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Further notes on cell decomposition in closed ordered differential fields

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ABSTRACT

In [T. Brihaye, C. Michaux, C. Rivière, Cell decomposition and dimension function in the theory of closed ordered differential fields, Ann. Pure Appl. Logic (in press).] the authors proved a cell decomposition theorem for the theory of closed ordered differential fields (*CODF*) which generalizes the usual Cell Decomposition Theorem for o-minimal structures. As a consequence of this result, a well-behaving dimension function on definable sets in *CODF* was introduced. Here we continue the study of this cell decomposition in *CODF* by proving three additional results. We first discuss the relation between the δ-cells introduced in the above-mentioned reference and the notion of Kolchin polynomial (or dimensional polynomial) in differential algebra. We then prove two generalizations of classical decomposition theorems in o-minimal structures. More exactly we give a theorem of decomposition into definably d-connected components (d-connectedness is a weak differential generalization of usual connectedness w.r.t. the order topology) and a differential cell decomposition theorem for a particular class of definable functions in *CODF*.

0. Outline

This paper is in direct filiation with paper [1]. Even though we recall in Section 1 some of the developments of the previous paper, it is certainly helpful to have a look at it before reading this one. In the sequel, we will denote by L the language $\{+, -, *, <, 0, 1\}$ of ordered rings and by L' the language $\{+, -, *, ', <, 0, 1\}$ of ordered differential rings.

The first section of this paper contains a brief summary of the work presented in [1]. In the latter, the authors study a differential analogue of o-minimality in the theory *CODF* of closed ordered differential fields. In particular we recall the statement of the differential cell decomposition theorem for definable sets in *CODF* (Theorem 1.6).

Section 2 was motivated by a question of T. Scanlon and contains the developments required to link the notion of δ-cell introduced in [1] with the *Kolchin polynomial* defined in partial[1] differential algebra [2, Theorem 6,p.115]. In the particular case of a differential field M equipped with a single derivation, the Kolchin polynomial describes, for any tuple $\bar{a}$ in an extension of M, the asymptotic behavior of the algebraic transcendence degree of the field $M(\bar{a}, \bar{a}', \ldots, \bar{a}^{(n)})$ over M (when n tends to ∞). Furthermore W.-Y. Pong proved in [4] that this polynomial has a very simple form $dX + b$ where d is the differential transcendence degree of $\bar{a}$ over M and b is a positive integer. Our aim here is to explain how some further investigations concerning δ-cells allow recovering the integers d and b (and then the Kolchin polynomial) in case M is a model of *CODF*. For this we first define a notion of *K-type* for a particular class of δ-cells called *engaged* δ-cells (Definitions 2.4–2.6). In fact the K-type provides a rank on δ-cells which is more precise than the δ-dimension and allows associating a *K-rank* with any tuple $\bar{a}$ in a differential extension of M (Definition 2.8). We finally prove that this K-rank is equivalent to the Kolchin polynomial associated to $\bar{a}$ in the sense that it easily permits computing the integers d and b described above (Theorem 2.10).

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[1] This means that the differential fields are equipped with finitely many commuting derivations.

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The third section contains a summary of our efforts to generalize a well-known consequence of the Cell Decomposition Theorem for o-minimal structures. This result asserts that any definable set in an o-minimal structure can be partitioned into finitely many *definably connected components* (see Theorem 3.2). We first quickly remark that the analogue of this result has no chance of holding if we consider a model of *CODF* and the δ-connectedness (i.e. connectedness w.r.t. the δ-topology, see Definition 1.4). This forces us to introduce a weaker notion of connectedness (*d-connectedness*, Definition 3.3) for which we can prove a result of decomposition for any L'-definable set in *CODF* (Theorem 3.5). We conclude with a result showing that the number of definably d-connected components of any L'-definable set is strongly related to the number of definably connected components of its different L-definable sources (Theorem 3.7).

Finally we consider in Section 4 a possible differential analogue of the Cell Decomposition Theorem for definable functions (see [9, 2.11 (II_m)]). In other words: given an L'-definable function $f : A \to M$ where M is a model of *CODF*, can we find a finite partition $\mathcal{C}$ of A into δ-cells such that the restriction of f to any of these δ-cells is δ-continuous?. After some preliminary definitions and results, we give a positive *partial* answer for a restricted class of L'-definable functions in *CODF* called *admissible* functions (Definition 4.5, Theorem 4.11). Unfortunately, even for an admissible function $f : A \to M$, Theorem 4.11 does not ensure the δ-continuity of f on a partition of A. This theorem only asserts that for any positive integer n there exists a finite partition $\mathcal{C}_n$ of A into δ-cells such that the restriction of f to each of these δ-cells is *continuous at order n* (Definition 4.7) which is a weaker result than the δ-continuity. Nevertheless in the (very) particular case where the admissible L'-definable function commutes with the derivation, we obtain a stronger result (Theorem 4.15) which is the exact differential analogue of [9, 2.11 (II_m)]. We finish this paper with a simple example showing that the hypothesis of commutativity in Theorem 4.15 is not a necessary condition.

1. Preliminaries [1]

The theory *CODF* is the complete L'-theory of an ordered differential field. This theory has quantifier elimination in L' and a model M of *CODF* is called a **closed ordered differential field** [7]. Note that any model of *CODF* is a real closed field (we denote by *RCF* the L-theory of real closed fields).

Definition 1.1. Let M be a model of *CODF*. For any $k \in \mathbb{N}$ and any $(n_1, \ldots, n_k) \in \mathbb{N}^k$, we define the $(n_1; \ldots; n_k)$**-jet-space** of M^k to be the following L'-definable set:

$$J_{(n_1;\ldots;n_k)}(M^k) := \{(x_1, x_1', \ldots, x_1^{(n_1)}; \ldots; x_k, x_k' \ldots, x_k^{(n_k)}) \mid (x_1; \ldots; x_k) \in M^k\}$$
$$= J_{n_1}(M) \times \cdots \times J_{n_k}(M).$$

Let A be a L'-definable subset of M^k. By quantifier elimination there exists a *quantifier free* L'-formula $\varphi(\bar{x})$ such that $A = A_\varphi := \{\bar{x} \in M^k \mid \varphi(\bar{x})\}$. For each $i \in \{1, \ldots, k\}$, assume that the highest derivative of the variable X_i appearing non-trivially in φ is $X_i^{(n_i)}$. The L'-formula φ can then be considered as a quantifier free L-formula φ^L in the differential variables $X_1, X_1', \ldots, X_1^{(n_1)}; \ldots; X_k, X_k', \ldots, X_k^{(n_k)}$ with:

$$\forall X_1, \ldots, X_k \big(\varphi(X_1, \ldots, X_k) \Leftrightarrow \varphi^L(X_1, X_1', \ldots, X_1^{(n_1)}; \ldots; X_k, X_k', \ldots, X_k^{(n_k)})\big).$$

