Dual actions for massless, partially-massless and massive gravitons in (A)dS

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A R T I C L E   I N F O

A B S T R A C T

We provide a unified treatment of electric–magnetic duality, at the action level and with manifest Lorentz invariance, for massive, massless as well as partially-massless gravitons propagating in maximally symmetric spacetimes of any dimension $n > 3$. For massive and massless fields, we complete previous analyses that use parent-action techniques by giving dual descriptions that enable direct counting of physical degrees of freedom in the flat and massless limit. The same treatment is extended to the partially-massless case, where the duality has been previously discussed in covariant form only at the level of the equations of motion. The nature of the dual graviton is therefore clarified for all values of the mass and of the cosmological constant.

1. Introduction

In this note, we complete and extend previous analyses on dual formulations of massive and (partially) massless spin-2 theories in (A)dS backgrounds of arbitrary dimension $n > 3$. We resort to the parent-action technique employed in the papers [1–10] in order to derive equivalent, dual actions in the sense of Fradkin and Tseytlin [11]. In brief, in this framework one obtains two equivalent second-order actions — whose field equations are related by electric–magnetic duality — by eliminating different sets of fields from a common "parent" first-order action. In (A)dS$_n$ these techniques have been employed for massless and massive gravitons, while the partially-massless case has been discussed recently only in $n = 3$ [12]. The same setup has also been used in the context of Hořava–Lifshitz gravity [13].

For all values of the mass and of the cosmological constant, we furnish dual formulations at the action level and in a manifestly Lorentz-invariant way. The dual actions that we built are such that the flat and massless limits are smooth, thereby making the identification of the physical degrees of freedom and of the helicities straightforward. In the partially-massless case [14], we obtain for

the first time a dual, manifestly covariant action principle featuring a mixed-symmetry gauge field.

At the level of the field equations, (self-)duality symmetry, often named pseudo (self-)duality, has been studied in flat spacetime for linearised gravity in [15,16]; see also [17]. In (A)dS$_4$, pseudo-duality symmetry for partially massless spin-2 fields was studied in [18,19]. These are first steps towards the establishment of an equivalence between theories, for which an off-shell duality relation is necessary. In flat spacetime, the duality between the massless Fierz–Pauli action and the Curtright action [20,21] was proven in the series of works [2–4].

The action principles that we present feature both the original spin-2 field and its dual, in a manifestly Lorentz-invariant fashion. On the other hand, the pair of dual fields does not enter the action in a duality-symmetric way. For such a democratic appearance of electric and magnetic fields inside the action, the price to pay is the loss of manifest spacetime covariance, as explained for massless spin-2 and higher-spin theories around flat spacetime in the papers [22–24] and references therein. In the same, non manifestly Lorentz-covariant framework, a double-potential formulation of linearised gravity around (A)dS$_n$ spacetime was studied in [25, 26] for $n = 4$ and in [27] for $n > 4$. As for what concerns partially-massless fields of maximal depth, the paper [28] provides an off-shell formulation exhibiting a nearly manifest electric–magnetic duality symmetry. Interestingly enough, manifest duality-invariant formulations of linearised gravity, in the presence of sources, have been given in [29] and in an alternative way in [30]; in the

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partially-massless case, see also [31]. Finally, the integrability properties of duality-symmetric systems were studied in [32].

In more details, the unified treatment of spin-2 duality presented in this note leads to the following results:

- In the case of a massless graviton in (A)dS, we complete the programme sketched in [10] by linking the dual action obtained therein to its Stueckelberg formulation admitting a smooth flat limit;
- For partially massless spin-2 field in (A)dS, we obtain a dual description at the action level, thereby elevating the duality from a pseudo to a genuine off-shell duality;
- In the massive case in (A)dS, we clarify the flat limit of the dual model presented in [6] in that we have Stueckelberg gauge fields representing the dual spin-2, spin-1 and scalar sectors. Therefore, our actions admit a smooth flat limit in *both* electric and magnetic formulations.

In sections 2 and 3 we recall the main features of the first-order description of massive spin-two fields, that we use as a parent action. Section 4 collects our original results on dual formulations for spin-2 fields in (A)dS.

2. The parent action

We consider as parent action the first-order Stueckelberg action describing, for generic values of the parameters, the propagation of a massive spin-2 field in a constant curvature background [33]. It is obtained by coupling the free actions for massless fields of spin two, one and zero. It thus comprises the kinetic terms for these fields,[1]

$$\mathcal{L}^{(2)} = -\frac{\epsilon_{abcp[n-3]}}{2(n-3)!} \left(\omega^{ab} \nabla h^c + \frac{1}{n-2} \omega^a{}_q \omega^{qb} \bar{e}^c \right) H^{p[n-3]}, \quad (1)$$

$$\mathcal{L}^{(1)} = \frac{\epsilon_{abc[n-2]}}{2(n-2)!} F^{ab} \left(\nabla A - \frac{1}{4} F_{kl} \bar{e}^k \bar{e}^l \right) H^{c[n-2]}, \quad (2)$$

