Anyons in quantum mechanics with a minimal length

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Abstract

The existence of anyons, *i.e.* quantum states with an arbitrary spin, is a generic feature of standard quantum mechanics in $(2+1)-$dimensional Minkowski spacetime. Here it is shown that relativistic anyons may exist also in quantum theories where a minimal length is present. The interplay between minimal length and arbitrary spin effects are discussed.

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I. INTRODUCTION

The existence of a minimal observable length in Nature is suggested by string theory and quantum gravity, see *e.g.* [1]. An economical way of introducing such a minimal length at the level of nonrelativistic quantum mechanics is to use a suitably modified Heisenberg algebra [2–4]. This proposition has been extended to covariant quantum mechanics in [5] and further investigated in many relevant cases including the Klein-Gordon [6] and Dirac [7] equations.

Although the study of quantum theories characterized by a minimal length has become an active field of research, some comments remain to be done about the case of a $(2+1)-$dimensional spacetime with a minimal length. Indeed, it is well-known that spin is no longer quantized in $2+1$ dimensions, allowing the appearance of anyons, *i.e.* paricles with arbitrary spin. To our knowledge, the question of the existence (or not) of anyons in a $(2+1)-$dimensional spacetime with a minimal length has never been addressed. It is the topic of the present paper.

II. POINCARÉ ALGEBRA IN $2+1$ DIMENSIONS

A. Generalities

The Poincaré algebra in $2+1$ dimensions can be written under the form [8]

$$[J^\mu, J^\nu] = -i\varepsilon^{\mu\nu\rho} J_\rho \ , \quad [J^\mu, P^\nu] = -i\varepsilon^{\mu\nu\rho} P_\rho; \ , \quad [P^\mu, P^\nu] = 0 \ , \tag{1}$$

where P^μ are the translation generators and where

$$J^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho} L_{\nu\rho}, \tag{2}$$

$L_{\mu\nu}$ being the Lorentz generators. The convention $\epsilon^{012} = 1$ is used, as well as the Minkowski metric η =diag$(+,-,-)$. The two Casimir operators associated to (1) read, for massive representations,

$$M^2 = \boldsymbol{P}^2, \quad s = \frac{\boldsymbol{P}\cdot\boldsymbol{J}}{M}. \tag{3}$$

They represent the squared mass and the spin of a given state respectively, while the notation $\boldsymbol{X} = (X^0, X^1, X^2)$ for a 3−vector.

It is useful to note that, in the rest frame of a given state, s reduces to $J^0 = \pm L_{12}$, that is the generator of spatial rotations. One recovers then a maybe more intuitive definition of spin as the phase factor associated to rotations.

The unitary irreducible representations of the Lorentz algebra in $2+1$ dimensions – the first commutator of (1) – have been built in [8], while the unitary irreducible representations of the full Poincaré algebra have been obtained in [9]. The conclusions shared by both studies are identical in what concerns the spin of the representations: It can be an arbitrary real number. This is a peculiar feature of quantum mechanics in $2+1$ dimensions and strongly differs from the $(3+1)-$ dimensional case, in which the spin of a state can only be integer of half-integer. The properties of such particles with arbitrary spin, called anyons, have been studied in a considerable amount of works. The interested reader may find relevant informations in the pioneering works [10] and [11, 12], as well as in the reviews [13, 14].

It is known that finite-dimensional representations of the Lorentz algebra are non-unitary in $2+1$ dimensions. However, this does not imply that bosons and fermions are fobidden. It is indeed worth mentioning that bosonic and fermionic states can be built in $2+1$ dimensions from the tensor product of non-unitary infinite-dimensional representations of the Lorentz algebra. That subtle issue has been studied in [15], to which we refer the interested reader.

