j!}(j-1)^2\right) + \sum_{j=6}^{\infty}\frac{j^{n-2}}{j!}2^p(j+1)^p\left(2^{n-3}(j-2)^3-(j-1)^2\right)$$

$$= 12^p\left(\frac{1}{6}2^{n-3}3^{n-2} + \frac{8}{24}2^{n-3}4^{n-2} + \frac{27}{120}2^{n-3}5^{n-2} - \frac{1}{2}2^{n-2} - \frac{4}{6}3^{n-2} - \frac{9}{24}4^{n-2} - \frac{16}{120}5^{n-2}\right) +$$

$$\sum_{j=6}^{\infty}\frac{j^{n-2}}{j!}2^p(j+1)^p\left(2^{n-3}(j-2)^3-(j-1)^2\right).$$

It is easy to check that

$$\frac{1}{6}2^{n-3}3^{n-2} + \frac{8}{24}2^{n-3}4^{n-2} + \frac{27}{120}2^{n-3}5^{n-2} - \frac{1}{2}2^{n-2} - \frac{4}{6}3^{n-2} - \frac{9}{24}4^{n-2} - \frac{16}{120}5^{n-2} > 0$$

for all $n \geq 4$, and $2^{n-3}(j-2)^3 - (j-1)^2 > 0$ for all $n \geq 4$ and $j \geq 4$. Hence $f(n,p) > 0$. $\square$

As shown in the above proof, the above theorem is equivalent to the following inequalities for the Bell numbers.

Corollary 12. *If $n \geq 4$ and $p \geq 0$ then*

$$\sum_{i=0}^{p}\binom{p}{i}B_{n+i+1}\left(\sum_{\ell=0}^{p}\binom{p}{\ell}(B_{n+\ell-1}-B_{n+\ell-2})\right) > \sum_{i=0}^{p}\binom{p}{i}B_{n+i}\left(\sum_{\ell=0}^{p}\binom{p}{\ell}(B_{n+\ell}-B_{n+\ell-1})\right).$$

Example 13. For $p = 0$ and $n \geq 4$ we get the following inequality for the Bell numbers:

$$B_{n+1}(B_{n-1}-B_{n-2}) > B_n(B_n - B_{n-1}).$$

We now compare the average number of colors in colorings of $C_n \cup pK_1$ with the average number of colors in colorings of $H_{3,n-3} \cup pK_1$.

Lemma 14. *If $n \geq 3$, $r \geq 0$ and $p \geq 0$ then*

$$\mathcal{B}(H_{n,r}\cup pK_1) = \begin{cases} \sum_{i=0}^{\frac{n-3}{2}}\binom{\frac{n-3}{2}}{i}\mathcal{B}(H_{3,2i+r}\cup pK_1), & \textit{if } n \textit{ is odd;} \\ \sum_{i=0}^{\frac{n-4}{2}}\binom{\frac{n-4}{2}}{i}\mathcal{B}(H_{3,2i+r+1}\cup pK_1) + \mathcal{B}(P_{2+r}\cup pK_1), & \textit{if } n \textit{ is even} \end{cases}$$

and

$$\mathcal{T}(H_{n,r}\cup pK_1)=\begin{cases}\displaystyle\sum_{i=0}^{\frac{n-3}{2}}\binom{\frac{n-3}{2}}{i}\mathcal{T}(H_{3,2i+r}\cup pK_1), & \textit{if } n \textit{ is odd;}\\ \displaystyle\sum_{i=0}^{\frac{n-4}{2}}\binom{\frac{n-4}{2}}{i}\mathcal{T}(H_{3,2i+r+1}\cup pK_1)+\mathcal{T}(P_{2+r}\cup pK_1), & \textit{if } n \textit{ is even.}\end{cases}$$

