Algebraic Classification of Weyl Anomalies in Arbitrary Dimensions

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Conformally invariant systems involving only dimensionless parameters are known to describe particle physics at very high energy. In the presence of an external gravitational field, the conformal symmetry may generalize to the Weyl invariance of classical massless field systems in interaction with gravity. In the quantum theory, the latter symmetry no longer survives: A Weyl anomaly appears. Anomalies are a cornerstone of quantum field theory, and, for the first time, a general, purely algebraic understanding of the universal structure of the Weyl anomalies is obtained, in arbitrary dimensions and independently of any regularization scheme.

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At very high energies, such as, e.g., in the early Universe, all of the particles can be considered as massless, and renormalized matter models are invariant under the conformal group. Since the Weyl transformations are the generalization to curved space of conformal transformations in flat space, there are good reasons to anticipate the Weyl symmetry as a symmetry of a fundamental theory incorporating gravity [1].

On the other hand, symmetries may be broken at the quantum level: Anomalies then appear. The cancellation of anomalies puts severe constraints on the physical content of a theory, as is the case with the standard model (for a review on anomalies in quantum field theory, see, e.g., [2]). In the case of (super)string theory, the critical dimensions correspond to the absence of the two-dimensional Weyl anomaly [3].

The Weyl (or conformal, or trace) anomalies were discovered about 30 years ago [4,5] and still occupy a central position in theoretical physics, partly because of their important rôles within the anti-de Sitter/conformal field theory correspondence and their many applications in cosmology, particle physics, higher-dimensional conformal field theory, supergravity, and strings. The body of work devoted to this subject is, therefore, considerable. A very nonexhaustive list of references can be found, e.g., in [6–9].

The central equations which determine the candidate anomalies in quantum field theory are the Wess-Zumino (WZ) consistency conditions [10]. By using these conditions, the general structure of all of the known anomalies *except the Weyl ones* has been determined by purely algebraic methods featuring descent equations in the manner of Stora and Zumino [11,12]. Such algebraic treatments are crucial, since they are independent of any regularization scheme and very general. The algebraic analysis of anomalies can best be performed within the Becchi-Rouet-Stora-Tyutin (BRST) [13] formulation.

The BRST formulation for the determination of the Weyl anomalies was initiated in the pioneering works [14,15], with explicit results up to spacetime dimension $n = 6$ and the general structure guessed in an arbitrary even dimension. The authors of Refs. [14,15] found that the Weyl anomalies comprise (i) the integral over spacetime of the Weyl scaling parameter times the Euler density of the manifold plus (ii) terms that are given by (the integral of) the Weyl parameter times strictly Weyl-invariant scalar densities. Some of the terms from (ii) can trivially be obtained from contractions of products of the conformally invariant Weyl tensor, while the others are more complicated and involve covariant derivatives of the Riemann tensor. It was also mentioned in Ref. [15] that an algebraic analysis of the Weyl anomalies, similar to the Stora-Zumino treatment of the non-Abelian chiral anomaly in Yang-Mills theory, was unlikely to exist.

Somewhat later, by using dimensional regularization, the authors of Ref. [16] confirmed the structure of the Weyl anomalies found in Refs. [14,15] and extended the results to arbitrary (even) dimensions. The Euler term from class (i) was called a "type-*A* Weyl anomaly," while the terms of (ii) were called "type-*B* anomalies." Very interestingly, they discovered a similitude between the type-*A* Weyl anomaly and the non-Abelian chiral anomaly. Accordingly, they hinted at the existence of an algebraic treatment for the Weyl anomaly, featuring descent equations.

In this Letter, we provide for the Weyl anomalies the general, purely algebraic understanding in the manner of Stora-Zumino that all of the other known anomalies in quantum field theory enjoy, thereby filling a gap in the literature.