$$\mathcal{L}^{(0)} = \frac{\epsilon_{ab[n-1]}}{(n-1)!} \pi^a \left(\nabla \varphi - \frac{1}{2} \pi_k \bar{e}^k \right) H^{b[n-1]}, \quad (3)$$

together with cross couplings and mass terms:

$$\mathcal{L}_{\text{cross}} = \frac{\epsilon_{abc[n-2]}}{(n-1)!} \left((n-1) m \omega^{ab} A + m F^a{}_d h^d \bar{e}^b + \mu \pi^a A \bar{e}^b \right.$$
$$\left. - \frac{(n-2)\mu^2}{4} h^a h^b - m \mu \varphi h^a \bar{e}^b - \frac{m^2}{n-2} \varphi^2 \bar{e}^a \bar{e}^b \right) H^{c[n-2]}. \quad (4)$$

The full action is the integral of $\mathcal{L} = \sum_{s=0}^{2} \mathcal{L}^{(s)} + \mathcal{L}_{\text{cross}}$ and it is invariant under the gauge symmetries

$$\delta h^a = \nabla \xi^a - \Lambda^{ab} \bar{e}_b + \frac{2m}{n-2} \epsilon \bar{e}^a, \quad (5a)$$

$$\delta \omega^{ab} = \nabla \Lambda^{ab} + \frac{\mu^2}{n-1} \bar{e}^{[a} \xi^{b]}, \quad (5b)$$

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[1] We denote the background vielbein by $\bar{e}^a$, while ∇ is the Lorentz-covariant derivative on (A)dS$_n$. In our conventions, it satisfies $\nabla^2 V^c = -\sigma \lambda^2 \bar{e}^c \wedge \bar{e}_b V^b$, so that $\sigma = 1$ in AdS$_n$ and $\sigma = -1$ in dS$_n$. We define the Levi-Civita symbol $\epsilon_{a_1 \cdots a_n}$ such that $\epsilon_{01 \cdots n-1} = -1$ and we adopt the mostly-plus convention for the metric. In the following we omit wedge products and we substitute groups of antisymmetrised indices with a label denoting the total number of indices. For instance, we introduce the k-form $H^{a[k]} \equiv H^{a_1 \cdots a_k} = \bar{e}^{a_1} \cdots \bar{e}^{a_k}$. Indices enclosed between square brackets are antisymmetrised, and dividing by the number of terms involved is understood (strength-one convention). Finally, repeated indices also denote an antisymmetrisation, e.g., $A_a B_a \equiv A_{[a_1} B_{a_2]}$.

and

$$\delta A = \nabla \epsilon - m \xi^a \bar{e}_a, \qquad \delta F^{ab} = 2m \Lambda^{ab}, \quad (6)$$

$$\delta \varphi = -\mu \epsilon, \qquad \delta \pi^a = -m \mu \xi^a. \quad (7)$$

For later convenience, we introduced the constants m and μ, even if the action actually depends only on a single mass parameter (besides the (A)dS radius). Gauge invariance requires

$$\mu^2 = \frac{2(n-1)}{n-2} \left(2m^2 + \sigma(n-2)\lambda^2 \right). \quad (8)$$

When $m = 0$ the fields of spin one and zero decouple from the spin-two sector and one recovers the usual first-order formulation of linearised gravity in (A)dS. At $\mu = 0$, the sole scalar sector decouples and one obtains a first-order description of a partially-massless graviton, propagating helicities two and one in the flat limit. With the manifestly unitary conventions used in (1)–(3), one can set μ to zero by tuning the mass $m \in \mathbb{R}$ only in dS ($\sigma = -1$). In section 4.2 we shall show that partially-massless fields in AdS can be described in this formalism at the price of flipping the sign of the spin-one kinetic term, which makes their lack of unitarity manifest.

Eq. (1) can be expressed in terms of the field [2]

$$Y^{bc|a} = \omega^{a|bc} + g^{ab} \omega_d{}^{|cd} - g^{ac} \omega_d{}^{|bd}, \quad (9)$$

which is antisymmetric in its first two indices and transforms as

$$\delta Y^{bc}|_a = \nabla_a \Lambda^{bc} + 2 \bar{e}_a{}^{[b} \nabla_d \Lambda^{c]d} - \frac{(n-2)\mu^2}{n-1} \bar{e}_a{}^{[b} \xi^{c]}. \quad (10)$$

The spin-2 kinetic term can then be cast in the form (from now on we will omit the integration measure $d^n x \sqrt{-g}$ brought by $\bar{e}^{a_1} \cdots \bar{e}^{a_n} = \det(\bar{e}) \epsilon^{a_1 \cdots a_n} d^n x$)

$$\mathcal{L}^{(2)} = \nabla_b h_{c|}{}^a Y^{bc}|_a - \frac{1}{2} \left(Y^{bc|a} Y_{ab|c} + \frac{1}{n-2} Y^{ab|}{}_b Y_{ac|}{}^c \right), \quad (11)$$

while the cross couplings and mass terms read

$$\mathcal{L}_{\text{cross}} = -\frac{2m}{n-2} Y^{ab}|_b A_a - m F^{ab} h_{a|b} - \mu \pi^a A_a$$
$$- \frac{(n-2)\mu^2}{4(n-1)} \left(h_{a|b} h^{b|a} - h^2 \right) + m \mu h \varphi + \frac{n m^2}{n-2} \varphi^2, \quad (12)$$

where $h = h_{a|}{}^a$ denotes the trace of the linearised vielbein.

