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Lorentz Covariant Spin Two Superspaces

Chandrashekar Devchand[1] and Jean Nuyts[2]

devchand@mis.mpg.de, nuyts@umh.ac.be

[1] *Max-Planck-Institut für Mathematik in den Naturwissenschaften Inselstraße 22-26, 04103 Leipzig, Germany*

[2] *Physique Théorique et Mathématique, Université de Mons-Hainaut 20 Place du Parc, 7000 Mons, Belgium*

Abstract

Superalgebras including generators having spins up to two and realisable as tangent vector fields on Lorentz covariant generalised superspaces are considered. The latter have a representation content reminiscent of configuration spaces of (super)gravity theories. The most general canonical supercommutation relations for the corresponding phase space coordinates allowed by Lorentz covariance are discussed. By including generators transforming according to every Lorentz representation having spin up to two, we obtain, from the super Jacobi identities, the complete set of quadratic equations for the Lorentz covariant structure constants. These defining equations for *spin two Heisenberg superalgebras* are highly overdetermined. Nevertheless, non-trivial solutions can indeed be found. By making some simplifying assumptions, we explicitly construct several classes of these superalgebras.

1 Introduction

The super-Poincaré algebra is the extension of the Lorentz algebra by the supersymmetry algebra, a $\mathbb{Z}_2$-graded extension of the algebra of translation vector fields by Grassmann-odd Lorentz spinors. The Grassmann-even subspace $\mathcal{A}_0$ of the supersymmetry algebra contains the spin one translation generator $X_{\alpha\dot{\beta}}$ transforming according to the $(\frac{1}{2}, \frac{1}{2})$ Lorentz representation and the odd subspace $\mathcal{A}_1$ contains the spin $\frac{1}{2}$ representations, $X_{\dot{\alpha}}$ and X_α, transforming as $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$. They satisfy

$$\left\{X_\alpha, X_{\dot{\beta}}\right\} = 2i\ X_{\alpha\dot{\beta}}\ , \tag{1}$$

with all other supercommutators equal to zero.

The possibility of extending the supersymmetry algebra to include generators of spin greater than one, and thus going beyond the Haag-Łopusanski-Sohnius barrier, was broached by Fradkin and Vasiliev [1, 2]. These authors were motivated by physical considerations to realise such higher-spin algebras on de Sitter space fields. Consistency of the dynamics required the inclusion of *all* spins, yielding infinite dimensional algebras realised on infinite chains of fields having spins all the way up to infinity.

The approach we have taken recently [3, 4] has been more abstract. We considered extentions of the supersymmetry algebra by further representations of the Lorentz group to those given above, maintaining the $\mathbb{Z}_2$-grading, with all integer-spin representations in the even-statistics (bosonic) subspace $\mathcal{A}_0$ and all half-integer-spin representations in the odd-statistics (fermionic) subspace $\mathcal{A}_1$. Insisting on Lorentz covariance determines the space of the a priori allowed structure constants and solutions of the super Jacobi identities then provide concrete examples of Lorentz covariant generalisations of the super-Poincaré algebra. From the work of Fradkin and Vasiliev, it is clear that finite-dimensional examples of these algebras do not have non-trivial realisations on fields in standard four-dimensional space. In [3, 4], however, the possibility was raised of realising these algebras on higher-dimensional extensions of four-dimensional space, generalising the idea of superspace. These *hyperspaces* provide natural representation spaces for finite-dimensional examples of higher-spin superalgebras.

Recall that the super-Poincaré algebra is realisable as a superalgebra of infinitesimal translation vector fields on superspace, the quotient of the super Poincaré group by the Lorentz group, with supercommuting coordinates $\{Y^\alpha, Y^{\dot{\alpha}}, Y^{\alpha\dot{\alpha}}\}$, having the same Lorentz-transformation properties and statistics as the corresponding supersymmetry generators. The action of the supersymmetry algebra on superfields depending on these coordinates is determined by the Heisenberg superalgebra with non-zero canonical supercommutation relations,

$$\begin{gathered}\left[X_{\alpha\dot{\alpha}}, Y^{\beta\dot{\beta}}\right] = \delta_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}}\ , \quad \left\{X_\alpha, Y^\beta\right\} = \delta_\alpha^\beta\ , \quad \left\{X_{\dot{\alpha}}, Y^{\dot{\beta}}\right\} = \delta_{\dot{\alpha}}^{\dot{\beta}} \\ \left[X_\alpha, Y^{\beta\dot{\beta}}\right] = i\ \delta_\alpha^\beta Y^{\dot{\beta}}\ , \qquad \left[X_{\dot{\alpha}}, Y^{\beta\dot{\beta}}\right] = i\delta_{\dot{\alpha}}^{\dot{\beta}} Y^\beta\ \ .\end{gathered} \tag{2}$$

It is this construction which we generalised in [3, 4]. In these papers explicit examples of Lorentz covariant hyperspaces $\mathcal{M}$ with sets of coordinates $\{Y(s, \dot{s})\}$ transforming according to more general $(s,\dot{s})$ representations than the traditional spinorial and vectorial ones were presented. Such hyperspaces are graded vector spaces and the coordinates $\{Y(s,\dot{s})\}$ span a supercommutative $\mathbb{Z}_2$-graded algebra, $\mathcal{V}{=}\mathcal{V}_0{+}\mathcal{V}_1$, with $\mathcal{V}_0$ (resp. $\mathcal{V}_1$) containing bosonic (resp. fermionic) coordinates with $2(s{+}\dot{s})$ even (resp. odd). Each set of coordinates $Y(s,\dot{s})$ included increases the bosonic (resp. fermionic) dimension of $\mathcal{M}$ by $(2s{+}1)(2\dot{s}{+}1)$. If p is the maximum value of $s{+}\dot{s}$ occurring, we call the hyperspace a *spin p superspace.*

Tangent spaces of these hyperspaces $\mathcal{M}$ are $\mathbb{Z}_2$-graded vector spaces $\mathcal{A}{=}\mathcal{A}_0{+}\mathcal{A}_1$ spanned by infinitesimal generalised translation vector fields $\{X(s,\dot{s})\}$. These vector spaces are required to be superalgebras (generalising the supersymmetry algebra); i.e. they have a supersymmetric bilinear map (super commutator), $[\,.\,,.\,]:\mathcal{A}\times\mathcal{A}\rightarrow\mathcal{A}$, satisfying the super Jacobi identities.

The vector fields $X\in\mathcal{A}$ act as superderivations on functions of the Y's. We assume that the action of $\mathcal{A}$ on $\mathcal{V}$ corresponds to a linear transformation; the combined vector space $\mathcal{G}{=}\mathcal{A}{+}\mathcal{V}$ having the supercommutation relations of a generalised Heisenberg superalgebra,

$$\begin{array}{rcccl}\mathcal{A}\times\mathcal{A} & \ni & (X,X') \mapsto & [X,X'] & \in\ \mathcal{A} \\ \mathcal{A}\times\mathcal{V} & \ni & (X,Y) \mapsto & [X,Y] & \in\ \mathcal{V}\ +\ \mathcal{C} \\ \mathcal{V}\times\mathcal{V} & \ni & (Y,Y') \mapsto & [Y,Y'] & =\ 0\ .\end{array} \tag{3}$$

Here, $\mathcal{C}$ is a space of central charges determined by a pairing between $\mathcal{A}$ and $\mathcal{V}$ (see section 2.1). The super Jacobi identities are satisfied and the combined grading is preserved, i.e.

$$[\mathcal{A}_\alpha,\mathcal{A}_\beta]\subset\mathcal{A}_{\alpha+\beta}\ ,\quad [\mathcal{A}_\alpha,\mathcal{V}_\beta]\subset\mathcal{V}_{\alpha+\beta}+\mathcal{C}\delta_{\alpha,\beta}\ ,\ \text{with}\ \alpha,\beta\in\mathbb{Z}_2\ . \tag{4}$$

We call the algebra $\mathcal{G}$ with relations (3) a *spin p Heisenberg superalgebra* if p is the maximum value of $s{+}\dot{s}$ amongst the representations appearing in $\mathcal{A}{+}\mathcal{V}$.

If the elements appearing in the supercommutators in (3) transform respectively as $(s,\dot{s})$ and $(r,\dot{r})$ representations, Lorentz covariance requires that the *a priori* elements on the right-hand-sides transform according to $(v,\dot{v})$ representations occurring in the double Clebsch-Gordon decomposition of the direct product of the two Lorentz representations. Namely,

$$\begin{aligned} s\otimes r &= \sum\oplus\, v \;=\; (s{+}r)\oplus(s{+}r{-}1)\oplus\ldots\oplus|s-r| \\ \dot{s}\otimes\dot{r} &= \sum\oplus\, \dot{v} \;=\; (\dot{s}{+}\dot{r})\oplus(\dot{s}{+}\dot{r}{-}1)\oplus\ldots\oplus|\dot{s}-\dot{r}|\quad . \end{aligned} \tag{5}$$

With this algebraic structure, the hyperspaces $\mathcal{M}$ provide higher dimensional spaces having manifest four-dimensional Lorentz covariance. They are modeled on standard superspace used in supersymmetric field theories. Explicit examples of algebras $\mathcal{G}$ were presented in [3, 4] for spins $s{+}\dot{s}$ up to $\frac{3}{2}$. As an application gauge fields on $\mathcal{M}$ were considered: Associating a gauge potential A to each of these generalised derivatives, we defined, in a natural fashion, the covariant derivative $\mathcal{D}{=}X{+}A$ and the corresponding curvature tensors F. This allowed us to define generalised self-dualities in terms of Lorentz covariant constraints on components of the curvature. We thus obtained a different class of higher-dimensional generalisations of the self-duality equations to those presented in [5], having manifest four-dimensional Lorentz covariance and affording generalised twistor-like transforms. Moreover, a novel hierarchy of *light-like integrable systems* was also presented, whose simplest non-trivial member is the well-known $N{=}3$ super-Yang-Mills set of on-shell curvature constraints. These systems therefore provide an infinitely large hierarchy of gauge- and Lorentz-covariant solvable systems.