Let $N = (n_1 + 1) + \cdots + (n_k + 1)$, we associate two subsets of M^N with A:

$$A_\varphi{}^L := \{(x_{10}, \ldots, x_{1n_1}; \ldots; x_{k0}, \ldots, x_{kn_k}) \in M^N \mid M \models \varphi^L(\bar{x_1}; \ldots; \bar{x_k})\};$$
$$A_\varphi{}^* := \{(x_1, \ldots, x_1^{(n_1)}; \ldots; x_k, \ldots, x_k^{(n_k)}) \in M^N \mid M \models \varphi(x_1; \ldots; x_k)\}$$
$$= A_\varphi{}^L \cap J_{(n_1;\ldots;n_k)}(M^k).$$

We remark that $A_\varphi{}^L$ is L-definable and the second equality above holds because the L'-formula φ (and hence the L-formula φ^L) is quantifier free. For the same reason, A_φ is the projection of $A_\varphi{}^*$ onto some appropriate coordinates (namely $X_{10}, \ldots, X_{k0}$). We call the latter the **canonical projection** of $A_\varphi{}^*$ (or of $A_\varphi{}^L$ when the context is clear). We also say that the L-definable set $A_\varphi{}^L$ **gives rise to** (or is **a source of**) the L'-definable set A_φ.

Remark 1.2. In order to simplify the notation, we drop the subscript φ in the sets $A_\varphi, A_\varphi{}^L$ and $A_\varphi{}^*$ defined above and simply denote them by A, A^L and A^* respectively.

Unfortunately, the equivalence of two L'-formulas φ,ψ in *CODF* does *not* imply the equivalence of the corresponding L-formulas φ^L and ψ^L in *RCF*. We formalize this ambiguity between different sources of a given L'-definable set via the following definition.

Definition 1.3. Two L-definable sets are δ**-equivalent** (denoted by $\equiv_\delta$) if they both give rise to the same L'-definable set.

Any model M of *CODF* is equipped with the order topology. Unfortunately this topology appears to be not very efficient to study L'-definable sets in M. This is why we consider a possibly more natural topology on M [1, Section 3].

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Definition 1.4. An L'-definable subset A of M is a basic open set for the δ**-topology** (we say that A is a **basic** δ**-open set**) if $A^L \subseteq M^n$ is δ-equivalent to a basic open L-definable set for the product topology in M^n. In the sequel, we will use the prefix "δ-" before any topological object to specify that we consider it in the δ-topology (e.g. δ-closed, δ-interior, δ-continuous, etc.). Unless otherwise stated, all topological objects will be considered in the order topology.

Definition 1.5. Let $M \models CODF$ and C be an L'-definable subset of M^k. C is an $(i_1; \ldots; i_k)$-δ if C^L is δ-equivalent to an $(i_{10}, \ldots, i_{1n_1}; \ldots; i_{k0}, \ldots, i_{kn_k})$-cell D^L such that: for any $j \in \{1, \ldots, k\}$,

$$\begin{cases} i_j = 1 & \text{if } i_{jl} = 1 \text{ for each } l \in \{0, \ldots n_j\}, \\ i_j = 0 & \text{otherwise.} \end{cases}$$

The tuple $(i_1; \ldots; i_k)$ is called the δ**-type** of C.

It is proved in [1] that the δ-type of a δ-cell *does not depend* on the cell D^L appearing in Definition 1.5 and so is well-defined. Furthermore, as in the o-minimal case, the $(1; \ldots; 1)$-δ-cells are exactly the δ-cells which are δ-open in their ambient space.

Theorem 1.6 (*Differential Cell Decomposition Theorem [1, Theorem 4.9]*). *Let M be a closed ordered differential field. For any finite collection $\mathcal{A} = \{A_1, \ldots, A_l\}$ of L'-definable (over $P \subset M$) subsets of M^k there exists a finite δ-decomposition $\mathcal{C}$ of M^k (definable over P) compatible with $\mathcal{A}$ (i.e. partitioning each of the A_i).*

Recall that:

. *a* δ**-decomposition** *of M is a partition of M into finitely many δ-cells;*

. *a* δ**-decomposition** *of M^k ($k > 1$) is a partition $\mathcal{C}$ of M^k into finitely many δ-cells such that $\pi_{k-1}(\mathcal{C})$ is still a δ-decomposition of M^{k-1} (where π_{k-1} is the projection onto the $(k-1)$ first coordinates).*

2. δ-cells and the Kolchin polynomial

In [1, Section 4], Theorem 1.6 is used to define a notion of δ**-dimension** for any definable set in *CODF*. Although the latter enjoys a lot of nice properties, it can be interesting to obtain a finer notion of dimension (or rank) in *CODF*. This is the goal of this section.

2.1. Dimensional polynomial of Kolchin

The **differential dimensional polynomial** (or **Kolchin polynomial**) first appeared in [2, Theorem 6, p.115] in the general case of differential fields equipped with finitely many commuting derivations. Here we only consider it in the particular (and rather simple) case of *CODF*. If $\bar{a}$ is a tuple in some differential extension of a differential field M, the Kolchin polynomial of $\bar{a}$ describes the asymptotic behavior of the transcendence degree of the field extension $M(\bar{a}, \bar{a}', \ldots, \bar{a}^{(n)})$ over M $\big($denoted by $tr_M(M(\bar{a}, \bar{a}', \ldots, \bar{a}^{(n)}))\big)$ when n tends to ∞.

In our particular case of a differential field M equipped with a single derivative, W.-Y. Pong proved the following result.

Theorem 2.1 (*[4, Proposition 2.4]*). *Let M be a differential field and $\bar{a}$ be in a differential extension of M. Then there exist positive integers d and b such that*

$$tr_M(M(\bar{a}, \bar{a}', \ldots, \bar{a}^{(n)})) = d(n+1) + b$$

for all sufficiently large $n \in \mathbb{N}$. Furthermore d is the differential transcendence degree of $\bar{a}$ over M.

Example. Let (a_1, a_2) belong to a differential extension of M and assume that:

(i) a_1, a_1' are algebraically independent over M and a_1'' is algebraic over $M(a_1, a_1')$ $\big($remark that $a_1^{(3)}, a_1^{(4)}, \ldots$ are also algebraic over $M(a_1, a_1')$, see Lemma 2.11 below$\big)$.

(ii) a_2 is differentially transcendental over $M\langle a_1\rangle$.

Then

$$\begin{cases} tr_M(M(a_1, a_2)) & = 2; \\ tr_M(M(a_1, a_2, a_1', a_2')) & = 4; \\ tr_M(M(a_1, a_2, a_1', a_2', a_1'', a_2'')) & = 5; \\ tr_M(M(a_1, a_2, a_1', a_2', a_1'', a_2'', a_1^{(3)}, a_2^{(3)})) & = 6; \\ \ldots \end{cases}$$

Also, one can see that for any $n \geq 1$,

$$tr_M(M(a_1, a_2, a_1', a_2', \ldots, a_1^{(n)}, a_2^{(n)})) = 1 \cdot (n+1) + 2$$

where 1 is the differential transcendence degree of $M\langle a_1, a_2\rangle$ over M.

