Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
European Journal of Operational Research 167 (2005) 427–443
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH
www.elsevier.com/locate/dsw

Decision Aiding

A characterization of concordance relations

Denis Bouyssou [a,*], Marc Pirlot [b]

[a] *CNRS, LAMSADE, Université Paris Dauphine, Place du Maréchal de Lattre de Tassigny, F-75775 Paris Cedex 16, France*
[b] *Faculté Polytechnique de Mons, 9, rue de Houdain, B-7000 Mons, Belgium*

Received 25 August 2003; accepted 11 March 2004
Available online 8 June 2004

Abstract

The notion of concordance is central to many multiple criteria techniques relying on ordinal information, e.g. outranking methods. It leads to compare alternatives by pairs on the basis of a comparison of coalitions of attributes in terms of "importance". This paper proposes a characterization of the binary relations that can be obtained using such comparisons within a general framework for conjoint measurement that allows for intransitive preferences. We show that such relations are mainly characterized by the very rough differentiation of preference differences that they induce on each attribute.

Keywords: Multiple criteria analysis; Concordance; Outranking methods; Conjoint measurement; Nontransitive preferences

1. Introduction

A classical problem in the field of decision analysis with multiple attributes is to build a preference relation on a set of multi-attributed alternatives on the basis of preferences expressed on each attribute and "inter-attribute" information such as weights. The classical way to do so is to build a value function that aggregates into a real number the evaluations of each alternative on the set of attributes (see French, 1993; Keeney and Raiffa, 1976). The construction of such a value function requires a detailed analysis of the trade-offs between the various attributes. When such an analysis appears difficult, one may resort to techniques for comparing alternatives that have a more ordinal character. Several such techniques, the so-called outranking methods, were proposed by B. Roy (for presentations in English, see Bouyssou, 2001; Roy, 1991, 1996; Vincke, 1992, 1999). Most outranking methods use the notion of concordance. It leads to compare alternatives by pairs on the basis of a comparison of coalitions of attributes in terms of "importance". Such pairwise comparisons do not lead to preference relations having nice transitivity properties (Bouyssou, 1996). These relations, henceforth called *concordance relations*, are therefore quite distinct from the transitive structures usually dealt with in conjoint measurement (Krantz et al., 1971; Roberts, 1979; Wakker, 1989).

* Corresponding author. Tel.: +33-1-44054898; fax: +33-1-44054091.
E-mail addresses: bouyssou@lamsade.dauphine.fr (D. Bouyssou), marc.pirlot@fpms.ac.be (M. Pirlot).

0377-2217/$ - see front matter
doi:10.1016/j.ejor.2004.03.015

The aim of this paper is to propose a characterization of concordance relations within a general framework for conjoint measurement allowing for incomplete and/or intransitive relations that was introduced in Bouyssou and Pirlot (2002b). It will turn out that, within this framework, the main distinctive feature of concordance relations is the very rough differentiation of preference differences that they induce on each attribute. Our results extend to the case of—possibly incomplete—reflexive preference relations (interpreted as "at least as good as" relations), the results proposed in Bouyssou and Pirlot (2002a,c) for asymmetric relations (interpreted as "strict preference"). Pirlot (1997) proposes an alternative approach to the analysis of concordance relations that is not based on a conjoint measurement model.

The paper is organized as follows. Section 2 introduces our main definitions and notation. Concordance relations are defined and illustrated in Section 3. Section 4 characterizes concordance relations within our general framework for conjoint measurement. A final section compares our results with other approaches to concordance relations and presents directions for future research. All proofs are relegated in Appendix A.

2. Definitions and notation

A *binary relation* $\mathscr{R}$ on a set A is a subset of $A \times A$; we write $a\mathscr{R}b$ instead of $(a, b) \in \mathscr{R}$. A binary relation $\mathscr{R}$ on A is said to be:

- *reflexive* if $[a\mathscr{R}a]$,
- *complete* if $[a\mathscr{R}b$ or $b\mathscr{R}a]$,
- *symmetric* if $[a\mathscr{R}b] \Rightarrow [b\mathscr{R}a]$,
- *asymmetric* if $[a\mathscr{R}b] \Rightarrow [\text{Not}[b\mathscr{R}a]]$,
- *transitive* if $[a\mathscr{R}b$ and $b\mathscr{R}c] \Rightarrow [a\mathscr{R}c]$,
- *Ferrers* if $[(a\mathscr{R}b$ and $c\mathscr{R}d) \Rightarrow (a\mathscr{R}d$ or $c\mathscr{R}b)]$,
- *semi-transitive* if $[(a\mathscr{R}b$ and $b\mathscr{R}c) \Rightarrow (a\mathscr{R}d$ or $d\mathscr{R}c)]$

for all $a, b, c, d \in A$.

A *weak order* (respectively, an *equivalence*) is a complete and transitive (respectively, reflexive, symmetric and transitive) binary relation. If $\mathscr{R}$ is an equivalence on A, $A/\mathscr{R}$ will denote the set of equivalence classes of $\mathscr{R}$ on A. An *interval order* is a complete and Ferrers binary relation. A *semiorder* is a semi-transitive interval order.

In this paper $\succsim$ will always denote a *reflexive* binary relation on a set $X = \prod_{i=1}^{n} X_i$ with $n \geqslant 2$. Elements of X will be interpreted as alternatives evaluated on a set $N = \{1, 2, \ldots, n\}$ of attributes and $\succsim$ as an "at least as good as" relation between these alternatives. We note $\succ$ (respectively, $\sim$) the asymmetric (respectively, symmetric) part of $\succsim$. A similar convention holds when $\succsim$ is starred, superscripted and/or subscripted.

For any nonempty subset J of the set of attributes N, we denote by X_J (respectively, X_{-J}) the set $\prod_{i \in J} X_i$ (respectively, $\prod_{i \notin J} X_i$). With customary abuse of notation, (x_J, y_{-J}) will denote the element $w \in X$ such that $w_i = x_i$ if $i \in J$ and $w_i = y_i$ otherwise. When $J = \{i\}$ we shall simply write X_{-i} and (x_i, y_{-i}).

Let J be a nonempty set of attributes. We define the marginal preference $\succsim_J$ induced by $\succsim$ on X_J letting, for all $x_J, y_J \in X_J$

$$x_J \succsim_J y_J \quad \text{iff } (x_J, z_{-J}) \succsim (y_J, z_{-J}), \quad \text{for all } z_{-J} \in X_{-J}.$$

When $J = \{i\}$ we write $\succsim_i$ instead of $\succsim_{\{i\}}$.

If, for all $x_J, y_J \in X_J$, $(x_J, z_{-J}) \succsim (y_J, z_{-J})$, for some $z_{-J} \in X_{-J}$ implies $x_J \succsim_J y_J$, we say that $\succsim$ is independent for J. If $\succsim$ is independent for all nonempty subsets of attributes we say that $\succsim$ is *independent*. It is not difficult to see that a binary relation is independent if and only if it is independent for $N \setminus \{i\}$, for all $i \in N$ (Wakker, 1989). A relation is said to be *weakly independent* if it is independent for all subsets containing a single attribute; while independence implies weak independence, it is clear that the converse is not true (Wakker, 1989).

We say that attribute $i \in N$ is *influent* (for $\succsim$) if there are $x_i, y_i, z_i, w_i \in X_i$ and $x_{-i}, y_{-i} \in X_{-i}$ such that $(x_i, x_{-i}) \succsim (y_i, y_{-i})$ and $\text{Not}[(z_i, x_{-i}) \succsim (w_i, y_{-i})]$ and *degenerate* otherwise. It is clear that a degenerate attribute has no influence whatsoever on the comparison of the elements of X and may be suppressed from N.

We say that attribute $i \in N$ is weakly essential for $\succsim$ (respectively, essential) if $(x_i, a_{-i}) \succ (y_i, a_{-i})$, for some $x_i, y_i \in X_i$ and some $a_{-i} \in X_{-i}$ (respectively, if $\succ_i$ is not empty). For a weakly independent relation, weak essentiality and essentiality are equivalent. It is clear that an essential attribute is weakly essential and that a weakly essential attribute is influent. The reverse implications do not hold. In order to avoid unnecessary minor complications, we suppose henceforth that *all attributes in N are influent*. This does *not* imply that all attributes are weakly essential.

3. Concordance relations

3.1. Definition

The following definition, building on Bouyssou and Pirlot (2002a) and Fargier and Perny (2001), formalizes the idea of a concordance relation, i.e. a preference relation that has been obtained comparing alternatives by pairs on the basis of the "importance" of the attributes favoring each element of the pair.

Definition 1 (*Concordance relations*). Let $\succsim$ be a *reflexive* binary relation on $X = \prod_{i=1}^{n} X_i$. We say that $\succsim$ is a *concordance relation* (or, more briefly, that $\succsim$ is a CR) if there are

- a *complete* binary relation S_i on each X_i $(i = 1, 2, \ldots, n)$,
- a binary relation $\trianglerighteq$ between subsets of N having N for union that is *monotonic* w.r.t. inclusion, i.e. for all $A, B, C, D \subseteq N$ such that $A \cup B = N$ and $C \cup D = N$,

  $$[A \trianglerighteq B, C \supseteq A, B \supseteq D] \Rightarrow C \trianglerighteq D, \tag{1}$$

  such that, for all $x, y \in X$,

  $$x \succsim y \iff S(x,y) \trianglerighteq S(y,x), \tag{2}$$

  where $S(x,y) = \{i \in N : x_i S_i y_i\}$. We say that $\langle \trianglerighteq, S_i \rangle$ is a *representation* of $\succsim$.

Hence, when $\succsim$ is a CR, the preference between x and y only depends on the subsets of attributes favoring x or y in terms of the complete relation S_i. It does not depend on "preference differences" between the various levels on each attribute besides the distinction between levels indicated by S_i. As shown below, although our definition imposes a comparison between two coalitions of attributes in order to decide whether or not x is at least as good as y, it is sufficiently flexible to include the case in which x is declared at least as good as y as soon as the attributes in $S(x,y)$ are "sufficiently" important, as in ELECTRE I (see Roy, 1968).

Let $\succsim$ be a CR with a representation $\langle \trianglerighteq, S_i \rangle$. We denote by I_i (respectively, P_i) the symmetric part (respectively, asymmetric part) of S_i. We define the relations $\triangleq$, $\triangleright$ and $\bowtie$ between subsets of N having N for union letting, for all $A, B \subseteq N$ such that

$$A \cup B = N,$$
$$A \triangleq B \iff [A \trianglerighteq B \text{ and } B \trianglerighteq A],$$
$$A \triangleright B \iff [A \trianglerighteq B \text{ and Not}[B \trianglerighteq A]],$$
$$A \bowtie B \iff [\text{Not}[A \trianglerighteq B] \text{ and Not}[B \trianglerighteq A]].$$

The following lemma takes note of some elementary properties of concordance relations; it uses the hypothesis that all attributes are influent.