The purpose of this paper is to highlight another setting for the application of these higher-spin algebras. Our hyperspaces $\mathcal{M}$ in fact serve as models for configuration spaces, or for moduli spaces of solutions, of Lorentz invariant field theories; and the supercommutations relations for $\mathcal{G}$ provide canonical supercommutation relations for the corresponding phase spaces, providing an algebraic description of the local symplectic structure. With this application in mind, it is clear that algebras including spins up to two are of possible relevance for the canonical quantisation of gravity and supergravity theories. In [3, 4], a rather simple non-trivial example containing generators of spins $\frac{1}{2}, 1$ and $\frac{3}{2}$ was presented. In order to search for possibly interesting examples containing spin 2 generators, we use a modified notation to that of [3, 4], which is more convenient for the extraction of the complete set of algebraic equations for the structure constants from the super Jacobi identities. We describe these in section 2. In section 3, we restrict ourselves to superalgebras $\mathcal{G}$ containing all Lorentz tensors having spin less than or equal to 2. For unit multiplicity of each Lorentz representation $(s,\dot{s})$ for $0 \leq s{+}\dot{s} \leq 2$, we obtain, for the superalgebra $\mathcal{A}$, 1993 quadratic equations for 163 structure constants, which are supplemented by further 5732 quadratic equations for a total of $163 + 339$ structure constants and 15 central charges. Each solution of this overdetermined system of equations corresponds to a specific example of a spin 2 Heisenberg superalgebra $\mathcal{G}$; and we present some classes of solutions in section 4. In fact, the space of 163+339+15=517 structure constants subject to 1993+5732=7725 quadratic equations parametrises the moduli space of spin 2 Heisenberg superalgebras.

Remarks:

a) Our setting is basically complex: We consider representations of the complex extension of the Lorentz algebra, $so(4,\mathbb{C}) = sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$. This yields higher spin superalgebras with or without the 'chiral' symmetry interchanging dotted and undotted indices, which for the real Lorentzian case is an automatic consequence of complex conjugation. The broader complex framework thus affords more general possibilities, which may be of relevance in concrete physical settings requiring covariance under a Euclidean ($so(4) = su(2) \oplus su(2)$), Lorentzian ($so(3,1) = sl(2,\mathbb{C})$) or Kleinian ($sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$) real six-dimensional subalgebras of $so(4,\mathbb{C})$.

b) Although we remain in the realm of supercommutative geometry, with $[\mathcal{V},\mathcal{V}]{=}0$, a generalisation to non-supercommutative geometry is clearly a further possibility, with the simplest superalgebra variant having $[\,.\,,\,.\,] : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ such that $[\mathcal{V}_\alpha, \mathcal{V}_\beta] \subset \mathcal{V}_{\alpha+\beta}$. Further generalisations, replacing this superalgebra structure, for instance, by q-deformed supercommutation relations, may also be considered along the lines of the present investigation.

c) In this paper, we will consider an element of $\mathcal{A}, \mathcal{V}$ to be of bosonic type if its spin $(s{+}\dot{s})$ is an integer and of fermionic type if its spin is a genuine half-integer; and we use shall assume the corresponding statistics. We note, however, that the assignment of even (resp. odd) statistics to elements of $\mathcal{A}_0, \mathcal{V}_0$ (resp. $\mathcal{A}_1, \mathcal{V}_1$) is a purely conventional one, motivated by the spin-statistics theorem. This can indeed be lifted, if required, to yield Lie algebra (rather than superalgebra) extensions of the Poincaré algebra containing integer and half-integer spin elements, all of even statistics. Such algebras maintain, nevertheless, their $\mathbb{Z}_2$-graded nature [6]. Such a variant of the supersymmetry algebra was recently shown to be the target space symmetry of the N=2 string [7] and the space of string physical states was shown to be elegantly and compactly describable in terms of a field on a hyperspace with a vectorial and an even-spinorial coordinate.

d) In the context of application of our formalism to canonical quantisation of (super) gravity theories, we have only considered the simplest putative phase space coordinates: a metric represented by canonically conjugate variables X, Y transforming as $(0,0)+(1,1)$ coupled to single copies of other representations. A generalisation to higher multiplicities ($N > 1$ supergravities) follows on the lines of the discussion in appendix A of [4], with the variables acquiring a further 'internal' index labeling the inequivalent copies of any particular representation thus: $\{X(s,\dot{s})\}$, $\{Y(s,\dot{s})\} \rightarrow \{X(s,\dot{s}\,;n)\}$, $\{Y(s,\dot{s}\,;m)\}$.

e) A further interesting generalisation is to matrix-indexed variables X, Y. This is clearly a variant of the above-mentioned higher-multiplicity generalisation, with the variables having additional indices labeling a space of internal matrices rather than internal vectors. For instance, Ashtekar's canonical variables for gravity consist of $sl(2,\mathbb{C})$-indexed $X(1,0), Y(1,0)$.

2 Higher-spin superalgebras

In this section we give a more precise definition of *spin p Heisenberg superalgebras* $\mathcal{G}$. For simplicity, we restrict ourselves to the case of unit multiplicity of any tensor with given Lorentz behaviour. A generalisation to 'N-extended' cases is, in principle, straightforward; based on the discussion in the Appendix of [4].

Let us denote by Λ_p some freely specifiable lattice of doublets of half-integers $(s,\dot{s})$,

$$\Lambda_p = \{(s,\dot{s})\} \ \subset \mathbb{K}^2 \quad , \quad \text{with} \quad p = \max\{s+\dot{s}\} \quad , \tag{6}$$

where $\mathbb{K} = \frac{1}{2}\mathbb{N} \cup \{0\}$, the set of non-negative half-integers.

2.1 A basis for $\mathcal{G}$=$\mathcal{A}$+$\mathcal{V}$

For any point $(s,\dot{s})$ in Λ_p, consider the coordinate tensor $Y(s,\dot{s})$ transforming according to the $(s,\dot{s})$ representation of the Lorentz group. We label its $(2s+1)(2\dot{s}+1)$ components as $Y(s,s_3\,;\dot{s},\dot{s}_3)$, where s_3 (resp. $\dot{s}_3$) run from $-s$ to s (resp. from $-\dot{s}$ to $\dot{s}$) in integer steps.

We define the span of coordinates $\{Y(s,s_3\,;\dot{s},\dot{s}_3)\}$, for all $(s,\dot{s})$ in the chosen set Λ_p, to be a basis of the vector space $\mathcal{V}$. This provides a coordinate system for Lorentz-covariant spin p hyperspaces $\mathcal{M}$. The corresponding tangent space $\mathcal{A}$ is spanned by the components $\{X(s,s_3\,;\dot{s},\dot{s}_3)\}$ of vector fields $\{X(s,\dot{s})\}$. These vector fields are taken to be in one-to-one correspondence with the coordinate tensors, and therefore with the points on the lattice Λ_p. The specific choice of points making up this lattice therefore effectively determines the basis elements of both vector spaces $\mathcal{A}$ and $\mathcal{V}$. So, for instance, the superalgebra (1),(2) is based on the lattice of three points $\Lambda_p = \{(0,\frac{1}{2}),(\frac{1}{2},0),(\frac{1}{2},\frac{1}{2})\}$. Between vector field $X(s,\dot{s})$ and coordinate tensor $Y(s,\dot{s})$, which transform similarly under the Lorentz group, we assume a bilinear pairing given by

$$\begin{aligned} &< X(s,s_3\,;\dot{s},\dot{s}_3), Y(r,r_3\,;\dot{r},\dot{r}_3) > \\ &= \; c(s,\dot{s})\; C(s,s_3,s,-s_3\,;0,0)\; C(\dot{s},\dot{s}_3,\dot{s},-\dot{s}_3\,;0,0)\; \delta_{sr}\,\delta_{\dot{s}\dot{r}}\,\delta_{s_3+r_3,0}\,\delta_{\dot{s}_3+\dot{r}_3,0} \quad . \end{aligned} \tag{7}$$

Here the Clebsch-Gordon coefficients $C(s,s_3,s,-s_3\,;0,0)$ denote Wigner's 'metric' invariant in the representation space of fixed spin s (see e.g. [8]). This defines a pairing map

$$< .\,,\,. >: \mathcal{A}\times\mathcal{V}\rightarrow\mathcal{C} \;=\; \{c(s,\dot{s})\;;\;(s,\dot{s})\in\Lambda_p\}\;. \tag{8}$$

The coefficients $c(s,\dot{s})$, which will be called the central structure constants, can, in principle, be zero or, if non-zero, can be set to 1 by a suitable renormalisation of the X's and/or the Y's (provided, as is the case here, that representations do not occur multiply). Thus, $c\,:\,\Lambda_p\rightarrow\mathbb{Z}_2 = \{0,1\}$. We shall henceforth, without loss of generality, assume that the c's are thus renormalised.