Proof. The result is clearly valid for $n=3$. For $n=4$, Equations (3) and (4) give

$$\begin{aligned}\mathcal{B}(H_{4,r}\cup pK_1)&=\mathcal{B}(P_{4+r}\cup pK_1)-\mathcal{B}(H_{3,r}\cup pK_1)\\&=(\mathcal{B}(H_{3,r+1}\cup pK_1)+\mathcal{B}(P_{3+r}\cup pK_1))-(\mathcal{B}(P_{3+r}\cup pK_1)-\mathcal{B}(P_{2+r}\cup pK_1))\\&=\mathcal{B}(H_{3,r+1}\cup pK_1)+\mathcal{B}(P_{2+r}\cup pK_1).\end{aligned}$$

Similarly, $\mathcal{T}(H_{4,r}\cup pK_1)=\mathcal{T}(H_{3,r+1}\cup pK_1)+\mathcal{T}(P_{2+r}\cup pK_1)$ which shows that the result is valid for $n=4$. For larger values of n, we proceed by induction. Hence, it is sufficient to prove that $\mathcal{B}(H_{n,r}\cup pK_1)=\mathcal{B}(H_{3,n-3+r}\cup pK_1)+\mathcal{B}(H_{n-2,r}\cup pK_1)$ and $\mathcal{T}(H_{n,r}\cup pK_1)=\mathcal{T}(H_{3,n-3+r}\cup pK_1)+\mathcal{T}(H_{n-2,r}\cup pK_1)$. Using Equations (3) and (4), we get

$$\begin{aligned}\mathcal{B}(H_{n,r}\cup pK_1)&=\mathcal{B}(P_{n+r}\cup pK_1)-\mathcal{B}(H_{n-1,r}\cup pK_1)\\&=(\mathcal{B}(H_{3,n-3+r}\cup pK_1)+\mathcal{B}(P_{n+r-1}\cup pK_1))-(\mathcal{B}(P_{n+r-1}\cup pK_1)-\mathcal{B}(H_{n-2,r}\cup pK_1))\\&=\mathcal{B}(H_{3,n-3+r}\cup pK_1)+\mathcal{B}(H_{n-2,r}\cup pK_1).\end{aligned}$$

The proof for $\mathcal{T}(H_{n,r}\cup pK_1)$ is similar. □

We are now ready to compare $\mathcal{A}(C_n\cup pK_1)$ with $\mathcal{A}(H_{3,n-3}\cup pK_1)$.

Theorem 15. *$\mathcal{A}(C_n\cup pK_1)<\mathcal{A}(H_{3,n-3}\cup pK_1)$ for all $n\geq 3$ and $p\geq 0$.*

Proof. Lemma 14 (with $r=0$) gives the following:

$$\mathcal{B}(C_n\cup pK_1)=\begin{cases}\mathcal{B}(H_{3,n-3}\cup pK_1)+\displaystyle\sum_{i=0}^{\frac{n-5}{2}}\mathcal{B}(H_{3,2i}\cup pK_1), & \text{if } n \text{ is odd;}\\ \mathcal{B}(H_{3,n-3}\cup pK_1)+\displaystyle\sum_{i=0}^{\frac{n-6}{2}}\mathcal{B}(H_{3,2i+1}\cup pK_1)+\mathcal{B}(P_2\cup pK_1), & \text{if } n \text{ is even}\end{cases}$$

and

$$\mathcal{T}(C_n\cup pK_1)=\begin{cases}\mathcal{T}(H_{3,n-3}\cup pK_1)+\displaystyle\sum_{i=0}^{\frac{n-5}{2}}\mathcal{T}(H_{3,2i}\cup pK_1), & \text{if } n \text{ is odd;}\\ \mathcal{T}(H_{3,n-3}\cup pK_1)+\displaystyle\sum_{i=0}^{\frac{n-6}{2}}\mathcal{T}(H_{3,2i+1}\cup pK_1)+\mathcal{T}(P_2\cup pK_1), & \text{if } n \text{ is even.}\end{cases}$$

We know from Proposition 6 that $\mathcal{A}(P_2 \cup pK_1) < \mathcal{A}(C_3 \cup pK_1) = \mathcal{A}(H_{3,0} \cup pK_1)$. Since $H_{3,n-3} \cup pK_1$ is obtained from $H_{3,i} \cup pK_1$ ($i < n-3$) by repeatedly adding vertices of degree 1, we have $\mathcal{A}(P_2 \cup pK_1) < \mathcal{A}(H_{3,i} \cup pK_1) < \mathcal{A}(H_{3,n-3} \cup pK_1)$ for $i = 0, \ldots, n-5$. Proposition 7 therefore implies $\mathcal{A}(C_n \cup pK_1) < \mathcal{A}(H_{3,n-3} \cup pK_1)$. □

