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# An Action for Matter Coupled Higher Spin Gravity in Three Dimensions


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## ABSTRACT

We propose a covariant Hamiltonian action for the Prokushkin and Vasiliev's matter coupled higher spin gravity in three dimensions. The action is formulated on $\mathcal{X}_4 \times \mathcal{Z}_2$ where $\mathcal{X}_4$ is an open manifold whose boundary contains spacetime and $\mathcal{Z}_2$ is a noncommutative twistor space. We examine various consistent truncations to models of BF type in $\mathcal{X}_4$ and $\mathcal{Z}_2$ with $\mathrm{B}^2$ terms and central elements. They are obtained by integrating out the matter fields in the presence of a vacuum expectation value $\nu \in \mathbb{R}$ for the zero-form master field. For $\nu = 0$, we obtain a model on $\mathcal{X}_4$ containing Blencowe's action and a model on $\mathcal{Z}_2$ containing the Prokushkin–Segal–Vasiliev action. For generic $\nu$ (including $\nu = 0$), we propose an alternative model on $\mathcal{X}_4$ with gauge fields in the Weyl algebra of Wigner's deformed oscillator algebra and Lagrange multipliers in the algebra of operators acting in the Fock representation space of the deformed oscillators.

# Contents



# 1 Introduction

Three-dimensional gravity with negative cosmological constant, as defined by the $sl(2,\mathbb{R})\oplus sl(2,\mathbb{R})$ Chern–Simons (CS) action of [1, 2], provides a rich framework for testing various aspects of quantum gravity in a setting that is simpler than in higher dimensions, yet nontrivial. Although $\mathrm{AdS}_3$ gravities are topological, they admit black holes [3] and possess moduli spaces at conformal infinity governed by infinite-dimensional conformal symmetry algebras [4, 5]; see [6] for a review.

As for higher spin gravities, these are simpler as well in three dimensions, where the massless higher spin fields are topological, and hence the spectrum requirements on the gauge algebras simplify considerably. Topological higher spin gravities based on the principal embedding of $sl(2,\mathbb{R})$ into $sl(N,\mathbb{R})$ were shown in [7, 8] to have asymptotic $W_N$ symmetries. In [9] holographic correspondences were conjectured for higher spin gravities with infinite-dimensional gauge algebras $hs(2) \oplus hs(2)$ and their deformation $hs(\lambda) \oplus hs(\lambda)$ coupled to (complex) bulk scalars.

In the above works, the (classical) gauge sector is assumed to be described by various CS generalisations [10, 11, 12] of the Achucarro–Townsend supergravity Lagrangian [1]. On the other hand, one class of matter coupled higher spin gravities is described on-shell by the Prokushkin–Vasiliev (PV) equations [13], and off-shell by the Prokushkin–Segal–Vasiliev (PSV) action principle in twistor space [14]. However, as spacetime is absent in the latter, its relation to the CS formulations has remained unclear. Moreover, there exists a second class of matter coupled higher spin gravities based on action principles in three dimensions with an extra dynamical two-form [12], whose relation to the PV system is unclear.

In this paper, we provide the PV system with an action principle of covariant Hamiltonian type on a six-manifold given by the direct product of a closed twistor space $\mathcal{Z}_2$ and an open four-manifold $\mathcal{X}_4$ whose boundary $\mathcal{X}_3$ contains spacetime $\mathcal{M}_3$. When subjected to the variational principle combined with natural boundary conditions, the action yields the PV equations on its five-dimensional boundary $\mathcal{X}_3 \times \mathcal{Z}_2$. The action is constructed such that upon integrating out the matter fields in the presence of an expectation value $\nu$ for the PV zero-form, the effective action for the gauge fields can be consistently truncated to models of BF type on $\mathcal{X}_4$. These model contain B squared terms[1] containing the standard symplectic structure of three-dimensional Fronsdal fields.

For $\nu = 0$, we obtain a model on $\mathcal{X}_4$ containing Blencowe's action, as well as a model on $\mathcal{Z}_2$ containing the Prokushkin–Segal–Vasiliev action. In these models, the dual spaces that contain the gauge fields and Lagrange multipliers are isomorphic. For generic $\nu$ (including $\nu = 0$), we shall also consider a model on $\mathcal{X}_4$ in which the aforementioned two spaces are not isomorphic. However, its existence depends on the finiteness of the trace of the vacuum-to-vacuum projector of a deformed oscillator induced from six dimensions. If existing, such a model would provide

[1] The B field of the BF-like models originate from the even Lagrange multiplier in the covariant Hamiltonian action, denoted by $T$ in (3.27).

an alternative to the BF-like Blencowe model based on Vasiliev's supertrace [15], whose six-dimensional origin remains unclear.

The master action to be constructed here is analog of the one for the four-dimensional Vasiliev system found in [16]. In particular, it does *not* extend the closed and central two-form of the PV system into a dynamical field off-shell. The inclusion of a dynamical two-form in an action that is an analog of that for four-dimensional models given recently in [17] and which makes contact with both the PV system as well as the action proposed in [12] will be treated elsewhere.

The paper is structured as follows: In Section 2, we cast the PV system as a differential algebra on the direct product of twistor space and spacetime. After this preparation, we propose a covariant Hamiltonian action on $\mathcal{X}_4 \times \mathcal{Z}_2$, with a term quadratic in Lagrange multipliers, in Section 3. In Section 4, we examine the consistent truncations to the BF-like version of the Blencowe model on $\mathcal{X}_4$ and the PSV action on $\mathcal{Z}_2$. We summarize our results and provide an outlook in the Conclusions in Section 5. In Appendix A, we review the mass deformation. In Appendix B, we present the proposed $\nu$-deformed BF-like model on $\mathcal{X}_4$.

# 2 Prokushkin–Vasiliev models

In this section, we rewrite the PV equations [13] as a differential algebra generated by master fields on a noncommutative manifold valued in an associative algebra, or equivalently, as an associative bundle with fusion rules. In particular, we shall identify the minimal bosonic model and its massive deformation. For a recent, in-depth treatment of the weak-field perturbative analysis of PV systems to first nontrivial order in interactions, see [18].

## 2.1 Differential algebra

The master fields are

$$A = dx^\mu W_\mu(x,z|y;\Gamma_i) + dz^\alpha V_\alpha(x,z|y;\Gamma_i) \ , \quad B = B(x,z|y;\Gamma_i) \ , \tag{2.1}$$

defined locally on the direct product $\mathcal{M}_3 \times \mathcal{Z}_2$ of a commutative three-dimensional real manifold $\mathcal{M}_3$ with coordinates $x^\mu$, $\mu = 1,2,3$, and a non-commutative two-dimensional real manifold $\mathcal{Z}_2$ with coordinates $z^\alpha$, $\alpha = 1,2$. The fields are valued in an associative algebra generated by a real oscillator $y^\alpha$, $\alpha = 1, 2$, coordinatizing an internal noncommutative manifold $\mathcal{Y}_2$, and a set of elements $\Gamma_i$, $i = 1, \dots, N$, obeying

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \ , \tag{2.2}$$

thus coordinatizing the Clifford algebra $\mathcal{C}\ell_N$, that we shall denote by $\mathcal{C}_N$ for brevity. The dependence of the master fields on $(y^\alpha, z^\alpha)$ is treated using symbol calculus, whereby they belong to classes of functions (or distributions) on $\mathcal{Y}_2 \times \mathcal{Z}_2$ that can be composed using two associative products: the standard commutative product rule, denoted by juxtaposition, and an additional noncommutative product rule, denoted by a $\star$. In what follows, we shall use the normal ordered basis in which the star product rule is defined formally by

$$(f \star g)(y, z) := \int_{\mathbb{R}^4} \frac{d^2u d^2v}{(2\pi)^2} e^{iv^\alpha u_\alpha} f(y+u, z+u)\, g(y+v, z-v) \ , \tag{2.3}$$

whereas a more rigorous definition requires a set of fusion rules (see below). In particular, the above composition rule rigorously defines the associative Weyl algebra Aq(4). This algebra consists of arbitrary polynomials in $y^\alpha$ and $z^\alpha$, modulo

$$y_\alpha \star y_\beta = y_\alpha y_\beta + i\epsilon_{\alpha\beta} \ , \qquad y_\alpha \star z_\beta = y_\alpha z_\beta - i\epsilon_{\alpha\beta} \ , \tag{2.4}$$

$$z_\alpha \star y_\beta = z_\alpha y_\beta + i\epsilon_{\alpha\beta} \ , \qquad z_\alpha \star z_\beta = z_\alpha z_\beta - i\epsilon_{\alpha\beta} \ , \tag{2.5}$$

whose symmetric and anti-symmetric parts, respectively, define the normal order and the (ordering independent) commutation rules, *viz.*[2]

$$[y^\alpha, y^\beta]_\star = -[z^\alpha, z^\beta]_\star = 2i\epsilon^{\alpha\beta} \ , \quad [y^\alpha, z^\beta]_\star = 0 \ . \tag{2.6}$$

The basis one-forms $(dx^\mu, dz^\alpha)$ obey

$$[dx^\mu, f]_\star = 0 = [dz^\alpha, f]_\star \ , \tag{2.7}$$

where the graded star commutator[3] of differential forms is given by

$$[f, g]_\star = f \star g - (-1)^{\deg(f)\deg(g)} g \star f \ , \tag{2.8}$$

with deg denoting the total form degree on $\mathcal{M}_3 \times \mathcal{Z}_2$ . To describe bosonic models, we impose

$$\pi(A) = A \ , \quad \pi(B) = B \tag{2.9}$$

[2]The doublet variables $y^\alpha$ and $z^\alpha$ form Majorana spinors once the equations are cast into a manifestly Lorentz covariant form.