Remark 2.2 (*[5]*). The Kolchin polynomial of $\bar{a}$ is clearly determined by the type of $\bar{a}$ over M and hence one can define the Kolchin polynomial of an n-type p over M to be the Kolchin polynomial of any realization of p. On the other hand, the Kolchin polynomial is not a differential bi-rational invariant; i.e. two tuples may generate the same differential field extension over M even if their respective Kolchin polynomials are different. Consider for example the tuples a and (a, a') where a is differentially transcendental over M. Then a and (a, a') clearly generate the same differential field extension over M but the Kolchin polynomial of a is $X + 1$ while the one of (a, a') is $X + 2$.

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2.2. Recovering the Kolchin polynomial from the δ-decomposition

From now on, M is a closed ordered differential field.

Definition 2.3. Let C^L be an $(i_{10},\ldots,i_{1n_1};\ldots;i_{k0},\ldots,i_{kn_k})$-cell giving rise to a δ-cell $C\subseteq M^k$. The **type of algebraicity** of C^L (denoted by al-type(C^L)) is equal to $(t_1;\ldots;t_k)$ where t_j is the least $l\in\{0,\ldots,n_j\}$ such that $i_{jl}=0$ if such an l exists and $t_j=\infty$ otherwise ($j\in\{1,\ldots,k\}$). Furthermore, each t_i is called the type of algebraicity of C^L in variable X_i.

Examples.

- For any $i\in\mathbb{N}$, let $C_i^L:=\{(x_0,\ldots,x_i)\in M^{i+1}\mid x_i=0\}$. Then for each i, C_i^L is a $(\underbrace{1,\ldots,1}_{i\text{ times}},0)$-cell giving rise to the (0)-δ-cell $C_i=\{x\in M\mid x^{(i)}=0\}$ and al-type$(C_i^L)=i$.
- Let $C_1^L:=\{(x_0,x_1,x_2)\in M^3\mid x_0=0\wedge x_1=0\wedge x_2=0\}$ and $C_2^L:=\{(x_0,x_1,x_2)\in M^3\mid x_1=0\wedge x_0=x_2\}$. Then C_1^L (resp. C_2^L) is a $(0,0,0)$-cell (resp. $(1,0,0)$-cell) and al-type$(C_1^L)=0$ (resp. al-type$(C_2^L)=1$).

Remark that in the second example above, the cells C_1^L and C_2^L are δ-equivalent (since they both give rise to the singleton $\{0\}\subseteq M$) *but they do not have the same type of algebraicity*. This fact stops us from directly defining a similar notion of type of algebraicity for δ-cells.

Definition 2.4. Let C be a δ-cell <u>in M</u>. The K**-type** of C is equal to (t) where t is minimal w.r.t. the property that there exists a source cell of C which has the type of algebraicity t.

Examples.

- Consider again the second example above. Since al-type$(C_1^L)=0$, the (0)-δ-cell $C_1=\{0\}$ has K-type (0).
- Since any source cell of a δ-open δ-cell is open in its ambient space, the K-type of any δ-open δ-cell in M is $(+\infty)$. Furthermore one can see that any δ-cell in M with K-type $(+\infty)$ is δ-open in M.

The case where C is a δ-cell <u>in M^k</u> $(k>1)$ is a bit more complicated. In order to define the K-type of C we have to treat all coordinate axes independently.

Definition 2.5. Let $C\subseteq M^k$ be a δ-cell. The K**-type** of C is equal to $(t_1;\ldots;t_k)$ where, for each $i\in\{1,\ldots,k\}$, $t_i\in(\mathbb{N}\cup\{+\infty\})$ is minimal w.r.t. the property that there exists a cell $C_i{}^L$ giving rise to C and whose type of algebraicity in variable X_i is equal to t_i.

Hence to any δ-cell $C\subseteq M^k$ there may correspond k different source cells $C_1^L,\ldots,C_k^L$ which are necessary to determine the K-type of C. In order to get rid of this constraint we introduce the following definition.

Definition 2.6. A δ-cell C in M^k is **engaged** if its K-type is determined by a single source cell. In other words, C is engaged if there exists a source cell C^L of C such that, for any $i\in\{1,\ldots,k\}$, C^L has the same type of algebraicity in the variable X_i as the source cell C_i^L appearing in Definition 2.5.

We now apply the notion of K-type to the study of finitely generated differential extensions of M. For this we consider two models M,N of *CODF* where N is an $|\,M\,|^+$-saturated elementary extension of M and a tuple $\bar a=(a_1;\ldots;a_k)\in N^k$. For any definable (with parameters in M) set $A\subseteq M^k$ we denote by A_N the subset of N^k defined by the same formula as A. We are interested in the δ-cell C of minimal[2] K-rank which is definable over M and such that C_N contains $\bar a$. We say that C is K**-minimal w.r.t.** $\bar a$.

The next lemma ensures the existence of such a K-minimal δ-cell which furthermore is engaged. Together with Definitions 2.4 and 2.5, it will allow us to associate a rank with any tuple $\bar a\in N^k$.

Lemma 2.7. (i) *Let $C_1,C_2\subseteq M^k$ be two δ-cells such that $(C_1)_N$ and $(C_2)_N$ contain $\bar a$. Then there exists a δ-cell $C\subseteq M^k$ such that C_N also contains $\bar a$ and the K-type of C is not greater than the ones of C_1 and C_2 (for the product order in $(\mathbb{N}\cup\{+\infty\})^k$).*
(ii) *Let $C\subseteq M^k$ be a K-minimal δ-cell w.r.t. $\bar a$. Then there exists an <u>engaged</u> δ-cell $D\subseteq M^k$ such that $\bar a\in D_N$ and K-type$(C)=K$-type (D).*

Proof. (i) Let $(t_1;\ldots;t_k)$ and $(u_1;\ldots;u_k)$ be the K-types of C_1 and C_2 respectively. For each $i\in\{1,\ldots,k\}$, let C_{1i}^L (resp. C_{2i}^L) denote a source cell of C_1 (resp. C_2) whose type of algebraicity in variable X_i is equal to t_i (resp. u_i). We can assume that, for a sufficiently large[3] $m\in\mathbb{N}$, the non-empty L-definable set

$$A^L:=C_1^L\cap C_2^L\cap\bigcap_{i=1}^{k}(C_{1i}^L\cap C_{2i}^L)$$

2 We consider here the product order in $(\mathbb{N}\cup\{+\infty\})^k$.
3 It suffices to take m greater than the t_i,u_j's.

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is a subset of M^{km+k}. The Cell Decomposition Theorem for o-minimal structures then provides a cell partition $(\mathcal{C}^L)_N$ of $(A^L)_N$. Assume that $(C^L)_N \in (\mathcal{C}^L)_N$ contains $(a_1, \dots, a_1^{(m)}; \dots; a_k, \dots, a_k^{(m)})$. Since $(C^L)_N \subseteq (A^L)_N$, its (o-minimal) type is not greater *component by component* than the ones of $C_1^L, C_2^L, C_{11}^L, C_{21}^L, \dots, C_{1k}^L, C_{2k}^L$. The type of algebraicity of C^L in any variable X_i is then lower than t_i and u_i.