The above way of representing the $(2s+1)(2\dot{s}+1)$ components of an $(s,\dot{s})$-tensor is equivalent to the representation in standard two-spinor index notation with $2s$ (resp. $2\dot{s}$) symmetrised undotted (resp. dotted) indices, e.g. $X_{\alpha\ldots\alpha_{2s}}^{\dot{\alpha}\ldots\dot{\alpha}_{2\dot{s}}}$. The two-spinor notation, which was used in [3, 4], has the advantage of having the (double) Clebsch-Gordon decomposition readily expressible in terms of products of the invariant two-index ϵ-tensors, viz. $\epsilon^{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$. Obtaining the complete set of quadratic defining conditions for the structure constants, however, is not a very straightforward procedure. In the above alternative non-index notation, the quadratic equations may be found using a purely algorithmic procedure using the super Jacobi identities and explicit values for Clebsch-Gordan coefficients and $6j$ symbols from e.g. [8]. The extraction of *all* the quadratic conditions for the structure constants is then streamlined, allowing automation of the procedure using a symbolic manipulation language like REDUCE or MAPLE. Once a solution of these equations is found, the supercommutation relations for the algebra $\mathcal{G}$ may be written immediately in either notation. The correspondence between components in the two notations may easily be established. For instance, the index-notation component of $X(s,\dot{s})$ with n (resp. $\dot{n}$) indices taking the value 1 (resp. $\dot{1}$), with the remaining $2s{-}n$ undotted (resp. $2\dot{s}{-}\dot{n}$ dotted) indices taking the value 2 (resp. $\dot{2}$), denoted $X(1^n 2^{(2s-n)}\dot{1}^{\dot{n}}\dot{2}^{(2\dot{s}-\dot{n})})$, is related to the $s_3{=}n{-}s$, $\dot{s}_3{=}\dot{n}{-}\dot{s}$ component thus:

$$X(1^n 2^{(2s-n)}\dot{1}^{\dot{n}}\dot{2}^{(2\dot{s}-\dot{n})}) = P(s,\dot{s})\sqrt{(n!(2s-n)!\dot{n}!(2\dot{s}-\dot{n})!)}X(s,n-s;\dot{s},\dot{s}-\dot{n})\;, \tag{9}$$

where $P(s,\dot{s})$ is an arbitrary normalisation.

2.2 The supercommutation relations

We take the entire set of coordinates $\{Y(s, s_3\,; \dot{s}, \dot{s}_3)\}$, for all $(s, \dot{s}) \in \Lambda_p$, to be supercommutative:

$$[\ Y(s, s_3\,; \dot{s}, \dot{s}_3)\ ,\ Y(r, r_3\,; \dot{r}, \dot{r}_3)\ ]_{S\bullet R} = 0\ , \tag{10}$$

where we introduce the shorthand notation

$$S = (s, \dot{s})\ , \quad R = (r, \dot{r})\ , \tag{11}$$

in terms of which the sign of the graded bracket is defined as

$$S \bullet R = R \bullet S = (-1)^{4(s+\dot{s})(r+\dot{r})+1}\ . \tag{12}$$

We define another lattice in two dimensions, $\Gamma(S, R) \subset \mathbb{K}^2$, to be the set of representation labels $(v, \dot{v})$ arising in the Clebsch-Gordon product (5) of $(s, \dot{s})$ with $(r, \dot{r})$, namely,

$$\Gamma(S, R) = \{\ (v, \dot{v})\ ; \quad v \in \gamma(s, r)\ , \quad \dot{v} \in\ \gamma(\dot{s}, \dot{r})\}\ . \tag{13}$$

Here we denote by $\gamma(s, r) \subset \mathbb{K}$ the set of integers or half-integers arising in any single Clebsch-Gordon series,

$$\gamma(s, r)\ =\ \{\ s{+}r\ ,\ s{+}r{-}1\ ,\ \ldots,\ |s - r|\ \}\ , \tag{14}$$

so that the lattice in (13) $\Gamma(S, R) = \gamma(s, r) \otimes \gamma(\dot{s}, \dot{r})$.

We postulate that the vector fields $\{X(s, s_3; \dot{s}, \dot{s}_3)\}$ generate a Lorentz covariant superalgebra $\mathcal{A}$. The most general supercommutation relations allowed by Lorentz covariance have the form

$$\begin{aligned} &[\ X(s, s_3\,; \dot{s}, \dot{s}_3)\ ,\ X(r, r_3\,; \dot{r}, \dot{r}_3)\ ]_{S\bullet R} \\ =\ &\sum_{(v,\dot{v})\in\Gamma(S,R)\cap\Lambda_p} C(s, s_3, r, r_3\,; v, s_3 + r_3)\ C(\dot{s}, \dot{s}_3, \dot{r}, \dot{r}_3\,; \dot{v}, \dot{s}_3 + \dot{r}_3) \\ &\qquad\qquad \times t(s, \dot{s}\,, r, \dot{r}\,, v, \dot{v})\ X(v, s_3 + r_3\,; \dot{v}, \dot{s}_3 + \dot{r}_3)\quad . \end{aligned} \tag{15}$$

Here the Clebsch-Gordan coefficients $C(s, s_3, r, r_3\,; v, s_3 + r_3)$ have the symmetry property

$$C(s, s_3, r, r_3\,; v, s_3 + r_3)\ =\ (-1)^{s+r-v} C(r, r_3, s, s_3\,; v, s_3 + r_3)\quad . \tag{16}$$

The super Jacobi identities for the supercommutation relations (15) yield quadratic equations for the set of admissible structure constants $t(s, \dot{s}\,, r, \dot{r}\,, v, \dot{v})$. Solutions then define superalgebras $\mathcal{A}$. For any choice of Λ_p the admissible structure constants depend on six spin variables, integers or half-integers specifying a lattice of points in six dimensions, $\Omega_p \subset \mathbb{K}^6$, defined by

$$\Omega_p\ =\ \{\ (s, \dot{s}\,, r, \dot{r}\,, v, \dot{v})\ ; \quad (s, \dot{s}), (r, \dot{r}) \in \Lambda_p\ ,\ (v, \dot{v}) \in \Gamma(S, R) \cap \Lambda_p\ \} \tag{17}$$

The space of these structure constants is manifestly restricted by superskewsymmetry, namely,

$$t(r, \dot{r}\,, s, \dot{s}\,, v, \dot{v})\ =\ (-1)^{4(s+\dot{s})(r+\dot{r})+(s+\dot{s})+(r+\dot{r})-(v+\dot{v})+1}\, t(s, \dot{s}\,, r, \dot{r}\,, v, \dot{v})\quad . \tag{18}$$

This redundancy in the set of structure constants may be factored out with no loss of generality by imposing the restriction $\Omega_p \,|_{S\leq R}$, where the ordering $S \leq R$ denotes $s{+}\dot{s} \leq r{+}\dot{r}$ and for equality, $s \leq r$. Equation (18) also implies that certain parameters vanish, viz.,

$$t(s, \dot{s}\,, s, \dot{s}\,, v, \dot{v}) = 0 \quad \text{if} \quad 4(s{+}\dot{s})^2{+}2(s{+}\dot{s}){-}(v{+}\dot{v}){+}1 = 1 \bmod 2 \;, \tag{19}$$

i.e. if both $2(s{+}\dot{s})$ and $2(s{+}\dot{s}) - (v{+}\dot{v})$ are either even or odd.

We require that the vector space $\mathcal{V}$ carries a linear representation of $\mathcal{A}$. This then allows realisation of this superalgebra by vector fields satisfying (15) and acting as superderivations on functions of the Y's. The most general Lorentz covariant supercommutation relations between the X's and the Y's consistent with this requirement take the form

$$\begin{aligned}
&[\; X(s, s_3\,; \dot{s}, \dot{s}_3), Y(r, r_3\,; \dot{r}, \dot{r}_3) \;]_{S\bullet R} \\
&= \sum_{(v,\dot{v})\in\Gamma(S,R)\cap\Lambda_p} C(s, s_3, r, r_3\,; v, s_3 + r_3)\, C(\dot{s}, \dot{s}_3, \dot{r}, \dot{r}_3\,; \dot{v}, \dot{s}_3 + \dot{r}_3) \\
&\qquad\qquad \times u(s, \dot{s}\,, r, \dot{r}\,, v, \dot{v})\; Y(v, s_3 + r_3\,; \dot{v}, \dot{s}_3 + \dot{r}_3) \\
&\quad + \; C(s, s_3, s, -s_3\,; 0, 0)\; C(\dot{s}, \dot{s}_3, \dot{s}, -\dot{s}_3\,; 0, 0)\; c(s, \dot{s})\; \delta_{sr}\, \delta_{\dot{s}\dot{r}}\, \delta_{s_3+r_3,0}\, \delta_{\dot{s}_3+\dot{r}_3,0} \quad .
\end{aligned} \tag{20}$$

The u's are further structure constants, also depending on the lattice Ω_p, i.e. $u : \Omega_p \rightarrow \mathbb{C}$. They have no a priori symmetry properties under interchange of points on Ω_p . The X's thus transform the Y's linearly amongst themselves and the combined vector space $\mathcal{G}{=}\mathcal{A}{+}\mathcal{V}$ forms an enlarged superalgebra if the structure constants $\{t, u, c\}$ are subject to the quadratic equations tantamount to the satisfaction of the super Jacobi identities amongst the X's and the Y's.

2.3 The super Jacobi identities

Refining the shorthand notation (11),

$$\overline{S} = \{s, s_3\,; \dot{s}, \dot{s}_3\} \;, \quad \overline{R} = \{r, r_3\,; \dot{r}, \dot{r}_3\} \;, \quad \text{etc.,} \tag{21}$$

the super Jacobi identities for any three operators $A(\overline{S})$, $B(\overline{R})$ and $C(\overline{V})$ are given by

$$\begin{aligned}
&\Big[\Big[A(\overline{S}), B(\overline{R})\Big]_{S\bullet R}, C(\overline{V})\Big]_{(S+R)\bullet V} \\
&-(S \bullet R)\,\Big[B(\overline{R}), \Big[A(\overline{S}), C(\overline{V})\Big]_{S\bullet V}\Big]_{(S+V)\bullet R} \\
&-\Big[A(\overline{S}), \Big[B(\overline{R}), C(\overline{V})\Big]_{R\bullet V}\Big]_{(R+V)\bullet S} \quad = \quad 0
\end{aligned} \tag{22}$$

Since the Y's supercommute (10), the only non trivial (not automatically satisfied) super Jacobi identities are the ones for three X's and for two X's and a Y.