[3]We will explicitly use commutators and anticommutators, in Section 3.

where $\pi$ is the automorphism of the differential star product algebra defined by

$$\pi(x^\mu, dx^\mu, z^\alpha, dz^\alpha, y^\alpha, \Gamma_i) = (x^\mu, dx^\mu, -z^\alpha, -dz^\alpha, -y^\alpha, \Gamma_i) \ . \tag{2.10}$$

The hermitian conjugation is defined by

$$(f \star g)^\dagger = (-1)^{\mathrm{deg(f)deg(g)}} g^\dagger \star f^\dagger \ , \qquad (z_\alpha, dz^\alpha; y_\alpha, \Gamma_i)^\dagger = (-z_\alpha, -dz^\alpha; y_\alpha, \Gamma_i) \ . \tag{2.11}$$

and the reality conditions on the master fields read

$$A^\dagger = -A \ , \qquad B^\dagger = B \ . \tag{2.12}$$

Defining

$$F = dA + A \star A \ , \quad DB = dB + A \star B - B \star A \ , \quad d = dx^\mu \partial_\mu + dz^\alpha \frac{\partial}{\partial z^\alpha} \ , \tag{2.13}$$

where the differential obeys

$$d(f \star g) \ = \ (df) \star g + (-1)^{\mathrm{deg(f)}} f \star dg \ , \quad (df)^\dagger = d(f^\dagger) \ , \tag{2.14}$$

the PV field equations can be written as

$$F + B \star J = 0 \ , \qquad DB = 0 \ , \tag{2.15}$$

where

$$J := -\tfrac{i}{4} dz^\alpha dz_\alpha \, \kappa \qquad \kappa := e^{i y^\alpha z_\alpha} \ . \tag{2.16}$$

The element $J$ is closed and central in the space of $\pi$-invariant forms, *viz.*

$$dJ = 0 \ , \qquad J \star f = \pi(f) \star J \ , \tag{2.17}$$

as can be seen from the fact that $\kappa$, which is referred to as the inner Klein operator, obeys

$$\kappa \star f(x, dx, z, dz, y, \Gamma_i) \star \kappa = f(x, dx, -z, dz, -y, \Gamma_i) \ . \tag{2.18}$$

It follows that (2.15) defines a universally Cartan integrable system (*i.e.* a set of generalized curvature constraints compatible with $d^2 \equiv 0$ in any dimension). The Cartan gauge transformations take the form

$$\delta_\epsilon A = d\epsilon + [A, \epsilon]_\star \ , \quad \delta_\epsilon B = [B, \epsilon]_\star \ . \tag{2.19}$$

## 2.2 Lorentz covariance

Introducing

$$S_\alpha := z_\alpha - 2i\,V_\alpha\ , \qquad d_X := dx^\mu \partial_\mu\ , \tag{2.20}$$

the equations can be rewritten as

$$\begin{aligned} & d_X W + W \star W = 0\ , \quad d_X B + [W, B]_\star = 0\ , \quad d_X S_\alpha + [W, S_\alpha]_\star = 0\ , \\ & [S_\alpha, B]_\star = 0\ , \quad [S_\alpha, S_\beta]_\star = -2i\epsilon_{\alpha\beta}\left(1 - B \star \kappa\right)\ . \end{aligned} \tag{2.21}$$

In view of $\{S_\alpha, \kappa\}_\star = 0$, which follows from the bosonic projection, the above equations define a deformed oscillator algebra, fibered over $\mathcal{M}_3$, for which $B$ plays the role of deformation parameter. The equations can be cast into manifestly Lorentz covariant form [19, 20] by introducing a bona fide Lorentz connection $\omega^{\alpha\beta} = dx^\mu \omega_\mu^{\alpha\beta}$ on $\mathcal{M}_3$ and defining

$$\mathcal{W} = W - \frac{1}{4i}\omega^{\alpha\beta} M_{\alpha\beta}\ , \quad M_{\alpha\beta} = y_{(\alpha} \star y_{\beta)} - z_{(\alpha} \star z_{\beta)} + S_{(\alpha} \star S_{\beta)}\ . \tag{2.22}$$

in terms of which the master field equations on $\mathcal{M}_3$ take the form

$$\nabla\mathcal{W} + \mathcal{W} \star \mathcal{W} + \frac{1}{4i} r^{\alpha\beta} M_{\alpha\beta} = 0\ , \quad \nabla B + [\mathcal{W}, B]_\star = 0\ , \quad \nabla S_\alpha + [\mathcal{W}, S_\alpha]_\star = 0\ , \tag{2.23}$$

where

$$\nabla\mathcal{W} = d_X \mathcal{W} + [\omega, \mathcal{W}]_\star\ , \qquad \nabla B = d_X B + [\omega, B]_\star\ , \tag{2.24}$$

$$\nabla S_\alpha = d_X S_\alpha - \omega_\alpha{}^\beta S_\beta + [\omega, S_\alpha]_\star\ . \tag{2.25}$$

and

$$\omega = \frac{1}{4i} dx^\mu \omega_\mu^{\alpha\beta} (y_\alpha \star y_\beta - z_\alpha \star z_\beta)\ , \qquad r^{\alpha\beta} = d_X \omega^{\alpha\beta} - \omega^{\alpha\gamma}\omega_\gamma{}^\beta\ , \tag{2.26}$$

which are related by $r = \frac{1}{4i} r^{\alpha\beta}(y_\alpha \star y_\beta - z_\alpha \star z_\beta) = d_X\omega + \omega \star \omega$. The deformed Lorentz generators obey the algebra

$$[M_{\alpha\beta}, M_{\gamma\delta}]_\star = 4i\epsilon_{(\beta|(\gamma} M_{\delta)|\alpha)} - \delta_{\alpha\beta} M_{\gamma\delta} + \delta_{\gamma\delta} M_{\alpha\beta}\ , \tag{2.27}$$

where the induced transformations

$$\delta_{\alpha\beta} M_{\gamma\delta} = 4i\epsilon_{(\beta|(\gamma} M_{\delta)|\alpha)} - [y_{(\alpha} \star y_{\beta)} - z_{(\alpha} \star z_{\beta)}, M_{\gamma\delta}]_\star \tag{2.28}$$

act on the component fields of $M_{\gamma\delta}$. The above commutation rules are an example of a more general construction wherein a Lie algebra $L$ acts on a space $M$ via Lie derivatives and

$$T : L \times M \to \mathcal{A}\ , \qquad T : (X, p) \mapsto T_X(p)\ , \tag{2.29}$$

is a representation of $L$ in an associative algebra with product $\star$ obeying

$$[T_X, T_Y]_\star = T_{[X,Y]} - \mathcal{L}_X T_Y + \mathcal{L}_Y T_X \ , \tag{2.30}$$

which can be seen to obey the Jacobi identity using $[\mathcal{L}_X, \mathcal{L}_X] = \mathcal{L}_{[X,Y]}$ and the Leibniz' rule $\mathcal{L}_X(T_Y \star T_Z) = (\mathcal{L}_X T_Y) \star T_Z + T_Y \star (\mathcal{L}_X T_Z)$.

## 2.3 Original PV model and its truncations

By taking $N = 4$ and identifying

$$(k)_{\rm PV} = \Gamma \ , \quad (\nu)_{\rm PV} = -\nu \ , \quad (\rho)_{\rm PV} = \Gamma_1 \ , \quad (y_\alpha)_{\rm PV} = \Gamma_1 \, y_\alpha \ , \quad (z_\alpha)_{\rm PV} = \Gamma_1 \, z_\alpha \ , \tag{2.31}$$

$$(\psi_1)_{\rm PV} = i\Gamma_{23} \ , \qquad (\psi_2)_{\rm PV} = i\Gamma_{24} \ . \tag{2.32}$$

we recover the original PV system, in which $\psi_1$ is used to define the $\mathrm{AdS}_3$ translation operators. By imposing the following conditions on the master fields, conditions that will be justified later on from the existence of an action principle,

$$[\Gamma, A] = 0 \ , \qquad [\Gamma, B] = 0 \ , \tag{2.33}$$

*i.e.* by taking them to be valued in the subalgebra

$$\mathcal{C}_4^+ = \bigoplus_{\sigma=\pm} \Pi_\Gamma^\sigma \ \mathcal{C}_4 \ \Pi_\Gamma^\sigma \ , \qquad \Pi_\Gamma^\sigma = \frac{1}{2}(1 + \sigma\Gamma) \ , \qquad \Gamma = \Gamma_{1234} \ , \tag{2.34}$$

of $\mathcal{C}_4$, we obtain the $\rho$-projected PV system in which the master fields $(W, B)_{\rm PV}$ are $\rho$-independent and $(S_\alpha)_{\rm PV}$ depend linearly on $\rho$. The $B$ field consists of eight real zero-form master fields. Four of these describe real propagating scalar fields in $\mathrm{AdS}_3$. The remaining four provide topological deformation parameters. The following truncation

$$(A, B) = \Pi_\Gamma^+ (A, B) \ , \tag{2.35}$$

yields a model containing two real propagating scalars and two topological master fields. Truncating one last time by imposing

$$\tau(A, B) = (-A, B) \ , \tag{2.36}$$

using the anti-automorphism defined by

$$\tau(f \star g) = (-1)^{\rm deg(f) deg(g)} \tau(g) \star \tau(f) \ , \tag{2.37}$$

$$\tau(z^\alpha, dz^\alpha; y^\alpha, \Gamma_i) = (-iz^\alpha, -idz^\alpha; iy^\alpha, \epsilon_{(i)}\Gamma_i) \ , \qquad \epsilon_{(i)} = (+,+,-,-) \ , \tag{2.38}$$

yields a model with one propagating scalar and one topological master field. This model is identical to the original PV model based on the algebra $ho^+_{01}(1,0|4)$[4]. In all of the above models, the component along $\Gamma$ plays the role of a real mass parameter, denoted by $\nu$; see Appendix A.

## 2.4 Associative bundle

The master fields equations define an associative algebra bundle $\widehat{\mathcal{A}}$ over $\mathcal{M}_3$ [21, 22]. The fiber algebra $\widehat{\mathcal{A}}|_p$ at a generic point $p \in \mathcal{M}_3$ is related by a similarity transformation to that at a reference point $p_0 \in \mathcal{M}_3$. To describe the latter, it is convenient to separate the variables in $\Omega_{[0]}(\mathcal{Y}_2) \otimes \Omega(\mathcal{Z}_2)$, as (2.3) reduces to separate Weyl order on $\Omega_{[0]}(\mathcal{Y}_2)$ and $\Omega(\mathcal{Z}_2)$. One can then use the factorization formula

$$\kappa = \kappa_z \star \kappa_y \ , \qquad \kappa_y := 2\pi\, \delta^2(y^\alpha) \ , \qquad \kappa_z := 2\pi \delta^2(z^\alpha) \ , \tag{2.39}$$

to solve the deformed oscillator algebra at $p_0$ formally in terms of auxiliary integrals facilitated by analytical continuation methods in $\Omega(\mathcal{Z}_2)$ [23]. Thus,

$$\widehat{\mathcal{A}}|_p \cong \widehat{\mathcal{A}}|_{p_0} = \Omega(\mathcal{Z}_2) \otimes \mathcal{A} \otimes \mathcal{C}_N \ , \qquad \mathcal{A} = \bigoplus_\Sigma \mathrm{Aq}(2)[\Sigma] \ , \tag{2.40}$$

where $\mathrm{Aq}(2)[\Sigma]$ are vector spaces of symbols corresponding to a set of boundary conditions on $\mathcal{M}_3 \times \mathcal{Z}_2$ [24, 23]; for examples, see Section 4. Harmonic expansions, spectrum analysis and exact solutions show that the associative bundle contains nonpolynomial sectors obtainable from reference elements [24, 23, 17]

$$T_\Sigma \in \mathrm{Aq}(2)[\Sigma] \ , \tag{2.41}$$

by the left and right action of the Weyl algebra $\mathrm{Aq}(2)$. We write

$$\mathrm{Aq}(2)[\Sigma] = \mathrm{Aq}(2)[T_\Sigma; \lambda, \rho] \ , \tag{2.42}$$

indicating the properties of $\mathrm{Aq}(2)[\Sigma]$ as a left ($\lambda$) and right ($\rho$) module of $\mathrm{Aq}(2)$. The associative structure of $\mathcal{A}$ requires a fusion rule

$$\mathrm{Aq}(2)[\Sigma] \star \mathrm{Aq}(2)[\Sigma'] = \bigoplus_{\Sigma''} \mathcal{N}_{\Sigma\Sigma'}{}^{\Sigma''} \mathrm{Aq}(2)[\Sigma''] \ , \qquad \mathcal{N}_{\Sigma\Sigma'}{}^{\Sigma''} \in \{0,1\} \ , \tag{2.43}$$