Equations (3) give $\mathcal{B}(H_{3,n-3} \cup pK_1) = \mathcal{B}(P_n \cup pK_1) - \mathcal{B}(P_{n-1} \cup pK_1)$ and $\mathcal{T}(H_{3,n-3} \cup pK_1) = \mathcal{T}(P_n \cup pK_1) - \mathcal{T}(P_{n-1} \cup pK_1)$. Hence, Propositions 3 and 5 immediately give the following corollary.

Corollary 16. *If $n \geq 3$ and $p \geq 0$ then*

$$\left(\sum_{j=1}^{n-1}(-1)^{j+1}\sum_{i=0}^{p}\binom{p}{i}B_{n+i-j+1}\right)\left(\sum_{i=0}^{p}\binom{p}{i}(B_{n+i-1}-B_{n+i-2})\right)$$
$$< \left(\sum_{j=1}^{n-1}(-1)^{j+1}\sum_{i=0}^{p}\binom{p}{i}B_{n+i-j}\right)\left(\sum_{i=0}^{p}\binom{p}{i}(B_{n+i}-B_{n+i-1})\right).$$

Example 17. For $p = 0$ and $n \geq 3$, the above corollary provides the following inequalities for the Bell numbers:

$$(B_{n-1} - B_{n-2})\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j+1} < (B_n - B_{n-1})\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j}.$$

It is easy to check that this inequality is also valid for $n = 2$.

We now compare the average number of colors in colorings of paths with the average number of colors in colorings of cycles.

Theorem 18. *$\mathcal{A}(C_n \cup pK_1) > \mathcal{A}(P_n \cup pK_1)$ for all $n \geq 5$ and $p \geq 0$.*

Proof. We know from Theorems 11 and 15 that $\mathcal{A}(P_n \cup pK_1) > \mathcal{A}(H_{3,n-4} \cup pK_1) > \mathcal{A}(C_{n-1} \cup pK_1)$, which implies that

$$\mathcal{T}(C_{n-1} \cup pK_1) < \frac{\mathcal{T}(P_n \cup pK_1)\mathcal{B}(C_{n-1} \cup pK_1)}{\mathcal{B}(P_n \cup pK_1)}.$$

Equations (3) show that $\mathcal{B}(C_n \cup pK_1) = \mathcal{B}(P_n \cup pK_1) - \mathcal{B}(C_{n-1} \cup pK_1)$ and $\mathcal{T}(C_n \cup pK_1) = \mathcal{T}(P_n \cup pK_1) - \mathcal{T}(C_{n-1} \cup pK_1)$. Hence:

$$\begin{aligned}
\mathcal{A}(C_n \cup pK_1) &= \frac{\mathcal{T}(C_n \cup pK_1)}{\mathcal{B}(C_n \cup pK_1)} = \frac{\mathcal{T}(P_n \cup pK_1) - \mathcal{T}(C_{n-1} \cup pK_1)}{\mathcal{B}(P_n \cup pK_1) - \mathcal{B}(C_{n-1} \cup pK_1)} \\
&> \frac{\mathcal{T}(P_n \cup pK_1) - \frac{\mathcal{T}(P_n \cup pK_1)\mathcal{B}(C_{n-1} \cup pK_1)}{\mathcal{B}(P_n \cup pK_1)}}{\mathcal{B}(P_n \cup pK_1) - \mathcal{B}(C_{n-1} \cup pK_1)} \\
&= \frac{\mathcal{T}(P_n \cup pK_1)\left(\mathcal{B}(P_n \cup pK_1) - \mathcal{B}(C_{n-1} \cup pK_1)\right)}{\mathcal{B}(P_n \cup pK_1)\left(\mathcal{B}(P_n \cup pK_1) - \mathcal{B}(C_{n-1} \cup pK_1)\right)} = \frac{\mathcal{T}(P_n \cup pK_1)}{\mathcal{B}(P_n \cup pK_1)} = \mathcal{A}(P_n \cup pK_1).
\end{aligned}$$

□

Propositions 3 and 5 immediately give the following corollary.