Lemma 2. *If $\succsim$ is a CR with a representation $\langle \trianglerighteq, S_i \rangle$, then*:

1. *for all $i \in N$, P_i is nonempty*,
2. *for all $A, B \subseteq N$ such that $A \cup B = N$ exactly one of $A \triangleright B$, $B \triangleright A$, $A \triangleq B$ and $A \bowtie B$ holds and we have $N \triangleq N$*,
3. *for all $A \subseteq N$, $N \trianglerighteq A$*,
4. *$N \triangleright \varnothing$*,
5. *$\succsim$ is independent*,
6. *$\succsim$ is marginally complete, i.e., for all $i \in N$, all $x_i, y_i \in X_i$ and all $a_{-i} \in X_{-i}$, $(x_i, a_{-i}) \succsim (y_i, a_{-i})$ or $(y_i, a_{-i}) \succsim (x_i, a_{-i})$*,
7. *for all $i \in N$, either $\succsim_i = S_i$ or $x_i \sim_i y_i$ for all $x_i, y_i \in X_i$*,
8. *$\succsim$ has a unique representation.*

Proof. See Appendix A. □

We say that a CR $\succsim$ is *responsive* if, for all $A \subseteq N$, $[A \neq \varnothing] \Rightarrow N \triangleright N \setminus A$. As shown by the examples below, there are CR that are not responsive. It is not difficult to see that a CR is responsive if and only if all attributes are (weakly) essential on top of being influent. This implies $\succsim_i = S_i$. This shows that in our nontransitive setting, assuming that all attributes are (weakly) essential is far from being as innocuous an hypothesis as it traditionally is in conjoint measurement.

The main objective of this paper is to characterize CR within a general framework of conjoint measurement, using conditions that will allow us to isolate their specific features.

Remark 3. In most outranking methods, the concordance relation is modified by the application of the so-called discordance condition (Roy, 1991). Discordance amounts to refuse to accept the assertion $x \succsim y$ when y is judged "far better" than x on some attribute. This leads to defining a binary relation $V_i \subseteq P_i$ on each X_i and to accept the assertion $x \succsim y$ only when (2) holds and it is not true that $y_j V_j x_j$, for some $j \in N$. Our analysis does not take discordance into account.

3.2. Examples

The following examples show that CR arise with a large variety of ordinal aggregation models that have been studied in the literature.

Example 4 (*Simple majority preferences* (Sen, 1986)). The binary relation $\succsim$ is a simple majority preference relation if there is a weak order S_i on each X_i such that

$$x \succsim y \iff |\{i \in N : x_i S_i y_i\}| \geqslant |\{i \in N : y_i S_i x_i\}|.$$

A simple majority preference relation is easily seen to be a CR defining $\trianglerighteq$ letting, for all $A, B \subseteq N$ such that $A \cup B = N$,

$$A \trianglerighteq B \iff |A| \geqslant |B|.$$

It is easy to see that $\succsim$ is complete but that, in general, neither $\succsim$ nor $\succ$ are transitive. This CR is responsive. For all $A, B \subseteq N$ such that $A \cup B = N$, we have either $A \trianglerighteq B$ or $B \trianglerighteq A$.

Example 5 (*ELECTRE I* (Roy, 1968, 1991)). The binary relation $\succsim$ is an ELECTRE I preference relation if there are a real number $s \in [1/2; 1]$ and, for all $i \in N$,

- a semiorder S_i on X_i,
- a positive real number $w_i > 0$,

such that, for all $x, y \in X$,

$$x \succsim y \iff \frac{\sum_{i \in S(x,y)} w_i}{\sum_{j \in N} w_j} \geqslant s.$$

An ELECTRE I preference relation is easily seen to be a CR defining $\trianglerighteq$ letting, for all $A, B \subseteq N$ such that $A \cup B = N$,

$$A \trianglerighteq B \iff \frac{\sum_{i \in A} w_i}{\sum_{j \in N} w_j} \geqslant s.$$

Such a CR may not be responsive. It may well happen that, for some $A, B \subseteq N$ such that $A \cup B = N$, neither $A \trianglerighteq B$ nor $B \trianglerighteq A$, i.e. $A \bowtie B$. The importance relation $\trianglerighteq$ is such that, for all $A, B \subseteq N$, $A \trianglerighteq B \Rightarrow A \trianglerighteq N$. Simple examples show that, in general, $\succsim$ is neither complete nor transitive. It may happen that $\succ$ is not transitive and has circuits.

Example 6 (*Semiordered weighted majority* (Vansnick, 1986)). The binary relation $\succsim$ is a semiordered weighted majority preference relation if there are a real number $\varepsilon \geqslant 0$ and, for all $i \in N$,

- a semiorder S_i on X_i,
- a real number $w_i > 0$,

such that

$$x \succsim y \iff \sum_{i \in S(x,y)} w_i \geqslant \sum_{j \in S(y,x)} w_j - \varepsilon.$$

An semiordered weighted majority preference relation is easily seen to be a CR defining $\trianglerighteq$ letting, for all $A, B \subseteq N$ such that $A \cup B = N$:

$$A \trianglerighteq B \iff \sum_{i \in A} w_i \geqslant \sum_{j \in B} w_j - \varepsilon.$$

The relation $\succsim$ may not be transitive (the same is true for $\succ$). It is always complete. Unless in special cases, this CR is not responsive. Clearly, for all $A, B \subseteq N$ such that $A \cup B = N$, we have either $A \trianglerighteq B$ or $B \trianglerighteq A$.

4. A characterization of concordance relations

4.1. Concordance relations without attribute transitivity

Our general framework for conjoint measurement tolerating intransitive and incomplete relations is detailed in Bouyssou and Pirlot (2002b). We briefly recall here its main ingredients and its underlying logic. It mainly rests on the analysis of induced relations comparing preference differences on each attribute. The importance of such relations for the analysis of conjoint measurement models is detailed in Wakker (1988, 1989).

Definition 7 (*Relations comparing preference differences*). Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. We define the binary relations $\succsim_i^*$ and $\succsim_i^{**}$ on X_i^2 letting, for all $x_i, y_i, z_i, w_i \in X_i$,

$$(x_i, y_i) \succsim_i^* (z_i, w_i) \iff$$
$$[\text{for all } a_{-i}, b_{-i} \in X_{-i}, (z_i, a_{-i}) \succsim (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \succsim (y_i, b_{-i})]$$

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \iff$$
$$[(x_i, y_i) \succsim_i^* (z_i, w_i) \text{ and } (w_i, z_i) \succsim_i^* (y_i, x_i)].$$

The definition of $\succsim_i^*$ suggests that $(x_i, y_i) \succsim_i^* (z_i, w_i)$ can be interpreted as saying that the preference difference between x_i and y_i is at least as large as the preference difference between z_i and w_i. Indeed, as soon as $(z_i, a_{-i}) \succsim (w_i, b_{-i})$, $(x_i, y_i) \succsim_i^* (z_i, w_i)$ implies $(x_i, a_{-i}) \succsim (y_i, b_{-i})$. The definition of $\succsim_i^*$ does not imply that the two "opposite" differences (x_i, y_i) and (y_i, x_i) are linked. This is at variance with the intuition concerning preference differences and motivates the introduction of the relation $\succsim_i^{**}$. We have $(x_i, y_i) \succsim_i^{**} (z_i, w_i)$ when we have both $(x_i, y_i) \succsim_i^* (z_i, w_i)$ and $(w_i, z_i) \succsim_i^* (y_i, x_i)$. By construction, $\succsim_i^{**}$ is *reversible*, i.e. $(x_i, y_i) \succsim_i^{**} (z_i, w_i) \iff (w_i, z_i) \succsim_i^{**} y_i, x_i)$.

The asymmetric and symmetric parts of $\succsim_i^*$ are, respectively, denoted by $\succ_i^*$ and $\sim_i^*$, a similar convention holding for $\succsim_i^{**}$. By construction, $\succsim_i^*$ and $\succsim_i^{**}$ are reflexive and transitive. Therefore, $\sim_i^*$ and $\sim_i^{**}$ are equivalence relations (the hypothesis that attribute $i \in N$ is influent meaning that $\sim_i^*$ has at least two distinct equivalence classes). It is important to notice that $\succsim_i^*$ and $\succsim_i^{**}$ may not be complete. As will be apparent soon, interesting consequences obtain when this is the case.

We note below a few useful connections between $\succsim_i^*$, $\succsim_i^{**}$ and $\succsim$.

Lemma 8

(1) *$\succsim$ is independent if and only if $(x_i, x_i) \sim_i^* (y_i, y_i)$, for all $i \in N$ and all $x_i, y_i \in X_i$.*

(2) *For all $x, y \in X$ and all $z_i, w_i \in X_i$,*

$$[x \succsim y \text{ and } (z_i, w_i) \succsim_i^* (x_i, y_i)] \Rightarrow (z_i, x_{-i}) \succsim (w_i, y_{-i}), \tag{3}$$

$$[(z_i, w_i) \sim_i^* (x_i, y_i), \text{ for all } i \in N] \Rightarrow [x \succsim y \iff z \succsim w]. \tag{4}$$

Proof. See Bouyssou and Pirlot (2002b, Lemma 3). □

We now introduce two conditions, taken from Bouyssou and Pirlot (2002b), that will form the basis of our framework for conjoint measurement. Their main rôle is to ensure that $\succsim_i^*$ and $\succsim_i^{**}$ are complete.

Definition 9 (*Conditions RC1 and RC2*). Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. This relation is said to satisfy:

$RC1_i$ if

$$\left.\begin{array}{c}(x_i, a_{-i}) \succsim (y_i, b_{-i})\\ \text{and}\\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(x_i, c_{-i}) \succsim (y_i, d_{-i})\\ \text{or}\\ (z_i, a_{-i}) \succsim (w_i, b_{-i}),\end{array}\right.$$

$RC2_i$ if

$$\left.\begin{array}{c}(x_i, a_{-i}) \succsim (y_i, b_{-i})\\ \text{and}\\ (y_i, c_{-i}) \succsim (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(z_i, a_{-i}) \succsim (w_i, b_{-i})\\ \text{or}\\ (w_i, c_{-i}) \succsim (z_i, d_{-i}),\end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $\succsim$ satisfies $RC1$ (respectively, $RC2$) if it satisfies $RC1_i$ (respectively, $RC2_i$) for all $i \in N$.