For any particular choice of $\overline{S}, \overline{R}, \overline{V}$ the identity (22) yields, in general, several equations for the structure constants t, u and c, since the coefficients of all the linearly independent tensors have to vanish. These can be determined by using the recoupling or $6j$-symbols. In particular, we require the formula (see ([8], eq.(36), p.261))

$$\begin{aligned} & C(b,b_3,a,a_3\,;e,e_3)\,C(c,c_3,e,e_3\,;f,f_3) \qquad (23) \\ &= \sum_{s,s_3}(-1)^{2f}\begin{Bmatrix} a & b & e \\ c & f & s \end{Bmatrix}(2e+1)^{\frac{1}{2}}(2s+1)^{\frac{1}{2}}\,C(b,b_3,c,c_3\,;s,s_3)\,C(a,a_3,s,s_3\,;f,f_3) \end{aligned}$$

where the indices are restricted to their obvious allowed ranges, viz., on the left side,

$$\begin{aligned} e &\in \gamma(a,b) \quad , \quad e_3 = a_3+b_3 \\ f &\in \gamma(c,e) \quad , \quad f_3 = c_3+e_3 = a_3+b_3+c_3 \end{aligned} \qquad (24)$$

specifying the domain of definition of the $6j$-symbols; and on the right side, determining the ranges of the s, s_3 summation,

$$s \in \gamma(b,c)\cap\gamma(f,a) \quad , \quad s_3 = b_3+c_3 \quad . \qquad (25)$$

The super Jacobi identities between $X(\overline{S})$, $X(\overline{R})$ and $X(\overline{V})$ yields quadratic equations for the structure constants t in (15). Using (15) and (23) we obtain
<u>*the tt-equations:*</u>

$$\begin{aligned} & t(s,\dot{s}\,,r,\dot{r}\,,e,\dot{e})\,t(e,\dot{e}\,,v,\dot{v}\,,f,\dot{f}) \\ - & \sum_{g,\dot{g}}(S\bullet R)(-1)^{e+v+f+\dot{e}+\dot{v}+\dot{f}}\sqrt{(1+2g)(1+2e)(1+2\dot{g})(1+2\dot{e})} \\ & \times\begin{Bmatrix} v & s & g \\ r & f & e \end{Bmatrix}\begin{Bmatrix} \dot{v} & \dot{s} & \dot{g} \\ \dot{r} & \dot{f} & \dot{e} \end{Bmatrix} t(s,\dot{s}\,,v,\dot{v}\,,g,\dot{g})\,t(r,\dot{r}\,,g,\dot{g}\,,f,\dot{f}) \\ - & \sum_{h,\dot{h}}(-1)^{s+r+v+f+\dot{s}+\dot{r}+\dot{v}+\dot{f}}\sqrt{(1+2h)(1+2e)(1+2\dot{h})(1+2\dot{e})} \\ & \times\begin{Bmatrix} v & r & h \\ s & f & e \end{Bmatrix}\begin{Bmatrix} \dot{v} & \dot{r} & \dot{h} \\ \dot{s} & \dot{f} & \dot{e} \end{Bmatrix} t(r,\dot{r}\,,v,\dot{v}\,,h,\dot{h})\,t(s,\dot{s}\,,h,\dot{h}\,,f,\dot{f}) \quad = \quad 0\;. \end{aligned} \qquad (26)$$

These equations are to be imposed for every $S,R,V\in\Lambda_p$, corresponding to the three operators appearing in the super-Jacobi identities (22), and for every possible intermediate and final-state indices, viz.,

$$\begin{aligned} (e,\dot{e}) = E &\in \Gamma(S,R)\cap\Lambda_p \\ (f,\dot{f}) = F &\in \Gamma(E,V)\cap\Lambda_p \quad . \end{aligned} \qquad (27)$$

The ranges of the summations in (26) are given by

$$\begin{aligned} (g,\dot{g}) = G &\in \Gamma(S,V)\cap\Gamma(R,F)\cap\Lambda_p \\ (h,\dot{h}) = H &\in \Gamma(R,V)\cap\Gamma(S,F)\cap\Lambda_p \quad . \end{aligned} \qquad (28)$$

Interchanging the indices S, R and V clearly does not produce independent equations, so that these indices need to be restricted by some ordering, e.g. $S\le R\le V$. The space of parameters $t(s,\dot{s}\,,r,\dot{r}\,,v,\dot{v})$, with domain given by (17) and subject to (18) and (26) is the parameter space of superalgebras $\mathcal{A}$.

The super Jacobi identities between operators $X(\overline{S}), X(\overline{R})$ and $Y(\overline{V})$ yield
the tu-equations:

$$
\begin{aligned}
& t(s,\dot{s}\,,r,\dot{r}\,,e,\dot{e})\,u(e,\dot{e}\,,v,\dot{v}\,,f,\dot{f}) \\
& -\sum_{g,\dot{g}}(S\bullet R)(-1)^{e+v+f+\dot{e}+\dot{v}+\dot{f}}\sqrt{(1+2g)(1+2e)(1+2\dot{g})(1+2\dot{e})} \\
& \qquad \times \begin{Bmatrix} v & s & g \\ r & f & e \end{Bmatrix} \begin{Bmatrix} \dot{v} & \dot{s} & \dot{g} \\ \dot{r} & \dot{f} & \dot{e} \end{Bmatrix} u(s,\dot{s}\,,v,\dot{v}\,,g,\dot{g})\,u(r,\dot{r}\,,g,\dot{g}\,,f,\dot{f}) \\
& -\sum_{h,\dot{h}}(-1)^{s+r+v+f+\dot{s}+\dot{r}+\dot{v}+\dot{f}}\sqrt{(1+2h)(1+2e)(1+2\dot{h})(1+2\dot{e})} \\
& \qquad \times \begin{Bmatrix} v & r & h \\ s & f & e \end{Bmatrix} \begin{Bmatrix} \dot{v} & \dot{r} & \dot{h} \\ \dot{s} & \dot{f} & \dot{e} \end{Bmatrix} u(r,\dot{r}\,,v,\dot{v}\,,h,\dot{h})\,u(s,\dot{s}\,,h,\dot{h}\,,f,\dot{f}) \quad = \quad 0 \quad . \qquad (29)
\end{aligned}
$$

These equations hold for every $S, R, V \in \Lambda_p$, with an ordering $S \leq R$, and every allowed E, F given in (27). The summations are again over values of G, H in (28). The $(X(\overline{S}), X(\overline{R}), Y(\overline{V}))$-identities also yield
the tuc-equations:

$$
\begin{aligned}
& t(s,\dot{s}\,,r,\dot{r}\,,v,\dot{v})\,c(v,\dot{v}) \\
& \quad -(S\bullet R)(-1)^{2r+2\dot{v}}\sqrt{(2r+1)(2r+1)(2\dot{r}+1)(2\dot{v}+1)} \\
& \quad \times \begin{Bmatrix} v & s & r \\ r & 0 & v \end{Bmatrix} \begin{Bmatrix} \dot{v} & \dot{s} & \dot{r} \\ \dot{r} & 0 & \dot{v} \end{Bmatrix} u(s,\dot{s}\,,v,\dot{v}\,,r,\dot{r})\,c(r,\dot{r}) \\
& \quad -(-1)^{s+r+v+\dot{s}+\dot{r}+\dot{v}}\sqrt{(2s+1)(2r+1)(2\dot{s}+1)(2\dot{v}+1)} \\
& \quad \times \begin{Bmatrix} v & r & s \\ s & 0 & v \end{Bmatrix} \begin{Bmatrix} \dot{v} & \dot{r} & \dot{s} \\ \dot{s} & 0 & \dot{v} \end{Bmatrix} u(r,\dot{r}\,,v,\dot{v}\,,s,\dot{s})\,c(s,\dot{s}) \quad = \quad 0 \qquad (30)
\end{aligned}
$$

for every $S, R, V \in \Lambda_p$. Again, an ordering $S \leq R$ yields independent equations, which are in one-to-one correspondence with the number of inequivalent non-zero t's.

The space of parameters $t(s,\dot{s}\,,r,\dot{r}\,,v,\dot{v})$ and $u(s,\dot{s}\,,r,\dot{r}\,,v,\dot{v})$, with domain given by Ω_p and the t-equivalences (18) and zeroes (19) factored out, together with the central charges $c(s,\dot{s})$, with $S \in \Lambda_p$, subject to the quadratic constraints (26),(29) and (30), is the moduli space of spin p Heisenberg superalgebras. Particular solutions of the equations (26),(29) and (30) correspond to examples of superalgebras $\mathcal{G}$. In section 4, we construct some explicit classes of solutions for values of spin up to 2, i.e. for various choices of Λ_2 .

3 Superalgebras $\mathcal{G}$ including elements of spin up to two

With the values of $6j$-symbols taken from the tables of [8], we have used REDUCE to generate the complete set of quadratic equations for the structure constants $\{t, u, c\}$ of superalgebras $\mathcal{G}$ containing elements $\{X(s, \dot{s}), Y(s, \dot{s})\}$ for spins up to $s+\dot{s} = 2$, i.e. with generators having indices spanning the largest Λ_2 lattice,

$$\Lambda_2^{max} = \{(s, \dot{s})\; ;\; s, \dot{s} \in \mathbb{K}\; ,\; 0 \leq s+\dot{s} \leq 2\}\; . \tag{31}$$

Specifically, the representations we have taken into account, with multiplicity one, are

1. spin 0
   - a scalar $(0, 0)$. The variable $Y(0, 0)$ could correspond to the trace of the space time metric $g_{\mu\nu}$, representing the conformal weight. The corresponding vector field $X(0, 0)$ generally behaves like a dilatation operator, generating Weyl rescalings of the metric.
2. spin $\frac{1}{2}$
   - a Dirac spinor $(\frac{1}{2}, 0) + (0, \frac{1}{2})$ corresponding to the variables usually added to the Minkowski space (Lorentz vector) variables to construct standard superspace. These could represent Dirac spinor degrees of freedom coupled to gravity.
3. spin 1
   - a Lorentz vector $(\frac{1}{2}, \frac{1}{2})$. In standard superspace, the Minkowski space variables, these could correspond to Maxwell degrees of freedom.
   - $(1, 0) + (0, 1)$ representations corresponding to self- and anti-self-dual halves of an antisymmetric 4×4 matrix. The tangent vector fields may, for certain specific examples, be chosen to be the generators of the Lorentz group.
4. spin $\frac{3}{2}$
   - a Rarita-Schwinger representation $(1, \frac{1}{2}) + (\frac{1}{2}, 1)$. The corresponding X's and Y's are the putative gravitino phase-space variables.
   - a $(0, \frac{3}{2}) + (\frac{3}{2}, 0)$ representation.
5. spin 2
   - a $(1, 1)$ representation corresponding to a symmetric tracefree four dimensional matrix which together with the $(0, 0)$ representation corresponds to degrees of freedom transforming as the gravitational metric $g_{\mu\nu}$.
   - a $(\frac{3}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{3}{2})$ representation.
   - a $(0, 2) + (2, 0)$ Weyl tensor representation.

