Multiplicity and symmetry of positive solutions to semi-linear elliptic problems with Neumann boundary conditions

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The Lane-Emden problem

Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded, $N \geqslant 2$, and $2 < p < 2^* := \frac{2N}{N-2}$. We consider

$$(\mathcal{P}_p) \begin{cases} -\Delta u + u = |u|^{p-2} u, & \text{in } \Omega \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

Solutions are critical points of the functional

$$\mathcal{E}_p : H^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2} \int_\Omega |\nabla u|^2 + u^2 - \frac{1}{p} \int_\Omega |u|^p$$

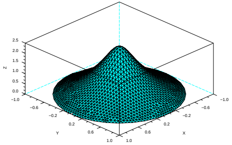
Notation: $1 = \lambda_1 < \lambda_2 < \cdots$ denote the eigenvalues of $-\Delta + 1$
E_i denote the corresponding eigenspaces

Remark: 0 and ± 1 are always (trivial) solutions.

Outline

Dirichlet boundary conditions

$$\begin{cases} -\Delta u + u = |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$



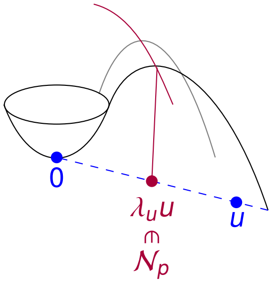
- The ground state solution is positive and is even w.r.t. any hyperplane leaving Ω invariant (when Ω is convex). In particular, it is radially symmetric on a ball.
- Uniqueness of the positive solution when Ω is a ball.
- If Ω is strictly starshaped and $p \geqslant 2^*$, no solution exist.

Existence of ground state solutions ($p < 2^*$)

Theorem (Z. Nehari, A. Ambrosetti, P.H. Rabinowitz)

For any $p \in]2, 2^*[$, $(\mathcal{P}_p)$ possesses

1. *a ground state solution to $(\mathcal{P}_p)$;*
2. *it is a one-signed function;*
3. *its Morse index is 1.*



$p \approx 2$: symmetry of ground state solutions

Theorem (D. Bonheure, V. Bouchez, C. Grumiau, C. T., J. Van Schaftingen, '08)

For p close to 2 and any $R \in O(N)$ s.t. $R(\Omega) = \Omega$, ground state solutions to $(\mathcal{P}_p)$ are symmetric w.r.t. R.

E.g. if Ω is radially symmetric, so must the the ground state solution be.

Remark that the seminal method of moving planes is not applicable.

Uniqueness of the positive solution

Theorem

1 *is the unique positive solution to* $-\Delta u + u = |u|^{p-2}u$ *with NBC for* p *small.*

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Let $v := P_{E_1} u$ (constant function) and $w := P_{E_1^\perp} u$ (zero mean).

$$\int_\Omega |\nabla w|^2 + w^2 = \int_\Omega |u|^{p-1} w$$

As $\lambda_1 = 1 < \lambda_2$, for $p \approx 2$, $w = 0$ and then $u = v = 1$.

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$$\lambda_2 \int_\Omega w^2 \leqslant \int_\Omega |\nabla w|^2 + w^2 = \int_\Omega |u|^{p-1} w = \int_\Omega \Big((v+w)^{p-1} - v^{p-1} \Big) w$$

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$$\begin{aligned}
\lambda_2 \int_\Omega w^2 \leqslant \int_\Omega |\nabla w|^2 + w^2 = \int_\Omega |u|^{p-1} w &= \int_\Omega \Bigl((v+w)^{p-1} - v^{p-1} \Bigr) w \\
&= \int_\Omega (p-1)(v + \vartheta_p w)^{p-2} w^2 \qquad (\vartheta_p \in]0,1[) \\
&\leqslant (p-1)(|v| + \|w\|_\infty)^{p-2} \int_\Omega w^2 \leqslant (p-1) K^{p-2} \int_\Omega w^2.
\end{aligned}$$

As $\lambda_1 = 1 < \lambda_2$, for $p \approx 2$, $w = 0$ and then $u = v = 1$.