[4] In the notation of Prokushkin and Vasiliev, the algebra $ho^+_{01}(1,0|4)$ arises in Section 9 of [13] by taking $n=1$, $m=0$, $\alpha=0$ and $\beta=1$ in Eqs. (9.6)–(9.11) followed by a $\mathcal{P}^+$-projection (which corresponds to our $\Pi^+_\Gamma$-projection) and the projection using the anti-automorphism $\sigma$ in Eq. (4.21).

such that if $\mathcal{N}_{\Sigma\Sigma'}{}^{\Sigma''} = 1$ then the left-hand side is to be computed using (2.3) with $z^\alpha = 0$ and expanded into the basis of $\mathrm{Aq}(2)[\Sigma'']$ such that all nontrivial products are finite and the resulting multiplication table is associative. For example, massless particles and various types of algebraically special exact solution spaces arise within Gaussian sectors. The Weyl algebra $\mathrm{Aq}(2) \equiv \mathrm{Aq}(2)[\mathbf{1}]$, with reference state being the identity operator, is also included (as the sector corresponding to twistor space plane waves), typically with[5] $\mathcal{N}_{1\Sigma}{}^{\Sigma} = \mathcal{N}_{\Sigma 1}{}^{\Sigma} = 1$.

# 3 Covariant Hamiltonian action

In this section we begin by discussing some generalities on covariant Hamiltonian actions on $\mathcal{X}_4 \times \mathcal{Z}_2$. We then determine the constraints on the Hamiltonian such that it leads to a master action in which the master field content, including the Lagrange multipliers, are extended to consist of sum of even and odd forms of appropriate degree, and central elements. This action yields a generalized version of the PV field equations.

## 3.1 Generalities

In order to formulate the theory within the AKSZ framework [26] using its adaptation to non-commutative higher spin geometries proposed in [27], we assume a formulation of the PV system that treats $\mathcal{Z}_2$ as being closed and introduce an open six-manifold $\mathcal{M}_6$ with boundary

$$\partial\mathcal{M}_6 = \mathcal{X}_3 \times \mathcal{Z}_2 \ , \tag{3.1}$$

where $\mathcal{X}_3$ is a closed manifold containing $\mathcal{M}_3$ as an open submanifold. On $\mathcal{M}_6$, we introduce a two-fold duality extended [28, 16, 29] [6] set of differential forms given by

$$A = A_{[1]} + A_{[3]} + A_{[5]} \ , \qquad B = B_{[0]} + B_{[2]} + B_{[4]} \ , \tag{3.2}$$

$$T = T_{[4]} + T_{[2]} + T_{[0]} \ , \qquad S = S_{[5]} + S_{[3]} + S_{[1]} \ , \tag{3.3}$$

[5] An exception is fractional spin gravity [25] whose fractional spin sector $\Psi$ and Lorentz singlet sector $U$ have $\mathcal{N}_{\Psi 1}{}^{\Psi} = 1$ and $\mathcal{N}_{1U}{}^{U} = \mathcal{N}_{U1}{}^{U} = 0$.

[6] Starting from a universally Cartan integrable system and replacing each $p$-form by a sum of forms of degrees $p$, $p+2$, ..., $p+2N$, and each structure constant by a function of off-shell closed and central terms, *i.e.* elements in the de Rham cohomology valued in the center of the fiber algebra, with a decomposition into degrees 0, 2, ..., $2N$, yields a new universally Cartan integrable system, referred to as the $N$-fold duality extension of the original system. More generally, one may consider on-shell duality extensions by including on-shell closed complex-valued functionals into the extension of the structure constants [30, 29].

valued in $\mathcal{A}\otimes\mathcal{C}_4$ and where the subscript denotes the form degree. We let $\{J^I\}$ denote the generators of the ring of off-shell closed and central terms, *i.e.* elements in the de Rham cohomology of $\mathcal{M}_6$ valued in the center of $\mathcal{A}\otimes\mathcal{C}_4$, which hence obey

$$dJ^I \;=\; 0\;, \quad \left[J^I,f\right]_\star \;=\; 0\;, \tag{3.4}$$

(off-shell) for any differential form $f$ on $\mathcal{M}_6$ valued in $\mathcal{A}\otimes\mathcal{C}_4$. Following the approach of [16], we consider actions of the form

$$S_{\rm H} \;=\; \int_{\mathcal{M}_6} {\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4}\left[S\star DB + T\star F + \mathcal{V}(S,T;B;J^I)\right] \tag{3.5}$$

$$\;=\; \int_{\mathcal{M}_6} {\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4}\left[S\star {\rm d}B + T\star {\rm d}A - \mathcal{H}(S,T;A,B;J^I)\right] \tag{3.6}$$

where ${\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4}$ denotes a cyclic trace operation on $\mathcal{A}\otimes\mathcal{C}_4$. We assume a structure group gauged by $A$ and that $S$, $T$ and $B$ belong to sections, and (3.6) makes explicit the covariant Hamiltonian form, with

$$\mathcal{H}(S,T;A,B;J^I) = -S\star[A,B]_\star - T\star A\star A - \mathcal{V}(S,T;B;J^I)\;. \tag{3.7}$$

Thus, the coordinate and momentum master fields, defined by

$$(X^\alpha;P_\alpha) := (A,B;T,S)\;, \tag{3.8}$$

lie in subspaces of $\mathcal{A}$ that are dually paired using ${\rm Tr}_{\mathcal{A}}$, which leads to distinct models depending on whether these subspaces are isomorphic or not. In the reductions that follow, we shall consider the first type of models, while a model with coordinates and momenta in non-isomorphic spaces is treated in Appendix B. Moreover, for definiteness, we shall assume that

$$\mathcal{M}_6 = \mathcal{X}_4\times\mathcal{Z}_2\;, \tag{3.9}$$

and the associative bundle $\widehat{\mathcal{A}}$ defined in (2.40) is chosen such that

$$\check{\mathcal{L}} = \oint_{\mathcal{Z}_2} {\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4}\left[S\star DB + T\star F + \mathcal{V}(S,T;B;J^I)\right]\;, \tag{3.10}$$

is finite (and globally defined on $\mathcal{X}_4$). The action can then be written as

$$S_{\rm H} = \int_{\mathcal{X}_4}\check{\mathcal{L}}\;. \tag{3.11}$$

We shall furthermore assume that

$$\int_{\mathcal{M}_6} \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4} df = \oint_{\partial\mathcal{M}_6} \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4} f \ , \tag{3.12}$$

and

$$\int_{\mathcal{M}_6} \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4} f \star g = (-1)^{\deg(f)\deg(g)} \int_{\mathcal{M}_6} \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4} g \star f \ , \tag{3.13}$$

$$\oint_{\partial\mathcal{M}_6} \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4} f \star g \ = \ \oint_{\partial\mathcal{M}_6} \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4} g \star f \ , \tag{3.14}$$

from which it follows that $\mathcal{H}$ is a graded cyclic $\star$-function.

## 3.2 Constraints on $\mathcal{H}$

The Hamiltonian is constrained by gauge invariance, or equivalently, by universal on-shell Cartan integrability[7]. In addition, it is constrained by the requirement that the equations of motion on $\mathcal{M}_6$ reduce to a desired set of equations of motion on $\partial\mathcal{M}_6$ upon assuming natural boundary conditions. To examine the above, we let

$$Z^i \equiv (X^\alpha; P_\alpha) \ , \tag{3.15}$$

and consider the total variation

$$\delta S_{\mathrm{H}} \ \equiv \ \int_{\mathcal{M}_6} \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4} \delta Z^i \star \mathcal{R}^j \, \Omega_{ij} + (-)^{\deg(P_\alpha)} \oint_{\partial\mathcal{M}_6} \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4} P_\alpha \star \delta X^\alpha \ , \tag{3.16}$$

where $\Omega_{ij}$ is a graded anti-symmetric constant matrix[8] and the Cartan curvatures are given by

$$\mathcal{R}^i \ := \ dZ^i + \mathcal{Q}^i(Z) \ \approx \ 0 \ , \quad \mathcal{Q}^i \ := \ \Omega^{ij}\partial_j \mathcal{H} \ , \tag{3.17}$$

where $\partial_i$ denotes the graded cyclic derivative defined by

$$\delta \int_{\mathcal{M}_6} \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4^+} \mathcal{U} \ = \ \int_{\mathcal{M}_6} \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4^+} \delta Z^i \star \partial_i \mathcal{U} \ , \tag{3.18}$$

for any graded cyclic $\star$-function $\mathcal{U}$. We find

$$\mathcal{R}^A \ = \ F + \partial_T \mathcal{V} \ , \qquad \mathcal{R}^B \ = \ DB + \partial_S \mathcal{V} \ , \tag{3.19}$$

$$\mathcal{R}^S \ = \ DS + \partial_B \mathcal{V} \ , \qquad \mathcal{R}^T \ = \ DT - [B,S]_\star \ . \tag{3.20}$$

[7] Covariant Hamiltonian actions are gauge invariant iff their equations of motion form universally Cartan integrable systems.

[8] Adopting the conventions of [27], we take $\Omega^{ik}\Omega_{kj} = -\delta^i_j$ .