It follows that C is a δ-cell such that C_N contains $\bar{a}$ and whose K-type is not greater component by component than the ones of C_1 and C_2.

(ii) The argument is similar to the one in part (i). Let C have K-type $(t_1; \dots; t_k)$ where each t_i is determined by a source cell C_i^L of C and assume that this K-type is minimal amongst all the δ-cells B such that $\bar{a} \in B_N$. Consider a cell D^L from a cell decomposition partitioning $A^L = \bigcap_{i=1}^k C_i^L$ such that $\bar{a} \in (D^L)_N$. This cell has an o-minimal type not greater (component by component) than the ones of each C_i^L. Hence the K-type of the δ-cell D is at most the one of C. The minimality of the latter implies that D has exactly the same K-type as C. This K-type is then entirely determined by a unique o-minimal cell D^L. □

Lemma 2.7 ensures the coherence of the following definition.

Definition 2.8. The K**-rank** of $\bar{a}$ over M is the k-tuple

$$K\text{-}rank(\bar{a}/K) := min\{K\text{-}rank(C) \mid \bar{a} \in C_N \text{ and } C \text{ is an engaged } \delta\text{-cell}\}$$

where the minimum is taken for the product order in $(\mathbb{N} \cup \{+\infty\})^k$.

Definition 2.9. We say that a δ-cell $C \subseteq M^k$ is **married** with the tuple $\bar{a} \in N^k$ if C is engaged and K-minimal w.r.t. $\bar{a}$.

The following theorem links the K-rank and the Kolchin polynomial.

Theorem 2.10. *Let $M, N \models CODF$ where N is an $|M|^+$-saturated elementary extension of M and $\bar{a} = (a_1; \dots; a_k) \in N^k$. Assume that the K-rank of $\bar{a}$ is $(t_1; \dots; t_k)$ with $t_i = +\infty$ iff $i \in \{j_1, \dots, j_d\} \subseteq \{1, \dots, k\}$. Then, for all sufficiently large n,*

$$tr_M M(\bar{a}, \bar{a}', \dots, \bar{a}^{(n)}) = d(n+1) + b$$

where b is the sum of all t_i's with $i \in I := \{1, \dots, k\} \setminus \{j_1, \dots, j_d\}$.

Before we prove Theorem 2.10, we recall the following result from [4].

Lemma 2.11 ([4, Lemma 2.3]). *Let $M \subset N$ be a differential field extension and $S \subseteq N$. If $a_1, \dots a_l \in N$ are algebraically dependent over $M(S)$ then $a_1', \dots a_l'$ are algebraically dependent over $M(S \cup S' \cup \{a_1, \dots, a_l\})$.*

Here is the proof of Theorem 2.10.

Proof. Let C be a δ-cell married with $\bar{a}$ and let $C^L \subseteq M^{km+k}$ be a source cell of C giving rise to the K-type of C. By Definitions 2.8 and 2.9, the K-type of C is equal to $(t_1; \dots; t_k)$ with $t_i = +\infty$ iff $i \in \{j_1, \dots, j_d\} \subseteq \{1, \dots, k\}$.

(1) Let $n > m$ and $\bar{n}$ be the k-tuple $(n, \dots, n)$. Since $a^* := (a_1, \dots, a_1^{(n)}; \dots; a_k, \dots, a_k^{(n)})$ belongs to the "swelled" cell $C^L{}_{\overline{n}}$ defined as in [1, Remark 2.5 (ii)], each component $a_i^{(t_i)}$ with $i \in I = \{1, \dots, k\} \setminus \{j_1, \dots, j_d\}$ is algebraically dependent over the field $\tilde{M}$ generated over M by the other components of a^* (since these components $a_i^{(t_i)}$ correspond to a digit 0 in the o-minimal type of $C^L{}_{\overline{n}}$). Furthermore, Lemma 2.11 implies that the successive derivatives of these components are also algebraically dependent over $\tilde{M}$. Hence

$$\begin{aligned} tr_M M(\bar{a}, \bar{a}', \dots, \bar{a}^{(n)}) &= tr_M M(a_1, \dots, a_1^{(n)}; \dots; a_k, \dots, a_k^{(n)}) \\ &\leq k(n+1) - \sum_{i \in I}(n+1-t_i) \\ &\leq k(n+1) - (k-d)(n+1) + \sum_{i \in I} t_i \\ &\leq d(n+1) + \sum_{i \in I} t_i. \end{aligned}$$

(ii) On the other hand, since K-type(C) is minimal amongst the engaged δ-cells containing $\bar{a}$, any cell which gives rise to an engaged δ-cell D containing $\bar{a}$ has a type of algebraicity greater than or equal to the one of C. In particular, for any $n \in \mathbb{N}$, any source cell $D^L \subseteq M^{n(k+1)}$ of D has an o-minimal type $(i_{10}, \dots, i_{1n}; \dots; i_{k0}, \dots, i_{kn})$ with:

$$\begin{cases} i_{j_1 0} = \cdots = i_{j_1 n} = \cdots = i_{j_d 0} = \cdots = i_{j_d n} = 1 \\ i_{l0} = \cdots = i_{l, t_l - 1} = 1 \end{cases}$$

for any $l \in I$. Since this is true for any engaged δ-cell D containing $\bar{a}$, the corresponding component of $a^* = (a_1, \dots, a_1^{(n)}; \dots; a_k, \dots, a_k^{(n)})$ is algebraically independent over the field generated by the other components of a^* over M. Hence

$$tr_M\, M(\bar{a}, \bar{a}', \dots, \bar{a}^{(n)}) \geq d(n+1) + \sum_{i \in I} t_i. \quad \square$$

3. δ-connectedness vs d-connectedness

3.1. Connectedness in o-minimal structures

Recall first that a subset A of a topological space X is **disconnected** if there exist two non-empty disjoint subsets U_1, U_2 of A which are open *in* A (w.r.t. the induced topology) and such that $U_1 \dot{\cup} U_2 = A$. Furthermore, if X is a first-order topological structure and A is a *definable* subset of M^k ($k \in \mathbb{N}$), we say that A is **definably disconnected** if A can be written as the disjoint union of two *definable* open sets in A. A definable set A is **(definably) connected** if it is not (definably) disconnected. A **definably connected component** of $A \subseteq M^k$ is a maximal definably connected subset of A.

Lemma 3.1. *If M is an o-minimal structure then each cell $C \subseteq M^k$ is definably connected (w.r.t. the order topology).*

Proof. See [6, Chapter 2]. □

Lemma 3.1 and the Cell Decomposition Theorem (1.6) lead to an important theorem of decomposition for definable sets in o-minimal structures.