Condition $RC1_i$ (inteR-attribute Cancellation) strongly suggests that, w.r.t. the relation $\succsim_i^*$, either the difference (x_i, y_i) is at least as large as the difference (z_i, w_i) or vice versa. Indeed, suppose that $(x_i, a_{-i}) \succsim (y_i, b_{-i})$ and $(z_i, c_{-i}) \succsim (w_i, d_{-i})$. If the preference difference between z_i and w_i is at least as large as the difference between x_i and y_i, we should obtain $(z_i, a_{-i}) \succsim (w_i, b_{-i})$. Similarly, if the preference difference between x_i and y_i is at least as large as the preference difference between z_i and w_i, we should obtain $(x_i, c_{-i}) \succsim (y_i, d_{-i})$. This is precisely what $RC1_i$ says.

Condition $RC2_i$ suggests that the preference difference (x_i, y_i) is linked to the "opposite" preference difference (y_i, x_i). Indeed, it amounts to saying that either the preference difference between x_i and y_i is at least as large as the preference difference between z_i and w_i or that the preference difference between w_i and z_i is at least as large as the preference difference between y_i and x_i. Taking $x_i = y_i$, $z_i = w_i$, $a_{-i} = c_{-i}$ and $b_{-i} = d_{-i}$ shows that $RC2_i$ implies that $\succsim$ is independent for $N \setminus \{i\}$ and, hence, independent.

The following lemma summarizes the main consequences of $RC1$ and $RC2$ on $\succsim_i^*$ and $\succsim_i^{**}$.

Lemma 10

(1) $RC1_i \iff [\succsim_i^*$ *is complete*],
(2) $RC2_i \iff$ [*for all* $x_i, y_i, z_i, w_i \in X_i$, Not$[(x_i, y_i) \succsim_i^* (z_i, w_i)] \Rightarrow (y_i, x_i) \succsim_i^* (w_i, z_i)$],
(3) [$RC1$ *and* $RC2_i$] $\iff [\succsim_i^{**}$ *is complete*].
(4) *In the class of reflexive relations*, $RC1$ *and* $RC2$ *are independent conditions*.

Proof. See Bouyssou and Pirlot (2002b, Lemmas 1 and 2). □

We envisage here binary relations $\succsim$ on X that can be represented as:

$$x \succsim y \iff F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geqslant 0, \qquad \text{(M)}$$

where p_i are real-valued functions on X_i^2 that are *skew symmetric* (i.e. such that $p_i(x_i, y_i) = -p_i(y_i, x_i)$ for all $x_i, y_i \in X_i$) and F is a real-valued function on $\prod_{i=1}^n p_i(X_i^2)$ being *nondecreasing* in all its arguments and such that, abusing notation, $F(\mathbf{0}) \geqslant 0$. The following lemma takes note of a few properties of binary relations satisfying model (M).

Lemma 11. *Let* $\succsim$ *be a binary relation on* $X = \prod_{i=1}^n X_i$ *that has a representation in model* (M). *Then*

(1) $\succsim$ *is reflexive, independent and marginally complete*,
(2) $[x_i \succ_i y_i$, *for all* $i \in J \subseteq N] \Rightarrow [x_J \succ_J y_J]$,
(3) $\succsim$ *satisfies* $RC1$ *and* $RC2$.

Proof. See Bouyssou and Pirlot (2002b, Proposition 1 and Lemma 2). □

The conditions envisaged above allow us to completely characterize model (M) when, for all $i \in N$, $X_i^2 / \sim_i^{**}$ is finite or countably infinite.

Theorem 12. *Let* $\succsim$ *be a binary relation on* $X = \prod_{i=1}^n X_i$. *If, for all* $i \in N$, $X_i^2 / \sim_i^{**}$ *is finite or countably infinite, then* $\succsim$ *has a representation* (M) *if and only if it is reflexive and satisfies* $RC1$ *and* $RC2$.

Proof. See Bouyssou and Pirlot (2002b, Theorem 1). □

Remark 13. It should be noticed that the framework offered by model (M) is quite flexible. It is not difficult to see that preference relations that have a representation in the additive value model (see Fishburn, 1970; Krantz et al., 1971; Wakker, 1989):

$$x \succsim y \iff \sum_{i=1}^n u_i(x_i) \geqslant \sum_{i=1}^n u_i(y_i) \qquad \text{(U)}$$

(where u_i is a real-valued function on X_i), or the additive difference model (see Fishburn, 1992; Tversky, 1969)

$$x \succsim y \iff \sum_{i=1}^n \Phi_i(u_i(x_i) - u_i(y_i)) \geqslant 0 \qquad \text{(ADM)}$$

(where Φ_i is increasing and odd), are all included in model (M). We show below that model (M) also contains all CR.

Remark 14. Following Bouyssou and Pirlot (2002b), it is not difficult to extend Theorem 12 to sets of arbitrary cardinality adding a, necessary, condition implying that the weak orders $\succsim_i^{**}$ have a numerical representation. This will not be useful here. We also refer the reader to Bouyssou and Pirlot (2002b) for an analysis of the, obviously very weak, uniqueness properties of the numerical representation in Theorem 12. Let us simply observe here that the proof of Theorem 12 shows that if $\succsim$ has a representation in model (M), it always has a *regular* representation, i.e. a representation such that:

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \iff p_i(x_i, y_i) \geqslant p_i(z_i, w_i). \qquad (5)$$

Although (5) may be violated in some representations, it is easy to see that we always have:

$$(x_i, y_i) \succ_i^{**} (z_i, w_i) \Rightarrow p_i(x_i, y_i) > p_i(z_i, w_i). \tag{6}$$

When an attribute is influent, we know that there are at least two distinct equivalence classes of $\sim_i^*$. When $RC1_i$ and $RC2_i$ hold, this implies that $\succsim_i^{**}$ must have at least three distinct equivalence classes. Therefore, when all attributes are influent, the functions p_i in any representation of $\succsim$ in model (M) must take at least three distinct values.

Consider a binary relation $\succsim$ that has a representation in model (M) in which all functions p_i take at most three distinct values. Intuition suggests that such a relation $\succsim$ is quite close from a concordance relation. We formalize this intuition below.

The following two conditions aim at capturing the ordinal character of the aggregation underlying CR and, hence, at characterizing CR within the framework of model (M).

Definition 15 (*Conditions UC and LC*). Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^n X_i$. This relation is said to satisfy:

UC_i if

$$\left.\begin{array}{c}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim (x_i, b_{-i}) \\ \text{or} \\ (x_i, c_{-i}) \succsim (y_i, d_{-i}),\end{array}\right.$$

LC_i if

$$\left.\begin{array}{c}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (y_i, c_{-i}) \succsim (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \succsim (x_i, b_{-i}) \\ \text{or} \\ (z_i, c_{-i}) \succsim (w_i, d_{-i}),\end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $\succsim$ satisfies UC (respectively, LC) if it satisfies UC_i (respectively, LC_i) for all $i \in N$.

The interpretation of these two conditions is easier considering their consequences on the relations $\succsim_i^*$ and $\succsim_i^{**}$.

Lemma 16

(1) $UC_i \iff [\text{Not}[(y_i, x_i) \succsim_i^* (x_i, y_i)] \Rightarrow (x_i, y_i) \succsim_i^* (z_i, w_i),$ *for all* $x_i, y_i, z_i, w_i \in X_i]$.

(2) $LC_i \iff [\text{Not}[(y_i, x_i) \succsim_i^* (x_i, y_i)] \Rightarrow (z_i, w_i) \succsim_i^* (y_i, x_i),$ *for all* $x_i, y_i, z_i, w_i \in X_i]$.

(3) $[RC2_i, UC_i$ *and* $LC_i] \Rightarrow RC1_i$.

(4) $[RC2_i, UC_i$ *and* $LC_i] \Rightarrow [\succsim_i^{**}$ *has at most three equivalence classes*].

(5) *In the class of reflexive relations*, $RC2$, UC *and* LC *are independent conditions.*

(6) $[RC2_i, UC_i, LC_i] \Rightarrow$ *all* $x_i, y_i \in X_i$ *such that* $(x_i, y_i) \succsim_i^{**} (y_i, y_i)$ *satisfy one and the same of the following*:

I $(x_i, y_i) \succ_i^* (y_i, y_i) \succ_i^* (y_i, x_i)$,
II $(x_i, y_i) \succ_i^* (y_i, y_i)$ and $(y_i, y_i) \sim_i^* (y_i, x_i)$,
III $(x_i, y_i) \sim_i^* (y_i, y_i)$ and $(y_i, y_i) \succ_i^* (y_i, x_i)$.

Proof. See Appendix A. □

Hence, condition UC amounts to saying that if a preference difference (y_i, x_i) is not larger than its opposite (x_i, y_i), it is the smallest possible preference difference in that every other preference is at least as large as (y_i, x_i). Condition LC has an obvious dual interpretation.

Together with $RC2_i$, conditions UC_i and LC_i imply that $\succsim_i^{**}$ has at most three equivalence classes, that $RC1_i$ holds and that each attribute has type I, II or III as defined in part 6. In presence of $RC2$, these two conditions seem to adequately capture the ordinal character of the aggregation at work in a CR. Indeed, when $RC2$, LC and UC hold, a preference difference is either "positive", "null" or "negative"; there is no possibility to further differentiate the size of preference differences. When an attribute has type I, it has the above three distinct types of preference differences. For type II attributes, it is only possible to distinguish between positive and nonpositive differences. For type III it is only possible to distinguish negative and nonnegative differences.

The following lemma shows that all CR satisfy UC and LC while having a representation in model (M).

Lemma 17. *Let* $\succsim$ *be a binary relation on a set* $X = \prod_{i=1}^n X_i$. *If* $\succsim$ *is a CR then,*

(1) $\succsim$ *satisfies* $RC1$ *and* $RC2$,
(2) $\succsim$ *satisfies* UC *and* LC.

Proof. See Appendix A. □

We are now in position to present our general characterization of CR.

Theorem 18. *Let $\succsim$ be a binary relation on $X = \prod_{i=1}^{n} X_i$. Then $\succsim$ is a CR iff it is reflexive and satisfies RC2, UC and LC.*

Proof. See Appendix A. □

Remark 19. An easy corollary of the above result is that a binary relation is a CR if and only if it has a representation in model (M) in which all functions p_i take at most three distinct values.

4.2. Concordance relations with attribute transitivity

Our definition of CR relations in Section 3 does not require the relations S_i to possess any remarkable property besides completeness. This is at variance with what is done in most ordinal aggregation methods (see the examples in Section 3.2). We show here how to characterize CR with all relations S_i being semiorders. Our results are easily extended, using conditions introduced in Bouyssou and Pirlot (2003), to cover the case in which all relations S_i are weak orders.

We first show, following Bouyssou and Pirlot (2003), how to introduce a linear arrangement of the elements of each X_i within the framework of model (M).