As recalled in section 3, eliminating the auxiliary fields $Y^{bc}|_a$, F^{ab} and π^a from the parent action $\mathcal{L}$ one obtains a second-order description of a massive spin-2 field in terms of the linearised metric and the fields A_μ and φ, which reduces to the Fierz–Pauli action for $m = 0$. In section 4 we will instead show how eliminating the fields $h_{a|}{}^b$, A_a and φ leads to its dual description, involving mixed-symmetry fields for generic values of n.

3. Electric reduction

The equations of motion for $Y^{bc}|_a$, F^{ab} and π^a arising from $\mathcal{L}[h, Y, A, F, \varphi, \pi]$ allow to solve for them algebraically. E.g.

$$Y_{ab|c} = \nabla_c h_{[a|b]} - \nabla_a h_{(b|c)} + \nabla_b h_{(a|c)}$$
$$+ 2 g_{c[a} \left(\nabla^d h_{b]|d} - \nabla_{b]} h + 2m A_{b]} \right). \quad (13)$$

By plugging this and the similar expressions for F^{ab} and π^a into the parent Lagrangian $\mathcal{L}$, the latter reduces, modulo a total derivative, to the second-order Stueckelberg Lagrangian for a symmetric spin-2 field [34,35]:

$$\mathcal{L}[h, A, \varphi] = -\tfrac{1}{2} \nabla_a h_{(b|c)} \nabla^a h^{(b|c)} + \nabla_a h_{(b|c)} \nabla^c h^{(b|a)}$$
$$+ \tfrac{1}{2} \nabla_a h \nabla^a h - \nabla_a h \nabla_b h^{(a|b)} - \tfrac{(n-1)\sigma\lambda^2}{2} \left(2 h_{(a|b)} h^{(a|b)} - h^2 \right)$$
$$- \nabla_{[a} A_{b]} \nabla^{[a} A^{b]} - (n-1) \sigma \lambda^2 A_a A^a - \tfrac{1}{2} \nabla_a \varphi \nabla^a \varphi \qquad (14)$$
$$- 2m A_a \left(\nabla^a h - \nabla_b h^{(a|b)} \right) + \mu \varphi \nabla_a A^a$$
$$- m^2 \left(h_{(a|b)} h^{(a|b)} - h^2 \right) + \tfrac{n\,m^2}{n-2} \varphi^2 + m\,\mu\,h\,\varphi \, .$$

The resulting action is invariant under the gauge transformations (5a) for $h_{(a|b)}$, to be identified with the linearised metric, together with (6) and (7) for the Stueckelberg fields A_a and φ. The antisymmetric part of the vielbein, $h_{[a|b]}$, enters the reduced Lagrangian only through a total derivative, consistently with the shift symmetry it enjoys under Lorentz transformations.

The first two lines of (14) gives the Fierz–Pauli Lagrangian for a massless spin-2 field in (A)dS. For $\mu = 0$ one obtains a description of a partially-massless spin-2 field in dS in terms of the Stueckelberg coupling of the Fierz–Pauli and Proca Lagrangians. The field A_a can be gauged away using ξ^a, and the resulting action is invariant under

$$\delta h_{(a|b)} = \tfrac{1}{\lambda} \left(\nabla_{(a} \nabla_{b)} \epsilon + \lambda^2 g_{ab} \epsilon \right). \qquad (15)$$

In this context, the partially-massless gauge symmetry thus follows because gauge transformations (5a) and (6) with $\nabla_a \epsilon - m \xi_a = 0$ preserve the gauge fixing $A_a = 0$.

4. Magnetic reduction

4.1. Massless case

When $m = 0$ the fields of spin one and zero decouple and one can consider the parent Lagrangian

$$\mathcal{L}_0[h, Y] = \mathcal{L}^{(2)}[h, Y] - \tfrac{(n-2)\sigma\lambda^2}{2} \left(h_{a|b} h^{b|a} - h^2 \right), \qquad (16)$$

with $\mathcal{L}^{(2)}$ given in (11). Its gauge symmetries are obtained by setting $m = 0$ in (5a) and (10). Contrary to flat space [4] (where it enters the action linearly), in (A)dS the linearised vielbein is an auxiliary field thanks to the mass term in (16): it can thus be eliminated through its own equation of motion [5]. This leads to an action depending only on the traceless projection of $Y^{bc}|_a$:

$$\hat{Y}^{bc}|_a = Y^{bc}|_a + \tfrac{2}{n-1} \bar{e}_a{}^{[b} Y^{c]d}|_d \, . \qquad (17)$$