Theorem 3.2. *Let M be an o-minimal structure and A a non-empty definable subset of M^k. Then A has finitely many definably connected components and furthermore, these definably connected components form a partition of A.*

Proof. See [9, Proposition 2.18, Chapter 3]. □

3.2. δ-connectedness, a deception

It would certainly be interesting to get an analogue of Theorem 3.2 in the case where M is a closed ordered differential field equipped with the δ-topology.[4] Unfortunately, according to the following basic example, this hope quickly goes up in smoke.

Example. Let C be the δ-cell defined by the formula $X' = 1$. Then we can split C into $(C \cap O_1)\dot{\cup}(C \cap O_2)$ where $O_1 = \{x \in M \mid x > 0\}$ and $O_2 = \{x \in M \mid x < 0\}$ are two δ-open subsets of M. Hence even the analogue of Lemma 3.1 does not hold anymore in this context. Furthermore C have *infinitely many* definably δ-connected components since C is dense and co-dense in M and the only definably δ-connected subsets of C are its singletons. This produces a counter-example to Theorem 3.2 in *CODF*.

In fact, since any open set is δ-open, there is no hope to find other definably δ-connected sets (i.e. sets which are definably connected for the δ-topology) than those which are already definably connected for the order topology. In other words, *any definably δ-connected set is definably connected*.

Consequently, in order to write down a generalization of Theorem 3.2, we have to slightly reconsider our approach and study a weaker notion of connectedness, namely the ***d*-connectedness**.

3.3. d-connectedness, a theorem of decomposition

Definition 3.3. Let M be a closed ordered differential field and A an L'-definable subset of M^k. A is **definably d-connected** if A^L is δ-equivalent to a definably connected set.

Examples

(i) By Definition 1.5 and Lemma 3.1, any δ-cell is definably d-connected.
(ii) Any definably connected set is definably d-connected.

Lemma 3.4. *The union of two definably d-connected sets having non-empty intersection is also definably d-connected.*

Proof. Let A, B be definably d-connected and such that $A \cap B \neq \emptyset$. Without any loss of generality we can assume that A^L and B^L are definably connected subsets of their respective ambient space. Furthermore, since Cartesian products of connected sets are still connected [3, Theorem 1.6, Ch. 3] and M is definably connected, we can apply the "swelling procedure" [1, Remark 4.4] and assume that A^L and B^L lie in the same ambient space M^N. Since $A^L \cap B^L$ is δ-equivalent to $(A \cap B)^L$ and $A \cap B \neq \emptyset$, $A^L \cap B^L$ is non-empty. Hence $A^L \cup B^L$ is a definably connected subset of M^N [3, Theorem 1.3, Ch. 3]. But $A^L \cup B^L$ is δ-equivalent to $(A \cup B)^L$ and hence $A \cup B$ is definably d-connected. □

As in the previous section, we define a **definably d-connected component** of an L'-definable set $A \subseteq M^k$ to be a maximal definably d-connected subset of A.

We are now able to state a generalization of Theorem 3.2.

[4] In what follows, "definably connected" means L-definably connected for the order topology and "definably δ-connected" means L'-definably connected for the δ-topology.

Theorem 3.5. *Every non-empty L'-definable subset A of M^k has finitely many definably d-connected components which furthermore form a partition of A.*

The proof is just a slightly modified version of the proof of [9, Proposition 2.18, Chapter 3].

Proof. Let $\{C_1, \ldots, C_l\}$ be a δ-decomposition of A and consider, for each subset I of $\{1, \ldots, l\}$, the L'-definable set $C_I = \cup_{i \in I} C_i$. Consider now the non-empty sets $C^1, \ldots, C^s$ which are maximal amongst the C_I w.r.t. the property of being definably d-connected ($s \leq 2^l - 1$). Since $\{C_1, \ldots, C_l\}$ is a δ-decomposition of A and each δ-cell is definably d-connected, $\cup_{j=1}^s C^j = A$.

We show that this union forms the wanted partition of A into definably d-connected components. For this, let us fix a j in $\{1, \ldots, s\}$.

Claim. *If B is a definably d-connected subset of A such that $B \cap C^j \neq \emptyset$ then $B \subseteq C^j$.*

Assume the claim is true. Then C^j is maximal amongst the definably d-connected subsets of A and hence is a definably d-connected component of A. Furthermore, for any $C^{j'}$ with $j' \in \{1, \ldots, s\} \setminus \{j\}$, $C^j \cap C^{j'} = \emptyset$ so that the C^j's form a partition of A. Finally, if D is any definably d-connected component of A, there exists $j \in \{1, \ldots, l\}$ such that $D \cap C^j \neq \emptyset$. Thus, by the claim and the maximality of D, $D = C^j$. Hence the C^j's are the only[5] definably d-connected components of A, completing the proof of Theorem 3.5.

Proof of the claim. Let

$$C_B := \bigcup_{i=1}^{l} \{C_i \mid C_i \cap B \neq \emptyset\}.$$

Since the C_i's form a partition of A, $B \subseteq C_B$. Hence there exists $r \in \{1, \ldots, l\}$ such that

$$C_B = B \cup (C_{i_1} \cup \cdots \cup C_{i_r})$$

with $B \cap C_{i_j} \neq \emptyset$ $\big(j \in \{1, \ldots r\}\big)$. By Lemma 3.4, C_B is definably d-connected and, since

$$C^j \cap C_B \supseteq C^j \cap B \neq \emptyset,$$

$C^j \cup C_B$ is also definably d-connected. It follows from the maximality of C^j that $C^j \cup C_B = C^j$ i.e. $C_B \subseteq C^j$. Hence $B \subseteq C^j$ and the proof of the claim is complete. □

In fact the decomposition into definably d-connected components of a given L'-definable set A is strongly related to the decompositions into definably connected components of all possible sources of A. In order to make this more precise, we first introduce the following definition.

Definition 3.6. The **index of d-connectedness** of A $\big($denoted by $I_c(A)\big)$ is the minimum, amongst all the L-definable sets B^L which are δ-equivalent to A^L, of the number of definably connected components[6] of B^L.

Remark that A is definably d-connected iff $I_c(A) = 1$.