A priori bounds for positive solutions

Lemma

Positive solutions (u_p) are bounded in L^∞ as $p \approx 2$.

- Integration & Hölder: $\displaystyle\int_\Omega u_p^{p-1} = \int_\Omega u_p \leqslant |\Omega|$ (recall $u_p > 0$).
- Brezis-Strauss: from the bound on $\displaystyle\int_\Omega u_p^{p-1}$, we deduce a bound on $\|u_p\|_{W^{1,q}(\Omega)}$, $1 \leqslant q < N/(N-1)$.
- Sobolev embedding: (u_p) bounded in $L^r(\Omega)$, $1 < r < N/(N-2)$.
- Bootstrap: $\|u_p\|_{W^{2,r}(\Omega)}$ is bounded for some $r > N/2$ when $p \approx 2$.

A priori bounds for positive solutions

Proposition

Let $2 < \bar{p} < 2^*$. There exists $C_{\bar{p}} > 0$ such that any positive solution to $(\mathcal{P}_p)$ with $2 < p \leqslant \bar{p}$ satisfies $\max\left\{\|u\|_{H^1}, \|u\|_{L^\infty}\right\} \leqslant C_{\bar{p}}$.

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It remains to obtain a bound for $2 < \underline{p} < \bar{p} < 2^*$ in L^∞. Blow up argument (Gidas-Spruck). Suppose on the contrary that there is a sequence $(p_n) \subseteq [\underline{p}, \bar{p}]$ and (u_{p_n}) s.t.

$$u_{p_n}(x_{p_n}) := \|u_{p_n}\|_{L^\infty} \to +\infty \qquad \text{and} \qquad p_n \to p^* \in [\underline{p}, \bar{p}].$$

(Drop index n.) Define

$$v_p(y) := \mu_p u_p\Big(\mu_p^{(p-2)/2} y + x_p\Big) \qquad \text{where } \mu_p := 1/\|u_p\|_{L^\infty} \to 0.$$

Note: $v_p(0) = \|v_p\|_{L^\infty} = 1$.

A priori bounds for positive solutions

The rescaled function v_p satisfies

$$-\Delta v_p + \mu_p^{p-2} v_p = v_p^{p-1} \quad \text{on } \Omega_p := (\Omega - x_p)/\mu_p^{(p-2)/2}$$

with NBC. By elliptic regularity, (v_p) is bounded in $W^{2,r}$ and $C^{1,\alpha}$, $0 < \alpha < 1$ on any compact set. Thus, taking if necessary a subsequence,

$$v_n \to v^* \quad \text{in } W^{2,r} \text{ and } C^{1,\alpha} \text{ on compact sets of } \Omega^* = \mathbb{R}^N \text{ or } \mathbb{R}^{N-1} \times \mathbb{R}_{>a}.$$

One has $v^* \geqslant 0$, $v^*(0) = 1 = \|v\|_{L^\infty}$ and v^* satisfies

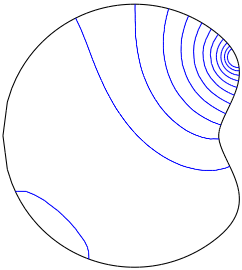
$$-\Delta v^* = (v^*)^{p^*-1} \quad \text{in } \mathbb{R}^N \qquad \text{or} \qquad \begin{cases} -\Delta v^* = (v^*)^{p^*-1} & \text{in } \mathbb{R}^{N-1} \times \mathbb{R}_{>a} \\ \partial_N v^* = 0 & \text{when } x_N = a \end{cases}$$

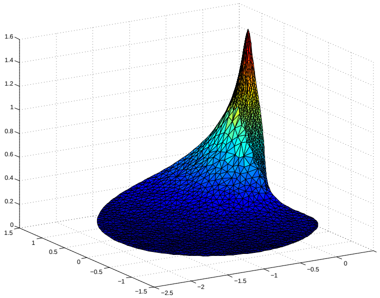
Liouville theorems imply $v^* = 0$. □

Symmetry breaking of the ground state

Theorem (W.-M. Ni, I. Takagi, '93; Adimurthi, F. Pacella, S.L. Yadava '93)

When R is sufficiently large, ground state solutions possess a unique maximum point $P_R \in \partial(R\Omega)$. Moreover, $u_R \to 0$ outside a small neighborhood of P_R. P_R is situated at the "most curved" part of $\partial(R\Omega)$.





p large: symmetry breaking of the ground state

Corollary

1 *cannot remain the ground state for all p on "large" domains.*

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Lemma

1 *cannot remain the ground state solution for* $p > 1 + \lambda_2$.