The Weyl anomaly being a local functional, i.e., the integral over the n-dimensional spacetime manifold $\mathcal{M}_n$ of a local n form a_1^n at ghost-number unity $gh(a_1^n) = 1$ (cf. [17]), the WZ consistency conditions for the Weyl anomalies [14,15] can be written in terms of local forms:

$$s_W a_1^n + db_2^{n-1} = 0, \qquad a_1^n \neq s_W p_0^n + df_1^{n-1}, \quad (1)$$

$$s_D a_1^n + dc_2^{n-1} = 0, \qquad s_D p_0^n + dh_1^{n-1} = 0. \qquad (2)$$

The BRST differentials s_W and s_D implement the Weyl transformations and the diffeomorphisms, respectively, whereas d denotes the exterior total derivative. Together with the invertible spacetime metric $g_{\mu\nu}$, the other fields of the problem are the Weyl ghost ω and the diffeomorphism ghosts ξ^μ, $gh(\xi^\mu) = gh(\omega) = 1$. The BRST transformations on the fields $\Phi^A = \{g_{\mu\nu}, \omega, \xi^\mu\}$ read

$$s_D g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho},$$
$$s_W g_{\mu\nu} = 2\omega g_{\mu\nu}, \qquad s_D \xi^\mu = \xi^\rho \partial_\rho \xi^\mu,$$
$$s_D \omega = \xi^\rho \partial_\rho \omega, \qquad s_W \xi^\mu = 0 = s_W \omega.$$

It should be understood, throughout this Letter, that the space in which BRST cohomologies are to be computed is the space of local p forms b^p, that is, the (jet) space of spacetime p forms that depend on the fields Φ^A and their derivatives up to some finite (but otherwise unspecified) order, which one denotes [17] by $b^p = \frac{1}{p!} dx^{\mu_1} \dots dx^{\mu_p} b_{\mu_1 \dots \mu_p}(x, [\Phi^A])$.

One unites the differentials $s = s_W + s_D$ and d into a single differential $\tilde{s} = s + d$, therefore working with local total forms. The latter are, by definition, formal sums of local forms with different form degrees and ghost numbers $\alpha = \sum_{p=0}^n a_{G-p}^p$, the total degree G being simply the sum of the form degree and the ghost number.

Powerful techniques for the computation of local BRST cohomologies in top form degree are exposed in Ref. [18] and allow one to consider local total forms depending only on a subset $\mathcal{W}$ of the set of local total forms, such that $\tilde{s}\mathcal{W} \subset \mathcal{W}$. For the general class of theories studied here, the corresponding space $\mathcal{W}$ was obtained in Ref. [19].

Accordingly, denoting $\tilde{s}_W = s_W + d$ and similarly for s_D, the problem (1) and (2) amounts to determining the $\tilde{s}_D$-invariant $(n+1)$-local total forms $\alpha(\mathcal{W})$ satisfying

$$\tilde{s}_W \alpha(\mathcal{W}) = 0, \qquad \alpha(\mathcal{W}) \neq \tilde{s}_W \zeta(\mathcal{W}) + \text{const}, \qquad (3)$$

where $\zeta(\mathcal{W})$ must be $\tilde{s}_D$-invariant.

Thanks to very general results explained in Ref. [18], we know that the solution of (3) will take the form

$$\alpha(\mathcal{W}) = 2\omega \tilde{C}^{N_1} \dots \tilde{C}^{N_n} a_{N_1 \dots N_n}(\mathcal{T}). \qquad (4)$$

The space $\mathcal{T}$ is generated by [19] the (invertible) metric $g_{\mu\nu}$ together with the W tensors $\{W_{\Omega_i}\}$, $i \in \mathbb{N}$, whose precise form will not be needed here. For the purposes of the present Letter, it suffices to know that they contain the conformally invariant Weyl tensor $W^\mu{}_{\nu\rho\sigma}$ and its first covariant derivative $\nabla_\tau W^\mu{}_{\nu\rho\sigma}$. The symbol ∇ denotes the usual torsion-free metric-compatible covariant differential associated with the Christoffel symbols $\Gamma^\mu{}_{\nu\rho}$. The Ricci tensor is $\mathcal{R}_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$, where $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} + \cdots$ is the Riemann tensor. The scalar curvature is given by $\mathcal{R} = g^{\alpha\beta}\mathcal{R}_{\alpha\beta}$. Then one can write the Weyl tensor as $W^\mu{}_{\nu\rho\sigma} = R^\mu{}_{\nu\rho\sigma} - 2(\delta^\mu_{[\rho} K_{\sigma]\nu} - g_{\nu[\rho} K_{\sigma]}{}^\mu)$, where the tensor $K_{\mu\nu} = \frac{1}{n-2}(\mathcal{R}_{\mu\nu} - \frac{1}{2(n-1)} g_{\mu\nu}\mathcal{R})$ plays a key rôle in the classification of the Weyl anomalies, as we will see. Square brackets denote strength-one complete antisymmetrization. We also need to recall the definition of the Cotton tensor: $C_{\alpha\rho\sigma} = 2\nabla_{[\sigma} K_{\rho]\alpha}$.