Theorem 3.7. *If A is an L'-definable subset of M^k then the number of definably d-connected components of A is equal to $I_c(A)$.*

Proof. We denote the number of definably d-connected components of A by d_c.
(i) Let B^L be an L-definable set which is δ-equivalent to A^L and assume $B^L = \bigcup_{i=1}^s (C^i)^L$ is the decomposition of B^L into definably connected components. Then

$$A = C^1 \dot{\cup} \cdots \dot{\cup} C^s$$

where the C^i's are definably d-connected (but not necessarily maximal w.r.t. this property). Since any definably d-connected component of A which intersects one of the C^i's already contains it (see the proof of Theorem 3.5), $d_c \leq s$. In particular $d_c \leq I_c(A)$.
(ii) Assume now that $C^1 \dot{\cup} \ldots \dot{\cup} C^{d_c}$ is the decomposition of A into definably d-connected components. Since each C^i is a disjoint union of δ-cells belonging to the same δ-decomposition of A (see the proof of Theorem 3.5), we can consider each $(C^i)^L$ as the disjoint union of usual o-minimal cells lying in the same ambient space. Note that the $(C^i)^L$ are not necessarily definably connected.[7] However, by Definition 3.3, for each i in $\{1, \ldots, d_c\}$ there exists an L-definable set which is δ-equivalent to $(C^i)^L$ and definably connected in its ambient space. By another abuse of notation, we also use $(C^i)^L$ to denote this definably connected set. Using the swelling procedure [1, Remark 2.5] if necessary, we can assume that all the $(C^i)^L$ lie in the same ambient space. Remark that the $(C^i)^L$ are not necessarily pairwise disjoint anymore. However, since the C^i's are

[5] This shows that the decomposition into definably d-connected components built in this proof is independent of the δ-decomposition $\{C_1, \ldots, C_l\}$ of A we consider.

[6] For each B^L this number is well-defined by Theorem 3.2.

[7] For example, let $C_1 = \{x \in M \mid x' = 1 \wedge x < 0\}$ and $C_2 = \{x \in M \mid x' = 1 \wedge x > 0\}$ be a δ-decomposition of $A = \{x \in M \mid x' = 1\}$, then $C_1 \cup C_2$ is still definably d-connected $\big($since $(C_1 \cup C_2)^L$ is δ-equivalent to the connected line $y = 1$ in the plane $M^2\big)$ but the union $(C_1)^L \cup (C_2)^L$ is not definably connected.

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pairwise disjoint, there exists a point in the jet-space which belongs to $(C^i)^L$ and does not belong to $(C^j)^L$ for any $i \neq j$, so that the $(C^i)^L$'s are all distinct. The L-definable set

$$B^L := \bigcup_{i=1}^{d_c} (C^i)^L$$

is δ-equivalent to A^L and contains d_c distinct definably connected subsets. Hence, since the o-minimal analogue of the claim in the proof of Theorem 3.5 holds (see the proof of [9, Proposition 2.18, Chapter 3]), B^L has at most d_c definably connected components. It follows that $I_c(A) \leq d_c$. □

4. Looking for a theorem of δ-decomposition for L'-definable functions

4.1. δ-continuity and continuity at order n

Let $M \models CODF$ and A be an L'-definable subset of M^k. A function $f : A \to M$ is L'-definable if its graph

$$\Gamma(f) := \{(x_1; \ldots; x_k; y) \in M^{k+1} \mid (x_1; \ldots; x_k) \in A \wedge f(x_1; \ldots; x_k) = y\}$$

is L'-definable.

In order to use the notation introduced in Definition 1.1 and develop the same kind of argument as in [1], we have to make a restriction on the class of L'-definable functions we consider.

Definition 4.1. Let $f : A \to M$ be an L'-definable function. If φ is a quantifier free L'-formula defining $\Gamma(f)$ then we associate to f the relation $\bar{f}_\varphi : {A_\varphi}^L \to M^{r+1}$ whose graph is equal to the set $(\Gamma(f))_\varphi{}^L$ defined by the L-formula φ^L (cf. Definition 1.1). More precisely,

$$\begin{aligned}\Gamma(\bar{f}_\varphi) &= \{(x_{10}, \ldots, x_{1n_1}; \ldots; x_{k0}, \ldots, x_{kn_k}; y_0, \ldots, y_r) \in {A_\varphi}^L \times M^k \mid \\ &\qquad M \models \varphi^L(x_{10}, \ldots, x_{1n_1}; \ldots; x_{k0}, \ldots, x_{kn_k}; y_0, \ldots, y_r)\}.\end{aligned}$$

Assumption $(*)$**:** We will always assume that there exists a quantifier free L'-formula φ defining f such that $\Gamma(\bar{f}_\varphi)$ is really a graph of a function; i.e. such that *$\bar{f}_\varphi$ is an L-definable function on ${A_\varphi}^L$*. From now on, we fix one such φ and simply denote $\bar{f}_\varphi$ by $\bar{f}$ (and similarly for ${A_\varphi}^L$).

Fact (★): This assumption implies that for any $(x_1; \ldots; x_k) \in A$ and $y \in M$:

$$\begin{aligned} f(x_1; \ldots; x_k) = y \ &\text{iff } (x_1; \ldots; x_k; y) \in \Gamma(f) \\ &\text{iff } M \models \varphi(x_1; \ldots; x_k; y) \\ &\text{iff } M \models \varphi^L(x_1, \ldots, x_1^{(n_1)}; \ldots; x_k, \ldots, x_k^{(n_k)}; y, \ldots, y^{(r)}) \\ &\text{iff } (x_1, \ldots, x_1^{(n_1)}; \ldots; x_k, \ldots, x_k^{(n_k)}; y, \ldots, y^{(r)}) \in \Gamma(\bar{f}) \\ &\text{iff } \bar{f}(x_1, \ldots, x_1^{(n_1)}; \ldots; x_k, \ldots, x_k^{(n_k)}) = (y, \ldots, y^{(r)}). \end{aligned}$$

Remark 4.2. If the formula φ has order 0 in variable Y then Assumption $(*)$ becomes vacuous. In fact, one can consider the L-definable set

$$B^L := \{(x_{10}, \ldots, x_{1n_1}; \ldots; x_{k0}, \ldots, x_{kn_k}) \in A^L \mid \exists! y \, \varphi^L(x_{10}, \ldots, x_{1n_1}; \ldots; x_{k0}, \ldots, x_{kn_k}; y)\}$$

which contains the elements of A^L where φ^L defines a function. Note that $A^* \subseteq B^L \subseteq A^L$ and then B^L is δ-equivalent to A^L (this is not necessarily true anymore if φ has ≥ 1 in variable Y). By quantifier elimination for real closed fields, B^L is definable by a quantifier free L-formula

$$\psi^L(X_{10}, \ldots, X_{1n_1}; \ldots; X_{k0}, \ldots, X_{kn_k}).$$

Hence the L'-formula

$$(\psi \wedge \varphi)(X_1; \ldots; X_k; Y)$$

defines f and the L-formula

$$(\psi \wedge \varphi)^L(X_{10}, \ldots, X_{1n_1}; \ldots; X_{k0}, \ldots, X_{kn_k}; Y)$$

defines the function $\bar{f} : B^L \to M$ and satisfies Assumption $(*)$.

Differential polynomial maps

$$p : M^k \to M : (X_1; \ldots; X_k) \mapsto Y = p(X_1; \ldots; X_k)$$

are natural examples of such L'-definable functions which satisfy both Assumption $(*)$ and Fact ★.

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Definition 4.3. Let $f : A \to M$ be an L'-definable function. The **derivative** of f is the L'-definable function

$$f' : A \to M : (x_1; \ldots; x_k) \mapsto (f(x_1; \ldots; x_k))'.$$

We define similarly all the higher derivatives $f'', f^{(3)}, \ldots$ of f.