After the elimination of $h_{a|b}$, the trace of $Y^{bc|a}$ indeed contributes to the action only via a boundary term, consistently with the shift symmetry generated by ξ^a in (10), which is still present for $m = 0$. One can cast the resulting Lagrangian in the form

$$\mathcal{L}_0[Y] = \tfrac{\sigma}{2(n-2)\lambda^2} \left[\nabla_a \hat{Y}^{cd}|_b \nabla_c \hat{Y}^{ab}|_d + \sigma \lambda^2 \hat{Y}_{bc|a} \hat{Y}^{ba|c} \right], \qquad (18)$$

in agreement with the result obtained by eliminating the vielbein from the linearised McDowell–Mansouri action [10].

Introducing the Hodge dual $T_{a[n-2]|b} = \tfrac{1}{2} \epsilon_{a[n-2]cd} \hat{Y}^{cd}|_b$ (which satisfies $\epsilon^{a[n-2]bc} T_{a[n-2]|b} = 0$ on account of $\hat{Y}^{ab}|_b = 0$), one obtains a dual description of a massless graviton in (A)dS$_n$. The field T, however, has the same structure as a massive graviton in flat space [1]; when $\lambda = 0$, the dual of a massless spin-two field is instead a $GL(n)$ Young-projected[2] $[n-3, 1]$ field [2–4]. As discussed in [10],

the different nature of the dual graviton in (A)dS and flat space can be explained as follows: massless mixed-symmetry fields display less gauge symmetries in (A)dS than in flat space. This is the Brink–Metsaev–Vasiliev (BMV) mechanism conjectured in [36], proved for AdS$_n$ in [37–39] and for dS$_n$ in [40]. It is also discussed in [41] from the point of view of reducibility conditions. As a result, in the flat limit, mixed-symmetry gauge fields decompose in multiplets of gauge fields. In this case, in the limit $\lambda \to 0$ the field T decomposes into a "proper" $[n-3, 1]$ dual graviton plus an additional field of type $[n-2, 1]$ that does not carry any local degrees of freedom.

This phenomenon can be described by introducing a suitable set of Stueckelberg fields. In the current example, following [42] one can introduce a new field, antisymmetric in its first three indices and traceless, implementing the shift

$$\hat{Y}^{bc}|_a \to \hat{Y}^{bc}|_a + \tfrac{1}{\lambda} \nabla_d W^{bcd}|_a \, , \qquad W^{abc}|_c \equiv 0 \, , \qquad (19)$$

either in the parent action (16) or in (17). This leads to the Lagrangian

$$\mathcal{L}_0[Y, W] = \tfrac{1}{\lambda^2} \Big[\tfrac{1}{2} \nabla_c W^{abc|d} \nabla^e W_{dbe|a} + \lambda \hat{Y}^{ab|c} \nabla^e W_{cbe|a}$$
$$+ \tfrac{\sigma}{2(n-2)} \nabla_b \hat{Y}^{ab|c} \nabla^d \hat{Y}_{cd|a} + \tfrac{\lambda^2}{2} \hat{Y}^{ab|c} \hat{Y}_{ac|b} \Big], \qquad (20)$$

that is invariant up to total derivatives under[3]

$$\delta \hat{Y}^{bc}|_a = \nabla_d \zeta^{bcd}|_a + \nabla_a \Lambda^{bc} + \tfrac{2}{n-1} \bar{e}_a{}^{[b} \nabla_d \Lambda^{c]d} + \tfrac{(n-3)\lambda}{\sigma} \chi^{bc}{}_a \, , \quad (21)$$

$$\delta W^{bcd}|_a = \nabla_e v^{bcde}|_a + \nabla_a \chi^{bcd} - \tfrac{3}{n-2} \bar{e}_a{}^{[b} \nabla_e \chi^{cd]e} - \lambda \zeta^{bcd}|_a \, . \quad (22)$$

Note that the new field can be gauged away using the shift symmetry generated by the traceless $\zeta^{bcd|a}$, while it also brings its own differential symmetries generated by $v^{bcde|a}$ (which is traceless and antisymmetric in the first four indices) and by the fully antisymmetric χ^{abc}.

Introducing the Hodge dual $C_{a[n-3]|b} = \tfrac{1}{3!} \epsilon_{a[n-3]cde} W^{cde}|_b$ (that is a $GL(n)$ Young-projected $[n-3, 1]$ field, since $W^{cde}|_b$ is traceless) and denoting $C'_{a[n-4]} = C_{a[n-4]|b}{}^b$ together with $T'_{a[n-3]} = T_{a[n-3]b}{}^b$, one obtains the dual Lagrangian

$$\mathcal{L}_0[C, T] = -\tfrac{1}{2\lambda^2(n-3)!} \left[\mathcal{L}[C] + \widehat{\mathcal{L}}_{\text{cross}} + \tfrac{\sigma}{(n-2)^2} \mathcal{I}[T] \right], \qquad (23)$$