Requiring $A$ and $B$ to be free to fluctuate on $\partial\mathcal{M}_6$ , the variational principle implies

$$P_\alpha|_{\partial\mathcal{M}_6} = 0 \ . \tag{3.21}$$

The Cartan integrability requires

$$\overrightarrow{\mathcal{Q}} \star \mathcal{Q}^i \ \equiv \ 0 \ , \tag{3.22}$$

using a notation in which $\star$-vector fields $\overrightarrow{V} \equiv V^i \overrightarrow{\partial}_i$ act on $\star$-functions as follows: [9]

$$\overrightarrow{V} \star (\mathcal{U}_1 \star \mathcal{U}_2) = (\overrightarrow{V} \star \mathcal{U}_1) \star \mathcal{U}_2 + (-1)^{\deg(\overrightarrow{V})\deg(\mathcal{U}_1)}\, \mathcal{U}_1 \star (\overrightarrow{V} \star \mathcal{U}_2) \ , \qquad \overrightarrow{V} Z^i = V^i \ . \tag{3.23}$$

Moreover, imposing

$$\partial_i \mathcal{V}|_{P_\alpha=0} \ = \ (0, 0; \mathcal{F}, 0) \ , \tag{3.24}$$

the set of boundary equations is a two-fold duality extension of the PV system, *viz.*

$$F + \mathcal{F}(B; J^I) = 0 \ , \qquad DB = 0 \ , \tag{3.25}$$

where

$$\mathcal{F}(B; J^I) \ := \ \sum_{n\geqslant 0} \mathcal{F}_n(J^I) \star B^{\star n} \ , \quad \mathcal{F}_n(J^I) \ = \ \sum_{k\geqslant 0} \mathcal{F}_{n,I_1\dots I_k} J^{I_1} \star \cdots \star J^{I_k} \ , \tag{3.26}$$

for a set of complex constants $\mathcal{F}_{n,I_1\dots I_k}$.

## 3.3 The master action

In order to obtain a model that admits consistent truncations to three-dimensional CS higher spin gravities, we need to assume that $\mathcal{V}$ contains a term that is quadratic in $T$. The simplest possible such action is given by

$$\boxed{S_{\rm H} = \int_{\mathcal{M}_6} \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4} \left[S \star DB + T \star \left[F + g + h \star (B - \tfrac{1}{2}\mu \star T)\right] + \mu \star B \star S \star S\right]} \tag{3.27}$$

where

$$g = g(J^I) \ , \qquad h = h(J^I) \ , \qquad \mu = \mu(J^I) \tag{3.28}$$

are even closed and central elements on $\mathcal{M}_6$ in degrees

$$\deg(g, h, \mu) = (2 \bmod 2, 2 \bmod 2, 0 \bmod 2) \ . \tag{3.29}$$

[9] If $\mathcal{U}_{\rm symm}$ is a totally symmetric $\star$-function, then $\partial_i \mathcal{U}_{\rm symm} \ = \ \overrightarrow{\partial}_i \mathcal{U}_{\rm symm}$.

The reality conditions are given by

$$(A, B; T, S; g, h, \mu)^\dagger = (-A, B; -T, S; -g, -h, -\mu) \ , \tag{3.30}$$

The total variation

$$\begin{aligned}\delta S_{\rm H} &= \int_{\mathcal{M}_6} {\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4}\Big(\delta T \star \mathcal{R}^A + \delta S \star \mathcal{R}^B + \delta A \star \mathcal{R}^T + \delta B \star \mathcal{R}^S\Big) \\ &\quad + \oint_{\partial\mathcal{M}_6} {\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4}(T \star \delta A - S \star \delta B) \ ,\end{aligned} \tag{3.31}$$

where the Cartan curvatures read

$$\begin{aligned}&\mathcal{R}^A = F + g + h \star (B - \mu \star T) \approx 0 \\ &\mathcal{R}^B = DB + \mu \star [S, B]_\star \approx 0 \\ &\mathcal{R}^T = DT + [S, B]_\star \approx 0 \\ &\mathcal{R}^S = DS + h \star T + \mu \star S \star S \approx 0\end{aligned} \tag{3.32}$$

The generalized Bianchi identities are

$$D\mathcal{R}^A \equiv h \star (\mathcal{R}^B - \mu \star \mathcal{R}^T) \ , \tag{3.33}$$

$$D\mathcal{R}^B \equiv [(\mathcal{R}^A + \mu \star \mathcal{R}^S), B]_\star - \mu \star \{\mathcal{R}^B, S\}_\star \ , \tag{3.34}$$

$$D\mathcal{R}^T \equiv [\mathcal{R}^A, T]_\star + [\mathcal{R}^S, B]_\star - \{\mathcal{R}^B, S\}_\star \ , \tag{3.35}$$

$$D\mathcal{R}^S \equiv [\mathcal{R}^A, S]_\star + \mu \star [\mathcal{R}^S, S]_\star + h \star \mathcal{R}^T \ . \tag{3.36}$$

The gauge transformations

$$\delta_{\epsilon,\eta} A = D\epsilon^A - h \star (\epsilon^B - \mu \star \eta^T) \ , \tag{3.37}$$

$$\delta_{\epsilon,\eta} B = D\epsilon^B - [\epsilon^A, B]_\star - \mu \star [\eta^S, B]_\star + \mu \star \{S, \epsilon^B\}_\star \ , \tag{3.38}$$

$$\delta_{\epsilon,\eta} T = D\eta^T - [\epsilon^A, T]_\star - [\eta^S, B]_\star + \{S, \epsilon^B\}_\star \ , \tag{3.39}$$

$$\delta_{\epsilon,\eta} S = D\eta^S - [\epsilon^A, S]_\star - \mu \star [\eta^S, S]_\star - h \star \eta^T \ , \tag{3.40}$$

which transform the Cartan curvatures into each other, induce

$$\delta_{\epsilon,\eta} S_{\rm H} = \int_{\mathcal{M}_6} {\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4}\Big(\eta^T \star [F + g + h \star B] + \eta^S \star DB\Big) \ . \tag{3.41}$$

We take $(\epsilon^B;\eta^T,\eta^S)$ to belong to sections of the structure group and impose[10]

$$(\eta^T,\eta^S)|_{\partial\mathcal{M}_6}=0\ . \tag{3.42}$$

We have also assumed that $(A,B)$ fluctuate on $\partial\mathcal{M}_6$, which implies[11]

$$T|_{\partial\mathcal{M}_6}\approx 0\approx S|_{\partial\mathcal{M}_6}\ . \tag{3.43}$$

The resulting boundary equations of motion

$$F+g+h\star B\approx 0\ ,\quad DB\approx 0 \tag{3.44}$$

thus provide a duality extended version of the Prokushkin–Vasiliev equations, that is free from any interaction ambiguity, following a variational principle.

In the action (3.27), the relative coefficient of the $BSS$ and $TT$ terms is fixed uniquely by Cartan integrability. The action is invariant under $(B,S;\mu,h)\to(\lambda\star B,\lambda^{-1}\star S;\lambda\star\mu,\lambda^{-1}\star h)$ for closed and central elements $\lambda=\lambda(J^i)$ of degree 0 mod 2 that are real and invertible. The canonical transformation $(A,B)\to(A-\frac{1}{2}\mu\star S,B+\frac{1}{2}\mu\star T)$ leads to replacement of $\int_{\mathcal{M}_6}\mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4}\left[-\frac{1}{2}\mu\star T\star T+\mu\star B\star S\star S\right]$ in (3.27) by $\frac{1}{4}\int_{\mathcal{M}_6}\mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4}\mu\star T\star S\star S-\frac{1}{2}\oint_{\partial\mathcal{M}_6}\mathrm{Tr}_{\mathcal{A}\otimes\mathcal{C}_4}\mu\star T\star S$. However, as we shall see, the form of the Hamiltonian action for the PV system that lends itself most straightforwardly to consistent truncations of the $B$ field is given by (3.27).

# 4 Consistent truncations

In this section we perform consistent truncations of the covariant Hamiltonian master action in six dimensions down to various models on $\mathcal{X}_4$ and $\mathcal{Z}_2$. The truncations consist of integrating out the fluctuations in $B$ around its vacuum expectation value $\nu\Gamma$ followed by reductions on $\mathcal{Z}_2$ and $\mathcal{X}_4$. On $\mathcal{X}_4$, we reach $BF$-like models with Lagrangian forms containing Blencowe's action for $\nu=0$ and a $\nu$-deformed version thereof that we present in Appendix B. For $\nu=0$, the reduction to $\mathcal{Z}_2$ yields the Prokushkin–Segal–Vasiliev (PSV) action.

A consistent truncation a system with action $S[\varphi]$ and equations of motion $E(\varphi)=0$ amounts to an Ansatz $\varphi=\varphi(\varphi')$ off-shell such that $E(\varphi(\varphi'))=0$ are equivalent to a set of equations

[10] Following the AKSZ approach, the Batalin–Vilkovisky classical master equation requires that the ghosts corresponding to $(\eta^T,\eta^S)$ vanish at $\partial\mathcal{M}_6$ off-shell.

[11] Following the AKSZ approach, the Batalin–Vilkovisky classical master equation requires that (3.43) holds off-shell.

$E'(\varphi')=0$ that i) are integrable without any algebraic constraints on $\varphi'$; and ii) follow by applying the variational principle to the reduced action $S_{\rm red}[\varphi']:=S[\varphi(\varphi')]$.

### 4.1 Reduction to BF-like extension of Blencowe's action

Starting from the equations of motion (3.32) and setting $B=0$ yields

$$F+g-h\star\mu\star T=0\ ,\qquad DT=0\ ,\tag{4.1}$$

and

$$DS+h\star T+\mu\star S\star S=0\ ,\tag{4.2}$$

which together form a Cartan integrable system containing (4.1) as a subsystem, *i.e.* the free differential algebra generated by $(A,T,S)$ contains a subalgebra generated by $(A,T)$. Assuming $\partial\mathcal{M}_6$ to consist of a single component, it follows from $S|_{\partial\mathcal{M}_6}=0$ that $S$ can be reconstructed from $(A,T)$ on-shell [12] from (4.2). Therefore, the system (4.1) is a consistent truncation of the original system (3.32) on-shell.