Remark 4.4. By Definition 4.3, $f^{(n)}$ is the function whose graph equals

$$\Gamma(f^{(n)}) := \{(x_1; \ldots; x_k; y) \in M^{k+1} \mid (x_1; \ldots; x_k) \in A \wedge y = (f(x_1; \ldots; x_k))^{(n)}\}.$$

If the graph of f is definable by the quantifier free L'-formula φ then

$$\Gamma(f^{(n)}) = \{(x_1; \ldots; x_k; y) \in M^{k+1} \mid \exists z\, \varphi(x_1; \ldots; x_k; z) \wedge y = z^{(n)}\}.$$

Hence $f^{(n)}$ is an L'-definable function and, by quantifier elimination in *CODF*, there exists a quantifier free L'-formula ψ defining $\Gamma(f^{(n)})$.

We will make the same assumption on $f', \ldots f^{(n)}, \ldots$ as on f.

Definition 4.5. An L'-definable function f is **admissible** if f and all its derivatives satisfy Assumption ($*$). This means that we can consider $\bar{f}, \overline{f'}, \ldots, \overline{f^{(n)}}, \ldots$ as L-definable functions on A^L.

Remark 4.6. Using A^L to denote the domain of the functions $\overline{f'}, \overline{f''}, \ldots$ could be a bit misleading as seen in the following example. If

$$f : M \to M : X \mapsto Y = X$$

then $f = \bar{f}$ and

$$f' : M \to M : X \mapsto Y = X'$$

so that $\bar{f}'$ is the function sending a pair (X_0, X_1) to $Y = X_1$. Hence the functions $\bar{f}, \overline{f'}, \ldots$ need not have the same domain. However, using the usual swelling procedure, we will always assume that, for a fixed $n \in \mathbb{N}$, the functions $\bar{f}, \bar{f}', \ldots, \overline{f^{(n)}}$ have the same domain A^L.

On the other hand, remark that

$$\begin{aligned} f'(x_1; \ldots; x_k) = z \;\; & \text{iff } z = y' \text{ with } f(x_1; \ldots; x_k) = y \\ & \text{iff } (z, \ldots, z^{(r)}) = (y', \ldots, y^{(r+1)}) \end{aligned}$$

with $\bar{f}(x_1, \ldots, x_1^{(n_1)}; \ldots; x_k, \ldots, x_k^{(n_k)}) = (y, \ldots, y^{(r)})$. Hence we can also assume that the functions $\bar{f}, \bar{f}', \ldots$ have the same range M^{r+1}.

The assumption of admissibility allows us to define a partial notion of differential continuity.

Definition 4.7. Let $f : A \to M$ be an admissible L'-definable function. We say that f is **continuous at order** n if $\bar{f}, \overline{f'}, \ldots, \overline{f^{(n)}}$ are continuous w.r.t. the order topology.

The next lemma justifies the introduction of Definition 4.7.

Lemma 4.8. *If a δ-open subset U of M is defined by a quantifier free L'-formula φ of order at most n, then its pre-image by an admissible L'-definable function $f : M^k \to M$ which is continuous at order n is δ-open.*

Proof. Without loss of generality we can assume that U is a basic δ-open subset of M and that U^L is an open box $I_0 \times \cdots \times I_n$ in M^{n+1}. Furthermore, using the swelling procedure if necessary, we assume that $\bar{f}, \overline{f'}, \ldots, \overline{f^{(n)}}$ are L-definable functions from $M^{k(m+1)}$ to M^{n+1} where m is a sufficiently[8] large integer. Let

$$\begin{aligned} f^{-1}(U) \; &:= \{(x_1; \ldots; x_k) \mid f(x_1; \ldots; x_k) \in U\} \\ &= \; \{(x_1; \ldots; x_k) \mid (f(x_1; \ldots; x_k), \ldots, f^{(n)}(x_1; \ldots; x_k)) \in U^L\}. \end{aligned}$$

For any $j \in \{1, \ldots, k\}$, let $x_j{}^* = (x_j, \ldots, x_j^{(m)})$ and $\overline{x_j} = (x_{j0}, \ldots, x_{jm})$. Then

$$\begin{aligned} f^{-1}(U) \; &= \left\{(x_1; \ldots; x_k) \mid (\bar{f}(x_1{}^*; \ldots; x_k{}^*), \ldots, \overline{f^{(n)}}(x_1{}^*; \ldots; x_k{}^*)) \in (I_0 \times M^n) \times \cdots \times (I_n \times M^n)\right\} \\ &= \left\{(x_1; \ldots; x_k) \mid \bigwedge_{i=0}^{n} \overline{f^{(i)}}(x_1^*; \ldots; x_k^*) \in I_i \times M^n\right\}. \end{aligned}$$

[8] This means that we assume that all the orders of the variables in any formula appearing in the proof are less than or equal to m.

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Let

$$\tilde{U} := \left\{ (\overline{x_1}, \ldots, \overline{x_r}) \mid \bigwedge_{i=0}^{n} \overline{f^{(i)}}(\overline{x_1}; \ldots; \overline{x_k}) \in I_i \times M^n \right\}.$$

Since $\bar{f}, \ldots, \overline{f^{(n)}}$ are continuous, the set

$$\tilde{U} = \bigcap_{i=0}^{n} \overline{f^{(i)}}^{-1} (I_i \times M^n)$$

is open and hence

$$f^{-1}(U) = \pi_{(10;\ldots;k0)}\big(\tilde{U} \cap J_{(n;\ldots;n)}(M^k)\big)$$

is a δ-open subset of M^k (cf. Definition 1.4). □

Since each definable δ-open subset of M is defined by a formula of finite order, Lemma 4.8 directly implies the following result which gives a useful criterion to determine whether an L'-definable function is δ-continuous.

Corollary 4.9. *An admissible L'-definable function $f : A \to M$ is δ-continuous on A as soon as each $\overline{f^{(i)}}$ ($i \in \mathbb{N}$) is continuous on A^L w.r.t. the order topology.*

Proof. By Lemma 4.8. □

The following corollary explains why we consider the δ-topology as the "natural" topology in *CODF*. In fact it shows that an ordered differential field equipped with the δ-topology satisfies the properties of a *topological system* as defined in [8, Definition 2.12]. Let us remark that in our case the δ-topology is definable by an *infinite* conjunction of L'-formulas.

Corollary 4.10. *Any differential polynomial $p \in M\{X_1, \ldots, X_k\}$ (considered as an L'-definable function from M^k to M) is δ-continuous on M^k.*

Proof. Let $p(X_1; \ldots; X_k) \in M\{X_1, \ldots, X_k\}$ be a differential polynomial in the variables $X_1, \ldots, X_k$. Remark first that $p', p'', \ldots$ are still differential polynomials in $X_1, \ldots, X_k$. Furthermore, for any positive integer n, $\overline{p^{(n)}}$ is the algebraic polynomial obtained from $p^{(n)}$ by replacing each differential variable $X_i^{(j)}$ by an ordinary variable X_{ij} ($i \in \{1, \ldots, k\}$ and $j \in \{0, \ldots, r_i\}$ where r_i is the order of $p^{(n)}$ in the differential variable X_i). The result now directly follows from Corollary 4.9 and the continuity of ordinary w.r.t. the order topology. □

We can now state a partial theorem of δ-decomposition for L'-definable functions.