Definition 20 (*Conditions AC*1, *AC*2 *and AC*3). We say that $\succsim$ satisfies:

$AC1_i$ if

$$\left.\begin{array}{c} x \succsim y \\ \text{and} \\ z \succsim w \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (z_i, x_{-i}) \succsim y \\ \text{or} \\ (x_i, z_{-i}) \succsim w, \end{array}\right.$$

$AC2_i$ if

$$\left.\begin{array}{c} x \succsim y \\ \text{and} \\ z \succsim w \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x \succsim (w_i, y_{-i}) \\ \text{or} \\ z \succsim (y_i, w_{-i}), \end{array}\right.$$

$AC3_i$ if

$$\left.\begin{array}{c} z \succsim (x_i, a_{-i}) \\ \text{and} \\ (x_i, b_{-i}) \succsim y \end{array}\right\} \Rightarrow \left\{\begin{array}{c} z \succsim (w_i, a_{-i}) \\ \text{or} \\ (w_i, b_{-i}) \succsim y, \end{array}\right.$$

for all $x, y, z, w \in X$, all $a_{-i}, b_{-i} \in X_{-i}$ and all $x_i, w_i \in X_i$. We say that $\succsim$ satisfies $AC1$ (respectively, $AC2$, $AC3$) if it satisfies $AC1_i$ (respectively, $AC2_i$, $AC3_i$) for all $i \in N$.

These three conditions are transparent variations on the theme of the Ferrers ($AC1$ and $AC2$) and semi-transitivity ($AC3$) conditions that are made possible by the product structure of X. The rationale for the name "AC" is that these conditions are "intrA-attribute Cancellation" conditions. Condition $AC1_i$ suggests that the elements of X_i (instead of the elements of X had the original Ferrers condition been invoked) can be linearly ordered considering "upward dominance": if x_i "upward dominates" z_i then $(z_i, c_{-i}) \succsim w$ entails $(x_i, c_{-i}) \succsim w$. Condition $AC2_i$ has a similar interpretation considering now "downward dominance". Condition $AC3_i$ ensures that the linear arrangements of the elements of X_i obtained considering upward and downward dominance are not incompatible. The study of the impact of these new conditions on model (M) will require an additional definition.

Definition 21 (*Linearity*, Doignon et al. (1988)). Let $\mathscr{R}$ be a binary relation on a set A^2. We say that

- $\mathscr{R}$ is *right-linear* iff $[\text{Not}[(b, c) \mathscr{R} (a, c)] \Rightarrow (a, d) \mathscr{R} (b, d)]$,
- $\mathscr{R}$ is *left-linear* iff $[\text{Not}[(c, a) \mathscr{R} (c, b)] \Rightarrow (d, b) \mathscr{R} (d, a)]$,
- $\mathscr{R}$ is *strongly linear* iff $[\text{Not}[(b, c) \mathscr{R} (a, c)]$ or $\text{Not}[(c, a) \mathscr{R} (c, b)]] \Rightarrow [(a, d) \mathscr{R} (b, d)$ and $(d, b) \mathscr{R} (d, a)]$

for all $a, b, c, d \in A$.

We have the following:

Lemma 22
(1) $AC1_i \iff \succsim_i^*$ *is right-linear*.
(2) $AC2_i \iff \succsim_i^*$ *is left-linear*.

(3) $AC3_i \iff [\text{Not}[(x_i, z_i) \succsim_i^* (y_i, z_i)]$ *for some* $z_i \in X_i \Rightarrow (w_i, x_i) \succsim_i^* (w_i, y_i)$, *for all* $w_i \in X_i]$.
(4) $[AC1_i, AC2_i$ *and* $AC3_i] \iff \succsim_i^*$ *is strongly linear* $\iff \succsim_i^{**}$ *is strongly linear*.
(5) In the class of reflexive relations satisfying $RC1$ and $RC2$, $AC1$, $AC2$ and $AC3$ are independent conditions.

Proof. See Bouyssou and Pirlot (2003, Lemma 4). □

We envisage binary relations $\succsim$ on X that can be represented as:

$$x \succsim y \iff F(\varphi_1(u_1(x_1), u_1(y_1)), \ldots, \varphi_n(u_n(x_n), u_n(y_n))) \geqslant 0, \qquad (\text{M}*)$$

where u_i are real-valued functions on X_i, φ_i are real-valued functions on $u_i(X_i)^2$ that are *skew symmetric*, *nondecreasing* in their first argument (and, therefore, *nonincreasing* in their second argument) and F is a real-valued function on $\prod_{i=1}^n \varphi_i(u_i(X_i)^2)$ being *nondecreasing* in all its arguments and such that $F(\mathbf{0}) \geqslant 0$. We summarize some useful consequences of model (M*) in the following:

Lemma 23. *Let* $\succsim$ *be a binary relation on* $X = \prod_{i=1}^n X_i$. *If* $\succsim$ *has a representation in* (M*), *then*

(1) *it satisfies* $AC1$, $AC2$ *and* $AC3$,
(2) *for all* $i \in N$, *the binary relation* T_i *on* X_i *defined by* $x_i T_i y_i \iff (x_i, y_i) \succsim_i^{**} (x_i, x_i)$ *is a semiorder.*

Proof. See Bouyssou and Pirlot (2003, Lemma 4). □

The conditions introduced so far allow to characterize model (M*) when each X_i is denumerable.

Theorem 24. *Let* $\succsim$ *be a binary relation on a finite or countably infinite set* $X = \prod_{i=1}^n X_i$. *Then* $\succsim$ *has a representation* (M*) *if and only if it is reflexive and satisfies* $RC1$, $RC2$, $AC1$, $AC2$ *and* $AC3$.

Proof. See Bouyssou and Pirlot (2003, Theorem 2). □

Remark 25. Note that, contrary to Theorems 12 and 24 is only stated here for finite or countably infinite sets X. This is no mistake: we refer to Bouyssou and Pirlot (2003) for details and for the analysis of the extension of this result to the general case.

Many variants of model (M*) are studied in Bouyssou and Pirlot (2003) including the ones in which φ is *increasing* in its first argument (and, thus, decreasing in its second argument) and F is odd. Clearly, although model (M*) is a particular case of model (M), it is still flexible enough to contain as particular cases models (U) and (ADM). We show below that it also contain all CR in which the relations S_i are semiorders.

The following lemma shows that all CR obtained on the basis of semiorders satisfy the conditions of model (M*).

Lemma 26. *Let* $\succsim$ *be a binary relation on* $X = \prod_{i=1}^n X_i$. *If* $\succsim$ *is a CR with a representation* $\langle \trianglerighteq, S_i \rangle$ *in which* S_i *is a semiorder then* $\succsim$ *satisfies* $AC1_i$, $AC2_i$ *and* $AC3_i$.

Proof. See Appendix A. □

Although Lemma 22 shows that, in the class of reflexive binary relations satisfying $RC1$ and $RC2$, $AC1$, $AC2$ and $AC3$ are independent conditions, the situation is more delicate when we bring conditions UC and LC into the picture since they impose strong requirements on $\succsim_i^*$ and $\succsim_i^{**}$. We have:

Lemma 27

(1) *Let* $\succsim$ *be a reflexive binary relation on a set* $X = \prod_{i=1}^n X_i$ *satisfying* $RC2$, UC *and* LC. *Then* $\succsim$ *satisfies* $AC1_i$ *iff it satisfies* $AC2_i$.
(2) *In the class of reflexive binary relations satisfying* $RC2$, UC *and* LC, *conditions* $AC1$ *and* $AC3$ *are independent.*

Proof. See Appendix A. □

This leads to our characterization of CR in which all relations S_i are semiorders.

Theorem 28. *Let $\succsim$ be a binary relation on $X = \prod_{i=1}^{n} X_i$. Then $\succsim$ is a CR having a representation $\langle \trianglerighteq, S_i \rangle$ in which all S_i are semiorders iff it is reflexive and satisfies $RC2$, UC, LC, $AC1$ and $AC3$.*

Proof. See Appendix A.

Remark 29. An easy corollary of the above result is that a binary relation on a finite or countably infinite set X is a CR with a representation $\langle \trianglerighteq, S_i \rangle$ in which all relations S_i are semiorders if and only if it has a representation in model (M*) in which all functions φ_i take at most three distinct values.

5. Discussion and comments

A number of recent papers (see Dubois et al., 2002; Dubois et al., 2001, 2003; Fargier and Perny, 2001; Greco et al., 2001) have close connections with the results proposed here. We briefly analyze them below and give possible directions for future research.

5.1. Relation to Greco et al. (2001)

Greco et al. (2001) have proposed a characterization of concordance relations in which all attributes are of type III in the sense of Lemma 16. Their analysis is based on a very clever condition limiting the number of equivalence classes of $\succsim_i^*$. We say that $\succsim$ is *super-coarse* on attribute $i \in N$ if, for all $x_i, y_i, z_i, w_i, r_i, s_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$,

$$\left.\begin{array}{c}(x_i, a_{-i}) \succsim (y_i, b_{-i})\\ \text{and}\\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(x_i, c_{-i}) \succsim (y_i, d_{-i})\\ \text{or}\\ (r_i, a_{-i}) \succsim (s_i, b_{-i}).\end{array}\right.$$

This condition is clear strengthening of $RC1_i$. It is not difficult to see that a $\succsim$ is super-coarse on attribute $i \in N$ if and only if $\succsim_i^*$ is complete and $\sim_i^*$ has at most two equivalence classes.

Note, however, that super-coarseness, on its own, does not imply independence. Therefore nothing prevents (x_i, x_i) and (y_i, y_i) from belonging to distinct equivalence classes of $\sim_i^*$. Greco et al. (2001) attain their aim, imposing, on top of super-coarseness, a strong condition imposing at the same time independence and the fact that the null differences (x_i, x_i) belong to the first equivalence class of $\succsim_i^*$ on each attribute. On top of super-coarseness, this additional condition is necessary and sufficient to characterize concordance relations in which all attributes are of type III. Since this additional condition implies $RC2$, the results in Greco et al. (2001) are in the same spirit as ours as they allows to characterize concordance relations within the framework of the broader model (M).

Greco et al. (2001) have shown how to extend their characterization to cope with discordance effects as in outranking methods. This is a major advantage of their approach. This appears to be much more difficult within our framework (note, however, that when discordance is introduced, it is clear that all relations $\succsim_i^{**}$ have at most five equivalence classes, see Bouyssou and Pirlot (2002a)). We have no satisfactory answer at this time.

5.2. Relation to Fargier and Perny (2001)

Fargier and Perny (2001) (closely related results appear in Dubois et al. (2001, 2002, 2003)) have proposed an alternative characterization of CR. The central condition in this approach is a condition that extends the ''noncompensation'' condition proposed in Fishburn (1975, 1976, 1978) to reflexive relations. It says that, for all $x, y, z, w \in X$,

$$\left.\begin{array}{l}\succsim(x, y) = \succsim(z, w)\\ \succsim(y, x) = \succsim(w, z)\end{array}\right\} \Rightarrow [x \succsim y \Longleftrightarrow z \succsim w], \tag{7}$$

where $\succsim(x, y) = \{i \in N : x \succsim_i y\}$.