Even for this rather restricted set of 15 representations, the number of quadratic equations for the set of structure constants t, u, c is rather formidable. For the choice of $\Lambda_2 = \Lambda_2^{max}$, the lattice Ω_2 defined in (17) has 339 points, yielding this number of u structure constants. The ordered lattice $\Omega_2 \mid_{S \leq R}$ has 196 points, in one-to-one correspondence with the *a priori* t-structure constants, of which 33 are zero in virtue of (19). We therefore have *a priori* a total of $163\,t + 339\,u + 15\,c = 517$ parameters, for which there are 1993 tt-equations (26), 5569 tu-equations (29) and 163 tuc-equations (30), the latter being in one-to-one correspondence with the number of t's. It is remarkable that this highly over-determined set of 7725 equations for a total of 517 parameters has any solutions at all. In fact the space of solutions is far from trivial; and in spite of the phenomenal over-determination the structure of solutions is rather intricate. We have archived these equations in an electronic appendix at the URL given in Appendix A.

4 Examples

A full discussion of all the allowed solutions would be rather involved and perhaps not entirely interesting. We restrict ourselves to a discussion of some classes of solutions of potential physical interest, based on subsets $\Lambda_2 \subset \Lambda_2^{max}$. One simplifying restriction is to choose *a priori* the values of the c's to be either 0 or 1. It is then convenient to label the points of Λ_p by the corresponding chosen values of c, writing $\overline{\Lambda}_p = \{\ (s,\dot{s})_{c(s,\dot{s})} \in (\mathbb{K}\otimes\mathbb{K})_{\mathbb{Z}_2}\}$. We note that if any c is chosen to be 0, the corresponding Y effectively decouples from $\mathcal{G}$. In fact, this means that $X(0,1) + X(1,0)$ generating Lorentz transformations (with $c(0,1) = c(1,0) = 0$) are always implicit.

4.1 A purely bosonic example: $\overline{\Lambda}_2 = \{(0,0)_1\,,\ (1,1)_1\,,\ (\frac{1}{2},\frac{1}{2})_1\}$

This simple restriction to a configuration space containing variables transforming like a gravitational metric and a Maxwell field allows solution of the Jacobi identities in full generality. The tt and tuc equations imply that only two non-zero t's are allowed, namely, $t(0,0,1,1,1,1)$ and $t(0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$. All tt equations are then automatically satisfied. There are nine u parameters which are possibly non-zero. We abbreviate them thus:

$$\begin{array}{rclrclrcl} u(0,0,0,0,0,0) &=& u_0\,, & u(0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) &=& v_{-1}\,, & u(0,0,1,1,1,1) &=& u_{-1}\,, \\ & & & u(\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2}) &=& v_{-2}\,, & u(1,1,0,0,1,1) &=& u_{-2}\,, \\ u(\frac{1}{2},\frac{1}{2},1,1,\frac{1}{2},\frac{1}{2}) &=& w_{-2}\,, & u(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0) &=& v_{-3}\,, & u(1,1,1,1,0,0) &=& u_{-3}\,. \end{array} \tag{32}$$

The tuc equations then fix the two non-zero t's in terms of the u's,

$$t(0,0,1,1,1,1) = u_{-3} - u_{-1} \quad , \quad t(0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) = v_{-3} - v_{-1} \quad , \tag{33}$$

Only tu equations remain. They are of two types, namely

$$u_{-2}u_{-3} \;=\; u_{-2}v_{-3} \;=\; u_{-2}w_{-2} \;=\; v_{-2}u_{-3} \;=\; v_{-2}v_{-3} \;=\; 0 \tag{34}$$

and

$$\begin{array}{l} u_{-2}(u_0 - 2u_{-1}) \;=\; 0\,, \quad u_{-3}(u_0 - u_{-3}) \;=\; 0\,, \\ v_{-2}(u_0 - 2v_{-1}) \;=\; 0\,, \quad v_{-3}(u_0 - v_{-3}) \;=\; 0\,, \quad w_{-2}(u_{-1} - 2v_{-1} + v_{-3}) \;=\; 0\,. \end{array} \tag{35}$$

The structure constants involving $X(0,0)$ (i.e. both t's in (33) and the u's in the first row of (32)) merely determine the scaling properties of all the X's and Y's. These constraints can be resolved in 12 independent ways. We list these in Table A of Appendix B. For instance, case 12 on the table corresponds to the Lie algebra with non-zero t and u structure constants,

$$\begin{array}{llll} t(0,0,1,1,1,1) = \mathrm{U}_0 - u_{-1} \,, & t(0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) = \tfrac{1}{2}(\mathrm{U}_0 - u_{-1}) \,, \\ u(0,0,1,1,1,1) = u_{-1} \,, & u(0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) = \tfrac{1}{2}(u_{-1}+\mathrm{U}_0) \,, \\ u(\tfrac{1}{2},\tfrac{1}{2},1,1,\tfrac{1}{2},\tfrac{1}{2}) = \mathrm{W}_{-2} \,, & \\ u(0,0,0,0,0,0) = u(1,1,1,1,0,0) = & u(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0,0) = \mathrm{U}_0 \,, \end{array} \tag{36}$$

with free parameters $\mathrm{U}_0, \mathrm{W}_{-2} \in \mathbb{C}\backslash\{0\}$ and $u_{-1} \in \mathbb{C}$.

4.2 The simplest 'super' example: $\overline{\Lambda}_2 = \{(\frac{1}{2},1)_1\,,\,(1,\frac{1}{2})_1\,,\,(0,0)_1\,,\,(1,1)_1\}$

This restriction to the representations occurring in simple supergravity, namely the graviton and gravitino representations, is of potential interest for canonical formulations of simple N=1 supergravity theories. Again, the complete set of solutions to the super Jacobi identities can be found without any simplifying assumptions. The tt, tu and tuc equations reduce the set of allowed structure constants to the following u's, which are possibly zero,

$$\begin{array}{llll} u(0,0,0,0,0,0) = u_0 & u(0,0,1,1,1,1) = u_{-1} & u(1,1,0,0,1,1) = u_{-2} & u(1,1,1,1,0,0) = u_{-3} \\ u(0,0,\tfrac{1}{2},1,\tfrac{1}{2},1) = u_1 & u(\tfrac{1}{2},1,0,0,\tfrac{1}{2},1) = u_2 & u(\tfrac{1}{2},1,\tfrac{1}{2},1,0,0) = u_3 & u(\tfrac{1}{2},1,1,1,\tfrac{1}{2},1) = u_4 \\ u(0,0,1,\tfrac{1}{2},1,\tfrac{1}{2}) = \widetilde{u}_1 & u(1,\tfrac{1}{2},0,0,1,\tfrac{1}{2}) = \widetilde{u}_2 & u(1,\tfrac{1}{2},1,\tfrac{1}{2},0,0) = \widetilde{u}_3 & u(1,\tfrac{1}{2},1,1,1,\tfrac{1}{2}) = \widetilde{u}_4 \end{array}$$

where the u's with a non positive index are invariants under the $\mathbb{Z}_2$ chiral transformation, $s \leftrightarrow \dot{s}$, which relates the u's with positive indices thus: $u_i \leftrightarrow \widetilde{u}_i$. In virtue of the tuc-equations, the three non-trivial t's are given in terms of the u's by

$$t(0,0,1,1,1,1) = u_{-3}-u_{-1} \,, \quad t(0,0,\tfrac{1}{2},1,\tfrac{1}{2},1) = u_3-u_1 \,, \quad t(0,0,1,\tfrac{1}{2},1,\tfrac{1}{2}) = \widetilde{u}_3-\widetilde{u}_1 \,. \tag{37}$$

The remaining tu conditions are of two types, namely

$$\begin{array}{l} u_{-2}u_{-3} = u_{-3}u_2 = u_{-2}u_3 = u_{-2}u_4 = u_2u_3 = 0 \,, \\ u_{-2}\widetilde{u}_3 = u_{-2}\widetilde{u}_4 = u_{-3}\widetilde{u}_2 = u_2\widetilde{u}_3 = \widetilde{u}_2u_3 = \widetilde{u}_2\widetilde{u}_3 = 0 \end{array} \tag{38}$$

and

$$\begin{array}{lll} u_2(u_0 - 2u_1) = 0 \,, & \widetilde{u}_2(u_0 - 2\widetilde{u}_1) = 0 \,, & u_{-2}(u_0 - 2u_{-1}) = 0 \,, \\ u_3(u_0 - u_3) = 0 \,, & \widetilde{u}_3(u_0 - \widetilde{u}_3) = 0 \,, & u_{-3}(u_0 - u_{-3}) = 0 \,, \\ u_4(u_3 - 2u_1 + u_{-1}) = 0 \,, & \widetilde{u}_4(\widetilde{u}_3 - 2\widetilde{u}_1 + u_{-1}) = 0 \,. & \end{array} \tag{39}$$

These constraints have a set of 48 independent solutions given in the two parts of Table B in Appendix B.