where (denoting antisymmetrisations with repeated indices)

$$\mathcal{L}[C] = \nabla_a C_{c[n-3]|b} \nabla^a C^{c[n-3]|b} - \nabla_a C_{b[n-3]|}{}^a \nabla^c C^{b[n-3]|}{}_c$$
$$- (n-3) \Big[\nabla_a C'_{b[n-4]} \nabla^a C'^{b[n-4]} + \nabla_b C_{ab[n-4]|}{}^c \nabla^a C^{b[n-3]|}{}_c$$
$$- 2(-1)^n \nabla^a C_{b[n-3]|a} \nabla^b C'^{b[n-4]}$$
$$- (n-4) \nabla_b C'_{cb[n-5]} \nabla^c C'^{b[n-4]} \Big], \qquad (24)$$

$$\widehat{\mathcal{L}}_{\text{cross}} = 2\lambda \Big[T_{a[n-3]b|}{}^c \nabla^b C^{a[n-3]|}{}_c - T'_{a[n-3]} \nabla^b C^{a[n-3]|}{}_b$$
$$+ (-1)^n (n-3) T'_{a[n-3]} \nabla^a C'^{a[n-4]} \Big], \qquad (25)$$

and

$$\mathcal{I}[T] = \mathcal{L}[T] + \sigma (n-2) \lambda^2 \times$$
$$\times \Big[T^{a[n-2]|b} T_{a[n-2]|b} - (n-2) T'^{a[n-3]} T'_{a[n-3]} \Big]. \qquad (26)$$

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[2] Two-column, $GL(n)$-irreducible fields are denoted by $[p, q]$, where p and q stand for the lengths of the first and second column of the corresponding Young tableau, respectively.

[3] If one implements the Stueckelberg shift (19) already in the parent action (16), the vielbein acquires the new transformation $\delta_\chi h^{a|b} = \tfrac{n-3}{(n-2)\lambda} \nabla_c \chi^{abc}$.

The expression for $\mathcal{L}[T]$ is obtained from $\mathcal{L}[C]$ in (24) by replacing everywhere in the latter expression the symbols C and n by T and $n+1$, respectively.

Lagrangian (23) is invariant, up to total derivatives, under

$$\delta T_{a[n-2]|b} = (-1)^{n-1}(n-2)\left[\nabla_a \tilde{\zeta}_{a[n-3]|b} + (n-3)\sigma\lambda\, g_{ba}\, \tilde{\chi}_{a[n-3]} \right.$$
$$\left. + \tfrac{(-1)^{n-1}}{n-1}\left(\nabla_b \tilde{\Lambda}_{a[n-2]} + (-1)^{n-1}\nabla_a \tilde{\Lambda}_{a[n-3]b}\right)\right], \quad (27)$$

$$\delta C_{a[n-3]|b} = (-1)^{n-1}(n-3)\,\nabla_a \tilde{\upsilon}_{a[n-4]|b} - \lambda\, \tilde{\zeta}_{a[n-3]|b}$$
$$+ \tfrac{n-3}{n-2}\left(\nabla_b \tilde{\chi}_{a[n-3]} + (-1)^n \nabla_a \tilde{\chi}_{a[n-4]b}\right), \quad (28)$$

where the parameters $\tilde{\zeta}_{a[n-3]|b}$, $\tilde{\Lambda}_{a[n-2]}$, $\tilde{\upsilon}_{a[n-4]|b}$ and $\tilde{\chi}_{a[n-3]}$ are the Hodge duals of those entering the transformations (21) and (22) (the dualisation always involves only the group of anti-symmetrised indices). In the limit $\lambda \to 0$ the field T decouples and does not propagate any degrees of freedom, while one retains the gauge field $C_{a[n-3]|b}$, the dual graviton in flat space [4].

In a spacetime of any dimension $D > n$, the action (23) — featuring one of the two possible BMV couples of fields including $T_{a[n-2]|b}$ — would give a non-unitary propagation in dS. This is manifested by the σ-dependent relative sign between the kinetic terms that we obtained. In this specific case, the relative sign is irrelevant because Y is a topological field in flat space and, indeed, the massless theory is unitary for any value of the cosmological constant.

4.2. Partially-massless case

Partially-massless spin-2 fields exist for any non-vanishing values of the cosmological constant, although they are unitary only in dS [43]. To exhibit these facts, in this subsection we slightly modify our conventions, multiplying $\mathcal{L}^{(1)}$ by $-\sigma$. With this choice the factor σ in (8) is replaced by -1, so that one can reach the point $\mu = 0$ in both dS and AdS. This leads to the parent Lagrangian

$$\mathcal{L}_{\text{PM}}[h, Y, A, F] = h_{a|b}\,\mathcal{C}^{a|b} + \tfrac{\sigma}{\tilde{m}} A_a \nabla_b \mathcal{C}^{b|a}$$
$$- \tfrac{1}{2}\left(Y^{bc|a} Y_{ab|c} + \tfrac{1}{n-2} Y^{ab|}{}_b Y_{ac|}{}^c\right) - \tfrac{\sigma}{4} F_{ab} F^{ab}, \quad (29)$$

where we defined

$$\mathcal{C}^{a|b} = \nabla_c Y^{ac|b} - \tilde{m} F^{ab}, \qquad \tilde{m} = \pm\lambda\sqrt{\tfrac{n-2}{2}}. \quad (30)$$