The so-called generalized connections $\tilde{C}^N$ in (4) are obtained from Ref. [19] after setting the diffeomorphism ghosts ξ^μ to zero. They read explicitly

$$\{\tilde{C}^N\} = \{2\omega, dx^\nu, \tilde{C}^\mu{}_\nu, \tilde{\omega}_\alpha\}, \qquad \tilde{C}^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho} dx^\rho,$$
$$\tilde{\omega}_\alpha = \omega_\alpha - K_{\alpha\rho} dx^\rho, \qquad \omega_\alpha = \partial_\alpha \omega.$$

As anticipated, the generalized connections $\tilde{\omega}_\alpha$ play a crucial rôle in the classification of the Weyl anomalies. They decompose into a ghost part ω_α and a "connection" one-form component $\mathcal{A}_\alpha = -K_{\alpha\rho} dx^\rho$. The decomposition of $\tilde{s}_W$ with respect to the $\tilde{\omega}_\alpha$ degree is at the core of the descent giving the type-A Weyl anomalies. The differential $\tilde{s}_W$ decomposes into a part noted $\tilde{s}_\flat$ which lowers the $\tilde{\omega}_\alpha$ degree by one unit, a part $\tilde{s}_\natural$ which does not change the $\tilde{\omega}_\alpha$ degree, and a part noted $\tilde{s}_\sharp$ which raises the $\tilde{\omega}_\alpha$ degree by one unit.

Before displaying the action of $\tilde{s}_W$ on $\mathcal{W}$, we need to introduce some further objects: (i) the two-forms $W^\mu{}_\nu = \frac{1}{2} dx^\rho dx^\sigma W^\mu{}_{\nu\rho\sigma}$, $R^{\mu\nu} = \frac{1}{2} dx^\rho dx^\sigma R^{\mu\nu}{}_{\rho\sigma}$, and $C_\alpha = \frac{1}{2} dx^\rho dx^\sigma C_{\alpha\rho\sigma}$, (ii) the symbol $\mathcal{P}^{\mu\alpha}_{\rho\nu} = (-g^{\mu\alpha} g_{\rho\nu} + \delta^\mu_\rho \delta^\alpha_\nu + \delta^\mu_\nu \delta^\alpha_\rho)$, (iii) the generators $\Delta^\mu{}_\nu$ of $GL(n)$ transformations of world indices acting on a type-(1, 1) tensor T^β_α as $\Delta^\mu{}_\nu T^\beta_\alpha = \delta^\mu_\alpha T^\beta_\nu - \delta^\beta_\nu T^\mu_\alpha$, and (iv) the Weyl-covariant operator $\mathcal{D}_\mu = \nabla_\mu + K_{\mu\alpha}\Gamma^\alpha$. The definition of the generators Γ^α is not needed here and can be found in Ref. [19]. These generators enter the formula for the Weyl transformation of the W tensors [19]: $s_W W_{\Omega_i} = \omega_\alpha \Gamma^\alpha W_{\Omega_i}$. Both the Cotton two-form C_α and the generalized connection $\tilde{\omega}_\alpha$ take their values along the generators Γ^α, $\mathbf{C} = C_\alpha \Gamma^\alpha$, and $\tilde{\omega} = \omega_\alpha \Gamma^\alpha$. The Weyl two-form takes its values along the $GL(n)$ generators: $\mathbf{W} = W^\mu{}_\nu \Delta^\nu{}_\mu$. Finally, we denote by $\varepsilon^{\mu_1 \dots \mu_n}$ the totally antisymmetric Levi-Civita tensor density (of weight 1).

Then the action of $\tilde{s}_W$ on $\mathcal{W}$ is given in Table I, following a decomposition with respect to the $\tilde{\omega}_\alpha$ degree.