Proof. The Morse index of 1 is the sum of the dimension of the eigenspaces corresponding to negative eigenvalues λ of

$$\begin{cases} -\Delta v + v = (p-1)v + \lambda v, & \text{in } \Omega, \\ \partial_\nu v = 0, & \text{on } \partial\Omega. \end{cases}$$

i.e. eigenvalues of $-\Delta + 1$ less than $p - 1$. When $p > 1 + \lambda_2$, the Morse index of the solution 1 is > 1.

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Proposition (Lopez, '96)

On radial domains, the ground state is either constant or (e.g. when $p > 1 + \lambda_2$) not radially symmetric.

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Proposition

When Ω is a ball or an annulus, the Morse index of a non-constant positive radial solution is at least $N + 1$. In particular, when $p > 1 + \lambda_2$, ground state solutions cannot be radial.

Based on: A. Aftalion, F. Pacella, Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains, CRAS, 339(5), '04.

Let u be non-constant positive radial solution of $(\mathcal{P}_p)$. We have to show that

$$Lv := -\Delta v + v - (p-1)|u|^{p-2} v$$

with NBC possesses $N + 1$ negative eigenvalues.

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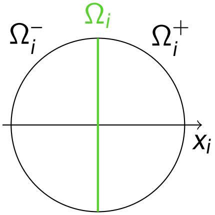
u radial $\Rightarrow \partial_{x_i} u = 0$ on $\partial\Omega$ and on Ω_i.



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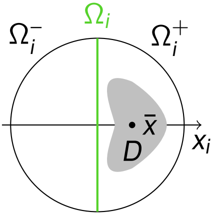
Let $\bar{x} \in \Omega_i^+$ s.t. $\partial_{x_i} u(\bar{x}) \neq 0$. Let D be the connected component of $\{\partial_{x_i} u(\bar{x}) \neq 0\}$ containing $\bar{x}$. $D \subseteq \Omega_i^+$.



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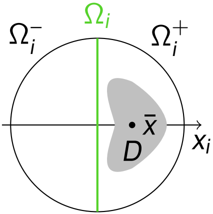
$$L(\partial_{x_i} u) = 0, \quad \text{on } D; \qquad \partial_{x_i} u = 0, \quad \text{on } \partial D.$$



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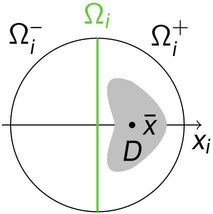

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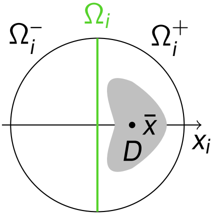

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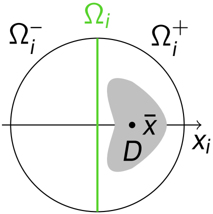

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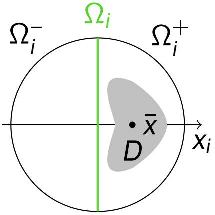

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All ψ_j^*, $j \neq i$ vanish on the axis $x_i \Rightarrow$ the family $(\psi_j^*)_{j=1}^N$ is lin. indep.

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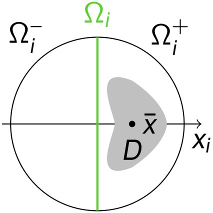

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All ψ_j^*, $j \neq i$ vanish on the axis $x_i \Rightarrow$ the family $(\psi_j^*)_{j=1}^N$ is lin. indep.
None of the $(\psi_j^*)_{j=1}^N$ is a first eigenfunction.