In the conventions adopted in this subsection, the gauge symmetries of the action are

$$\delta h_a{}^b = \nabla_a \xi^b + \Lambda_a{}^b + \tfrac{2\tilde{m}}{n-2} \bar{e}_a{}^b \epsilon, \quad (31)$$

$$\delta Y^{bc|}{}_a = \nabla_a \Lambda^{bc} + 2\,\bar{e}_a{}^{[b} \nabla_d \Lambda^{c]d}, \quad (32)$$

$$\delta A_a = \nabla_a \epsilon + \sigma\,\tilde{m}\,\xi_a, \quad (33)$$

$$\delta F^{ab} = -2\,\sigma\,\tilde{m}\,\Lambda^{ab}. \quad (34)$$

In (29) we stressed that the fields $h_{a|b}$ and A_a are both Lagrange multipliers when $\mu = 0$ (although the constraint imposed by the latter field is not independent). The analysis of the partially-massless case therefore follows closely that of a massless graviton in flat space [4], rather than those presented in sections 4.1 and 4.3. The constraint $\mathcal{C}_{a|b} = 0$ is solved by

$$Y^{bc|}{}_a = \tfrac{1}{\lambda}\nabla_d W^{bcd|}{}_a - \tfrac{\sigma}{2\tilde{m}}\left(\nabla_a F^{bc} + 2\,\bar{e}_a{}^{[b}\nabla_d F^{c]d}\right), \quad (35)$$

where $W^{bcd|}{}_a$ has the same structure as the field introduced in the Stueckelberg shift (19). In particular, it is traceless. Substituting (35) in (29), one obtains

$$\mathcal{L}_{\text{PM}}[W] = -\tfrac{1}{2\lambda^2}\nabla_d W^{bcd|a}\nabla^e W_{abe|c}$$
$$+ \nabla_a\left(F^{ab}\nabla^c F_{bc} - F_{bc}\nabla^c F^{ab} + \tfrac{4\sigma\tilde{m}}{\lambda} F_{bc}\nabla_d W^{abd|c}\right). \quad (36)$$

This Lagrangian actually depends only on the field $W^{bcd|a}$: F^{ab} contributes only via a total derivative consistently with the shift symmetry (34). It is still invariant under

$$\delta W^{bcd|}{}_a = \nabla_e \upsilon^{bcde|}{}_a, \quad (37)$$

while the other differential symmetry that was present in the massless case (cf. (22)) is absent.

All gauge symmetries that the field $W^{bcd|a}$ and, consequently, its Hodge dual would display in flat space can be recovered by implementing the Stueckelberg shift

$$W^{bcd|}{}_a \to W^{bcd|}{}_a + \tfrac{\tilde{m}^{-1}}{n-3}\left(\nabla_a U^{bcd} - \tfrac{3}{n-2}\bar{e}_a{}^{[b}\nabla_e U^{cd]e}\right). \quad (38)$$

Substituting in (36) one obtains the Lagrangian

$$\mathcal{L}_{\text{PM}}[W, U] = -\tfrac{1}{2\lambda^2}\nabla_d W^{bcd|a}\nabla^e W_{abe|c} + \tfrac{\sigma}{\tilde{m}} U_{abc}\nabla_d W^{abd|c}$$
$$- \tfrac{\sigma}{2(n-2)\tilde{m}^2}\nabla_c U^{abc}\nabla^d U_{abd} - \tfrac{\lambda^2}{2\tilde{m}^2} U_{abc} U^{abc}, \quad (39)$$

which is invariant up to total derivatives under

$$\delta W^{bcd|}{}_a = \nabla_e \upsilon^{bcde|}{}_a + \nabla_a \chi^{bcd} - \tfrac{3}{n-2}\bar{e}_a{}^{[b}\nabla_e \chi^{cd]e} - \tfrac{\sigma\lambda^2}{\tilde{m}}\rho^{bcd}{}_a, \quad (40)$$

$$\delta U^{abc} = \nabla_d \rho^{abcd} - (n-3)\tilde{m}\,\chi^{abc}. \quad (41)$$

The contribution in ρ in (40) (that was absent in (22)) is necessary because, contrary to the massless case, the field U does not enter the action only via its divergence.

As in the massless case, the sign of one of the two kinetic terms depends on σ. This is consistent with the observation that, after Hodge dualisation, one obtains a BMV couple of fields which is unitary only in dS [40]. However, in this case both fields propagate in the flat limit: the $[n-3, 1]$ dual of W carries the spin-2 helicities, while the $[n-3]$ dual of U carries the spin-1 helicities. Consequently, the sign flip of a kinetic terms does matter: recovering the BMV couple of fields that is not-unitary in AdS is just another way to see that partially-massless fields are not unitary in AdS.