Let us consider the case with the largest number of parameters not required to be zero, namely the case $\alpha = 5$ from Table B . It has non-trivial t and u structure constants,

$$\begin{array}{ll} t(0,0,1,1,1,1) = \mathrm{U}_0 - u_{-1} \,, & t(0,0,\tfrac{1}{2},1,\tfrac{1}{2},1) = \tfrac{1}{2}(\mathrm{U}_0 - u_{-1}) \,, \\ u(0,0,1,1,1,1) = u_{-1} \,, & u(0,0,\tfrac{1}{2},1,\tfrac{1}{2},1) = \tfrac{1}{2}(u_{-1}+\mathrm{U}_0) \,, \\ u(0,0,0,0,0,0) = u(\tfrac{1}{2},1,\tfrac{1}{2},1,0,0) = & u(1,1,1,1,0,0) = \mathrm{U}_0 \\ u(\tfrac{1}{2},1,1,1,\tfrac{1}{2},1) = \mathrm{U}_4 & \end{array} \tag{40}$$

together with further non-zero structure constants obtained by interchanging dotted and undotted arguments of fermionic index pairs $S=(s,\dot{s})$, e.g. $t(0,0,\frac{1}{2},1,\frac{1}{2},1)\mapsto t(0,0,1,\frac{1}{2},1,\frac{1}{2})$. The free parameters are $\mathrm{u}_0\,,\mathrm{u}_4\in\mathbb{C}\backslash\{0\}$ and $u_{-1}\in\mathbb{C}$.

4.3 $\overline{\Lambda}_2=\{(\frac{1}{2},0)_1\,,\,(0,\frac{1}{2})_1\,,\,(\frac{1}{2},1)_1\,,\,(1,\frac{1}{2})_1\,,\,(0,0)_1\,,\,(1,1)_1\}$

The Rarita-Schwinger vector-spinor contains not only the gravitino representations $(\frac{1}{2},1)+(1,\frac{1}{2})$ included in 4.2, but also spinorial $(\frac{1}{2},0)+(0,\frac{1}{2})$ auxiliary (gauge) degrees of freedom. Including these significantly alters the structure of the solution space. In particular the tt-equations have more solutions. These can be classified in a straightforward albeit lengthy fashion as follows.

The tuc-equations immediately imply that

$$\begin{aligned}&t(\tfrac{1}{2},1,1,1,\tfrac{1}{2},1)=0\ ,\quad t(1,\tfrac{1}{2},1,1,1,\tfrac{1}{2})=0\ ,\\&u(1,1,\tfrac{1}{2},1,\tfrac{1}{2},1)=0\ ,\quad u(1,1,1,\tfrac{1}{2},1,\tfrac{1}{2})=0\,,\quad u(1,1,1,1,1,1)=0\ .\end{aligned}\tag{41}$$

The tuc-equations are then entirely resolved and the discussion of the tt-equations may be carried out most conveniently in terms of the following parameters, which may possibly take zero value,

$$\begin{aligned}&t(0,\tfrac{1}{2},1,\tfrac{1}{2},1,1)\ =\ t_1\quad,\quad t(0,\tfrac{1}{2},1,1,1,\tfrac{1}{2})\ =\ t_2\quad,\quad t(\tfrac{1}{2},1,1,1,\tfrac{1}{2},0)\ =\ t_3\quad,\\&t(\tfrac{1}{2},0,\tfrac{1}{2},1,1,1)\ =\ \widetilde{t}_1\quad,\quad t(\tfrac{1}{2},0,1,1,\tfrac{1}{2},1)\ =\ \widetilde{t}_2\quad,\quad t(1,\tfrac{1}{2},1,1,0,\tfrac{1}{2})\ =\ \widetilde{t}_3\quad.\end{aligned}\tag{42}$$

The tt-equations yield the following 10 quadratic constraints among t_i and $\widetilde{t}_i, i=1,2,3$:

$$t_i\,\widetilde{t}_j=0\ ,\ i\neq j\ ,\quad t_1\,t_j=0\ ,\ j\neq 1\ ,\quad \widetilde{t}_1\,\widetilde{t}_j=0\ ,\ j\neq 1\ .\tag{43}$$

Introducing the shorthand for the remaining t-parameters,

$$\begin{aligned}&t(0,0,\tfrac{1}{2},1,\tfrac{1}{2},1)\ =\ w_1\quad,\quad t(0,0,1,\tfrac{1}{2},1,\tfrac{1}{2})\ =\ \widetilde{w}_1\quad,\\&t(0,0,1,1,1,1)\ =\ t_{-1}\quad,\quad t(0,0,0,\tfrac{1}{2},0,\tfrac{1}{2})\ =\ v_1\quad,\quad t(0,0,\tfrac{1}{2},0,\tfrac{1}{2},0)\ =\ \widetilde{v}_1\quad,\end{aligned}\tag{44}$$

we obtain twelve disjoint solutions of the tt-equations which we present in tabulated form: Table C of Appendix B. Every line of the table is a solution of the entire set of tt-equations.

Our classification of all solutions of the tu-equations is too lengthy to include here. A file containing this may be obtained from the authors by e-mail. Here, we concentrate on one case which seems particularly interesting: Case 4 on Table C . This has $\mathbb{Z}_2$ chiral symmetry under $s\leftrightarrow\dot{s}$ and the property that $X(1,1)$ commutes with all fermionic X's, namely

$$\begin{aligned}&t(0,\tfrac{1}{2},1,\tfrac{1}{2},1,1)\ =\ t_1\ \neq\ 0\ ,\qquad t(\tfrac{1}{2},0,\tfrac{1}{2},1,1,1)\ =\ \widetilde{t}_1\ \neq\ 0\ ,\\&t(0,\tfrac{1}{2},1,1,1,\tfrac{1}{2})\ =\ t(\tfrac{1}{2},0,1,1,\tfrac{1}{2},1)\ =\ t(\tfrac{1}{2},1,1,1,\tfrac{1}{2},0)\ =\ t(1,\tfrac{1}{2},1,1,0,\tfrac{1}{2})\ =\ 0\ .\end{aligned}\tag{45}$$

The tu, tuc equations then imply that the only possibly non-zero u's remaining are:

$$\begin{aligned}
u(0,0,0,0,0,0) &= u(0,0,1,1,1,1) = u_0 \\
u(0,0,0,\tfrac{1}{2},0,\tfrac{1}{2}) &= u(0,0,\tfrac{1}{2},0,\tfrac{1}{2},0) = \tfrac{u_0}{2} \\
u(0,0,\tfrac{1}{2},1,\tfrac{1}{2},1) &= u(0,0,1,\tfrac{1}{2},1,\tfrac{1}{2}) = \tfrac{u_0}{2} \\
u(0,\tfrac{1}{2},1,1,1,\tfrac{1}{2}) &= t_1 \ , \quad u(\tfrac{1}{2},0,1,1,\tfrac{1}{2},1) = \widetilde{t}_1 \\
u(\tfrac{1}{2},1,0,0,\tfrac{1}{2},1) &= u_2 \ , \quad u(1,\tfrac{1}{2},0,0,1,\tfrac{1}{2}) = \widetilde{u}_2 \\
u(\tfrac{1}{2},1,1,1,\tfrac{1}{2},1) &= u_4 \ , \quad u(1,\tfrac{1}{2},1,1,1,\tfrac{1}{2}) = \widetilde{u}_4 \\
u(\tfrac{1}{2},1,\tfrac{1}{2},1,0,0) &= u(1,\tfrac{1}{2},1,\tfrac{1}{2},0,0) = u(1,1,1,1,0,0) = u_3
\end{aligned} \tag{46}$$

All tt, tuc-equations are then solved and the remaining tu-equations may be solved in two possible ways:

a) $u_3 = 0$,

b) $u_3 = u_0$, $\quad u_2 = \widetilde{u}_2 = u_4 = \widetilde{u}_4 = 0$.

The latter, for instance, corresponds to the simple superalgebra with non-zero t and u structure constants,

$$\begin{aligned}
t(0,0,0,\tfrac{1}{2},0,\tfrac{1}{2}) &= -\tfrac{1}{2}u_0 \ , & t(0,0,\tfrac{1}{2},1,\tfrac{1}{2},1) &= \tfrac{1}{2}u_0 \ , \\
t(0,\tfrac{1}{2},1,\tfrac{1}{2},1,1) &= t_1 \ , & u(0,\tfrac{1}{2},1,1,1,\tfrac{1}{2}) &= t_1 \ , \\
u(0,0,0,\tfrac{1}{2},0,\tfrac{1}{2}) &= \tfrac{1}{2}u_0 \ , & u(0,0,1,\tfrac{1}{2},1,\tfrac{1}{2}) &= \tfrac{1}{2}u_0 \ , \\
u(1,\tfrac{1}{2},1,\tfrac{1}{2},0,0) &= u(1,1,1,1,0,0) = & u(0,0,0,0,0,0) &= u_0 \ , \\
u(0,0,1,1,1,1) &= u_0
\end{aligned} \tag{47}$$

and further structure constants obtained by interchanging dotted and undotted arguments of fermionic index pairs together with the replacement of t_1 by $\widetilde{t}_1$. Here $t_1, \widetilde{t}_1$ are non-zero free parameters and u_0 is arbitrary.