The close relation between CR and noncompensatory preferences in the sense of Fishburn (1976) was already noted in Bouyssou (1986, 1992) and Bouyssou and Vansnick (1986).

Although condition (7) may seem an obvious way to pinpoint the ordinal character of the ordinal aggregation at work in a CR and may look more transparent than our conditions LC and UC, its use raises problems. Indeed, as shown by the following two examples, it will be violated in CR in which some attributes are not essential.

Example 30. Let $X = \mathbb{R}^4$. Let $p_1 = p_2 = p_3 = p_4 = 1/4$. For all $i \in N$, let $S_i = \geqslant$. Consider the relation $\succsim$ on X defined by

$$x \succsim y \iff \sum_{i \in S(x,y)} p_i \geqslant \sum_{j \in S(y,x)} p_j - 1/4.$$

It is easy to see that such a relation is a CR (see Example 5 above).

Observe that, for all $i \in N$, any two elements of X_i are linked by $\sim_i$. Therefore, for all $x, y \in X$, we have $\succsim(x,y) = N$. While all attributes are influent, none is essential. Now, we clearly have $(10,10,10,10) \succsim (0,0,0,0)$ and $\text{Not}[(0,0,0,0) \succsim (10,10,10,10)]$. Hence, condition (7) is violated.

Example 31. Let $X = \mathbb{R}^4$. Let $p_1 = p_2 = p_3 = p_4 = 1/4$. For all $i \in N$, let $S_i = \geqslant$. Consider the relation $\succsim$ on X defined by

$$x \succsim y \iff \sum_{i \in S(x,y)} p_i \geqslant 3/4.$$

It is easy to see that such a relation is a CR (see Example 6 above).

Observe that, for all $i \in N$, any two elements of X_i are linked by $\sim_i$. Therefore, for all $x, y \in X$, we have $\succsim(x,y) = N$. While all attributes are influent, none is essential. We clearly have $(10,10,10,10) \succsim (0,0,0,0)$ and $\text{Not}[(0,0,0,0) \succsim (10,10,10,0)]$. Hence, condition (7) is violated.

Condition (7) uses the marginal relations $\succsim_i$ to model "ordinality". The above examples show that this is problematic as soon as one deals with CR in which some attributes may not be essential. Our analysis amounts to using, instead of $\succsim_i$, an appropriately defined "trace" on each attribute (see Bouyssou and Pirlot (2003) for a detailed analysis of traces in models (M) and (M*)). In our results, the central relation on each attribute is not $\succsim_i$ but the relation T_i such that $x_i T_i y_i \iff (x_i, y_i) \succsim_i^{**} (y_i, y_i)$. It may well happen that $\succsim_i$ is trivial while T_i is not. The use of trace allows us to deal with all CR whether or not attributes are essential. The price to pay for this is that, apparently, our conditions may appear less transparent than conditions like (7).

The various characterizations of CR proposed in Fargier and Perny (2001) and Dubois et al. (2001, 2002, 2003) all use condition (7) (called "ordinal invariance") or a strengthening of this condition incorporating a notion of monotonicity (inspired from "neutrality and monotonicity" conditions used in Social Choice Theory (see Sen, 1986). The above examples show that these results do not characterize the class of all CR. It is not difficult to see that, in fact, they characterize the class of CR in which all attributes are essential. Furthermore, condition (7) appears to be *very specific* to concordance relations in which all attributes are essential. In view of comparing concordance relations with other types of relations, as is possible via models (M) and (M*), this seems a serious defect. For a more detailed comparison between our approach and the one following the idea of noncompensation, we refer to Bouyssou and Pirlot (2002a).

It should finally be observed that the characterization of CR is not the central point in Fargier and Perny (2001) and Dubois et al. (2001, 2002, 2003). These papers mostly aim at underlining the limitation of "ordinal approaches" to MCDM as modelled by (7). Indeed, supposing at the same time that a binary relation $\succsim$ satisfies (7) and has "nice" properties (e.g. being such that $\succ$ is transitive) leads to a very uneven distribution of importance between the various attributes. This should be no surprise in that (7) is nothing but the classical "neutrality" condition used in social choice theory (see Sen, 1986) which is well-known to be instrumental in precipitating impossibility results. We show in Bouyssou and Pirlot (2002a,c) that similar results can be obtained for *all* CR using the broader framework of this paper.

5.3. Final comments

This paper has proposed a characterization of CR within the framework of a general model for nontransitive conjoint measurement. This characterization makes it possible to recast CR relations within a general class of relations and to isolate their specific features. Following the analysis in Bouyssou and Pirlot (2002a,c), it is not difficult to extend the proposed results to:

- analyze the case in which $\trianglerighteq$ is supposed to have some transitivity properties;
- analyze the, sweeping, consequences of supporting that $\succsim$ has nice transitivity properties (see also Bouyssou, 1992; Fargier and Perny, 2001; Fishburn, 1975).

Further work is clearly needed in order to characterize CR in which all attributes have the same type (in the sense of part 6 of Lemma 16) and to include in our analysis the possibility of discordance.

We would like to conclude with a note on the purpose of axiomatic analysis as we understand it. Our aim in providing an axiomatic analysis of CR was *not* to find properties that would characterize them; this would be an easy and somewhat futile exercise. Rather, our main aim was to take take advantage of this characterization to compare CR with other types of relations so as to underline their specific features. This explains our central use of a general framework for conjoint measurement in this analysis. More generally, we would like to emphasize the role of axiomatic analysis as a tool to uncover *structures* rather than a tool to achieve characterizations.

Acknowledgements

We wish to thank Salvatore Greco, Patrice Perny and Roman Słowiński for helpful discussions on the subject of this paper. The usual caveat applies.

Part of this work was accomplished while Denis Bouyssou was visiting the *Service de Mathématique de la Gestion* at the *Université Libre de Bruxelles* (Brussels, Belgium). The warm hospitality of the *Service de Mathématique de la Gestion*, the support of the Belgian *Fonds National de la Recherche Scientifique* and the Brussels-Capital Region through a "Research in Brussels" action grant are gratefully acknowledged.

Appendix A

Proof of Lemma 2. *Part 1*. If P_i is empty, then, since S_i is complete, for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$,

$$S((x_i, a_{-i}), (y_i, b_{-i})) = S((z_i, a_{-i}), (w_i, b_{-i})) \quad \text{and}$$

$$S((y_i, b_{-i}), (x_i, a_{-i})) = S((w_i, b_{-i}), (z_i, a_{-i})).$$

This implies, using (2), that attribute $i \in N$ is degenerate, contrarily to our hypothesis.

Part 2. Since all relations P_i are nonempty, for all A, $B \subseteq N$ such that $A \cup B = N$, there are x, $y \in X$ such that $S(x,y) = A$ and $S(y,x) = B$. We have, by construction, exactly one of $x \succ y$, $y \succ x$, $x \sim y$ and [Not[$x \succsim y$] and Not[$y \succsim x$]]. Hence, using (2), we have exactly one of $A \rhd B$, $B \rhd A$, $A \triangleq B$ and $A \bowtie B$. Since the relations S_i are complete, we have $S(x,x) = N$. Using the reflexivity of $\succsim$, we know that $x \sim x$, so that (2) implies $N \triangleq N$.

Parts 3 and 4. Let $A \subseteq N$. Because $N \triangleq N$, the monotonicity of $\trianglerighteq$ implies $N \trianglerighteq A$. We thus have $N \trianglerighteq \varnothing$. Suppose now that $\varnothing \trianglerighteq N$. Then the monotonicity of $\trianglerighteq$ would imply that $A \trianglerighteq B$, for all $A, B \subseteq N$ such that $A \cup B = N$. This would contradict the fact that each attribute is influent. Hence, we have $N \rhd \varnothing$.

Part 5. Using the completeness of all S_i, we have, for all x_i, $y_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$,

$$S((x_i, a_{-i}), (x_i, b_{-i})) = S((y_i, a_{-i}), (y_i, b_{-i})) \quad \text{and}$$

$$S((x_i, b_{-i}), (x_i, a_{-i})) = S((y_i, b_{-i}), (y_i, a_{-i})).$$

Using (2), this implies that, for all $i \in N$, all $x_i, y_i \in X_i$ and all a_{-i}, $b_{-i} \in X_{-i}$ $(x_i, a_{-i}) \succsim (x_i, b_{-i}) \iff (y_i, a_{-i}) \succsim (y_i, b_{-i})$. Therefore, $\succsim$ is independent for $N \setminus \{i\}$ and, hence, independent.

Part 6. Follows from the fact that S_i is complete, $N \triangleq N$ and $N \trianglerighteq N \setminus \{i\}$ for all $i \in N$.

Part 7. Let $i \in N$. We know that $N \trianglerighteq N \setminus \{i\}$. If $N \triangleq N \setminus \{i\}$, then (2) implies $x_i \succsim_i y_i$ for all $x_i, y_i \in X_i$. Otherwise we have $N \rhd N \setminus \{i\}$ and $N \triangleq N$. It follows that $x_i S_i y_i \Rightarrow x_i \succsim_i y_i$ and $x_i P_i y_i \Rightarrow x_i \succ_i y_i$. Since S_i and $\succsim_i$ are complete, it follows that $S_i = \succsim_i$.

Part 8. Suppose that $\succsim$ is a CR with a representation $\langle \trianglerighteq, S_i \rangle$. Because $i \in N$ is influent, there are x_i, y_i, z_i, $w_i \in X_i$ and $a_{-i}, b_{-i} \in X_{-i}$ such that $(x_i, a_{-i}) \succsim (y_i, b_{-i})$ and Not[$(z_i, a_{-i}) \succsim (w_i, b_{-i})$]. Since $\succsim$ is a CR, we must have either:

[$x_i P_i y_i$ and $w_i P_i z_i$] or [$x_i P_i y_i$ and $w_i I_i z_i$] or

[$x_i I_i y_i$ and $w_i P_i z_i$].