Rewriting the full action (3.27) by integrating by parts in its $SDB$-term yields

$$S_{\rm H}=\int_{\mathcal{M}_6}{\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4}\left[T\star(F+g-\tfrac{1}{2}h\star\mu\star T)+B\star(DS+h\star T+\mu\star S\star S)\right]\ .\tag{4.3}$$

It follows that $B=0$ is a saddle point of the path integral at which $B$ and $S$ can be integrated out in a perturbative expansion. Schematically, modulo gauge fixing, one has

$$\int_{\langle B\rangle=0}[DB][DS]e^{\frac{i}{\hbar}S_{\rm H}}\sim e^{\frac{i}{\hbar}S_{\rm eff}[A,T]}\ ,\tag{4.4}$$

where the effective action

$$S_{\rm eff}[A,T]=S_{\rm red}[A,T]+O(\hbar)\ ,\tag{4.5}$$

consists of loop corrections (comprising attendant functional determinants on noncommutative manifolds) and

$$S_{\rm red}=\int_{\mathcal{M}_6}{\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4}T\star(F+g-\tfrac{1}{2}h\star\mu\star T)\ .\tag{4.6}$$

The latter is a consistently reduced classical action in the sense that it reproduces the subsystem (4.1). The reduced system, which thus consists if the gauge sector of the original system, is a topological theory with local symmetries

$$\delta A=D\epsilon+\mu\star h\star\eta\ ,\quad \delta T=D\eta-[\epsilon,T]_\star\ ,\tag{4.7}$$

[12] Since $T|_{\partial\mathcal{M}_6}=0$ on-shell as well it follows that both $S$ and $T$ can be taken to vanish on $\mathcal{M}_6$ on-shell.

and equations of motion and boundary conditions given by

$$F + g - \mu \star h \star T = 0 \ , \quad DT = 0 \ , \tag{4.8}$$

$$T|_{\partial \mathcal{M}_6} = 0 \ . \tag{4.9}$$

The boundary equations are thus given by

$$(F + g)|_{\partial \mathcal{M}_6} = 0 \ . \tag{4.10}$$

To address Blencowe's theory, we truncate once more by reducing (4.1) under the assumptions that

$$g = \check{g}_{[2]} - \mu_0 J \star \check{g}'_{[2]} \ , \quad \mu = \mu_{[0]} \equiv \mu_0 \ , \quad h = J \ , \tag{4.11}$$

where $\mu_0$ is an imaginary constant, and that

$$A = \check{W}_{[1]} - \check{K}_{[1]} - \mu_0 J \star \check{K}_{[1]} \ , \tag{4.12}$$

$$T = \check{T}_{[2]} + \check{K}_{[1]} \star \check{K}_{[1]} - \mu_0 J \star \check{T}_{[2]} \ , \tag{4.13}$$

where by definition

$$\check{f} \in \Omega(\mathcal{X}_4) \otimes 1_{\Omega(\mathcal{Z}_2)} \otimes \check{\mathcal{A}} \otimes \check{\mathcal{C}}_4 \ , \tag{4.14}$$

in terms of an associative algebra $\check{\mathcal{A}}$ of $\pi$-projected symbols of $y^\alpha$ (to be specified below). Thus

$$d\check{f} = d_X \check{f} \ , \qquad \pi(\check{f}) = \check{f} \ , \tag{4.15}$$

as required for $\pi(A, T) = (A, T)$. Defining

$$\check{F} = d_X \check{W} + \check{W} \star \check{W} \ , \quad \check{D}\check{K} = d_X \check{K} + [\check{W}, \check{K}]_\star \ , \quad \check{D}\check{T} = d_X \check{T} + [\check{W}, \check{T}]_\star \ , \tag{4.16}$$

suppressing the subscripts indicating form degrees, the reduction of (4.1) yields

$$\check{F} + \check{T} + \check{g} + \check{g}' = 0 \ , \qquad \check{D}\check{T} = 0 \ , \tag{4.17}$$

$$D\check{K} - \check{K} \star \check{K} + \check{T} + \check{g}' = 0 \ , \tag{4.18}$$

which is a Cartan integrable system containing (4.17) as a subsystem. From (4.9) and (4.13), we deduce the boundary conditions

$$\check{T}|_{\partial \mathcal{X}_4} = 0 = (\check{K} \star \check{K})|_{\partial \mathcal{X}_4} \ , \tag{4.19}$$

which are compatible with (4.18) since $[\check{g}', \check{K}]_\star = 0$. Substituting (4.12) and (4.13) into (4.6) and using (4.19) we obtain

$$\check{S}_{\rm red}[\check{W}, \check{T}] = -\mu_0 \int_{\mathcal{X}_4} \int_{\mathcal{Z}_2} {\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4} J \star \check{T} \left(\check{F} + \check{g} + \check{g}' + \tfrac{1}{2}\check{T}\right) \ , \tag{4.20}$$

which reproduces (4.17) on-shell, implying that truncation (4.12)–(4.14) is indeed consistent.

There are two independent embeddings of Blencowe's model into the above master action. They can be obtained by choosing the fiber algebras

$$m = 0: \quad \check{\mathcal{A}} \otimes \check{\mathcal{C}}_4 = \left({\rm Aq}^+(2) \oplus ({\rm Aq}^+(2) \star \kappa_y)\right) \otimes \mathcal{C}_4 \ , \tag{4.21}$$

$$m = 1: \quad \check{\mathcal{A}} \otimes \check{\mathcal{C}}_4 = \left({\rm Aq}^+(2) \oplus ({\rm Aq}^+(2) \star \kappa_y)\right) \otimes \mathcal{C}_4^+ \ , \tag{4.22}$$

and equipping $\mathcal{A} \otimes \mathcal{C}_4$ with trace operations as follows[13]:

$${\rm Tr}^m_{\mathcal{A}\otimes\mathcal{C}_4}(f_0 + f_1 \star \kappa_y) := \int_{\mathcal{Y}_2} \frac{d^2y}{2\pi} {\rm Tr}_{\mathcal{C}_4}(\Gamma)^m \kappa_y \star f_m \equiv {\rm STr}_{{\rm Aq}(2)} {\rm Tr}_{\mathcal{C}_4}(\Gamma)^m f_m \ , \qquad m = 0, 1 \ , \tag{4.23}$$

where $f_m \in {\rm Aq}^+(2) \otimes \mathcal{C}_4$ and

$$\int_{\mathcal{Y}_2} \frac{d^2y}{2\pi} = \int_{\mathbb{R}^2} \frac{d^2y}{2\pi} \ , \qquad {\rm Tr}_{\mathcal{C}_4} \sum_{k=0}^{4} f_{i_1..i_k} \Gamma^{[i_1}..\Gamma^{i_k]} := f_0 \ . \tag{4.24}$$

The factorization formula (2.39) then yields[14]

$$\int_{\mathcal{Z}_2} {\rm Tr}^m_{\mathcal{A}\otimes\mathcal{C}_4} J \star (\check{f}_0 + \check{f}_1 \star \kappa_y) = i\pi \, {\rm STr}_{{\rm Aq}(2)} {\rm Tr}_{\mathcal{C}_4}(\Gamma)^m \check{f}_{1-m} \ . \tag{4.25}$$

We truncate the models further as follows:

$$m = 0 \quad : \qquad \check{W} = \Pi^+_{\kappa_y} \star W_+ + \Pi^-_{\kappa_y} \star W_- \ , \qquad \check{T} = \Pi^+_{\kappa_y} \star T_+ + \Pi^-_{\kappa_y} \star T_- \ , \tag{4.26}$$

$$m = 1 \quad : \qquad \check{W} = \Pi^+_\Gamma W_+ + \Pi^-_\Gamma W_- \ , \qquad \check{T} = \Pi^+_\Gamma T_+ + \Pi^-_\Gamma T_- \ , \tag{4.27}$$

where $W_\pm$ and $T_\pm$ are independent of $\Gamma_i$ and $\kappa_y$, and

$$\Pi^\pm_{\kappa_y} = \frac{1 \pm \kappa_y}{2} \ . \tag{4.28}$$

Inserting (4.26) and (4.27) into (4.20) and using

$$m = 0: \qquad \int_{\mathcal{Z}_2} {\rm Tr}^0_{\mathcal{A}\otimes\mathcal{C}_4} J \star \Pi^\pm_{\kappa_y} f = \pm \tfrac{i\pi}{2} \, {\rm STr}_{{\rm Aq}(2)} f \ , \tag{4.29}$$

[13] The (graded cyclic) supertrace obeys ${\rm STr}_{{\rm Aq}(2)} f \star g = {\rm STr}_{{\rm Aq}(2)} g \star \pi(f)$ and ${\rm STr}_{{\rm Aq}(2)} f = f(0)$ provided that $f(y)$ is the symbol of $f$ defined in Weyl order.

[14] We use the normalizations $dz^\alpha dz_\alpha = -2dz^1 dz^2 = -2d^2z$ and $\int d^2y d^2z \kappa \star f(y) = 4\pi^2 f(0)$.

$$m = 1: \quad \int_{\mathcal{Z}_2} \mathrm{Tr}^1_{\mathcal{A}\otimes\mathcal{C}_4} J \star \Pi^{\pm}_{\Gamma} f = \pm \tfrac{i\pi}{2} \, \mathrm{STr}_{\mathrm{Aq}(2)} \, f \ , \tag{4.30}$$

for $f$ independent of $\Gamma_i$ and $\kappa_y$, yields the following four-dimensional Hamiltonian extension of Blencowe's action:

$$S_{\mathrm{Bl}} = -\tfrac{i\pi}{2} \, \mu_0 \int_{\mathcal{X}_4} \mathrm{STr}_{\mathrm{Aq}(2)} \left[ T_+ (F_+ + \check{g} + \check{g}' + \tfrac{1}{2} T_+) - T_- (F_- + \check{g} + \check{g}' + \tfrac{1}{2} T_-) \right] \ , \tag{4.31}$$

which is thus reached for both $m = 0$ and $m = 1$.

Assuming that $\mathcal{X}_4 = \mathcal{X}_3 \times [0, \infty[$ and that all fields fall off at $\mathcal{X}_3 \times \infty$, and assuming furthermore that $\mathcal{X}_3$ has a simple topology such that

$$\check{g} + \check{g}' = 0 \ , \tag{4.32}$$

the elimination of the Lagrange multipliers yields

$$S_{\mathrm{Bl}} = \tfrac{i\pi}{2} \, \mu_0 \left( S_{\mathrm{CS}}[W_+] - S_{\mathrm{CS}}[W_-] \right) \ , \tag{4.33}$$

with

$$S_{\mathrm{CS}}[W] = \oint_{\mathcal{X}_3} \mathrm{STr}_{\mathrm{Aq}(2)} \left[ \tfrac{1}{2} W \star dW + \tfrac{1}{3} W \star W \star W \right] , \tag{4.34}$$

where now $d$ denotes the exterior derivative on $\mathcal{X}_3$. Equivalently,

$$S_{\mathrm{Bl}} = i\mu_0 \pi \oint_{\mathcal{X}_3} \mathrm{STr}_{\mathrm{Aq}(2)} \left[ E \star (d\Omega + \Omega \star \Omega) + \tfrac{1}{3} \, E \star E \star E \right] , \qquad W_\pm = \Omega \pm E \ , \tag{4.35}$$

from which we identify

$$\mu_0 = -\frac{4i}{\pi^2} \frac{\ell_{\mathrm{AdS}}}{G_{\mathrm{N}}} \tag{4.36}$$

using the conventions of [31]. Relaxing the assumption on $\check{g} + \check{g}'$ by taking it to be a nontrivial element in the de Rham cohomology of $\mathcal{X}_3$, Blencowe's action is accompanied by the extra term

$$\begin{aligned} S_g &= 2i\pi\mu_0 \int_{\mathcal{X}_4} \mathrm{STr}_{\mathrm{Aq}(2)} (\check{g} + \check{g}') \star \check{F} = 2i\pi\mu_0 \oint_{\mathcal{X}_3} \mathrm{STr}_{\mathrm{Aq}(2)} (\check{g} + \check{g}') \star \check{W} \\ &= i\pi\mu_0 \oint_{\mathcal{X}_3} \mathrm{STr}_{\mathrm{Aq}(2)} (\check{g} + \check{g}') \star E \ , \end{aligned} \tag{4.37}$$

which is the flux of the central gauge fields in $\check{W}$ through the two-cycle dual to $\check{g} + \check{g}'$. Thus, the modified Blencowe equations of motion take the form

$$d\Omega + \Omega \star \Omega + E \star E = -(\check{g} + \check{g}') \ , \qquad dE + \Omega \star E + E \star \Omega = 0 \ . \tag{4.38}$$