Corollary 19. *If $n \geq 5$ and $p \geq 0$ then*

$$\left(\sum_{i=0}^{p}\binom{p}{i}B_{n+i-1}\right)\left(\sum_{j=1}^{n-1}(-1)^{j+1}\sum_{i=0}^{p}\binom{p}{i}B_{n+i-j+1}\right) > \left(\sum_{i=0}^{p}\binom{p}{i}B_{n+i}\right)\left(\sum_{j=1}^{n-1}(-1)^{j+1}\sum_{i=0}^{p}\binom{p}{i}B_{n+i-j}\right).$$

Example 20. For $p = 0$ and $n \geq 5$ the above corollary provides the following inequality for the Bell numbers:

$$B_{n-1}\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j+1} > B_n\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j}.$$

Finally, we compare $\mathcal{A}(C_n \cup pK_1)$ with $\mathcal{A}(C_{n-2} \cup (p+2)K_1)$.

Lemma 21. *If $n \geq 3$ and $p \geq 0$ then*

$$\begin{aligned}\mathcal{B}(C_n \cup (p+2)K_1) &= \mathcal{B}(H_{n,2} \cup pK_1) + 2\mathcal{B}(H_{n,1} \cup pK_1) + \mathcal{B}(H_{n,0} \cup pK_1)\\ \mathcal{T}(C_n \cup (p+2)K_1) &= \mathcal{T}(H_{n,2} \cup pK_1) + 2\mathcal{T}(H_{n,1} \cup pK_1) + \mathcal{T}(H_{n,0} \cup pK_1).\end{aligned}$$

Proof. Equations (4) give

$$\begin{aligned}\mathcal{B}(C_n \cup (p+2)K_1) &= \mathcal{B}(H_{n,1} \cup (p+1)K_1) + \mathcal{B}(C_n \cup (p+1)K_1)\\ &= (\mathcal{B}(H_{n,2} \cup pK_1) + \mathcal{B}(H_{n,1} \cup pK_1)) + (\mathcal{B}(H_{n,1} \cup pK_1) + \mathcal{B}(H_{n,0} \cup pK_1))\\ &= \mathcal{B}(H_{n,2} \cup pK_1) + 2\mathcal{B}(H_{n,1} \cup pK_1) + \mathcal{B}(H_{n,0} \cup pK_1).\end{aligned}$$

The proof is similar for $\mathcal{T}(C_n \cup (p+2)K_1)$. □

Theorem 22. $\mathcal{A}(C_n \cup pK_1) > \mathcal{A}(C_{n-2} \cup (p+2)K_1)$ *for all $n \geq 5$ and $p \geq 0$.*

Proof. We divide the proof into two cases, according to the parity of n.

Case 1: n is odd. Lemma 14 (with $r = 0$) shows that

$$\mathcal{B}(C_n \cup pK_1) = \sum_{i=0}^{\frac{n-3}{2}}\mathcal{B}(H_{3,2i} \cup pK_1) \text{ and } \mathcal{T}(C_n \cup pK_1) = \sum_{i=0}^{\frac{n-3}{2}}\mathcal{T}(H_{3,2i} \cup pK_1)$$