This, respectively, implies the existence of two subsets of attributes A and B such that $A \cup B \cup \{i\} = N$, $i \notin A$, $i \notin B$ and either:

$$A \cup \{i\} \trianglerighteq B \quad \text{and} \quad \text{Not}[A \trianglerighteq B \cup \{i\}] \quad \text{or} \qquad (\text{A.1a})$$

$$A \cup \{i\} \trianglerighteq B \quad \text{and} \quad \text{Not}[A \cup \{i\} \trianglerighteq B \cup \{i\}] \quad \text{or} \qquad (\text{A.1b})$$

$$A \cup \{i\} \trianglerighteq B \cup \{i\} \quad \text{and} \quad \text{Not}[A \trianglerighteq B \cup \{i\}]. \qquad (\text{A.1c})$$

Since P_i is nonempty, consider any $a_i, b_i \in X_i$ such that $a_i P_i b_i$. Respectively using (A.1a), (A.1b) and (A.1c), we have either

$$(a_i, a_{-i}) \succsim (b_i, b_{-i}) \quad \text{and} \quad \text{Not}[(b_i, a_{-i}) \succsim (a_i, b_{-i})] \quad \text{or} \qquad (\text{A.2a})$$

$$(a_i, a_{-i}) \succsim (b_i, b_{-i}) \quad \text{and} \quad \text{Not}[(b_i, a_{-i}) \succsim (b_i, b_{-i})] \quad \text{or} \qquad (\text{A.2b})$$

$$(a_i, a_{-i}) \succsim (a_i, b_{-i}) \quad \text{and} \quad \text{Not}[(b_i, a_{-i}) \succsim (a_i, b_{-i})] \qquad (\text{A.2c})$$

for some $a_{-i}, b_{-i} \in X_{-i}$.

Suppose now that $\succsim$ has a representation $\langle \trianglerighteq', S'_i \rangle$. Suppose that $a'_i I'_i b_i$. Any of (A.2a), (A.2b) and (A.2c), implies the existence of two subsets of attributes C and D such that $C \cup D \cup \{i\} = N$, $i \notin C, i \notin D$ and $C' \cup \{i\} \trianglerighteq' D \cup \{i\}$ and $\text{Not}[C \cup \{i\} \trianglerighteq' D \cup \{i\}]$, which is contradictory. Suppose therefore that $b_i P'_i a_i$. Respectively, using (A.2a), (A.2b), (A.2c) together with the fact that $\succsim$ is a CR, implies the existence of two subsets of attributes C and D such that $C \cup D \cup \{i\} = N$, $i \notin C$, $i \notin D$ and either

$$C \trianglerighteq' D \cup \{i\} \quad \text{and} \quad \text{Not}[C \cup \{i\} \trianglerighteq' D] \quad \text{or} \qquad (\text{A.3a})$$

$$C \trianglerighteq' D \cup \{i\} \quad \text{and} \quad \text{Not}[C \cup \{i\} \trianglerighteq' D \cup \{i\}] \quad \text{or} \qquad (\text{A.3b})$$

$$C \cup \{i\} \trianglerighteq' D \cup \{i\} \quad \text{and} \quad \text{Not}[C \cup \{i\} \trianglerighteq' D]. \qquad (\text{A.3c})$$

In any of these three cases, the monotonicity of $\trianglerighteq'$ is violated. Hence we have shown that, for all a_i, $b_i \in X_i$, $a_i P_i b_i \Rightarrow a_i P'_i b_i$. A similar reasoning shows that the converse implication is true. Hence, we must have $S_i = S'_i$. Using (2), it follows that $\trianglerighteq = \trianglerighteq'$. □

Proof of Lemma 16. *Part 1*. By definition, we have $\text{Not}[UC_i] \iff [\text{Not}[(y_i, x_i) \succsim^*_i (x_i, y_i)]$ and $\text{Not}[(x_i, y_i) \succsim^*_i (z_i, w_i)]]$. The proof of part 2 is similar.

Part 3. Suppose that $RC1_i$ is violated so that $\text{Not}[(x_i, y_i) \succsim^*_i (z_i, w_i)]$ and $\text{Not}[(z_i, w_i) \succsim^*_i (x_i, y_i)]$ for some x_i, y_i, w_i, $z_i \in X_i$. Using $RC2_i$, we have $(y_i, x_i) \succsim^*_i (w_i, z_i)$ and $(w_i, z_i) \succsim^*_i (y_i, x_i)$, so that $(y_i, x_i) \sim^*_i (w_i, z_i)$. Suppose that $\text{Not}[(y_i, x_i) \succsim^*_i (x_i, y_i)]$; then UC_i implies $(x_i, y_i) \succsim^*_i (z_i, w_i)$, a contradiction. Similarly, if $\text{Not}[(x_i, y_i) \succsim^*_i (y_i, x_i)]$, then LC_i implies $(z_i, w_i) \succsim^*_i (x_i, y_i)$, a contradiction. Hence, we have $(x_i, y_i) \sim^*_i (y_i, x_i)$. In a similar way, using UC_i and LC_i, it is easy to show that we must have $(z_i, w_i) \sim^*_i (w_i, z_i)$. Now, using the transitivity of $\sim^*_i$, we have $(x_i, y_i) \sim^*_i (z_i, w_i)$, a contradiction.

Part 4. Using part 3, we know that $\succsim^{**}_i$ is complete. Since $\succsim^{**}_i$ is reversible, the conclusion will be false if and only if there are $x_i, y_i, z_i, w_i \in X_i$ such that $(x_i, y_i) \succ^{**}_i (z_i, w_i) \succ^{**}_i (x_i, x_i)$. There are four cases to examine.

(1) Suppose that $(x_i, y_i) \succ^*_i (z_i, w_i)$ and $(z_i, w_i) \succ^*_i (x_i, x_i)$. Using $RC2_i$, we know that $(x_i, x_i) \succsim^*_i (w_i, z_i)$. Using the fact that $\succsim^*_i$ is a weak order, we have $(z_i, w_i) \succ^*_i (w_i, z_i)$. This violates UC_i since $(x_i, y_i) \succ^*_i (z_i, w_i)$.
(2) Suppose that $(x_i, y_i) \succ^*_i (z_i, w_i)$ and $(x_i, x_i) \succ^*_i (w_i, z_i)$. Using $RC2_i$, we know that $(z_i, w_i) \succsim^*_i (x_i, x_i)$. This implies $(z_i, w_i) \succ^*_i (w_i, z_i)$. This violates UC_i since $(x_i, y_i) \succ^*_i (z_i, w_i)$.
(3) Suppose that $(w_i, z_i) \succ^*_i (y_i, x_i)$ and $(z_i, w_i) \succ^*_i (x_i, x_i)$. Using $RC2_i$, we know that $(x_i, x_i) \succsim^*_i (w_i, z_i)$ so that $(z_i, w_i) \succ^*_i (w_i, z_i)$. This violates LC_i since $(w_i, z_i) \succ^*_i (y_i, x_i)$.
(4) Suppose that $(w_i, z_i) \succ^*_i (y_i, x_i)$ and $(x_i, x_i) \succ^*_i (w_i, z_i)$. Using $RC2_i$ we have $(z_i, w_i) \succsim^*_i (x_i, x_i)$ so that $(z_i, w_i) \succ^*_i (w_i, z_i)$. This violates LC_i since $(w_i, z_i) \succ^*_i (y_i, x_i)$.

Part 5. We provide below the required three examples.

Example 32 (UC, LC, Not[$RC2$]). Let $X = \{a,b\} \times \{x,y\}$. Consider $\succsim$ on X linking any two elements of X except that $(a,x) \succ (b,y)$ and $(a,y) \succ (b,x)$. We have, abusing notation,

- $[(a,b),(a,a),(b,b)] \succ_1^* (b,a)$ and
- $[(x,x),(y,y)] \succ_2^* [(x,y),(y,x)]$.

It is easy to check that $RC2_1$, UC and LC hold. $RC2_2$ is violated since $(x,x) \succ_2^* (x,y)$ and $(x,x) \succ_2^* (y,x)$.

Example 33 ($RC2$, LC, Not[UC]). Let $X = \{a,b\} \times \{x,y,z\}$ and $\succsim$ on X be identical to the linear order (abusing notation in an obvious way):

$$(a,x) \succ (a,y) \succ (a,z) \succ (b,x) \succ (b,y) \succ (b,z),$$

except that $(a,z) \sim (b,x)$.

We have, abusing notation,

- $(a,b) \succ_1^* [(a,a),(b,b)] \succ_1^* (b,a)$ and
- $(x,z) \succ_2^* [(x,x),(y,y),(z,z),(x,y),(y,z)] \succ_2^* [(y,x),(z,x),(z,y)]$.

Using Lemma 10, it is easy to check that $\succsim$ satisfies $RC2$. It is clear that UC_1, LC_1 and LC_2 hold. UC_2 is violated since we have $(x,y) \succ_2^* (y,x)$ and Not$(x,y) \succsim_2^* (x,z)$.

Example 34 ($RC2$, UC, Not[LC]). Let $X = \{a,b\} \times \{x,y,z\}$ and $\succsim$ on X be identical to the linear order (abusing notation in an obvious way):

$$(a,x) \succ (b,x) \succ (a,y) \succ (b,y) \succ (a,z) \succ (b,z),$$

except that $(b,x) \sim (a,y)$. We have, abusing notation,

- $(a,b) \succ_1^* [(a,a),(b,b)] \succ_1^* (b,a)$ and
- $[(x,y),(x,z),(y,z)] \succ_2^* [(x,x),(y,y),(z,z)] \succ_2^* (y,x) \succ_2^* [(z,x),(z,y)]$.

Using Lemma 10, it is easy to check that $\succsim$ satisfies $RC2$. It is clear that UC_1, LC_1 and UC_2 hold. LC_2 is violated since we have $(x,y) \succ_2^* (y,x)$ and Not$(z,x) \succsim_2^* (y,x)$.

Part 6. Let $x_i, y_i, z_i, w_i \in X_i$ be such that $(x_i,y_i) \succ_i^{**} (y_i,y_i)$ and $(z_i,w_i) \succ_i^{**} (w_i,w_i)$. By construction, we have either $(x_i,y_i) \succ_i^* (y_i,y_i)$ or $(y_i,y_i) \succ_i^* (y_i,x_i)$.

(1) Suppose first that $(x_i,y_i) \succ_i^* (y_i,y_i)$ and $(y_i,y_i) \succ_i^* (y_i,x_i)$. Consider $z_i, w_i \in X_i$ such that $(z_i,w_i) \succ_i^{**} (w_i,w_i)$. If either $(z_i,w_i) \sim_i^* (w_i,w_i)$ or $(w_i,z_i) \sim_i^* (w_i,w_i)$, it is easy to see, using the independence of $\succsim$ and the definition of $\succsim_i^{**}$, that we must have

$$(x_i,y_i) \succ_i^{**} (z_i,w_i) \succ_i^{**} (y_i,y_i) \succ_i^{**} (w_i,z_i) \succ_i^{**} (y_i,x_i),$$

violating the fact that $\sim_i^{**}$ has at most three distinct equivalence classes. Hence we have, for all z_i, $w_i \in X_i$ such that $(z_i,w_i) \succ_i^{**} (w_i,w_i)$, $(z_i,w_i) \succ_i^* (w_i,w_i)$ and $(w_i,w_i) \succ_i^* (w_i,z_i)$.