## 4.2 Reduction to PSV action

Instead of reducing (3.27) on $\mathcal{Z}_2$ one may consider a reduction on $\mathcal{X}_4$ under the assumption that

$$\partial\mathcal{X}_4 = \emptyset \ , \tag{4.39}$$

as well as $\partial\mathcal{Z}_2 = \emptyset$. The absence of any boundary condition on $T$ implies that its integration constant in form degree zero contains local degrees of freedom. To exhibit the model on $\mathcal{Z}_2$, we first perform a consistent truncation by setting

$$B = \nu\Gamma \ , \tag{4.40}$$

leading to the reduced action

$$S_{\rm red}[A,T] = \int_{\mathcal{M}_6} {\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4}\Big[T\star(F+g+\nu\Gamma h-\tfrac{1}{2}h\star\mu\star T)\Big] \ . \tag{4.41}$$

We then proceed by introducing a volume form $\check{J}_{[4]}$ on $\mathcal{X}_4$ and background potentials $\check{W}^{(0)}_{[3]}$ and $V^{(0)}_{[1]}$ on $\mathcal{X}_4$ and $\mathcal{Z}_2$, respectively, defined by

$$d_X\check{W}^{(0)}_{[3]} = \check{J}_{[4]} \ , \qquad F^{(0)}_{[2]} + \nu\Gamma J = 0 \ , \qquad F^{(0)}_{[2]} := d_Z V^{(0)}_{[1]} + V^{(0)}_{[1]}\star V^{(0)}_{[1]} \ . \tag{4.42}$$

In particular, we take $\check{W}^{(0)}_{[3]}$ to be independent of the internal coordinates $y^\alpha$. We next perform a further truncation by taking

$$h = J + i\check{J}_{[4]} \ , \qquad \mu = \mu_0 \ , \qquad g = g'_{[2]} \ , \tag{4.43}$$

and considering the Ansatz

$$A = V^{(0)}_{[1]} + V'_{[1]} - \mu_0\,\check{W}^{(0)}_{[3]}\star(1+i[\alpha-\beta]J)\star C' - i\nu\Gamma\,\check{W}^{(0)}_{[3]} \ , \tag{4.44}$$

$$T = i\,(1+i\alpha J+\beta\check{J}_{[4]})\star C' \ , \tag{4.45}$$

with $\alpha,\beta\in\mathbb{R}$ and fluctuating fields

$$f'\in 1|_{\mathcal{X}_4}\otimes\Omega(\mathcal{Z}_2)\otimes\mathcal{A}'\otimes\mathcal{C}^+_4 \ , \qquad \pi(f') = f' \ , \qquad df' = d_Z f' \ , \tag{4.46}$$

where $\Omega(\mathcal{Z}_2)\otimes\mathcal{A}'$ consists of an algebra of $\pi$-invariant master fields. Defining

$$F' = d_Z V' + \{V^{(0)},V'\}_\star + V'\star V' \ , \qquad D'C' = d_Z C' + [V^{(0)}+V',C']_\star \tag{4.47}$$

and suppressing the subscripts denoting form degrees, one has

$$F = F^{(0)} + F' - i\nu\Gamma\,\check{J} - \mu_0\,\check{J}\star[1+i(\alpha-\beta)J]\star C' + \mu_0\check{W}^{(0)}\star D'C'\ . \tag{4.48}$$

where we used that $J\star D'C'\equiv 0$ being a 3-form on $\mathcal{Z}_2$ and $\{V^{(0)}+V',\check{W}^{(0)}\}_\star\equiv 0$ since $\check{W}^{(0)}$ is independent of $y^\alpha$ .

The equations of motion of (4.41) on the above Ansatz read

$$F' + g' - i\mu_0 J\star C' = 0\ ,\qquad D'C' = 0\ ,\qquad \mu_0\,(\beta-2\alpha)\,J\star\check{J}_{[4]}\star C' = 0\ , \tag{4.49}$$

while plugging the Ansatz back into the action (4.41) yields

$$S'_{\rm red} = i\int_{\mathcal{X}_4}\check{J}_{[4]}\int_{\mathcal{Z}_2}{\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4}\Big(C'\star\big[\,\beta(F'_{[2]}+g'_{[2]}) - i\,\mu_0\,\alpha\;J\star C'\,\big]\Big)\ , \tag{4.50}$$

from which it follows that the Ansatz leads to a nontrivial and consistent truncation provided

$$\beta = 2\alpha\ . \tag{4.51}$$

In order to define the combined integration over $\mathcal{Z}_2$ and trace operation $\mathcal{A}$, we may take

$$\nu = 0\ ,\qquad g' = 0\ . \tag{4.52}$$

The background connection $V^{(0)}$ thereby is flat. The simplest choice amounts to take $V^{(0)} = 0$. We then choose

$${\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4} = \int_{\mathcal{Y}_2}\frac{d^2y}{2\pi}{\rm Tr}_{\mathcal{C}_4} \tag{4.53}$$

and make the redefinition[15]

$$C' = \kappa\star b'\ ,\qquad s'_\alpha = z_\alpha - 2i\,V'_\alpha\ , \tag{4.54}$$

so that

$$F' = -\tfrac{1}{4}dz^\alpha\wedge dz^\beta\left(s'_\alpha\star s'_\beta + i\epsilon_{\alpha\beta}\right)\ , \tag{4.55}$$

and the reduced action now reads

$$S'_{\rm red} = \frac{\alpha}{2}\,{\rm Vol}(\mathcal{X}_4)\int_{\mathcal{Z}_2}d^2z\;{\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4}\;\kappa\star(is'^\alpha\star s'_\alpha\star b' + 2b' + i\mu_0\,b'\star b')\ , \tag{4.56}$$

where ${\rm Vol}(\mathcal{X}_4) = \int_{\mathcal{X}_4}\check{J}_{[4]}$. The above action is identified with the original action given in [14] upon taking $\mu_0 = -i$ and ${\rm Vol}(\mathcal{X}_4) = \frac{2}{\alpha}$.

[15]One could as well take the flat connection $V^{(0)} = -iz_\alpha\,dz^\alpha$ together with $s'_\alpha = -z_\alpha - 2iV'_\alpha$.

# 5 Conclusions

We have presented an action principle for the bosonic sector of Prokushkin and Vasiliev's three-dimensional matter coupled higher spin gravity. By integrating out the matter fields, in a fashion that amounts to a consistent truncation in the classical limit, followed by consistent dimensional reductions, we have found that the action contains various higher spin CS models as well as the action [14] of Prokushkin, Segal and Vasiliev on twistor space.

The construction rests on three ingredients: i) Cartan and Vasiliev's unfolded formulation of classical field theory in terms of vanishing curvatures whereby the local degrees of freedom arise via the Weyl zero-form (as captured by harmonic expansions taking place in associative bundles independently of the dimension of the base manifold); ii) the usage of noncommutative twistor spaces for describing massless Weyl zero-forms in constantly curved backgrounds; and iii) the AKSZ formulation of quantum field theories based on covariant Hamiltonian actions on open bulk manifolds (whose boundaries contain the classical Cartan integrable systems).

When applied to massless degrees of freedom in three dimensions with spin greater than one, the above approach naturally leads to actions in six dimensions containing two-dimensional noncommutative twistor spaces. Their reductions on twistor space yields BF-like actions on four-manifolds, given by spacetimes times the extra auxiliary AKSZ radius, in their turn containing the standard symplectic structures of the three-dimensional massless gauge fields of spin greater than one (which are CS theories). Thus, modulo technicalities having to do with consistency of the reduction schemes and the structure of the modules making up the associative bundles underlying higher spin gravities, there is a clear overlap in four dimensions between the standard CS formulation of three-dimensional higher spin gauge fields and the covariant Hamiltonian formulation in six dimensions.

Turning to applications, it would be interesting to see to what extent the action, possibly supplemented by boundary terms, can be used to compute the free energy and entropy of exact solutions of the PV system, such as the recent nontrivial examples found in [32]; for related proposals for on-shell actions, see [12] and [33]. The action could also have a bearing on the one-loop corrections from matter fields to the higher spin CS gauge sector. The above implementations may be useful in solidifying the Gaberdiel–Gopakumar (GG) conjecture [9]. In particular, radiative corrections may be of importance in matching symmetry algebras [34] beyond the realm of CS actions; for reviews of the CS approximation, see [35], and for existing works beyond the CS approximation, see [36, 37, 18].

As for alternatives to the PV system, an interesting action for matter coupled higher spin gravity has been presented in [12]. Its four-dimensional covariant Hamiltonian reformulation is given by the BF-like action

$$S = \int_{\mathcal{X}_4} \mathrm{STr}\left(T \star (F + B \star \widetilde{B} + \tfrac{1}{2}T) - \widetilde{T} \star (\widetilde{F} + \widetilde{B} \star B) + \widetilde{S} \star DB + S \star \widetilde{D}\widetilde{B}\right) , \tag{5.1}$$

where $(A, \widetilde{A}, B, \widetilde{B}; T, \widetilde{T}, \widetilde{S}, S)$ are forms of degrees $(1, 1, 0, 2; 2, 2, 3, 1)$ valued in $\mathrm{Aq}^+(2) \cong hs(\frac{1}{2})$ and

$$F = dA + A \star A , \qquad DB = dB + A \star B - B \star \widetilde{A} , \tag{5.2}$$

$$\widetilde{F} = d\widetilde{A} + \widetilde{A} \star \widetilde{A} , \qquad \widetilde{D}\widetilde{B} = d\widetilde{B} + \widetilde{A} \star \widetilde{B} - \widetilde{B} \star A . \tag{5.3}$$

The action, with its dynamical two-form $\tilde{B}$, cannot be obtained from the six-dimensional master action (3.27), as there is an obstruction due to the presence of the central term $h$. Instead, it is natural to seek a connection between (5.1) and the PV system via a six-dimensional model on $\mathcal{X}_4 \times \mathcal{Z}_2$ built along the same lines as the nine-dimensional Frobenius–Chern–Simons model in [17]. The construction of such a model will be presented elsewhere.