B. Minimal length representation of the Poincaré algebra

It has been shown in [5] that, for a $(D+1)-$dimensional spacetime with metric $\eta=$diag$(+,-,\ldots,-)$, the Poincaré algebra is represented by the generators

$$P_\mu = (1-\beta\hat{\boldsymbol{P}}^2)^{-1}\hat{P}_\mu, \quad L_{\mu\nu} = (1-\beta\hat{\boldsymbol{P}}^2)^{-1}(\hat{P}_\nu\hat{X}_\mu - \hat{P}_\mu\hat{X}_\nu) \tag{4}$$

provided that

$$\begin{aligned}
\left[\hat{X}^\mu, \hat{P}^\nu\right] &= -i\left[(1-\beta\hat{\boldsymbol{P}}^2)\eta^{\mu\nu} - \beta'\hat{P}^\mu\hat{P}^\nu\right], \\
\left[\hat{X}^\mu, \hat{X}^\nu\right] &= -i\left[(2\beta-\beta') - (2\beta+\beta')\beta\hat{\boldsymbol{P}}^2\right]\ L^{\mu\nu}, \\
\left[\hat{P}^\mu, \hat{P}^\nu\right] &= 0.
\end{aligned} \tag{5}$$

where it is assumed that $\beta,\ \beta' \in \mathbb{R}^+$. The modified Heisenberg algebra (5) leads to a minimal uncertainty on the measurement of a position that can be computed to be, assuming isotropicity [5],

$$\Delta X_{\min} = \sqrt{(D\beta+\beta')(1-\beta\langle(P^0)^2\rangle)}, \tag{6}$$

and is usually referred to as "minimal length".

III. RELATIVISTIC ANYONS WITH A MINIMAL LENGTH

It follows from the general results recalled in the previous section that, in $(2+1)$ dimensions, quantum theories which are both Poincaré invariant and formulated in a spacetime with minimal length are allowed. As a consequence, anyons may exist when a minimal length is present too. Wave equations describing an anyon $|\psi\rangle$ have been proposed in [16] in a form that is convenient regarding to our framework. They read

$$V_\mu|\psi\rangle = 0, \quad V_\mu = sP_\mu - i\epsilon_{\mu\nu\rho}P^\nu J^\rho + MJ_\mu. \tag{7}$$

They are valid whatever the representation chosen for P^μ and J^μ, so they can be used with the minimal length representation (4), (5). An extensive discussions of the solutions of (7) can be found in [16].

In view of explicit computations, it is convenient to note that the algebra (5) can be represented by the operators [5]

$$\begin{aligned} \hat{P}^\mu &= p^\mu, \\ \hat{X}^\mu &= (1-\beta\boldsymbol{p}^2)x^\mu - \beta' p^\mu \boldsymbol{p}\cdot\boldsymbol{x} + i\gamma p^\mu, \end{aligned} \tag{8}$$

where $x_\mu = -i\frac{\partial}{\partial p^\mu}$ and $\boldsymbol{p}$ are standard position and momentum operators, *i.e.* $[x^\mu, p^\nu] = -i\eta^{\mu\nu}$. A straightforward computation shows that, using this representation, the Lorentz generators take a more familiar form

$$L_{\mu\nu} = i(p_\mu\frac{\partial}{\partial p^\nu} - p_\nu\frac{\partial}{\partial p^\mu}). \tag{9}$$

Using the representation (8) in polar coordinates, namely $p^\mu = (p^0, p, \theta_p)$, one quickly shows that

$$J^0 = L_{12} = i\frac{\partial}{\partial_{\theta_p}}, \tag{10}$$

recovering a standard form for J^0 even if a modified Heisenberg algebra is considered.

A peculiarity of algebra (5) is that $\left[\hat{X}^\mu, \hat{X}^\nu\right] \propto L^{\mu\nu}$. Hence, the noncommutativity of spatial coordinates is such that $\left[\hat{X}^1, \hat{X}^2\right] \propto J^0$. Let us indeed consider an anyonic state of squared mass M^2 and spin s in its rest frame and denote $|\psi; M^2, s, j\rangle$ such a state. Then one is led to the following uncertainty relation on the spatial coordinates:

$$\Delta X^1 \, \Delta X^2 \geq \frac{1}{2} |(2\beta - \beta') - (2\beta + \beta')\beta M^2| \, s. \tag{11}$$

Anyons with larger spins then "feel" a larger spatial noncommutativity. Note however that, even for small values of s, one has the lower bound $\Delta X^1 \Delta X^2 \geq \Delta X^2_{\min}$ implied by the modified Heisenberg algebra.