(2) Suppose that $(x_i,y_i) \succ_i^* (y_i,y_i)$ and $(y_i,y_i) \sim_i^* (y_i,x_i)$ and consider any $z_i, w_i \in X_i$ such that $(z_i,w_i) \succ_i^{**} (w_i,w_i)$. If $(z_i,w_i) \succ_i^* (w_i, w_i)$ and $(w_i,w_i) \succ_i^* (w_i,z_i)$, we have, using the independence of $\succsim$ and the definition of $\succsim_i^{**}$,

$$(z_i,w_i) \succ_i^* (x_i,y_i) \succ_i^* (y_i,y_i) \succ_i^* (y_i,x_i) \succ_i^* (w_i,z_i),$$

violating the fact that $\sim_i^{**}$ has at most three distinct equivalence classes. If $(z_i,w_i) \sim_i^* (w_i,w_i)$ and $(w_i,w_i) \succ_i^* (w_i,z_i)$, then $RC2_i$ is violated since we have $(x_i,y_i) \succ_i^* (z_i,w_i)$ and $(y_i,x_i) \succ_i^* (w_i,z_i)$. Hence, it must be true that $(z_i,w_i) \succ_i^{**} (w_i,w_i)$ implies $(z_i,w_i) \succ_i^* (w_i,w_i)$ and $(w_i,w_i) \sim_i^* (w_i,z_i)$.

(3) Suppose that $(x_i,y_i) \sim_i^* (y_i,y_i)$ and $(y_i,y_i) \succ_i^* (y_i,x_i)$ and consider any $z_i, w_i \in X_i$ such that $(z_i,w_i) \succ_i^{**} (w_i,w_i)$. If $(z_i,w_i) \succ_i^* (w_i,w_i)$ and $(w_i,w_i) \succ_i^* (w_i,z_i)$, we have, using the independence of $\succsim$ and the definition of $\succsim_i^{**}$,

$$(z_i,w_i) \succ_i^* (x_i,y_i) \succ_i^* (y_i,y_i) \succ_i^* (y_i,x_i) \succ_i^* (w_i,z_i),$$

violating the fact that $\sim_i^{**}$ has at most three distinct equivalence classes. If $(z_i,w_i) \succ_i^* (w_i,w_i)$ and $(w_i,w_i) \sim_i^* (w_i,z_i)$, then $RC2_i$ is violated since we have $(z_i,w_i) \succ_i^* (x_i,y_i)$ and $(w_i,z_i) \succ_i^* (y_i,x_i)$. Hence, it must be true that $(z_i,w_i) \succ_i^{**} (w_i,w_i)$ implies $(z_i,w_i) \sim_i^* (w_i,w_i)$ and $(w_i,w_i) \succ_i^* (w_i,z_i)$. $\square$

Proof of Lemma 17. Let $\langle \trianglerighteq, S_i \rangle$ be a representation of $\succsim$ (this representation is unique by Lemma 2).

Part 1. Let us show that $RC1_i$ holds, i.e. that $(x_i, a_{-i}) \succsim (y_i, b_{-i})$ and $(z_i, c_{-i}) \succsim (w_i, d_{-i})$ imply $(z_i, a_{-i}) \succsim (w_i, b_{-i})$ or $(x_i, c_{-i}) \succsim (y_i, d_{-i})$. There are nine cases to envisage:

	$z_iP_iw_i$	$z_iI_iw_i$	$w_iP_iz_i$
$x_iP_iy_i$	(i)	(ii)	(iii)
$x_iI_iy_i$	(iv)	(v)	(vi)
$y_iP_ix_i$	(vii)	(viii)	(ix)

Cases (i), (v) and (ix) clearly follow from (2). All other cases easily follow from (2) and the monotonicity of $\trianglerighteq$. The proof for $RC2$ is similar.

Part 2. Let us show that UC_i holds, i.e. that $(x_i, a_{-i}) \succsim (y_i, b_{-i})$ and $(z_i, c_{-i}) \succsim (w_i, d_{-i})$ imply $(y_i, a_{-i}) \succsim (x_i, b_{-i})$ or $(x_i, c_{-i}) \succsim (y_i, d_{-i})$. If $x_iP_iy_i$ then, using (2) and the monotonicity of $\trianglerighteq$, we have $(z_i, c_{-i}) \succsim (w_i, d_{-i}) \Rightarrow (x_i, c_{-i}) \succsim (y_i, d_{-i})$. If $y_iP_ix_i$ then, using (2) and the monotonicity of $\trianglerighteq$, we have $(x_i, a_{-i}) \succsim (y_i, b_{-i}) \Rightarrow (y_i, a_{-i}) \succsim (x_i, b_{-i})$. If $x_iI_iy_i$, then $y_iI_ix_i$ so that, using (2), $(x_i, a_{-i}) \succsim (y_i, b_{-i}) \Rightarrow (y_i, a_{-i}) \succsim (x_i, b_{-i})$. The proof for LC_i is similar. $\square$

Proof of Lemma 18. Necessity follows from Lemma 17. We show that if $\succsim$ satisfies $RC1$ and $RC2$ and is such that, for all $i \in N$, $\sim_i^{**}$ has at most three distinct equivalence classes then $\succsim$ is a CR. In view of part 4 of Lemma 16, this will establish sufficiency.

For all $i \in N$, define S_i letting, for all $x_i, y_i \in X_i$, $x_iS_iy_i \iff (x_i, y_i) \succsim_i^{**} (y_i, y_i)$. By hypothesis, we know that $\succsim_i^{**}$ is complete and $\succsim$ is independent. It easily follows that S_i is complete.

Since attribute $i \in N$ has been supposed influent, it is easy to see that P_i is nonempty. Indeed, $\succsim_i^{*}$ being complete, the influence of $i \in N$ implies that there are $z_i, w_i, x_i, y_i \in X_i$ such that $(x_i, y_i) \succ_i^{*} (z_i, w_i)$. Since $\succsim_i^{**}$ is complete, this implies $(x_i, y_i) \succ_i^{**} (z_i, w_i)$. If $(x_i, y_i) \succ_i^{**} (y_i, y_i)$ then $x_iP_iy_i$. If not, then $(y_i, y_i) \succsim_i^{**} (x_i, y_i)$ so that $(y_i, y_i) \succ_i^{**} (z_i, w_i)$ and, using the reversibility of $\succsim_i^{**}$ and the independence of $\succsim$, $w_iP_iz_i$. Therefore P_i is not empty. This implies that $\succsim_i^{**}$ has exactly three distinct equivalence classes, since $x_iP_iy_i \iff (x_i, y_i) \succ_i^{**} (y_i, y_i) \iff (y_i, y_i) \succ_i^{**} (y_i, x_i)$. Therefore, $x_iP_iy_i$ if and only if (x_i, y_i) belongs to the first equivalence class of $\succsim_i^{**}$ and (y_i, x_i) to its last equivalence class. Consider any two subsets $A, B \subseteq N$ such that $A \cup B = N$ and let

$$A \trianglerighteq B \iff [x \succsim y, \text{ for some } x, y \in X \text{ such that } S(x, y) = A \quad \text{and} \quad S(y, x) = B].$$

If $x \succsim y$ then, by construction, we have $S(x, y) \trianglerighteq S(y, x)$. Suppose now that $S(x, y) \trianglerighteq S(y, x)$. This implies that there are $z, w \in X$ such that $z \succsim w$, $S(z, w) = S(x, y)$ and $S(w, z) = S(y, x)$. The last two conditions imply $(x_i, y_i) \sim_i^{**} s(z_i, w_i)$, for all $i \in N$. Using (4), we have $x \succsim y$. Hence (2) holds. The monotonicity of $\trianglerighteq$ easily follows from (3). This completes the proof. $\square$

Proof of Lemma 26. $[AC1_i]$. Suppose that $(x_i, x_{-i}) \succsim (y_i, y_{-i})$ and $(z_i, z_{-i}) \succsim (w_i, w_{-i})$. We want to show that either $(z_i, x_{-i}) \succsim (y_i, y_{-i})$ or $(x_i, z_{-i}) \succsim (w_i, w_{-i})$.

If $y_iP_ix_i$ or $w_iP_iz_i$, the conclusion follows from the monotonicity of $\trianglerighteq$.

If $x_iP_iy_i$ and $z_iP_iw_i$, we have, using the fact that P_i is Ferrers, $z_iP_iy_i$ or $x_iP_iw_i$. In either case the desired conclusion follows using the fact that $\succsim$ is a CR.

This leaves three exclusive cases: $[x_iI_iy_i$ and $z_iP_iw_i]$ or $[x_iP_iy_i$ and $z_iI_iw_i]$, or $[x_iI_iy_i$ and $z_iI_iw_i]$. Using Ferrers, either case implies $x_iS_iw_i$ or $z_iS_iy_i$. If either $x_iP_iw_i$ or $z_iP_iy_i$, the desired conclusion follows from monotonicity. Suppose therefore that $x_iI_iw_i$ and $z_iI_iy_i$. Since we have either $x_iI_iy_i$ or $z_iI_iw_i$, the conclusion follows using the fact that $\succsim$ is a CR.

Hence $AC1_i$ holds. The proof for $AC2_i$ is similar, using Ferrers.

$[AC3_i]$. Suppose that $(z_i, z_{-i}) \succsim (x_i, a_{-i})$ and $(x_i, b_{-i}) \succsim (y_i, y_{-i})$. We want to show that either $(z_i, z_{-i}) \succsim (w_i, a_{-i})$ or $(w_i, b_{-i}) \succsim (y_i, y_{-i})$.

If either $y_iP_ix_i$ or $x_iP_iz_i$, the conclusion follows from monotonicity.

If $x_iP_iy_i$ and $z_iP_ix_i$, then semi-transitivity implies $w_iP_iy_i$ or $z_iP_iw_i$. In either case, the conclusion follows from monotonicity.

This leaves three exclusive cases: $[x_iI_iy_i$ and $z_iP_ix_i]$ or $[x_iP_iy_i$ and $z_iI_ix_i]$ or $[x_iI_iy_i$ and $z_iI_ix_i]$. In either case, semi-transitivity implies $w_iS_iy_i$ or $z_iS_iw_i$. If either $w_iP_iy_i$ or $z_iP_iw_i$, the desired conclusion follows from monotonicity. Suppose therefore that $w_iI_iy_i$ and $z_iI_iw_i$. Since in each of the remaining cases we have either $x_iI_iy_i$ or $z_iI_ix_i$, the conclusion follows because $\succsim$ is a CR. $\square$

Proof of Lemma 27. *Part 1*. We prove that $AC1_i \Rightarrow AC2_i$, the proof of the reverse implication being similar. Suppose $AC2_i$ is violated so that there are $x_i, y_i, z_i, w_i \in X_i$ such that $(x_i, y_i) \succ_i^* (x_i, w_i)$ and $(z_i, w_i) \succ_i^* (z_i, y_i)$. Using Lemma 16, we know that attribute i has a type. We analyze each type separately. If $i \in N$ has type II or III, then $\sim_i^*$ has only two distinct equivalence classes. We therefore have: $[(x_i, y_i) \sim_i^* (z_i, w_i)] \succ_i^* [(x_i, w_i) \sim_i^* (z_i, y_i)]$. This implies $(x_i, y_i) \succ_i^* (z_i, y_i)$. Using $AC1_i$, we have $(x_i, w_i) \succsim_i^* (z_i, w_i)$, a contradiction.