Comparing higher spin gravities in three and four dimensions, the latter admit covariant Hamiltonian actions in nine dimensions [16, 17], though it remains unclear whether they contain the standard symplectic structure for Fronsdal fields [38] [16]. Essentially, this is due to the presence of extra auxiliary fields in the unfolded description of Fronsdal fields on-shell, whose inclusion into a strictly four-dimensional off-shell formulation remains problematic. However, the on-shell actions receive contributions as well from boundary terms [21, 27] given by topological invariants that reduce on-shell to higher spin invariants [21]. These invariance, which are inserted on the eight-dimensional boundaries, are given by integrals over closed $p$-cycles in spacetime of on-shell (de Rham) closed $p$-forms in their turn given by integrals over twistor space of constructs built from spacetime curvatures. In terms of these observables, spacetime emerges in limits where physical states labelled by spacetime points become separated from each other [39]; see also the Conclusions of [23]. In particular, for $p = 0$, the resulting zero-form charges [40] of the (minimal bosonic) Type A and Type B models [41, 20] were shown in [42, 43] to provide free theory

[16] A related issue is whether four-dimensional higher spin gravity contains boundary states arising in its gauge sector, corresponding to an enhancement of the rigid higher spin symmetry algebra to the algebra of conformal higher spin currents.

correlation functions at the leading classical order, in accordance with the proposal made in [44, 45, 46, 47]. For boundary conditions corresponding to free fields, this proposal requires that correlation functions with separated points do not receive any radiative corrections, in agreement with the covariant Hamiltonian approach [16][17]; for a similar usage of zero-form charges in 3D, see [12].

To conclude, we view higher spin gravity as a useful laboratory for exploring the treatment of quantum field theory with local degrees of freedom by combining the AKSZ approach [26] to topological field theories on manifolds with boundaries and Cartan and Vasiliev's formulation of nonlinear partial differential equations as free differential algebras with infinite zero-form towers. To question its universality, it would be desirable to treat models in which nontrivial radiative corrections arise in an as simple context as possible. To this end, three-dimensional models might prove to be fruitful and we hope that our action will be helpful in this endeavour.

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[17]Feynman diagrams in the nine-dimensional bulk with external zero- and one-forms cannot be built using only vertices containing central and closed forms on the twistor $Z$ space; see also the Conclusions of [27]. The effects on radiative corrections from more general boundary conditions (including closed and central terms in $X$-space), nontrivial duality extensions (whereby massless degrees of freedom are carried also by forms in higher degrees) and other topological effects remain to be seen, however.

# A Massive vacuum of PV system

In this Appendix, we rewrite the massive vacuum of the PV system [13] using the Clifford algebra variables. The vacuum solution reads

$$B^{(0)} = \nu\,\Gamma\ ,\quad A^{(0)} = W^{(0)} + V^{(0)}\ ,\quad \nu \in \mathbb{R}\ ,\quad \Gamma = \Gamma^\dagger \in \mathcal{C}_4^+\ , \tag{A.1}$$

obeying

$$d_X W^{(0)} + W^{(0)} \star W^{(0)} = 0\ ,\qquad d_X S^{(0)}_\alpha + [W^{(0)}, S^{(0)}_\alpha]_\star = 0\ , \tag{A.2}$$

$$[\Gamma, W^{(0)}] = 0\ ,\qquad [\Gamma, S^{(0)}_\alpha]_\star = 0\ ,\qquad [S^{(0)}_\alpha, S^{(0)}_\beta]_\star = -2i\,\epsilon_{\alpha\beta}\big(1 - \nu\,\Gamma\kappa\big)\ , \tag{A.3}$$

where $S^{(0)}_\alpha = z_\alpha - 2iV^{(0)}_\alpha$. The constraints along $\mathcal{M}_3$ can be solved using a gauge function that commutes to $\Gamma$, *viz.*

$$W^{(0)} = L^{-1} \star dL\ ,\qquad S^{(0)}_\alpha = L^{-1} \star \tilde{z}_\alpha \star L\ ,\qquad [\Gamma, L]_\star = 0\ , \tag{A.4}$$

where $\tilde{z}_\alpha$ obeys

$$d_X \tilde{z}_\alpha = 0\ ,\qquad [\Gamma, \tilde{z}_\alpha]_\star = 0\qquad [\tilde{z}_\alpha, \tilde{z}_\beta]_\star = -2i\,\epsilon_{\alpha\beta}\big(1 - \nu\,\Gamma\kappa\big)\ . \tag{A.5}$$

The integrability on $\mathcal{Z}_2$ implies the existence of a double $\tilde{y}_\alpha$ obeying

$$[\tilde{y}_\alpha, \tilde{z}_\beta]_\star = 0\ ,\qquad \{\Gamma, \tilde{y}_\alpha\} = 0\ ,\qquad \tilde{y}_\alpha|_{\nu=0} = Ay_\alpha\ ,\qquad \{A, \Gamma\} = 0\ , \tag{A.6}$$

for some matrix $A \in \mathcal{C}_4$. Thus one can take

$$L = L(\tilde{y}_\alpha, \Gamma_{ij})\ ,\qquad S^{(0)}_\alpha = \tilde{z}_\alpha\ . \tag{A.7}$$

Remarkably, as found in [13], the solution obeys

$$[\tilde{y}_\alpha, \tilde{y}_\beta]_\star = 2i\epsilon_{\alpha\beta}\big(1 - \nu\,\Gamma\big)\ . \tag{A.8}$$

At the level of the complexified algebra, a solution is given by

$$\tilde{z}^{\mathbb{C}}_\alpha = X\,z_\alpha + Y\,\sigma_\alpha\ ,\quad \tilde{y}^{\mathbb{C}}_\alpha = A\,y_\alpha + B\,\tau_\alpha\ , \tag{A.9}$$

where $X$, $Y$, $A$ and $B$ are built out of gamma matrices $\Gamma_i$ and the building blocks

$$\sigma_\alpha := \nu\int_0^1 dt\,t\,e^{ityz}(y_\alpha + z_\alpha)\ ,\qquad \tau_\alpha := \nu\int_0^1 dt\,(t-1)\,e^{ityz}(y_\alpha + z_\alpha)\ , \tag{A.10}$$

with $yz := y^\alpha z_\alpha$, obey

$$[z_{[\alpha}, \sigma_{\beta]}]_\star = -i\nu\, \epsilon_{\alpha\beta}\, \kappa \ , \quad [\sigma_\alpha, \sigma_\beta]_\star = 0 \ , \quad \{\sigma_\alpha, \tau_\beta\}_\star = 0 \ ,$$
$$[z_\alpha, \tau_\beta]_\star = \{\sigma_\alpha, y_\beta\}_\star \ , \quad \{y_{[\alpha}, \tau_{\beta]}\}_\star = i\nu\, \epsilon_{\alpha\beta} \ , \quad [\tau_\alpha, \tau_\beta]_\star = 0 \ . \tag{A.11}$$

For the above Ansatz, Eqs. (A.5), (A.6) and (A.8), respectively, are equivalent to

$$X^2 = 1 \ , \quad XY = YX = -\Gamma \ , \quad [\Gamma, X] = 0 = [\Gamma, Y] \ , \tag{A.12}$$

$$[A, X] = \{A, Y\} = [B, X] = \{B, Y\} = 0 \ , \quad \{\Gamma, X\} = 0 = \{\Gamma, Y\} \ , \tag{A.13}$$

$$A^2 = 1 \ , \qquad AB = -BA = -\Gamma \ . \tag{A.14}$$

A solution for $\tilde{z}_\alpha$ that commutes to $\mathcal{C}_4^+$ is obtained by taking [13][18]

$$\Gamma = \Gamma_{1234} \ , \quad X = 1 \ , \quad Y = -\Gamma \ , \quad A = \Gamma_1 \ , \quad B = -\Gamma_{234} = -\Gamma_1 \Gamma \ , \tag{A.15}$$

where $\Gamma_{i_1..i_k} := \Gamma_{[i_1}..\Gamma_{i_k]}$, and the relation between the PV generators and ours is given in (2.31), which yields

$$\tilde{z}_\alpha = z_\alpha - \nu(y_\alpha + z_\alpha) \int_0^1 dt\, t\, e^{it\, yz}\, \Gamma \ , \quad \tilde{y}_\alpha = \Gamma_1 \left[ y_\alpha - \nu(y_\alpha + z_\alpha) \int_0^1 dt\, (t-1)\, e^{it\, yz}\, \Gamma \right] \ , \tag{A.16}$$

This solution, however, does not satisfy the required reality conditions, *i.e.* $((\tilde{y}^{\mathbb{C}}_\alpha)^\dagger, (\tilde{z}^{\mathbb{C}}_\alpha)^\dagger)$ form a set of deformed oscillators that is linearly independent from $(\tilde{y}^{\mathbb{C}}_\alpha, \tilde{z}^{\mathbb{C}}_\alpha)$ This is remedied by a highly nontrivial modification found in [13] given by

$$\begin{aligned}
\tilde{z}^{\rm sym}_\alpha &= z_\alpha + \frac{\nu}{8} \int_{-1}^1 ds (1-s) \Big[ e^{\frac{i}{2}(s+1) yz} (y_\alpha + z_\alpha) \star \Phi(\tfrac{1}{2}, 2; -\Gamma\kappa\ \ln|s|^{-\nu}) \\
&\quad + e^{\frac{i}{2}(s+1) yz} (y_\alpha - z_\alpha) \star \Phi(\tfrac{1}{2}, 2; \Gamma\kappa\ \ln|s|^{-\nu}) \Big] \star \kappa\, \Gamma \\
\tilde{y}^{\rm sym}_\alpha &= \Gamma_1 \Big[ y_\alpha + \Gamma \frac{\nu}{8} \int_{-1}^1 ds (1-s)\, e^{\frac{i}{2}(s+1) yz} \Big( (y_\alpha + z_\alpha)\, \Phi(\tfrac{1}{2}, 2; -\Gamma\ \ln|s|^{-\nu}) \\
&\quad - (y_\alpha - z_\alpha)\, \Phi(\tfrac{1}{2}, 2; \Gamma\ \ln|s|^{-\nu}) \Big) \Big] \ ,
\end{aligned} \tag{A.17}$$

where $\Phi(a, b; z)$ is the confluent hypergeometric function.

[18] Using the gauge function, the first order fluctuation $B^{(1)}$ can be written as $B^{(1)} = L^{-1} \star B'^{(1)} \star L$ where $d_X B'^{(1)} = 0$ and $[\tilde{z}^\alpha, B'^{(1)}]_\star = 0$. Thus, if $\tilde{z}^\alpha$ commutes to $\mathcal{C}_4^+$ then $B'^{(1)} = B'^{(1)}(\tilde{y}^\alpha, \Gamma_{ij})$. Otherwise, as $\tilde{z}_\alpha = z_\alpha - 2i V'^{(0)}_\alpha$ where $V'^{(0)} = L \star (d_Z + V^{(0)}) \star L^{-1}$, one may equivalently solve $d_Z B'^{(1)} + [V'^{(0)}, B'^{(1)}]_\star = 0$ iteratively using a homotopy contractor in $\mathcal{Z}_2$-space.

# B BF-like formulation of modified Blencowe action

In this Appendix, modulo a technical assumption on a normalization coefficient (given in Eq. (B.22)), we present a consistent truncation of the master action (3.27) for nontrivial vacuum expectation value of $B$ leading to a model on $\mathcal{X}_4$ in which the gauge fields and the Lagrange multipliers belong non-isomorphic dual spaces.