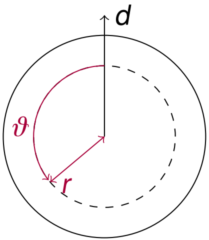
p large: symmetry breaking of the ground state

Theorem (Lopes, '96)

On radial domains, ground state solutions are symmetric w.r.t. any hyperplane containing a line L passing through the origin.

Theorem (J. Van Schaftingen, '04)

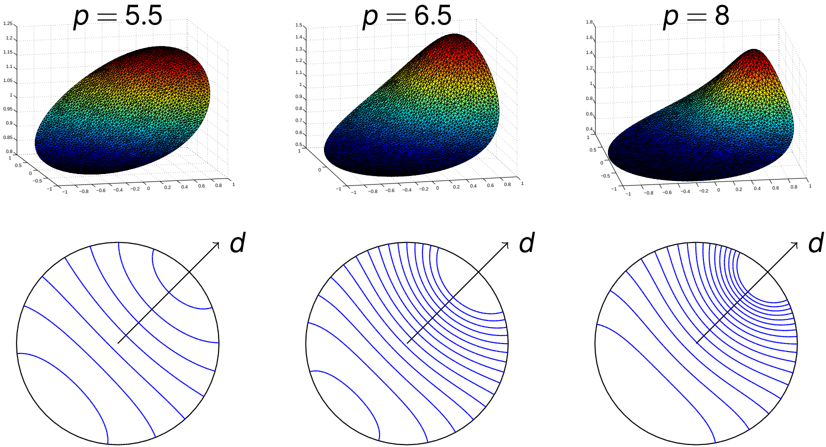
On radial domains, ground state solutions are foliated Schwarz symmetric.



There exists a unit vector d s.t. u depends only on $r = |x|$ and $\vartheta = \arccos(\frac{x}{|x|} \cdot d)$ and is non-increasing in ϑ.

p large: non-radially symmetric ground state

$$\Omega = B_1 \subseteq \mathbb{R}^2 \;\Rightarrow\; 1+\lambda_2 \approx 5.39$$



Ground states — summary

- When $p \approx 2$, 1 is the sole positive solution (hence the GS are ± 1).
- When $p > 1 + \lambda_2$,
  - 1 is not the GS anymore;
  - on a ball or an annulus, GS solutions are not radial but foliated Schwarz symmetric.

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Theorem (Lin, Ni, Takagi '88)

Let $\Omega_1 \subseteq \mathbb{R}^N$ be a bounded smooth domain and $p \in]2, 2^*[$. There exists $0 < R_0 \leqslant R_1$ such that the equation $-\Delta u + u = |u|^{p-2} u$ with NBC on $\Omega = R\Omega_1$ possesses

1. *only constant positive solutions for $R < R_0$;*
2. *a non-constant positive solution for $R \geqslant R_1$.*

Symmetry breaking at exactly $p = 1+\lambda_2$?

Conjecture

± 1 are the ground states of $-\Delta u + u = |u|^{p-2}u$ with NBC for all $p \leqslant 1+\lambda_2$.

- *If $1+\lambda_2 \geqslant 2^*$, no concentration therefore occurs when $p \to 2^*$.*
- *If $1+\lambda_2 < 2^*$, the GS solutions for $p \in]1+\lambda_2, 2^*[$ lie on the branch emanating from $(p,u) = (1+\lambda_2, 1)$ and concentrate on the boundary as $p \to 2^*$.*

Evidence for this conjecture: examine the bifurcation at $p = 1+\lambda_2$ on a ball.

Symmetry breaking at exactly $p = 1 + \lambda_2$?

The linearisation of the equation around $u = 1$,

$$Lv := -\Delta v + v - (p - 1)v$$

is not invertible iff $p = 1 + \lambda_i$, $i \geqslant 2$.

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is not invertible iff $p = 1 + \lambda_i$, $i \geqslant 2$.