If $i \in N$ has type I then $\sim_i^*$ has only three distinct equivalence classes. We distinguish several cases.

(1) Suppose that both (x_i, y_i) and (z_i, w_i) belong to the middle equivalence class of $\succsim_i^*$. This implies $[(x_i, y_i) \sim_i^* (z_i, w_i)] \succ_i^* [(x_i, w_i) \sim_i^* (z_i, y_i)]$, so that $(x_i, y_i) \succ_i^* (z_i, y_i)$. Using $AC1_i$, we have $(x_i, w_i) \succsim_i^* (z_i, w_i)$, a contradiction.
(2) Suppose that both (x_i, y_i) and (z_i, w_i) belong to the first equivalence class of $\succsim_i^*$. We therefore have $(x_i, y_i) \sim_i^* (z_i, w_i)$, $(x_i, y_i) \succ_i^* (x_i, w_i)$ and $(z_i, w_i) \succ_i^* (z_i, y_i)$. This implies $(x_i, y_i) \succ_i^* (z_i, y_i)$. Using $AC1_i$, we have $(x_i, w_i) \succsim_i^* (z_i, w_i)$, a contradiction.
(3) Suppose that (x_i, y_i) belongs to the first equivalence class of $\succsim_i^*$ and (z_i, w_i) belong to the central class of $\succsim_i^*$. This implies, using the reversibility of $\succsim_i^{**}$, $[(x_i, y_i) \sim_i^* (y_i, z_i)] \succ_i^* [(z_i, w_i) \sim_i^* (w_i, z_i)] \succ_i^* [(z_i, y_i) \sim_i^* (y_i, x_i)]$. Hence, we have $(y_i, z_i) \succ_i^* (w_i, z_i)$ and using $AC1_i$, we have $(y_i, x_i) \succsim_i^* (w_i, x_i)$. This implies that (x_i, w_i) must belong to the first equivalence class of $\succsim_i^*$ violating the fact that $(x_i, y_i) \succ_i^* (x_i, w_i)$".

Part 2. We provide below examples showing that, in the class of reflexive relations satisfying $RC2$, UC and LC, $AC1$ and $AC3$ are independent conditions.

Example 35 ($RC2$, UC, LC, $AC1$, Not[$AC3$]). Let $X = \{a, b, c, d\} \times \{x, y\}$. We build the CR in which:

- aP_1b, aI_1c, aP_1d, bI_1c, bP_1d, cI_1d,
- xP_2y,
- $\{1,2\} \rhd \varnothing$, $[1,2] \triangleq \{2\}$, $\{1,2\} \triangleq \{1\}$, $\{2\} \triangleq \{1\}$.

Therefore, $\succsim$ links any two elements of X except that we have: $(a, x) \succ (b, y)$, $(b, x) \succ (d, y)$ and $(a, x) \succ (d, y)$. It is easy to see that $AC1$ and $AC3_2$ hold. $AC3_1$ is violated since $(c, y) \succsim (a, x)$, $(d, y) \succsim (c, x)$ but neither $(b, y) \succsim (a, x)$ nor $(d, y) \succsim (b, x)$.

Example 36 ($RC2$, UC, LC, $AC3$, Not[$AC1$]). Let $X = \{a, b, c, d\} \times \{x, y\}$. We build the CR in which:

- aI_1b, aP_1c, aI_1d, bI_1c, bP_1d, cI_1d,
- xP_2y,
- $\{1,2\} \rhd \varnothing$, $\{1,2\} \triangleq \{2\}$, $\{1,2\} \triangleq \{1\}$, $\{2\} \triangleq \{1\}$.

Therefore, $\succsim$ links any two elements of X except that we have: $(a, x) \succ (c, y)$ and $(b, x) \succ (d, y)$. It is easy to see that $AC3$ and $AC1_2$ holds. $AC1_1$ is violated since $(d, y) \succsim (a, x)$ and $(c, y) \succsim (b, x)$ but neither $(c, y) \succsim (a, x)$ nor $(d, y) \succsim (b, x)$. $\square$

Proof of Lemma 28. The proof of Theorem 28 follows from combining Lemmas 23, 26 and 27 with the results in Section 4.1. $\square$

References

Bouyssou, D., 1986. Some remarks on the notion of compensation in MCDM. European Journal of Operational Research 26, 150–160.

Bouyssou, D., 1992. On some properties of outranking relations based on a concordance–discordance principle. In: Duckstein, L., Goicoechea, A., Zionts, S. (Eds.), Multiple Criteria Decision Making. Springer-Verlag, Berlin, pp. 93–106.

Bouyssou, D., 1996. Outranking relations: Do they have special properties? Journal of Multi-Criteria Decision Analysis 5, 99–111.

Bouyssou, D., 2001. Outranking methods. In: Floudas, C.A., Pardalos, P.M. (Eds.), Encyclopedia of Optimization, vol. 4. Kluwer, Dordrecht, pp. 249–255.

Bouyssou, D., Pirlot, M., 2002a. A characterization of strict concordance relations. In: Bouyssou, D., Jacquet-Lagrèze, E., Perny, P., Słowiński, R., Vanderpooten, D., Vincke, Ph. (Eds.), Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy. Kluwer, Dordrecht, pp. 121–145.

Bouyssou, D., Pirlot, M., 2002b. Nontransitive decomposable conjoint measurement. Journal of Mathematical Psychology 46, 677–703.

Bouyssou, D., Pirlot, M., 2002c. Ordinal aggregation and strict preferences for multi-attributed alternatives, Working Paper. Available from <http://www.lamsade.dauphine.fr/bouyssou>.

Bouyssou, D., Pirlot, M., 2003. 'Additive difference' models without additivity and subtractivity, Working Paper, submitted. Available from <http://www.lamsade.dauphine.fr/bouyssou>.

Bouyssou, D., Vansnick, J.-C., 1986. Noncompensatory and generalized noncompensatory preference structures. Theory and Decision 21, 251–266.

Doignon, J.-P., Monjardet, B., Roubens, M., Vincke, Ph, 1988. Biorder families, valued relations and preference modelling. Journal of Mathematical Psychology 30, 435–480.

Dubois, D., Fargier, H., Perny, P., 2002. On the limitation of ordinal approaches to decision making. In: Fensel, D., Guinchiglia, F., Williams, M.-A., McGuinness, D. (Eds.), Knowledge Representation 2002—Proceedings of the 8th International Conference (KR'02). Morgan Kaufmann, San Francisco, CA, pp. 133–144.

Dubois, D., Fargier, H., Perny, P., Prade, H., 2001. Towards a qualitative multicriteria decision theory. In: Proceedings of the EUROFUSE Workshop on Preference Modelling and Applications, Granada, Spain, April 25–27, 2001, pp. 121–129.

Dubois, D., Fargier, H., Perny, P., Prade, H., 2003. A characterization of generalized concordance rules in multicriteria decision-making. International Journal of Intelligent Systems 18 (7), 751–774.

Fargier, H., Perny, P., 2001. Modélisation des préférences par une règle de concordance généralisée. In: Colorni, A., Paruccini, M., Roy, B. (Eds.), A-MCD-A, Aide Multicritère *à* la Décision/Multiple Criteria Decision Aid. European Commission, Joint Research Centre, pp. 99–115.

Fishburn, P.C., 1970. Utility theory for decision-making. Wiley, New York.

Fishburn, P.C., 1975. Axioms for lexicographic preferences. Review of Economic Studies 42, 415–419.

Fishburn, P.C., 1976. Noncompensatory preferences. Synthese 33, 393–403.

Fishburn, P.C., 1978. A survey of multiattribute/multicriteria evaluation theories. In: Zionts, S. (Ed.), Multicriteria Problem Solving. Springer Verlag, Berlin, pp. 181–224.

Fishburn, P.C., 1992. Additive differences and simple preference comparisons. Journal of Mathematical Psychology 36, 21–31.

French, S., 1993. Decision Theory—An Introduction to the Mathematics of Rationality. Ellis Horwood, London.

Greco, S., Matarazzo, B., Słowiński, R., 2001. Axiomatic basis of noncompensatory preferences, communication to *FUR X*, 30 May–2 June, Torino, Italy.

Keeney, R.L., Raiffa, H., 1976. Decisions with Multiple Objectives: Preferences and Value Tradeoffs. Wiley, New-York.

Krantz, D.H., Luce, R.D., Suppes, P., Tversky, A., 1971. Foundations of Measurement. In: Additive and Polynomial Representations, vol. 1. Academic Press, New York.

Pirlot, M., 1997. A common framework for describing some outranking procedures. Journal of Multi-Criteria Decision Analysis 6, 86–92.

Roberts, F.S., 1979. Measurement theory with applications to decision making, utility and the social sciences. Addison-Wesley, Reading, MA.

Roy, B., 1968. Classement et choix en présence de points de vue multiples (la méthode ELECTRE). RIRO 2, 57–75.

Roy, B., 1991. The outranking approach and the foundations of ELECTRE methods. Theory and Decision 31, 49–73.

Roy, B., 1996. Multicriteria Methodology for Decision Aiding. Kluwer, Dordrecht (original version in French "*Méthodologie Multicritère d'Aide à la Décision*", Economica, Paris, 1985).

Sen, A.K., 1986. Social choice theory. In: Arrow, K.J., Intriligator, M.D. (Eds.), Handbook of Mathematical Economics, vol. 3. North-Holland, Amsterdam, pp. 1073–1181.

Tversky, A., 1969. Intransitivity of preferences. Psychological Review 76, 31–48.

Vansnick, J.-C., 1986. On the problems of weights in MCDM (the noncompensatory approach). European Journal of Operational Research 24, 288–294.

Vincke, Ph., 1992. Multi-criteria Decision Aid. Wiley, New York (original version in French "*L'Aide Multicritère à la Décision*", Éditions de l'Université de Bruxelles–Éditions Ellipses, Brussels, 1989).

Vincke, Ph, 1999. Outranking approach. In: Gal, T., Stewart, T., Hanne, T. (Eds.), Multicriteria Decision Making, Advances in MCDM Models, Algorithms, Theory and Applications. Kluwer, Dordrecht, pp. 11.1–11.29.

Wakker, P.P., 1988. Derived strength of preference relations on coordinates. Economics Letters 28, 301–306.

Wakker, P.P., 1989. Additive Representations of Preferences: A New Foundation of Decision Analysis. Kluwer, Dordrecht.