4.4 Adding a vector: $\overline{\Lambda}_2 = \{(\tfrac{1}{2},\tfrac{1}{2})_1 \ , (\tfrac{1}{2},0)_1 \ , (0,\tfrac{1}{2})_1 \ , (\tfrac{1}{2},1)_1 \ , (1,\tfrac{1}{2})_1 \ , (0,0)_1 \ , (1,1)_1\}$

Adding a vector, corresponding to a Maxwell degree of freedom, to the set of representations in 4.3, yields an example which has the additional interesting feature of combining the super-Poincaré representations in (1,2) with the simple supergravity representations in 4.2. The super-Jacobi identities immediately imply that

$$t(\tfrac{1}{2},\tfrac{1}{2},1,1,\tfrac{1}{2},\tfrac{1}{2}) = 0 \quad , \quad t(1,\tfrac{1}{2},1,1,1,\tfrac{1}{2}) = 0 \ . \tag{48}$$

The complete discussion of all the solutions of the super-Jacobi identities is rather detailed. The assumption of the *super-Poincaré condition*,

$$t(0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2}) \neq 0 \ , \tag{49}$$

considerably simplifies further discussion. In particular, apart from the six parameters of the form $t(0,0,a,b,a,b)$ encoding the scaling behaviours of the X's, only six further t's remain non-zero. The tt, tu, tuc-equations yield a solution with 19 parameters, $\{u_0, u_i, \widetilde{u}_i; i = 1, \ldots, 9\}$ invariant under the $\mathbb{Z}_2$ *chirality transformation*,

$$u_i \leftrightarrow \widetilde{u}_i \quad , \quad s \leftrightarrow \dot{s} \quad . \tag{50}$$

Modulo this symmetry, the non-zero t and u structure constants are

$$\begin{aligned}
&t(0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2}) = (u_2+\widetilde{u}_2)\,, && && t(0,\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{1}{2},\tfrac{1}{2}) = (u_6-u_3)\,,\\
&t(0,\tfrac{1}{2},1,\tfrac{1}{2},1,1) = (u_4+\widetilde{u}_8)\,, && && t(\tfrac{1}{2},1,1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) = (u_7+\widetilde{u}_7)\,,\\
&u(0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0) = u_2\,, && u(0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},1) = u_3\,, && u(0,\tfrac{1}{2},1,1,1,\tfrac{1}{2}) = u_4\,,\\
&u(\tfrac{1}{2},1,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2}) = u_6\,, && u(\tfrac{1}{2},1,\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{1}{2}) = u_7\,, && u(\tfrac{1}{2},1,0,0,\tfrac{1}{2},1) = u_5\,,\\
&u(\tfrac{1}{2},1,1,1,\tfrac{1}{2},0) = u_8\,, && u(\tfrac{1}{2},1,1,1,\tfrac{1}{2},1) = u_9\,, && u(0,\tfrac{1}{2},0,0,0,\tfrac{1}{2}) = u_1\,,
\end{aligned} \tag{51}$$

together with the scaling rules,

$$\begin{aligned}
&t(0,0,0,\tfrac{1}{2},0,\tfrac{1}{2}) = -\tfrac{u_0}{2}\,, && t(0,0,1,\tfrac{1}{2},1,\tfrac{1}{2}) = -\tfrac{u_0}{2}\,,\\
&t(0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}) = -u_0\,, && t(0,0,1,1,1,1) = -u_0\,,\\
&u(0,0,0,\tfrac{1}{2},0,\tfrac{1}{2}) = \tfrac{u_0}{2}\,, && u(0,0,1,\tfrac{1}{2},1,\tfrac{1}{2}) = \tfrac{u_0}{2}\,,\\
&u(0,0,\tfrac{1}{2},\tfrac{1}{2}\tfrac{1}{2},\tfrac{1}{2}) = u_0\,, && u(0,0,1,1,1,1) = u_0\,,\\
&u(0,0,0,0,0,0) = u_0\,.
\end{aligned} \tag{52}$$

4.5 An extension of the super-Poincaré algebra: $\Lambda_2 = \{(s,\dot{s})\,;\ 0 < s+\dot{s} \leq 2\}$

In [4] we constructed an explicit example of a spin $\frac{3}{2}$ superalgebra with the super-Poincaré algebra as a subalgebra. Following the procedure of section 4.3 of [4], we may extend that example to include elements of spin 2 as well, maintaining the super Poincaré embedding. This example has every representation with spin ≤ 2, except for the scalar $(0,0)$. The identification of $X(0,1) + X(1,0)$ with the Lorentz generators leads to the decoupling of $Y(0,1) + Y(1,0)$, which may be put to zero with $c(0,1) = c(1,0) = 0$. All other 12 c's are taken to be 1.

There are 46 free parameters, which we denote as follows:

$$\begin{aligned}
&u(\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2}) = u_1\,, && u(1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0) = u_2\,, && u(\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{1}{2}) = u_3\,,\\
&u(1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},0) = u_4\,, && u(\tfrac{3}{2},0,\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{1}{2}) = u_5\,, && u(1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},1) = u_6\,,\\
&u(\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{3}{2},0,\tfrac{3}{2}) = u_7\,, && u(\tfrac{1}{2},0,1,1,\tfrac{1}{2},1) = u_8\,, && u(\tfrac{1}{2},0,\tfrac{3}{2},\tfrac{1}{2},1,\tfrac{1}{2}) = u_9\,,\\
&u(\tfrac{1}{2},0,2,0,\tfrac{3}{2},0) = u_{10}\,, && u(1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{2},1) = u_{11}\,, && u(1,\tfrac{1}{2},1,1,0,\tfrac{1}{2}) = u_{12}\,,\\
&u(1,\tfrac{1}{2},1,1,0,\tfrac{3}{2}) = u_{13}\,, && u(1,\tfrac{1}{2},1,1,1,\tfrac{1}{2}) = u_{14}\,, && u(1,\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2},0) = u_{15}\,,\\
&u(1,\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2},1) = u_{16}\,, && u(1,\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{3}{2},0) = u_{17}\,, && u(1,\tfrac{1}{2},2,0,1,\tfrac{1}{2}) = u_{18}\,,\\
&u(\tfrac{3}{2},0,1,1,\tfrac{1}{2},1) = u_{19}\,, && u(\tfrac{3}{2},0,\tfrac{3}{2},\tfrac{1}{2},0,\tfrac{1}{2}) = u_{20}\,, && u(\tfrac{3}{2},0,\tfrac{3}{2},\tfrac{1}{2},1,\tfrac{1}{2}) = u_{21},\\
&u(\tfrac{3}{2},0,2,0,\tfrac{1}{2},0) = u_{22}\,, && u(\tfrac{3}{2},0,2,0,\tfrac{3}{2},0) = u_{23}\,,
\end{aligned} \tag{53}$$

together with further 23 parameters obtained from the above under the transformation (50). All other u's are zero, except for those concerning the Lorentz generators $X(0,1)+X(1,0)$, viz. $u(0,1\,,a,b\,,a,b)$ and $u(1,0\,,a,b\,,a,b)$, which together with the 'Lorentz' t's (those containing $(0,1)$ or $(1,0)$ in any of the three positions) are completely determined by Lorentz covariance. The remaining non-zero t's take the form,

$$\begin{array}{rclrcl} t(0,\frac{1}{2}\,,\frac{1}{2},0\,,\frac{1}{2},\frac{1}{2}) &=& u_1 + \widetilde{u}_1 \;, & t(\frac{1}{2},0\,,1,\frac{1}{2}\,,\frac{1}{2},\frac{1}{2}) &=& u_2 - u_3 \;, \\ t(\frac{1}{2},1\,,1,\frac{1}{2}\,,\frac{1}{2},\frac{1}{2}) &=& u_6 + \widetilde{u}_6 \;, & t(1,\frac{1}{2}\,,\frac{3}{2},0\,,\frac{1}{2},\frac{1}{2}) &=& u_4 + u_5 \;, \\ t(\frac{1}{2},0\,,0,\frac{3}{2}\,,\frac{1}{2},\frac{3}{2}) &=& u_7 + \widetilde{u}_{20} \;, & t(\frac{1}{2},0\,,\frac{1}{2},1\,,1,1) &=& u_8 + \widetilde{u}_{12} \;, \\ t(\frac{1}{2},0\,,1,\frac{1}{2}\,,\frac{3}{2},\frac{1}{2}) &=& u_9 + u_{15} \;, & t(\frac{1}{2},0\,,\frac{3}{2},0\,,2,0) &=& u_{10} + u_{22} \;, \\ t(\frac{1}{2},1\,,1,\frac{1}{2}\,,\frac{1}{2},\frac{3}{2}) &=& u_{11} - \widetilde{u}_{16} \;, & t(0,\frac{3}{2}\,,1,\frac{1}{2}\,,1,1) &=& u_{13} - \widetilde{u}_{19} \;, \\ t(1,\frac{1}{2}\,,\frac{3}{2},0\,,\frac{3}{2},\frac{1}{2}) &=& u_{21} - u_{17} \;, &&& \end{array} \tag{54}$$

together with 9 further t's obtained under the transformation (50) and using the relations (18) to recover t's in the ordered set. With the above structure constants, all the tt, tu, tuc-equations are resolved. The non-zero supercommutation relations may be read off directly from (53) and (54). Setting the parameters $\{u_i, \widetilde{u}_i\;; i = 7, \ldots, 23\}$ to zero may easily be seen to reduce this superalgebra to the 12-parameter spin $\frac{3}{2}$ extension of the super-Poincaré algebra obtained in [4]. Moreover, coordinate representations of this as well as the previous examples in this section may be found following that reference.

5 Concluding remarks

We have developed a framework for the construction and investigation of Lorentz covariant Heisenberg superalgebras with generators transforming according to representations of arbitrary values of spin. We have thus obtained a complete parametrisation of all such superalgebras, for the case of unit multiplicity. The parameter space is highly overdetermined. Closer investigation, however, reveals surprisingly non-trivial possibilities of resolving the constraints. As an example, we have obtained the most general set of constraints for superalgebras containing generators of spins up to two; and we have found several classes of explicit solutions. We have remained in a broader algebraic setting. Our spin two superalgebras, however, have the algebraic structure of phase spaces possibly underlying gravity and supergravity models. Concrete physical application of our algebras, for instance, to the canonical quantisation of supergravity theories, remains for future investigation.