and Lemmas 21 and 14 give

$$
\begin{aligned}
&\mathcal{B}(C_{n-2} \cup (p+2)K_1) \\
&= B(H_{n-2,2} \cup pK_1) + 2\mathcal{B}(H_{n-2,1} \cup pK_1) + \mathcal{B}(H_{n-2,0} \cup pK_1) \\
&= \sum_{i=0}^{\frac{n-5}{2}} \binom{\frac{n-5}{2}}{i} \mathcal{B}(H_{3,2i+2} \cup pK_1) + 2\sum_{i=0}^{\frac{n-5}{2}} \binom{\frac{n-5}{2}}{i} \mathcal{B}(H_{3,2i+1} \cup pK_1) + \sum_{i=0}^{\frac{n-5}{2}} \binom{\frac{n-5}{2}}{i} \mathcal{B}(H_{3,2i} \cup pK_1) \\
&= \sum_{i=1}^{\frac{n-3}{2}} \binom{\frac{n-5}{2}}{i-1} \mathcal{B}(H_{3,2i} \cup pK_1) + 2\sum_{i=0}^{\frac{n-5}{2}} \binom{\frac{n-5}{2}}{i} \mathcal{B}(H_{3,2i+1} \cup pK_1) + \sum_{i=0}^{\frac{n-5}{2}} \binom{\frac{n-5}{2}}{i} \mathcal{B}(H_{3,2i} \cup pK_1) \\
&= \sum_{i=1}^{\frac{n-5}{2}} \left(\binom{\frac{n-5}{2}}{i-1} + \binom{\frac{n-5}{2}}{i} \right) \mathcal{B}(H_{3,2i} \cup pK_1) + \mathcal{B}(H_{3,n-3} \cup pK_1) + \mathcal{B}(H_{3,0} \cup pK_1) \\
&+ 2\sum_{i=0}^{\frac{n-5}{2}} \binom{\frac{n-5}{2}}{i} \mathcal{B}(H_{3,2i+1} \cup pK_1) \\
&= \sum_{i=0}^{\frac{n-3}{2}} \binom{\frac{n-3}{2}}{i} \mathcal{B}(H_{3,2i} \cup pK_1) + 2\sum_{i=0}^{\frac{n-5}{2}} \binom{\frac{n-5}{2}}{i} \mathcal{B}(H_{3,2i+1} \cup pK_1) \\
&= \mathcal{B}(C_n \cup pK_1) + \sum_{i=1}^{n-4} \alpha_i \mathcal{B}(H_{3,i} \cup pK_1),
\end{aligned}
$$

where

$$
\alpha_i = \begin{cases} \binom{\frac{n-3}{2}}{\frac{i}{2}} - 1, & \text{if } i \text{ is even;} \\ 2\binom{\frac{n-5}{2}}{\frac{i-1}{2}}, & \text{if } i \text{ is odd.} \end{cases}
$$

Similarly, $\mathcal{T}(C_{n-2} \cup (p+2)K_1) = \mathcal{T}(C_n \cup pK_1) + \sum_{i=1}^{n-4} \alpha_i \mathcal{T}(H_{3,i} \cup pK_1)$. Moreover, we know from Theorems 11 and 18 that

$$
\mathcal{A}(H_{3,n-4} \cup pK_1) < \mathcal{A}(P_n \cup pK_1) < \mathcal{A}(C_n \cup pK_1).
$$

Also, given $i \in \{1, \ldots, n-5\}$, $H_{3,n-4} \cup pK_1$ is obtained from $H_{3,i} \cup pK_1$ by repeatedly adding vertices of degree 1, and it follows from Proposition 6 that

$$
\mathcal{A}(H_{3,i} \cup pK_1) < \mathcal{A}(H_{3,n-4} \cup pK_1) < \mathcal{A}(C_n \cup pK_1).
$$

Since all α_i are strictly positive, we can conclude from Proposition 7 that $\mathcal{A}(C_{n-2} \cup (p+2)K_1) < \mathcal{A}(C_n \cup (p+2)K_1)$.

Case 2: n is even. The proof is similar to the previous case. More precisely, Lemma 14 shows that

$$\mathcal{B}(C_n \cup pK_1) = \sum_{i=0}^{\frac{n-4}{2}} \mathcal{B}(H_{3,2i+1} \cup pK_1) + \mathcal{B}(P_2 \cup pK_1)$$

and

$$\mathcal{T}(C_n \cup pK_1) = \sum_{i=0}^{\frac{n-4}{2}} \mathcal{T}(H_{3,2i+1} \cup pK_1) + \mathcal{T}(P_2 \cup pK_1)$$