TABLE I. Action of $\tilde{s}_W$, decomposed with respect to the $\tilde{\omega}_\alpha$ degree.

	$\tilde{s}_\flat$	$\tilde{s}_\natural$	$\tilde{s}_\sharp$
$\tilde{\omega}_\alpha$	C_α	$\tilde{C}^\beta{}_\alpha \tilde{\omega}_\beta$	0
ω	0	0	$dx^\mu \tilde{\omega}_\mu$
W_{Ω_i}	0	$\tilde{C}^\mu{}_\nu \Delta^\nu{}_\mu W_{\Omega_i} + dx^\mu \mathcal{D}_\mu W_{\Omega_i}$	$\tilde{\omega}_\alpha \Gamma^\alpha W_{\Omega_i}$
$g_{\alpha\beta}$	0	$\tilde{C}^\mu{}_\nu \Delta^\nu{}_\mu g_{\alpha\beta} + 2\omega g_{\alpha\beta}$	0
$\tilde{C}^\mu{}_\nu$	0	$W^\mu{}_\nu - \tilde{C}^\mu{}_\alpha \tilde{C}^\alpha{}_\nu$	$\mathcal{P}^{\mu\alpha}_{\rho\nu} \tilde{\omega}_\alpha dx^\rho$

We can now state the following two theorems, the central results reported in this Letter.

Theorem 1.—Let $\psi_{\mu_1\ldots\mu_{2p}}$ be the local total form

$$\psi_{\mu_1\ldots\mu_{2p}} = \frac{\omega}{\sqrt{-g}}\varepsilon^{\alpha_1\ldots\alpha_r}{}_{\nu_1\ldots\nu_r\mu_1\ldots\mu_{2p}}\tilde{\omega}_{\alpha_1}\ldots\tilde{\omega}_{\alpha_r}dx^{\nu_1}\ldots dx^{\nu_r},$$
$$p = m - r, \qquad m = n/2, \qquad 0 \le r \le m,$$

and $W^{\mu\nu}$ the tensor-valued two-form $W^{\mu\nu} = W^{\mu}{}_{\rho}g^{\rho\nu}$. Then the local total forms $\Phi_r^{[n-r]}$ $(0 \le r \le m)$

$$\Phi_r^{[n-r]} = \frac{(-1)^p}{2^p}\frac{m!}{r!p!}\psi_{\mu_1\ldots\mu_{2p}}W^{\mu_1\mu_2}\ldots W^{\mu_{2p-1}\mu_{2p}}$$

obey the descent of equations

$$\tilde{s}_\flat\Phi_r^{[n-r]} + \tilde{s}_\natural\Phi_{r-1}^{[n-r+1]} = 0,$$
$$\tilde{s}_\sharp\Phi_r^{[n-r]} = 0, \qquad (1 \le r \le m)$$
$$\tilde{s}_\flat\Phi_1^{[n-1]} = 0 = \tilde{s}_W\Phi_0^{[n]},$$

so that the following relations hold: $\tilde{s}_W\alpha = 0 = \tilde{s}_W\beta$, with $\alpha = \sum_{r=1}^m \Phi_r^{[n-r]}$ and $\beta = \Phi_0^{[n]}$.

Theorem 2.—(A) The top form-degree component a_1^n of α (cf. Theorem 1) satisfies the WZ consistency conditions for the Weyl anomalies. The WZ conditions for a_1^n give rise to a nontrivial descent, and a_1^n is the *unique* anomaly with such a property, up to the addition of trivial terms and anomalies satisfying a trivial descent.

(B) The anomaly $\beta = \Phi_0^{[n]}$ satisfies a trivial descent and is obtained by taking contractions of products of Weyl tensors (m of them in dimension $n = 2m$). The top form-degree component e_1^n of $(\alpha + \beta)$ is proportional to the Euler density of the manifold $\mathcal{M}_n$:

$$e_1^n = \frac{(-1)^m}{2^m}\sqrt{-g}\,\omega(R^{\mu_1\nu_1}\ldots R^{\mu_m\nu_m})\varepsilon_{\mu_1\nu_1\ldots\mu_m\nu_m}.$$