Eigenfunctions of $-\Delta + 1$ with NBC have the form:

$$u(x) = r^{-\frac{N-2}{2}} J_\nu(\sqrt{\mu} r)\, P_k\Big(\frac{x}{|x|}\Big), \qquad \text{where } \nu = k + \frac{N-2}{2},$$

$r = |x|$, and $P_k : \mathbb{R}^N \to \mathbb{R}$ is an harmonic homogenous polynomial of degree k for some $k \in \mathbb{N}$. To satisfy the boundary conditions:

$$\sqrt{\mu} R \text{ is a root of } z \mapsto (k-\nu)J_\nu(z) + z\partial J_\nu(z) = kJ_\nu(z) - zJ_{\nu+1}(z).$$

$\Rightarrow \lambda_i = 1 + \mu$

Symmetry breaking at exactly $p = 1 + \lambda_2$?

In particular, a basis of E_2 is

$$x \mapsto r^{-\frac{N-2}{2}} J_{N/2}(\sqrt{\mu} r) \frac{x_j}{|x|}, \qquad j = 1, \dots, N.$$

There is single function (up to a multiple) that is invariant under rotation in $(x_2, \dots, x_N)$.

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Theorem (Crandall-Rabinowitz '71)

Let X and Y two Banach spaces, $u^* \in X$, and a function $F : \mathbb{R} \times X \to Y : (p, u) \mapsto F(p, u)$ such that $\forall p \in \mathbb{R}$, $F(p, u^*) = 0$. Let $p^* \in \mathbb{R}$ be such that $\ker(\partial_u F(p^*, u^*)) = \operatorname{span}\{\varphi^*\}$ has a dimension 1 and $\operatorname{codim}\big(\operatorname{Im}(\partial_u F(p^*, u^*))\big) = 1$. Let $\psi : Y \to \mathbb{R}$ be a continuous linear map such that $\operatorname{Im}(\partial_u F(p^*, u^*)) = \{y \in Y : \langle \psi, y \rangle = 0\}$.

In our case $F(p, u) = -\Delta u + u - |u|^{p-2} u$, $p^* = 1 + \lambda_2$, $u^* = 1$, $\varphi^* = \varphi_2$.

Symmetry breaking at exactly $p = 1 + \lambda_2$?

Theorem (Crandall-Rabinowitz (cont'd))

If $a := \langle \psi, \partial_{pu} F(p^*, u^*)[\varphi^*] \rangle \neq 0$, then (p^*, u^*) is a bifurcation point for F. In addition, the set of non-trivial solutions of $F = 0$ around (p^*, u^*) is given by a unique $\mathcal{C}^1$ curve $p \mapsto u_p$. The local behavior of the branch (p, u_p) for p close to p^* is as follows.

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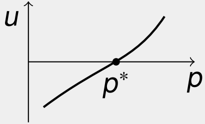
- *If $b := -\frac{1}{2a}\big\langle \psi, \partial_u^2 F(p^*, u^*)[\varphi^*, \varphi^*] \big\rangle \neq 0$ then the branch is transcritical and*

$$u_p = u^* + \frac{p - p^*}{b}\varphi^* + o(p - p^*).$$



In our case,

$$a = -\int_\Omega \varphi_2^2 = -1 \qquad \text{and} \qquad b = -\tfrac{1}{2}\lambda_2(\lambda_2 - 1)\int_\Omega \varphi_2^3 = 0.$$

Symmetry breaking at exactly $p = 1 + \lambda_2$?

Theorem (Crandall-Rabinowitz — extended)

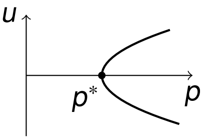
- *If $b = 0$, let us define*

$$c := -\frac{1}{6a}\Big(\big\langle \psi, \partial_u^3 F(p^*, u^*)[\varphi^*, \varphi^*, \varphi^*]\big\rangle + 3\big\langle \psi, \partial_u^2 F(p^*, u^*)[\varphi^*, w]\big\rangle\Big)$$

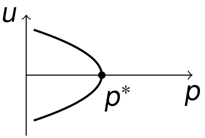
where $w \in X$ is any solution of the equation $\partial_u F(p^*, u^*)[w] = -\partial_u^2 F(p^*, u^*)[\varphi^*, \varphi^*]$. If $c \neq 0$ then

$$u_p = u^* \pm \left(\frac{p - p^*}{c}\right)^{1/2} \varphi^* + o\big(|p - p^*|^{1/2}\big).$$

In particular, the branch is supercritical if $c > 0$ and subcritical if $c < 0$.