and lemmas 21 and 14 give

$$\begin{aligned}
&\mathcal{B}(C_{n-2} \cup (p+2)K_1) = B(H_{n-2,2} \cup pK_1) + 2\mathcal{B}(H_{n-2,1} \cup pK_1) + \mathcal{B}(H_{n-2,0} \cup pK_1) \\
&= \left(\sum_{i=0}^{\frac{n-6}{2}} \binom{\frac{n-6}{2}}{i} \mathcal{B}(H_{3,2i+3} \cup pK_1) + \mathcal{B}(P_4 \cup pK_1) \right) + 2 \left(\sum_{i=0}^{\frac{n-6}{2}} \binom{\frac{n-6}{2}}{i} \mathcal{B}(H_{3,2i+2} \cup pK_1) + \mathcal{B}(P_3 \cup pK_1) \right) \\
&+ \left(\sum_{i=0}^{\frac{n-6}{2}} \binom{\frac{n-6}{2}}{i} \mathcal{B}(H_{3,2i+1} \cup pK_1) + \mathcal{B}(P_2 \cup pK_1) \right) \\
&= \sum_{i=1}^{\frac{n-4}{2}} \binom{\frac{n-6}{2}}{i-1} \mathcal{B}(H_{3,2i+1} \cup pK_1) + 2 \sum_{i=0}^{\frac{n-6}{2}} \binom{\frac{n-6}{2}}{i} \mathcal{B}(H_{3,2i+2} \cup pK_1) + \sum_{i=0}^{\frac{n-6}{2}} \binom{\frac{n-6}{2}}{i} \mathcal{B}(H_{3,2i+1} \cup pK_1) \\
&+ \mathcal{B}(P_4 \cup pK_1) + 2\mathcal{B}(P_3 \cup pK_1) + \mathcal{B}(P_2 \cup pK_1) \\
&= \sum_{i=1}^{\frac{n-6}{2}} \left(\binom{\frac{n-6}{2}}{i-1} + \binom{\frac{n-6}{2}}{i} \right) \mathcal{B}(H_{3,2i+1} \cup pK_1) + \mathcal{B}(H_{3,n-3} \cup pK_1) + \mathcal{B}(H_{3,1} \cup pK_1) \\
&+ 2 \sum_{i=0}^{\frac{n-6}{2}} \binom{\frac{n-6}{2}}{i} \mathcal{B}(H_{3,2i+2} \cup pK_1) + \mathcal{B}(P_4 \cup pK_1) + 2\mathcal{B}(P_3 \cup pK_1) + \mathcal{B}(P_2 \cup pK_1) \\
&= \sum_{i=0}^{\frac{n-4}{2}} \binom{\frac{n-4}{2}}{i} \mathcal{B}(H_{3,2i+1} \cup pK_1) + 2 \sum_{i=0}^{\frac{n-6}{2}} \binom{\frac{n-6}{2}}{i} \mathcal{B}(H_{3,2i+2} \cup pK_1) \\
&+ \mathcal{B}(P_4 \cup pK_1) + 2\mathcal{B}(P_3 \cup pK_1) + \mathcal{B}(P_2 \cup pK_1) \\
&= \mathcal{B}(C_n \cup pK_1) + \sum_{i=2}^{n-4} \alpha_i \mathcal{B}(H_{3,i} \cup pK_1) + \mathcal{B}(P_4 \cup pK_1) + 2\mathcal{B}(P_3 \cup pK_1)
\end{aligned}$$

where

$$\alpha_i = \begin{cases} \binom{\frac{n-4}{2}}{\frac{i-1}{2}} - 1, & \text{if } i \text{ is odd;} \\ 2\binom{\frac{n-6}{2}}{\frac{i-2}{2}}, & \text{if } i \text{ is even.} \end{cases}$$

Similarly,

$$\mathcal{T}(C_{n-2} \cup (p+2)K_1) = \mathcal{T}(C_n \cup pK_1) + \sum_{i=2}^{n-4} \alpha_i \mathcal{T}(H_{3,i} \cup pK_1) + \mathcal{T}(P_4 \cup pK_1) + 2\mathcal{T}(P_3 \cup pK_1).$$