Proofs.—The existence part of the nontrivial descent problem for the Weyl anomalies is given in Theorem 1 and part B of Theorem 2. It is proved by direct computation. Only part A of Theorem 2, the uniqueness part of the problem, is not straightforward. The detailed proof will be published elsewhere [20]. It follows lines of reasonings as in, e.g., Refs. [17,21,22] and uses general results given in Ref. [23]. The essential point is to determine the most general expression at the bottom of the nontrivial descents associated with the Weyl anomalies. (The anomalies that satisfy the trivial descent $s_W a_1^n = 0$ are the type-B Weyl anomalies [16]; they can be classified along the lines of Refs. [19,24]). It turns out [20] that the most general element at the bottom of these descents is the component of $\Phi_m^{[m]}$ with maximal ghost number $m + 1$.

We now illustrate our two theorems with the descents corresponding to $n = 2$, 4, and 6. The general case can readily be understood from these three examples.

The case $n = 2$ is a bit special. Although $K_{\mu\nu}$ is not determined $(\sim\frac{0}{0})$, its trace $K_\rho{}^\rho = \mathcal{R}/(2n-2)$ is well defined. Theorem 1 gives $(n = 2m = 2)$ $\Phi_0^{[2]} = 0$ and $\alpha = \Phi_1^{[1]} = (\omega/\sqrt{-g})\varepsilon^{\mu\rho}g_{\rho\nu}\tilde{\omega}_\mu dx^\nu = \omega\sqrt{-g}\varepsilon_{\rho\nu}g^{\rho\mu}\tilde{\omega}_\mu dx^\nu$. Taking the top form degree of α and recalling $\tilde{\omega}_\mu = \omega_\mu + \mathcal{A}_\mu = \omega_\mu - dx^\nu K_{\nu\mu}$, we find $a_1^2 = \frac{\omega}{2}\sqrt{-g}\mathcal{R}d^2x$, the well-known result for the Weyl anomaly in two dimensions.

Next, using Theorem 1 in the case $n = 4$ gives

$$\Phi_0^{[4]} = \frac{\omega}{4}\sqrt{-g}\varepsilon_{\mu_1\ldots\mu_4}W^{\mu_1\mu_2}W^{\mu_3\mu_4},$$
$$\Phi_1^{[3]} = -\omega\sqrt{-g}\varepsilon^{\alpha}{}_{\nu\rho\sigma}\tilde{\omega}_\alpha dx^\nu W^{\rho\sigma},$$
$$\Phi_2^{[2]} = \omega\sqrt{-g}\varepsilon^{\alpha\beta}{}_{\rho\sigma}\tilde{\omega}_\alpha\tilde{\omega}_\beta dx^\rho dx^\sigma.$$

The top form-degree component of $(\alpha + \beta)$ is e_1^4:

$$e_1^4 = \frac{\omega}{4}\sqrt{-g}\varepsilon_{\mu\nu\rho\sigma}(W^{\mu\nu} - 2\mathcal{A}^\mu dx^\nu)(W^{\rho\sigma} - 2\mathcal{A}^\rho dx^\sigma),$$

which obviously reproduces the expression for the Euler term of Theorem 2 because of the following identities:

$$R^{\mu\nu} = W^{\mu\nu} - 2\mathcal{A}^{[\mu}dx^{\nu]}, \qquad \mathcal{A}^\mu = -g^{\mu\nu}K_{\nu\rho}dx^\rho. \tag{5}$$

The descent for $n = 4$ thus reads

$$s_We_1^4 + db_2^3 = 0, \qquad s_Wb_2^3 + db_3^2 = 0, \qquad s_Wb_3^2 = 0,$$

with

$$b_2^3 = -2\omega\sqrt{-g}\varepsilon^{\alpha}{}_{\nu\rho\sigma}\omega_\alpha K^\nu{}_\mu dx^\mu dx^\rho dx^\sigma,$$
$$b_3^2 = \omega\sqrt{-g}\varepsilon^{\alpha\beta}{}_{\rho\sigma}\omega_\alpha\omega_\beta dx^\rho dx^\sigma.$$

Finally, in dimension 6, Theorems 1 and 2 give (a representative of) the unique Weyl anomaly satisfying a nontrivial descent of equations:

$$e_1^6 = \frac{-\omega}{8}\sqrt{-g}\varepsilon_{\mu_1\ldots\mu_6}R^{\mu_1\mu_2}\ldots R^{\mu_5\mu_6}. \tag{6}$$