Supercritical



Subcritical

Symmetry breaking at exactly $p = 1 + \lambda_2$?

In our case,

$$c = \frac{1}{6}\lambda_2(\lambda_2 - 1)\Big(-(\lambda_2 - 2)\int_{B_R}\varphi_2^4 - 3\lambda_2(\lambda_2 - 1)\int_{B_R}\varphi_2^2 w\Big)$$

where $(-\Delta + 1 - \lambda_2)w = \varphi_2^2$ with NBC on B_R.

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where $(-\Delta + 1 - \lambda_2)w = \varphi_2^2$ with NBC on B_R.

$$= \frac{1}{6}\bar{\mu}_2 R^{-(N+2)}\Big(1 + \frac{\bar{\mu}_2}{R^2}\Big)\Big((\beta - \alpha)\frac{\bar{\mu}_2}{R^2} + \beta + \alpha\Big)$$

where $\alpha := \int_{B_1}\bar{\varphi}_2^4, \quad \beta := -3\bar{\mu}_2\int_{B_1}\bar{\varphi}_2^2\bar{w},$

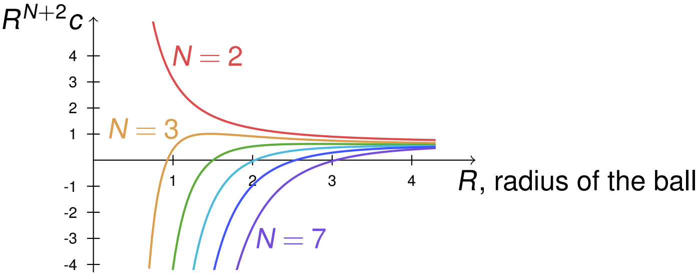
$(-\Delta - \bar{\mu}_2)\bar{w} = \bar{\varphi}_2^2$ with NBC on B_1,

$\bar{\varphi}_2$ and $\bar{\mu}_2 > 0$ are "the" second eigenfunction and eigenvalue of $-\Delta$ with NBC on B_1 s.t. $|\bar{\varphi}_2|_{L^2} = 1$.

Symmetry breaking at exactly $p = 1 + \lambda_2$?

We numerically have

N	α	β	$\beta - \alpha$	$\beta + \alpha$
2	0.5577	0.5884	0.0306	1.1461
3	0.4632	0.3096	−0.1536	0.7728
4	0.4222	0.1694	−0.2528	0.5916
5	0.4171	0.0858	−0.3313	0.5029
6	0.4421	0.0250	−0.4171	0.4671



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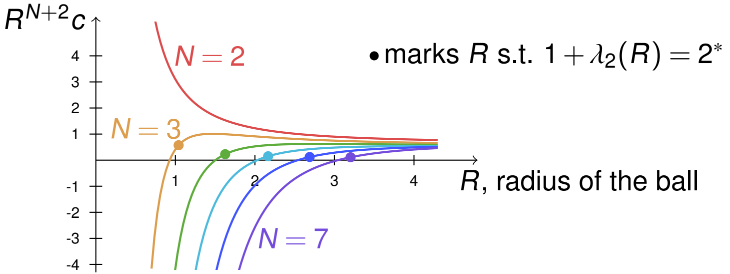

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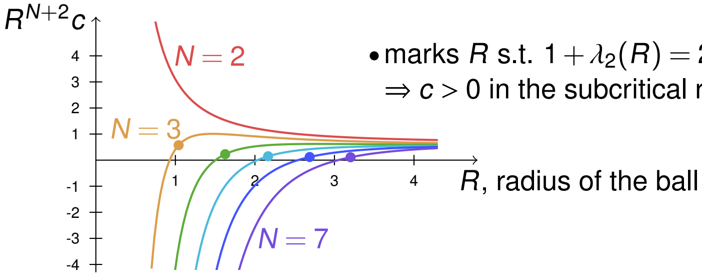

• marks R s.t. $1 + \lambda_2(R) = 2^*$
$\Rightarrow c > 0$ in the subcritical range

Symmetry breaking at exactly $p = 1 + \lambda_2$?