IV. NONRELATIVISTIC LIMIT

The modified Heisenberg algebra initially proposed by Kempf in a nonrelativistic version [2] can be recovered from algebra (5) and from (4) by neglecting the terms in $\beta(P^0)^2$, *i.e.* by assuming that the typical energy of the system under study is negligible compared to $1/\sqrt{\beta}$, and by only considering the spatial commutators. The only nonvanishing commutators in the case of two spatial dimensions are then

$$\begin{aligned} \left[\hat{X}^i, \hat{P}^j\right] &= i \left[(1 + \beta \hat{P}^2)\delta^{ij} + \beta' \hat{P}^i \hat{P}^j\right], \\ \left[\hat{X}^1, \hat{X}^2\right] &= -i \left[(2\beta - \beta') + (2\beta + \beta')\beta \hat{P}^2\right] J^0, \end{aligned} \tag{12}$$

where $\hat{P}^2 = (P^1)^2 + (P^2)^2$. The momentum representation (8) has the nonrelativistic counterpart

$$\begin{aligned} \hat{P}^j &= p^j, \\ \hat{X}^j &= i(1 + \beta p^2)\frac{\partial}{\partial_{p^j}} + i\beta' p^j \vec{p} \cdot \vec{\nabla}_p + i\gamma p^j. \end{aligned} \tag{13}$$

It can be check from the above representation that any Hamiltonian of the form $T(\hat{P}^2) + V(\hat{X}^2)$, being spherically symmetric in momentum representation, will commute with J^0, according to (10). Moreover, in the rest frame of the eigenstate $|\chi\rangle$ under study, J^0 is identified with the spin; its eigenvalue s can then be arbitrary. It follows

$$\langle \vec{p} | \chi \rangle = \mathrm{e}^{is\theta_p} \psi(p), \tag{14}$$

which fixes the angular dependence of the state. This last relation also shows that, if $\vec{\hat{P}}$ and $\vec{\hat{X}}$ are the relative momentum and position of a two-body system made of two identical bodies, the wave function acquires a phase factor $\mathrm{e}^{is\pi}$ under permutation of the two bodies. This exotic phase factor is the signature of a braid statistics, in agreement with the generalized spin-statistics theorem [17].

As an illustration, let us consider the Hamiltonian $\frac{\hat{P}^2}{2\mu} + \frac{1}{2}\mu\omega^2\hat{X}^2$, whose energy spectrum is analytically known in D spatial dimensions [18]. When $D = 2$, the spin is arbitrary and, using the results of this last reference, the energy spectrum is given by

$$\frac{E}{\omega} = (2n + |s| + 1)\sqrt{1 + \left[\beta^2 s^2 + \frac{(2\beta + \beta')^2}{4}\right]\mu^2\omega^2} + \left[(\beta + \beta')(2n + |s| + 1)^2 + (\beta - \beta')(s^2 + 1) + \beta'\right]\frac{\mu\omega}{2}, \tag{15}$$

which is a generalization of the anyonic harmonic oscillator, that can be recovered with $\beta = \beta' = 0$, see *e.g.* [14, 19]. The main effect of the modified Heisenberg algebra is to break to well-known degeneracy in $2n + |s| + 1$.

It is worth pointing out that there exists a deformation of the Heisenberg algebra which is inequivalent to (12) but which can also be related to anyons. It reads

$$[a^-, a^+] = 1 + \nu K,$$
$$\{K, a^\pm\} = 0, \quad K^2 = 1,$$

where the real parameter ν and the kleinian K are responsible for the deformation. $a^\pm$ are creation and annihilation operators. As shown in [20], this algebra can be used to describe in an elegant way relativistic quantum states with arbitrary spin. In the nonrelativistic limit, anyons built from such a formalism reduce to free massive particles in the noncommutative plane [21]: As in Eq. (11), states with arbitrary spin are indeed associated to noncommutative spatial coordinates.

V. SUMMARY AND OUTLOOK

Anyons may exist in 2 + 1 dimensions as representations of the Poincaré algebra. It follows that they may be present in any Poincaré invariant spacetime, including those where a minimal length is present. In such spacetimes, an anyon wave equation can be defined.