As already mentioned, we know that

$$\mathcal{A}(H_{3,i} \cup pK_1) < \mathcal{A}(H_{3,n-4} \cup pK_1) < \mathcal{A}(P_n \cup pK_1) < \mathcal{A}(C_n \cup pK_1)$$

for $i = 2, \ldots, n-4$. Also, $P_n \cup pK_1$ is obtained from $P_3 \cup pK_1$ by repeatedly adding vertices of degree 1, and it follows from Proposition 6 that

$$\mathcal{A}(P_3 \cup pK_1) < \mathcal{A}(P_4 \cup pK_1) < \mathcal{A}(P_n \cup pK_1) < \mathcal{A}(C_n \cup pK_1).$$

Since all α_i are strictly positive, we conclude from Proposition 7 that $\mathcal{A}(C_{n-2} \cup (p+2)K_1) < \mathcal{A}(C_n \cup (p+2)K_1)$. □

Proposition 5 immediately gives the following corollary.

Corollary 23. *If $n \geq 5$ and $p \geq 0$ then*

$$\begin{aligned} &\left(\sum_{j=1}^{n-3}(-1)^{j+1}\sum_{i=0}^{p+2}\binom{p+2}{i}B_{n+i-j-1}\right)\left(\sum_{j=1}^{n-1}(-1)^{j+1}\sum_{i=0}^{p}\binom{p}{i}B_{n+i-j}\right) \\ &< \left(\sum_{j=1}^{n-3}(-1)^{j+1}\sum_{i=0}^{p+2}\binom{p+2}{i}B_{n+i-j-2}\right)\left(\sum_{j=1}^{n-1}(-1)^{j+1}\sum_{i=0}^{p}\binom{p}{i}B_{n+i-j+1}\right). \end{aligned}$$

Example 24. For $p = 0$ and $n \geq 5$, the above corollary provides the following inequalities for the Bell numbers:

$$\begin{aligned} &\sum_{j=1}^{n-3}(-1)^{j+1}(B_{n-j-1} + 2B_{n-j} + B_{n-j+1})\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j} \\ &< \sum_{j=1}^{n-3}(-1)^{j+1}(B_{n-j-2} + 2B_{n-j-1} + B_{n-j})\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j+1} \\ \iff &\left(\sum_{j=2}^{n-2}(-1)^{j}B_{n-j} + 2\sum_{j=1}^{n-3}(-1)^{j+1}B_{n-j} + \sum_{j=0}^{n-4}(-1)^{j}B_{n-j}\right)\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j} \\ &< \left(\sum_{j=2}^{n-2}(-1)^{j}B_{n-j-1} + 2\sum_{j=1}^{n-3}(-1)^{j+1}B_{n-j-1} + \sum_{j=0}^{n-4}(-1)^{j}B_{n-j-1}\right)\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j+1} \\ \iff &(B_n + B_{n-1} + 7(-1)^n)\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j} < (B_{n-1} + B_{n-2} + 3(-1)^n)\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j+1}. \end{aligned}$$

It is easy to check that this inequality is also valid for $n = 4$.

4 Conclusion

We have shown how the average number of colors in the non-equivalent colorings of a graph G helps to derive inequalities for the Bell numbers. Among the inequalities, we have shown that

- $B_n^2 < B_{n-1}B_{n+1}$ for all $n \geq 1$,
- $B_n(B_n + B_{n+1}) < B_{n-1}(B_{n+1} + B_{n+2})$ for all $n \geq 1$,
- $B_n(B_n - B_{n-1}) < B_{n+1}(B_{n-1} - B_{n-2})$ for all $n \geq 4$,
- $(B_{n-1} - B_{n-2})\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j+1} < (B_n - B_{n-1})\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j}$ for all $n \geq 2$,
- $B_n\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j} < B_{n-1}\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j+1}$ for all $n \geq 5$,
- $(B_n + B_{n-1} + 7(-1)^n)\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j} < (B_{n-1} + B_{n-2} + 3(-1)^n)\sum_{j=1}^{n-1}(-1)^{j+1}B_{n-j+1}$ for all $n \geq 4$.

We have no doubt that other inequalities for the Bell numbers can be generated by comparing the average numbers of colors in the non-equivalent colorings of other types of graphs.

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