Using the dual field $C_{a[n-3]|b}$ defined as in the massless case and introducing the Hodge dual field $A_{a[n-3]} = \tfrac{1}{3!}\epsilon_{a[n-3]bcd}U^{bcd}$, the Stueckelberg Lagrangian we obtain for the dual partially massless spin-2 field in (A)dS$_n$ is

$$\mathcal{L}_{\text{PM}}[C, A] = \tfrac{-1}{2(n-3)!\lambda^2}\left[\mathcal{L}[C] - \tfrac{2\sigma\lambda^2}{(n-2)\tilde{m}^2}\mathcal{L}[A] + \tfrac{4\sigma\lambda^2}{\tilde{m}}\widetilde{\mathcal{L}}_{\text{cross}}\right], \quad (42)$$

where $\mathcal{L}[C]$ is given in (24),

$$\mathcal{L}[A] = \nabla_a A_{b[n-3]}\nabla^a A^{b[n-3]} - (n-3)\nabla^a A_{c[n-4]a}\nabla_b A^{c[n-4]b}$$
$$+ 3\sigma\lambda^2 A_{a[n-3]} A^{a[n-3]}, \quad (43)$$

and the cross terms are

$$\widetilde{\mathcal{L}}_{\text{cross}} = A^{a[n-3]}\left(\nabla_b C_{a[n-3]|}{}^b + (-1)^{n-1}(n-3)\nabla_a C'_{a[n-2]}\right). \quad (44)$$

The action is invariant under

$$\delta C_{a[n-3]|b} = (-1)^{n-1}(n-3)\left(\nabla_a \tilde{\upsilon}_{a[n-4]|b} - \tfrac{\sigma\lambda^2}{\tilde{m}} g_{ba}\,\tilde{\rho}_{a[n-4]}\right)$$
$$+ \tfrac{n-3}{n-2}\left(\nabla_b \tilde{\chi}_{a[n-3]} + (-1)^n \nabla_a \tilde{\chi}_{a[n-4]b}\right), \quad (45)$$

$$\delta A_{a[n-3]} = (n-3)\left((-1)^{n-1}\nabla_a \tilde{\rho}_{a[n-4]} - \tilde{m}\,\tilde{\chi}_{a[n-3]}\right), \quad (46)$$

where the parameters $\tilde{\upsilon}_{a[n-4]|b}$, $\tilde{\chi}_{a[n-3]}$ and $\tilde{\rho}_{a[n-4]}$ are the Hodge duals of those entering the transformations (40) and (41).

4.3. Massive case

We now consider the full Stueckelberg action presented in section 2. The elimination of the fields $h_{a|b}$, A_a and φ has been considered in [6,9]. In the spirit of our discussion of the special points $m = 0$ and $\mu = 0$, we complement these works by exhibiting a dual description with a smooth massless and flat limit. For generic values of m, $h_{a|b}$ is an auxiliary field and it can be eliminated through its equation of motion as in section 4.1. The field A_a is instead a Lagrange multiplier enforcing the constraint

$$\nabla_b F^{ba} - \tfrac{2m}{n-2} Y^{ab|}{}_b - \mu\,\pi^a = 0, \quad (47)$$

which can be solved by expressing π^a in terms of the other fields. The equation of motion of φ does not bring any new information, since it is not independent on account of the Noether identity associated with the gauge symmetry generated by ϵ.

Substituting the on-shell values of $h_{a|b}$ and π^a in the Stueckelberg Lagrangian leads to [6]

$$\mathcal{L}[\hat{Y}, F] = \tfrac{1}{\mu^2}\left[\tfrac{n-1}{n-2}\nabla_b \hat{Y}^{ab|c}\nabla^d \hat{Y}_{cd|a} + \tfrac{\mu^2}{2}\hat{Y}^{ab|c}\hat{Y}_{ac|b}\right. \quad (48)$$
$$\left. + \tfrac{1}{2}\nabla_b F^{ab}\nabla^c F_{ac} - \tfrac{2(n-1)m}{n-2} F_{ab}\nabla_c \hat{Y}^{bc|a} + \left(\tfrac{\mu^2}{4} - \tfrac{(n-1)m^2}{n-2}\right) F_{ab}F^{ab}\right],$$

where we recall that the parameters m and μ are related by (8). One can then consider the Hodge duals of the fields $\hat{Y}$ and F and obtain a dual theory for a massive graviton in terms of the Stueckelberg coupling of a massless spin-2 field (accounted by the $[n-2,1]$ dual of $\hat{Y}$) with a Proca field (accounted by the $[n-2]$ dual of F). Its gauge symmetries are those inherited from (6) and (10) after dualisation.