To this end, we observe that the equations of motion (3.32) admit a consistent truncation given by

$$B = \nu\Gamma \in \mathbb{R} \; , \tag{B.1}$$

as can be seen from the fact that the resulting field equations,

$$F + \nu\Gamma h + g - h \star \mu \star T = 0 \; , \quad DT = 0 \; , \quad DS + h \star T + \mu \star S \star S = 0 \; , \tag{B.2}$$

form a Cartan integrable system. Inserting (B.1) into the master action (3.27), the resulting consistently truncated action is given by

$$S_{\text{red}}[A;T] = \int_{\mathcal{M}_6} \text{Tr}_{\mathcal{A}\otimes\mathcal{C}_4} T \star \left(F + \nu\Gamma h + g - \tfrac{1}{2} h \star \mu \star T\right) \; . \tag{B.3}$$

Its gauge symmetries take the form

$$\delta A = D\epsilon + \mu \star h \star \eta \; , \quad \delta T = D\eta - [\epsilon, T]_\star \; , \tag{B.4}$$

and the equations of motion and boundary conditions are given by

$$F + \nu\Gamma h + g - \mu \star h \star T = 0 \; , \quad DT = 0 \; , \quad T|_{\partial\mathcal{M}_6} = 0 \; . \tag{B.5}$$

To reach a Blencowe type action, we assume Eq. (4.11) and make the reduction

$$A = V^{(0)}_{[1]} + \widetilde{W}_{[1]} - \widetilde{K}_{[1]} - \mu_0 J \star \widetilde{K}_{[1]} \; , \tag{B.6}$$

$$T = \widetilde{T}_{[2]} + \widetilde{K}_{[1]} \star \widetilde{K}_{[1]} - \mu_0 J \star \widetilde{T}_{[2]} \; , \tag{B.7}$$

where the subscripts indicating the form degree will be suppressed henceforth, and $V^{(0)}$ is a twistor space background connection obeying

$$dV^{(0)} + V^{(0)} \star V + \nu\Gamma J = 0 \; , \qquad V^{(0)}|_{\nu=0} = 0 \; , \tag{B.8}$$

and the reduced fields

$$\tilde{f} \in \Omega(\mathcal{X}_4) \otimes \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{C}}_4 \; , \tag{B.9}$$

where $\widetilde{\mathcal{A}}$ is an associative algebra generated by the deformed oscillator $\tilde{y}^\alpha$ obeying

$$d\tilde{y}^\alpha + [V^{(0)}, \tilde{y}^\alpha]_\star = 0 \ , \qquad [\tilde{y}^\alpha, \tilde{y}^\beta]_\star = 2i\epsilon^{\alpha\beta}(1-\nu\Gamma) \ , \qquad \tilde{y}_\alpha|_{\nu=0} = \Gamma_1 y_\alpha \ ; \tag{B.10}$$

see the Appendix A for further details. Thus,

$$\pi(\tilde{f}) = \tilde{f} \ , \qquad d\tilde{f} + [V^{(0)}, \tilde{f}]_\star = d_X \tilde{f} \ , \tag{B.11}$$

and the reduced equations of motion and boundary conditions are given by the counterparts of Eqs. (4.17)–(4.19) with all quantities now valued in (B.9), which form a Cartan integrable system. The consistently reduced action reads

$$\widetilde{S}_{\rm red}[\widetilde{W}, \widetilde{T}] = -\mu_0 \int_{\mathcal{X}_4} \int_{\mathcal{Z}_2} {\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4} J \star \widetilde{T} \left( \widetilde{F} + \breve{g} + \breve{g}' + \tfrac{1}{2}\widetilde{T} \right) \ . \tag{B.12}$$

To obtain the alternative model, we make the choice

$${\rm Tr}_{\mathcal{A}\otimes\mathcal{C}_4} = \int_{\mathcal{Y}_2} \frac{d^2 y}{2\pi} {\rm Tr}_{\mathcal{C}_4} \Gamma \ , \tag{B.13}$$

corresponding to $m = 1$ in Section 4.1. We also take

$$\widetilde{W} \in \Omega_{[1]}(\mathcal{X}_4) \otimes {\rm Aq}^+(2;\nu) \otimes \mathcal{C}_4^+ \ , \tag{B.14}$$

$$\widetilde{T} \in \Omega_{[2]}(\mathcal{X}_4) \otimes \rho({\rm End}^+(\mathcal{F}_\nu^\sigma)) \otimes \mathcal{C}_4^+ \ , \tag{B.15}$$

where $\mathcal{F}_\nu^\sigma$ is the Fock representation space of ${\rm Aq}(2;\nu)$ with ground state having eigenvalue $\sigma$ of the Klein operator $-\Gamma$, and

$$\rho : {\rm End}(\mathcal{F}_\nu^\sigma) \to {\rm Aq}(2;\nu) \tag{B.16}$$

is a monomorphism given by the deformed oscillator realization of ${\rm End}(\mathcal{F}_\nu^\sigma)$. The space ${\rm End}^+(\mathcal{F}_\nu^\sigma)$ consists of the endomorphisms that commute to $\Gamma$. Its oscillator realization

$$\rho({\rm End}^+(\mathcal{F}_\nu^\sigma)) = \bigoplus_{\sigma'=\pm} {\rm Aq}^{\sigma'}(2;\nu) \star P_\nu^\sigma \star {\rm Aq}^{\sigma'}(2;\nu) \ , \tag{B.17}$$

where

$${\rm Aq}^\sigma(2;\nu) = \bigoplus_{\sigma'=\pm} \Pi_\Gamma^{\sigma'} \star {\rm Aq}(2;\nu) \star \Pi_\Gamma^{\sigma\sigma'} \ , \tag{B.18}$$

and

$$P_\nu^\sigma = \frac{2}{1+\nu\sigma} \Pi_\Gamma^{-\sigma} \star \left[ {}_1F_1(\tfrac{3}{2}; \tfrac{3+\nu\sigma}{2}; -2w) \right]^{\rm W} \ , \tag{B.19}$$

where

$$w = \{a^+, a^-\}_\star \ , \qquad a^\pm = u^\pm_\alpha y^\alpha \ , \quad u^{-\alpha} u^+_\alpha = \tfrac{i}{2} \ , \tag{B.20}$$

is the symbol of the oscillator realization of the ground state projector in $\mathrm{End}(\mathcal{F}^\sigma_\nu)$ given in Weyl order; for details, see [25, 31]. The reduced action reads

$$\check{S}_{\mathrm{II}_\nu} = -\mu_0 \int_{\mathcal{X}_4} \int_{\mathcal{Z}_2 \times \mathcal{Y}_2} \frac{d^2 y}{2\pi} \,\mathrm{Tr}_{\mathcal{C}_4} \Gamma J \star \widetilde{T} \star (\widetilde{F} + g + g' + \tfrac{1}{2}\widetilde{T}) \ , \tag{B.21}$$

where $\widetilde{T} \star (\widetilde{F} + g + g')$ and $\widetilde{T} \star \widetilde{T}$ lie in $\rho(\mathrm{End}^+(\mathcal{F}_\nu))$. The Lagrangian is finite provided that

$$\mathcal{N}^\sigma_\nu = \int_{\mathcal{Z}_2 \times \mathcal{Y}_2} \frac{d^2 y}{2\pi} \,\mathrm{Tr}_{\mathcal{C}_4} \Gamma J \star P^\sigma_\nu \ , \tag{B.22}$$

is finite. If so, the dual pairing displayed in (B.14) and (B.15) is non-degenerate, and the equations of motion and boundary conditions read

$$\widetilde{F} + g + g' + \widetilde{T} \approx 0 \ , \quad \widetilde{D}\widetilde{T} \approx 0 \ , \qquad \widetilde{T}|_{\partial \mathcal{X}_4} = 0 \ , \tag{B.23}$$

where the first equation is valued in $\mathrm{Aq}^+(2;\nu)$ and the second equation in $\rho(\mathrm{End}^+(\mathcal{F}_\nu))$. Eliminating $\check{T}$ via the first equation (by inverting the monomorphism), yields

$$\check{S}_{\mathrm{II}_\nu} \approx \frac{1}{2}\mu_0 \int_{\mathcal{X}_4} \int_{\mathcal{Z}_2 \times \mathcal{Y}_2} \frac{d^2 y}{2\pi} \,\mathrm{Tr}_{\mathcal{C}_4} \Gamma J \star (\widetilde{F} + g + g') \star (\widetilde{F} + g + g') \ , \tag{B.24}$$

which is formally divergent for gauge fields given by finite polynomials, unless

$$\widetilde{F} + g + g' \approx 0 \ . \tag{B.25}$$

It is possible to construct a deformation of Blencowe's action by making use of Vasilievs (graded cyclic) supertrace operation $\mathrm{STr}_\nu$ on the Weyl algebra $\mathrm{Aq}(2;\nu)$ of the deformed oscillator algebra (B.10) [15], which is uniquely characterized by $\mathrm{STr}_\nu 1 = 1$ and $\mathrm{STr}_\nu \Gamma = \nu$ (and hence differs from the trace operation proposed above). Using this operation, it straightforward to deform Blencowe's action in $\mathcal{X}_3$ and uplift it to BF-type model in $\mathcal{X}_4$ with action

$$\check{S}_{\mathrm{I}_\nu} = -\mu_0 \int_{\mathcal{X}_4} \mathrm{STr}_\nu \,\mathrm{Tr}_{\mathcal{C}^+_4} \Gamma \,\widetilde{T} \star (\widetilde{F} + g + g' + \tfrac{1}{2}\widetilde{T}) \ , \tag{B.26}$$

where $\widetilde{T}$ and $\widetilde{W}$ are valued in $\mathrm{Aq}^+(2;\nu) \otimes \mathcal{C}^+_4$.

Whether there exists a modification of (B.13) that yields the $\mathrm{STr}_\nu$ operation starting from the master action in six dimensions, possibly by using the trace operation (4.23) for $m = 0$, remains to be seen[19].

[19] The operation $\int_{\mathcal{Y}_2} \mathrm{Tr}_{\mathcal{C}_4}$ is formally graded cyclic on $\mathrm{Aq}(2;\nu)$, but since it sends 1 and $\Gamma$ to 1 and 0, respectively, it cannot be proportional to $\mathrm{STr}_\nu$ and hence it must be ill-defined.

Thus, provided that $\mathcal{N}_\nu^\sigma$ is finite, we have found a covariant Hamiltonian action in which the gauge fields and the Lagrange multipliers belong non-isomorphic dual spaces that is an alternative to $\nu$-deformed Blencowe's action. Whether this action admits a coupling to matter is unknown, while the action presented above appears to be amenable to such couplings on $\mathcal{M}$.

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