As an outlook, let us mention that previously known analytical results about the spectra of Schrödinger-like Hamiltonians with modified Heisenberg algebra could be re-analysed in the particular case of two spatial dimensions to better appraise the interplay between of an arbitrary spin and a minimal length.

An other way of producing anyons is to minimally couple a particle to a vortex-like gauge field [11]. It could be interesting to see what is the generalization of such a gauge field configuration in electrodynamic with a minimal length, which has been proposed recently in [22]; we leave such a task for future works.

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[1] D. J. Gross and P. F. Mende, Nucl. Phys. B **303**, 407 (1988); D. Amati, M. Ciafaloni, and G. Veneziano, Phys. Lett. B **216**, 41 (1989); M. Maggiore, Phys. Lett. B **319**, 83 (1993) [hep-th/9309034]; A. Connes, Commun. Math. Phys. **182**, 155 (1996) [hep-th/9603053]; G. Amelino-Camelia, J. Ellis, N. E. Mavromatos, and D. V. Nanopoulos, Mod. Phys. Lett. A **12**, 2029 (1997) [hep-th/9701144]; N. Seiberg and E. Witten, JHEP **9909**, 032 (1999) [hep-th/9908142], and references therein.

[2] A. Kempf, J. Math. Phys. **35**, 4483 (1994) [hep-th/9311147].

[3] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D **52**, 1108 (1995) [hep-th/9412167]; H. Hinrichsen and A. Kempf, J. Math. Phys. **37**, 2121 (1996) [hep-th/9510144].

[4] A. Kempf, J. Phys. A **30**, 2093 (1997) [hep-th/9604045].

[5] C. Quesne and V. M. Tkachuk, J. Phys. A **39**, 10909 (2006) [quant-ph/0604118].

[6] S. K. Moayedi, M. R. Setare and H. Moayeri, Int. J. Theor. Phys. **49**, 2080 (2010) [arXiv:1004.0563].

[7] S. K. Moayedi, M. R. Setare and H. Moayeri, Int. J. Mod. Phys. A **26**, 4981 (2011) [arXiv:1105.1900]; A. Boumali and H. Hassanabadi, Can. J. Phys. **93**, no. 5, 542 (2015).

[8] A. O. Barut and C. Fronsdal, Proc. Roy. Soc. London **A287**, 532-548 (1965).

[9] B. Binegar, J. Math. Phys. **23**, 1511 (1982).

[10] J. M. Leinaas and J. Myrheim, Nuovo Cim. B **37**, 1 (1977).

[11] F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982).

[12] J. Frohlich and P. A. Marchetti, Lett. Math. Phys. **16**, 347 (1988); G. W. Semenoff, Phys. Rev. Lett. **61**, 517 (1988).

[13] S. Forte, Rev. Mod. Phys. **64**, 193 (1992).

[14] A. Khare, *Fractional statistics and quantum theory* (2005, World Scientific Publishing, 2nd Ed.).

[15] P. A. Horvathy, M. S. Plyushchay and M. Valenzuela, Annals Phys. **325**, 1931 (2010) [arXiv:1001.0274].

[16] J. L. Cortes and M. S. Plyushchay, J. Math. Phys. **35**, 6049 (1994) [hep-th/9405193].

[17] J. Mund, Commun. Math. Phys. **286**, 1159 (2009) [arXiv:0801.3621].

[18] L. N. Chang, D. Minic, N. Okamura and T. Takeuchi, Phys. Rev. D **65**, 125027 (2002) [hep-th/0111181].

[19] R. MacKenzie and F. Wilczek, Int. J. Mod. Phys. A **3**, 2827 (1988).

[20] M. S. Plyushchay, Phys. Lett. B **320**, 91 (1994) [hep-th/9309148]; Annals Phys. **245**, 339 (1996) [hep-th/9601116].

[21] P. A. Horvathy and M. S. Plyushchay, Phys. Lett. B **595**, 547 (2004) [hep-th/0404137].

[22] S. K. Moayedi, M. R. Setare and B. Khosropour, [arXiv:1303.0100].