The elements of the corresponding descent are obtained, as before, via the $\Phi_r^{[n-r]}$'s of Theorem 1:

$$\beta = \Phi_0^{[6]} = \frac{-\omega}{8}\sqrt{-g}\varepsilon_{\mu_1\ldots\mu_6}W^{\mu_1\mu_2}W^{\mu_3\mu_4}W^{\mu_5\mu_6},$$
$$\Phi_1^{[5]} = \frac{3\omega}{4}\sqrt{-g}\varepsilon^{\alpha}{}_{\nu\mu_1\ldots\mu_4}\tilde{\omega}_\alpha dx^\nu W^{\mu_1\mu_2}W^{\mu_3\mu_4},$$
$$\Phi_2^{[4]} = \frac{-3\omega}{2}\sqrt{-g}\varepsilon^{\alpha\beta}{}_{\mu\nu\rho\sigma}\tilde{\omega}_\alpha\tilde{\omega}_\beta dx^\mu dx^\nu W^{\rho\sigma},$$
$$\Phi_3^{[3]} = \omega\sqrt{-g}\varepsilon^{\alpha\beta\gamma}{}_{\mu\nu\rho}\tilde{\omega}_\alpha\tilde{\omega}_\beta\tilde{\omega}_\gamma dx^\mu dx^\nu dx^\rho.$$

Extracting from $\alpha = \Phi_1^{[5]} + \Phi_2^{[4]} + \Phi_3^{[3]}$ its top form-degree component amounts to selecting everywhere the contribution $\mathcal{A}_\mu$ of $\tilde{\omega}_\mu = \omega_\mu + \mathcal{A}_\mu$. As a consequence, the top form-degree component of $(\alpha + \beta)$ reproduces the expression (6), making use of the identities (5). On the other hand, extracting the different ghost-number contributions of α provides us with the elements b_2^5, b_3^4, and b_4^3 of the descent for e_1^6:

$$s_W e_1^4 + db_2^5 = 0, \qquad s_W b_2^5 + db_3^4 = 0,$$
$$s_W b_3^4 + db_4^3 = 0, \qquad s_W b_4^3 = 0.$$

Without the addition of the type-B anomaly β, the top form-degree component a_1^6 of α, taken alone, gives

$$a_1^6 = \frac{-3\omega}{8}\sqrt{-g}\varepsilon_{\mu_1\ldots\mu_6}[(-2\mathcal{A}^{\mu_1}dx^{\mu_2})W^{\mu_3\mu_4}W^{\mu_5\mu_6} + (-2\mathcal{A}^{\mu_1}dx^{\mu_2})(-2\mathcal{A}^{\mu_3}dx^{\mu_4})W^{\mu_5\mu_6} + (-2\mathcal{A}^{\mu_1}dx^{\mu_2})(-2\mathcal{A}^{\mu_3}dx^{\mu_4})(-2\mathcal{A}^{\mu_5}dx^{\mu_6})].$$

As we have shown, adding $\beta = \Phi_0^{[n]}$ to a_1^n somehow "covariantizes" the latter, producing the Euler term e_1^n. The Weyl anomaly a_1^n is reminiscent of the consistent non-Abelian chiral anomaly. However, note that the descent for a_1^n stops at form-degree $\frac{n}{2} > 0$. Amusingly, the Euler form e_1^n looks like the non-Abelian singlet anomaly. The "trace over the internal indices" is taken with the Levi-Civita density.

Conclusions.—The universal structure of the Weyl anomalies is established in a purely algebraic manner, independently of any regularization scheme and in arbitrary dimensions. In particular, we do not resort to dimensional analysis. The type-A Weyl anomaly of Ref. [16] is the counterpart of the consistent non-Abelian chiral anomaly, in that it is the *unique* Weyl anomaly satisfying a nontrivial descent of equations. This proves a long-standing conjecture originally due to Deser and Schwimmer [16]. Since the Weyl anomalies associated with a trivial descent can be systematically built and classified as in Refs. [19,24], our analysis completes a general, purely algebraic classification of the Weyl anomalies in arbitrary spacetime dimensions.

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