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A The spin two structure equations

We have archived the complete set of structure equations for spin two superalgebras at the URL http://www.mis.mpg.de/preprints/98/preprint1598-addendum1.html
The files given there list a) the 7725 quadratic polynomials which need to vanish for the satisfaction of the super Jacobi identities; and b) the 196 t's and 339 u's, which together with the 15 c's yield the total of 517 constrained parameters. The equations given are labeled as follows

the tuc-equations:	$tcrl(i)$;	$i = 1, ..., 163$
the tt-equations:	$ttrl(i)$;	$i = 1, ..., 1993$
the tu-equations:	$tu0rl(i)$;	$i = 1, ..., 324$
	$tu1rl(i)$;	$i = 1, ..., 1020$
	$tu2rl(i)$;	$i = 1, ..., 1688$
	$tu3rl(i)$;	$i = 1, ..., 1694$
	$tu4rl(i)$;	$i = 1, ..., 843$

Here $tuNrl$ with $N = 0, \ldots, 4$ denote relations derived from super Jacobi identities involving $X(s, \dot{s})$, $X(r, \dot{r})$ and $Y(v, \dot{v})$ for $S \leq R$ and with $N = 2(s + \dot{s})$. The equations are given in a text format compatible with REDUCE or MAPLE.

B Solutions of the spin two structure equations

In this appendix we give details of the solutions discussed in section 4. The following table lists the 12 independent solutions of (34) and (35) ($\overline{\Lambda}_2 = \{(0,0)_1\ ,\ (1,1)_1\ ,\ (\frac{1}{2}, \frac{1}{2})_1\}$).

Table A

<table>
<tr><th></th><th>u_0</th><th>u_{-1}</th><th>u_{-2}</th><th>u_{-3}</th><th>v_{-1}</th><th>v_{-2}</th><th>v_{-3}</th><th>w_{-2}</th></tr>
<tr><td>1</td><td rowspan="6">u_0</td><td>u_{-1}</td><td>0</td><td rowspan="8">0</td><td rowspan="3">v_{-1}</td><td rowspan="3">0</td><td rowspan="6">0</td><td rowspan="2">0</td></tr>
<tr><td>2</td><td>$u_0/2$</td><td>U_{-2}</td></tr>
<tr><td>3</td><td>$2v_{-1}$</td><td rowspan="3">0</td><td rowspan="2">W_{-2}</td></tr>
<tr><td>4</td><td>u_0</td><td rowspan="3">$u_0/2$</td><td rowspan="3">V_{-2}</td></tr>
<tr><td>5</td><td>u_{-1}</td><td rowspan="3">0</td></tr>
<tr><td>6</td><td>$u_0/2$</td><td>U_{-2}</td></tr>
<tr><td>7</td><td rowspan="6">U_0</td><td rowspan="6">u_{-1}</td><td rowspan="6">0</td><td>v_{-1}</td><td rowspan="6">0</td><td rowspan="2">U_0</td></tr>
<tr><td>8</td><td>$(u_{-1}+\mathrm{U}_0)/2$</td><td>W_{-2}</td></tr>
<tr><td>9</td><td rowspan="4">U_0</td><td>v_{-1}</td><td rowspan="2">0</td><td>0</td></tr>
<tr><td>10</td><td>$u_{-1}/2$</td><td>W_{-2}</td></tr>
<tr><td>11</td><td>v_{-1}</td><td rowspan="2">U_0</td><td>0</td></tr>
<tr><td>12</td><td>$(u_{-1}+\mathrm{U}_0)/2$</td><td>W_{-2}</td></tr>
</table>

The entries give the parameters listed in (32) in terms of the free subset. If the u_x column is marked u_x it means that it is a completely free parameter. If it is marked in uppercase roman font, e.g., U_x, it means that it is a free parameter but necessarily non-zero. Also in this notation, the 48 distinct choices of unconstrained parameters satisfying (38) and (39) (section 4.2, $\overline{\Lambda}_2 = \{(\frac{1}{2}, 1)_1\ ,\ (1, \frac{1}{2})_1\ ,\ (0,0)_1\ ,\ (1,1)_1\}$) are listed in Table B .

Table B

α	u_0	u_{-1}	u_{-2}	u_{-3}	u_1	$\widetilde{u}_1$	u_2	$\widetilde{u}_2$	u_3	$\widetilde{u}_3$	u_4	$\widetilde{u}_4$
1	u_0	$u_0/2$	u_{-2}	0	$u_0/2$	$u_0/2$	u_2	$\widetilde{\mathrm{u}}_2$	0	0	0	0
2						$\widetilde{u}_1$		0				
3					u_1	$u_0/2$	0	$\widetilde{\mathrm{u}}_2$				
4						$\widetilde{u}_1$		0				
5	u_0	u_{-1}	0	u_0	$(\mathrm{u}_0+u_{-1})/2$	$(\mathrm{u}_0+u_{-1})/2$			u_0	u_0	u_4	$\widetilde{\mathrm{u}}_4$
6						$\widetilde{u}_1$						0
7					u_1	$(\mathrm{u}_0+u_{-1})/2$					0	$\widetilde{\mathrm{u}}_4$
8					$(\mathrm{u}_0+u_{-1})/2$	$u_{-1}/2$				0	u_4	
9					$u_{-1}/2$	$(\mathrm{u}_0+u_{-1})/2$			0	u_0		
10					u_1	$\widetilde{u}_1$			u_0		0	0
11					$(\mathrm{u}_0+u_{-1})/2$					0	u_4	
12					$u_{-1}/2$				0	u_0		
13					u_1	$u_{-1}/2$			u_0	0	0	$\widetilde{\mathrm{u}}_4$
14						$(\mathrm{u}_0+u_{-1})/2$			0	u_0		
15					$u_{-1}/2$	$u_{-1}/2$				0	u_4	
16					u_1	$\widetilde{u}_1$			u_0	0	0	0
17									0	u_0		
18					$u_{-1}/2$					0	u_4	
19					u_1	$u_{-1}/2$					0	$\widetilde{\mathrm{u}}_4$
20						$\widetilde{u}_1$						0
21	u_0	u_0		0	$u_0/2$	$u_0/2$	u_2	$\widetilde{\mathrm{u}}_2$			u_4	$\widetilde{\mathrm{u}}_4$
22												0
23											0	$\widetilde{\mathrm{u}}_4$
24		u_{-1}										0
25	u_0				$(\mathrm{u}_0+u_{-1})/2$	$(\mathrm{u}_0+u_{-1})/2$	0	0	u_0	u_0	u_4	$\widetilde{\mathrm{u}}_4$
26						$\widetilde{u}_1$						0
27					u_1	$(\mathrm{u}_0+u_{-1})/2$					0	$\widetilde{\mathrm{u}}_4$
28						$\widetilde{u}_1$						0
29	u_0	u_0			$u_0/2$	$u_0/2$	u_2		0	0	u_4	$\widetilde{\mathrm{u}}_4$
30						$\widetilde{u}_1$						0
31		$2\widetilde{u}_1$									0	$\widetilde{\mathrm{u}}_4$
32		u_{-1}										0
33		u_0			$u_0/2$	$u_0/2$	0	$\widetilde{\mathrm{u}}_2$			u_4	$\widetilde{\mathrm{u}}_4$
34		$2u_1$			u_1							0
35		u_0									0	$\widetilde{\mathrm{u}}_4$
36		u_{-1}										0
37	u_0				$(\mathrm{u}_0+u_{-1})/2$	$u_{-1}/2$		0	u_0		u_4	$\widetilde{\mathrm{u}}_4$
38						$\widetilde{u}_1$						0
39					u_1	$u_{-1}/2$					0	$\widetilde{\mathrm{u}}_4$
40						$\widetilde{u}_1$						0
41					$u_{-1}/2$	$(\mathrm{u}_0+u_{-1})/2$			0	u_0	u_4	$\widetilde{\mathrm{u}}_4$
42						$\widetilde{u}_1$						0
43					u_1	$(\mathrm{u}_0+u_{-1})/2$					0	$\widetilde{\mathrm{u}}_4$
44						$\widetilde{u}_1$						0
45	u_0				$u_{-1}/2$	$u_{-1}/2$				0	u_4	$\widetilde{\mathrm{u}}_4$
46						$\widetilde{u}_1$						0
47					u_1	$u_{-1}/2$					0	$\widetilde{\mathrm{u}}_4$
48						$\widetilde{u}_1$						0

For the examples of section 4.3 ($\overline{\Lambda}_2 = \{(\frac{1}{2},0)_1\,,\,(0,\frac{1}{2})_1\,,\,(\frac{1}{2},1)_1\,,\,(1,\frac{1}{2})_1\,,\,(0,0)_1\,,\,(1,1)_1\}$) the constraints (43) have the following 12 possible choices of free sets of t's.

Table C

	t_1	$\widetilde{t}_1$	t_2	$\widetilde{t}_2$	t_3	$\widetilde{t}_3$	w_1	$\widetilde{w}_1$
1	0	0	τ_2	$\widetilde{\tau}_2$	0	0	$t_{-1}+\widetilde{v}_1$	$t_{-1}+v_1$
2				0	τ_3		$\widetilde{v}_1-t_{-1}$	
3					0		w_1	
4	τ_1	$\widetilde{\tau}_1$	0				$t_{-1}-\widetilde{v}_1$	
5		0					w_1	
6	0			$\widetilde{\tau}_2$		$\widetilde{\tau}_3$	$\widetilde{v}_1+t_{-1}$	v_1-t_{-1}
7						0	v_1+t_{-1}	$\widetilde{w}_1$
8		$\widetilde{\tau}_1$		0			$t_{-1}-v_1$	
9		0			τ_3	$\widetilde{\tau}_3$	$\widetilde{v}_1-t_{-1}$	v_1-t_{-1}
10					0		w_1	
11					τ_3	0	$\widetilde{v}_1-t_{-1}$	$\widetilde{w}_1$
12					0		w_1	

We note that some of theses cases are related by the chiral $\mathbb{Z}_2$ symmetry : $(2 \leftrightarrow 6)$, $(3 \leftrightarrow 7)$, $(5 \leftrightarrow 8)$ and $(10 \leftrightarrow 11)$.

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