In order to obtain a smooth massless and flat limit, one should introduce two additional fields: the traceless $W^{bcd|}{}_a$ that we already encountered in section 4.1 and a 3-form U^{abc}. This will allow to recover all Curtright gauge symmetries for the Hodge dual of $\hat{Y}^{bc|}{}_a$ and the usual gauge symmetry for the massless $(n-2)$-form which is the Hodge dual of F^{ab}. Due to the coupling $F^{ab}h_{a|b}$ in (12), introducing the 3-form via a Stueckelberg shift of F^{ab} would modify the equation of motion for $h_{a|b}$ and, as a result, it would introduce second-order kinetic terms mixing U^{abc} with the fully antisymmetric projection of $Y^{ab|c}$. On the other hand, the shifts

$$Y^{bc|a} \rightarrow Y^{bc|a} + \tfrac{1}{\mu}\nabla_d W^{bcd|a} - \tfrac{m}{\mu} U^{abc}, \quad (49a)$$
$$F^{ab} \rightarrow F^{ab} + \tfrac{1}{\mu}\nabla_c U^{abc} \quad (49b)$$

do not modify the equation of motion for $h_{a|b}$ and therefore they cannot introduce any mixed kinetic term. The elimination of the fields $h_{a|b}$, A_a and φ then proceeds as above and one obtains the sum of the kinetic terms

$$\mathcal{K} = \tfrac{1}{\mu^2}\left[\tfrac{1}{2}\nabla_c W^{abc|d}\nabla^e W_{dbe|a} + \tfrac{n-1}{n-2}\nabla_b \hat{Y}^{ab|c}\nabla^d \hat{Y}_{cd|a}\right.$$
$$\left. + \tfrac{1}{4}\nabla_c U^{abc}\nabla^d U_{abd} + \tfrac{1}{2}\nabla_b F^{ab}\nabla^c F_{ac}\right], \quad (50)$$

with the cross couplings

$$\mathcal{L}_{\text{cross}}^{(1)} = \tfrac{1}{\mu}\left[\hat{Y}_{ab|c}\nabla_d W^{acd|b} + \tfrac{m}{\mu} U_{abc}\nabla_d W^{abd|c}\right.$$
$$\left. - \tfrac{2(n-1)m}{(n-2)\mu} F_{ab}\nabla_c \hat{Y}^{bc|a} + \tfrac{1}{2} F_{ab}\nabla_c U^{abc}\right] \quad (51)$$

and mass-like terms

$$\mathcal{L}_{\text{cross}}^{(2)} = \tfrac{1}{2}\hat{Y}^{ab|c}\hat{Y}_{ac|b} + \tfrac{m}{\mu}\hat{Y}^{ab|c}U_{abc}$$
$$- \tfrac{m^2}{2\mu^2} U^{abc}U_{abc} + \left(\tfrac{1}{4} - \tfrac{(n-1)m^2}{(n-2)\mu^2}\right) F^{ab}F_{ab}. \quad (52)$$

This Lagrangian is invariant up to total derivatives under the following gauge transformations[4]:

$$\delta W^{bcd|}{}_a = \nabla_e \upsilon^{bcde|}{}_a + \nabla_a \chi^{bcd} - \tfrac{3}{n-2}\bar{e}_a{}^{[b}\nabla_e \chi^{cd]e}$$
$$- \mu\,\zeta^{bcd|}{}_a - m\,\rho^{bcd}{}_a, \quad (53)$$

$$\delta \hat{Y}^{bc|}{}_a = \nabla_d \zeta^{bcd|}{}_a + \nabla_a \Lambda^{bc} + \tfrac{2}{n-1}\bar{e}_a{}^{[b}\nabla_d \Lambda^{c]d}$$
$$+ \tfrac{(n-3)\mu}{2(n-1)}\chi^{bc}{}_a - m\,\psi^{bc}{}_a, \quad (54)$$

$$\delta U^{abc} = \nabla_d \rho^{abcd} - \mu\,\psi^{abc} + \tfrac{2(n-3)m}{n-2}\chi^{abc}, \quad (55)$$

$$\delta F^{ab} = \nabla_c \psi^{abc} + 2m\,\Lambda^{ab}. \quad (56)$$

The action involving the Hodge duals of the previous fields now admits a smooth flat and massless limit, in which different helicities decouple. The spin-two ones are carried by the $[n-3,1]$ Hodge dual of W (as discussed in section 4.1), while spin-one and zero helicities are carried, respectively, by the fully-antisymmetric Hodge duals of U and F. We refrain from displaying this action explicitly, as it can straightforwardly be obtained by expressing all fields in terms of their Hodge duals in (50)–(52). One can also check that the appropriate Curtright gauge symmetries are recovered from (53)–(56) together with their gauge-for-gauge symmetries.

Acknowledgements

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4 Implementing the shift (49) before the elimination of $h_{a|b}$ etc. from the parent action does not modify the gauge transformations of A_a, φ and π^a. The variation of $h_{a|b}$ takes instead the same form as in the massless case and acquires a contribution $\delta_\chi h^{a|b} = \tfrac{n-3}{(n-2)\mu}\nabla_c \chi^{abc}$.

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