Theorem 4.11. *For any admissible L'-definable function $f : A \to M$ and any positive integer n, there exists a finite partition $\mathcal{C}_n$ of A into δ-cells such that the restriction of f to any of these δ-cells is continuous at order n.*

We first prove an easy intermediate result.

Lemma 4.12. *Let $f : A \to M$ ($A \subseteq M^k$) be an admissible L'-definable function. Then there exists a finite cell decomposition $\mathcal{C}^L$ of A^L such that $\bar{f}$ is continuous on each cell belonging to $\mathcal{C}^L$.*

Proof. We can assume that $\bar{f}$ is an L-definable function from M^{kn+k} to M^{r+1} and then write $\bar{f}$ as the tuple $(\bar{f}_0, \ldots, \bar{f}_r)$ where, for each $i \in \{0, \ldots, r\}$, $\bar{f}_i : M^{kn+k} \to M$. Recall that $\bar{f}$ is continuous iff each $\bar{f}_i$ is continuous (see for example [3, Theorem 8.5]). For each i in $\{0, \ldots, r\}$,

$$\begin{aligned} \Gamma(\bar{f}_i) = \{ & (x_{10}, \ldots, x_{1n}; \ldots; x_{k0}, \ldots, x_{kn}; y_i) \in A^L \times M \mid \\ & (\exists y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_r)\, \varphi^L(x_{10}, \ldots, x_{1n_1}; \ldots; x_{k0}, \ldots, x_{kn_k}; y_0, \ldots, y_r)\} \end{aligned}$$

where φ^L defines the graph of $\bar{f}$. Hence $\bar{f}_i$ is L-definable and, by [9, 2.11 (II_m)], there exists a cell decomposition ${\mathcal{C}_i}^L$ of A^L such that the restriction of $\bar{f}_i$ to any element of ${\mathcal{C}_i}^L$ is continuous. If $\mathcal{C}^L$ is a cell decomposition refining all the cell decompositions ${\mathcal{C}_i}^L$ then, for any $i \in \{0, \ldots, r\}$, $\bar{f}_i$ is continuous on each $C^L \in \mathcal{C}^L$. □

Remark 4.13. This result can be interpreted as a multi-variable generalization of the o-minimal Cell Decomposition Theorem for definable functions. It is equivalent to Theorem 4.11 in the special case where $n = 0$.

We now give the proof of Theorem 4.11.

Proof. Let $n \in \mathbb{N}$. By Lemma 4.12, there exist cell decompositions ${\mathcal{C}_i}^L, \ldots, {\mathcal{C}_n}^L$ of A^L such that $\overline{f^{(i)}}$ is continuous on each element of ${\mathcal{C}_i}^L$ ($i = 0, \ldots, n$).

We then conclude as in the preceding proof. Let $\mathcal{C}^L$ be a cell decomposition refining all the ${\mathcal{C}_i}^L$'s. Then the functions $\bar{f}, \overline{f'}, \ldots, \overline{f^{(n)}}$ are continuous on each $C^L \in \mathcal{C}^L$. Hence f is continuous at order n on each δ-cell C belonging to $\mathcal{C}$ which is a finite partition of A into δ-cells. □

Remark 4.14. Note that the continuity of $\overline{f^{(n)}}$ and $\overline{f^{(n+1)}}$ on a cell C^L does not imply the continuity of $\overline{f^{(n+2)}}$ on this cell. Hence the δ-decomposition obtained in the proof above strongly depends on the integer n and there is no indication about the asymptotic behavior of the sequence $\mathcal{C}_n$ when n tends to ∞.

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4.2. A particular case

In this section we show that with an (rather strong) additional hypothesis, it is possible to obtain a differential analogue of the o-minimal Cell Decomposition Theorem for definable functions. For this we assume that $f : M^k \to M$ is an L'-definable function *commuting with the derivation*. The function $f' : M^k \to M$ sends the tuple $(x_1; \ldots; x_k)$ to

$$f'(x_1; \ldots; x_k) := (f(x_1; \ldots; x_k))' = f((x_1; \ldots; x_k)') = f(x_1'; \ldots; x_k')$$

and similarly for each derivative $f^{(n)}$ defined as in 4.3. Hence the function f is admissible as soon as it satisfies Assumption $(*)$. Furthermore, for any $n \in \mathbb{N}$, the continuity at order n of f is equivalent to the continuity of the function $\bar{f}$. Hence, by Lemma 4.9, f is δ-continuous as soon as it is continuous at order 0 (i.e. as soon as $\bar{f}$ is continuous). This gives us the possibility of writing down the following theorem of δ-decomposition.

Theorem 4.15. *Let M be a closed ordered differential field. For any L'-definable function $f : A \to M$ satisfying Assumption $(*)$ and commuting with the derivation, there exists a finite partition of A into δ-*cells *such that the restriction of f to any of these δ-cells is δ-continuous.*

Proof. This follows immediately from the remark above and Lemma 4.12. □

In the following simple example the function $f : M \to M$ does *not* commute with the derivation but is δ-continuous on a given δ-decomposition of M.

Example. Let

$$f : M \to M : X \mapsto \begin{cases} 0 & \text{if } X' = 0 \\ X & \text{if } X' \neq 0. \end{cases}$$

The function f is L'-definable and does not commute with the derivation: if $a \in M$ is such that $a' = 1$ then $f(a') = 0$ $\big($since $(a')' = 1' = 0\big)$ and $(f(a))' = a' = 1 \neq 0$.

The corresponding L-definable function is

$$\bar{f} : M^2 \to M : (X_0, X_1) \mapsto \begin{cases} 0 & \text{if } X_1 = 0 \\ X_0 & \text{if } X_1 \neq 0. \end{cases}$$

It is easy to see that $\bar{f}$ is continuous on each of the following cells:

(i) ${C_1}^L = \{(x_0, x_1) \in M^2 \mid x_1 = 0\}$;
(ii) ${C_2}^L = \{(x_0, x_1) \in M^2 \mid x_1 > 0\}$;
(iii) ${C_3}^L = \{(x_0, x_1) \in M^2 \mid x_1 < 0\}$.

The δ-decomposition of M built from this cell decomposition is:

(i) $C_1 = \{x \in M \mid x' = 0\}$;
(ii) $C_2 = \{x \in M \mid x' > 0\}$;
(iii) $C_3 = \{x \in M \mid x' < 0\}$.

We remark that $f \equiv 0$ on C_1 and is the function $Id : X \mapsto X$ on C_2 and C_3. Hence the restriction of f to each of these δ-cells is δ-continuous.

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