Theorem (Crandall-Rabinowitz — extended)

Assume $F(p,u) = \partial_u \mathcal{E}(p,u)$. If (p, u_p) is the branch of nontrivial solutions emanating from (p^*, u^*), $b = 0$ and $c \neq 0$,

$$\mathcal{E}(p,u_p) - \mathcal{E}(p,u^*) = \frac{a}{6c}(p-p^*)^2 + o\big((p-p^*)^2\big) \qquad \text{when } \frac{p-p^*}{c} > 0.$$

In our case, $a = -1 < 0$ and $c > 0$. Consequence: the energy along the super-critical branch emanating from $(1+\lambda_2, 1)$ has lower energy than the trivial solution 1.

Multiplicity

Existence of infinitely many solutions

Well known: infinitely many sign changing solutions when the problem is invariant under an orthogonal symmetry group.

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Well known: infinitely many sign changing solutions when the problem is invariant under an orthogonal symmetry group.

Theorem (A Castro '03)

If Ω is a bounded region in $\mathbb{R}^N$ that can be tiled by prisms congruent to

$$\Big\{(x_1,\dots,x_N) : x_i \geqslant 0 \text{ for } i = 1,\dots,N, \\ x_N \leqslant \min\{x_i, a - x_i\} \text{ for } i = 1,\dots,N-1\Big\}$$

with $a > 0$ arbitrarily small, then the problem $-\Delta u + u = |u|^{p-2}u$ with NBC on Ω has infinitely many sign-changing solutions.

Many *positive* solutions

Theorem (C. Gui, J. Wei, M. Winter '00)

For any smooth bounded domain Ω_1, $p \in]2, 2^*[$, and any fixed $K \in \mathbb{N}_0$, there always exists a boundary K-peaked positive solution to

$$(\mathcal{P}_p)\begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \Omega \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

with $\Omega = R\Omega_1$, provided that R is large enough.

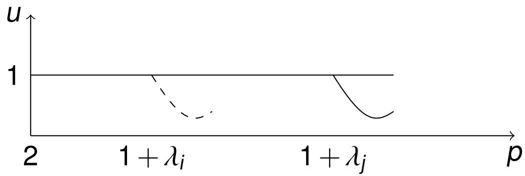
Peaks can also be inside Ω [M. Grossi, A. Pistoia, J. Wei '00] or both inside Ω and on $\partial\Omega$ [C. Gui, J. Wei '00].

p large: bifurcations from 1

Lemma

When $p > 2$ is increasing,

1. *a bifurcation **sequence** start from 1 **iff** p crosses $1 + \lambda_i$;*
2. *this is actually a continuum if λ_i has **odd** multiplicity.*



Go to KMB theorem

p large: transcritical *radial* bifurcations

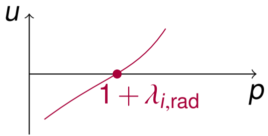
$\lambda_{i,\mathrm{rad}}$ eigenvalues that possess a radial eigenfunction (simple in H^1_{rad}).

Proposition

On balls, two branches radial solutions in $C^{2,\alpha}(\Omega)$ of

$$(\mathcal{P}_p)\begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \Omega \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

start from each $(p,u) = (1+\lambda_{i,rad}, 1)$, $i > 1$. Locally, these branches form a unique C^1-curve. Moreover, for i large enough independent of the measure of Ω, the bifurcation is transcritical.



p large: transcritical radial bifurcations

Proof. $\Omega = B_R$. Using Crandall-Rabinowitz' theorem, one has to show

$$b = -\tfrac{1}{2}\lambda_i(\lambda_i - 1)\int_{B_R} \varphi_{i,\text{rad}}^3 \neq 0.$$

Given that radial eigenfunctions are given by constant spherical harmonics ($k = 0$, $\nu = (N-2)/2$), this amounts to

$$\int_0^R \Bigl(r^{-\frac{N-2}{2}} J_\nu(r\sqrt{\bar\mu_{i,\text{rad}}}/R)\Bigr)^3 r^{N-1}\,\mathrm{d}r \neq 0 \quad \text{i.e.} \quad \int_0^{\sqrt{\bar\mu_{i,\text{rad}}}} t^{1-\nu} J_\nu^3(t)\,\mathrm{d}t \neq 0$$

where $\lambda_{i,\text{rad}} = 1 + \bar\mu_{i,\text{rad}}/R^2$. This is true for large i because

$$\int_0^\infty t^{1-\nu} J_\nu^3(t)\,\mathrm{d}t = \frac{2^{\nu-1}(3/16)^{\nu-1/2}}{\pi^{1/2}\Gamma(\nu+1/2)} > 0.$$

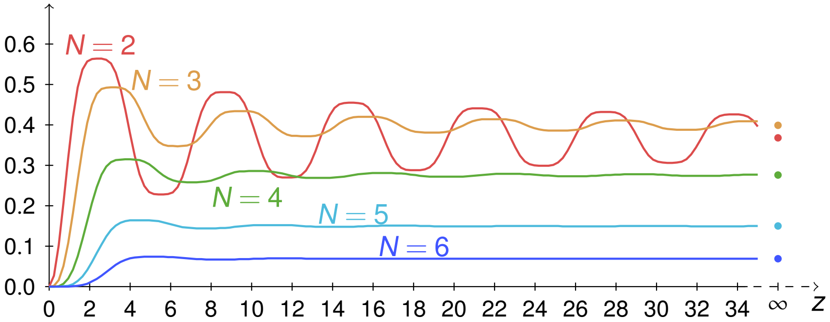
Thus $b < 0$.

p large: transcritical radial bifurcations

Numerical computations indicate that

$$\forall z \in \,]0, +\infty[, \quad \int_0^z t^{1-\nu} J_\nu^3(t)\, \mathrm{d}t > 0, \qquad \nu = (N-2)/2,$$

and therefore that radial bifurcations are transcritical for all i.

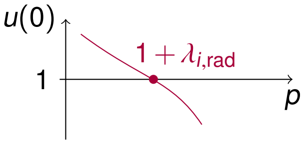


Shape of transcritical radial bifurcations

$$u_p = 1 + \frac{p-(1+\lambda_{i,\mathrm{rad}})}{b}\varphi_{i,\mathrm{rad}} + o\big(p-(1+\lambda_{i,\mathrm{rad}})\big)$$

where $\varphi_{i,\mathrm{rad}}(x) = |x|^{-\nu} J_\nu(\sqrt{\lambda_{i,\mathrm{rad}} - 1}\,|x|)$. Thus

- $u_p(0) > 1$ if $p < 1 + \lambda_{i,\mathrm{rad}}$
- $u_p(0) < 1$ if $p > 1 + \lambda_{i,\mathrm{rad}}$



These facts remain true along the whole banches.

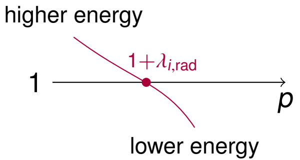
p large: positive transcritical radial bifurcations

Theorem (Crandall-Rabinowitz — extended)

Assume $F(p,u) = \partial_u \mathcal{E}(p,u)$. If (p,u_p) is the branch of nontrivial solutions emanating from (p^*,u^*) and $b \neq 0$,

$$\mathcal{E}(p,u_p) - \mathcal{E}(p,u^*) = \frac{a}{6b^2}(p-p^*)^3 + o\big((p-p^*)^3\big).$$

In our case $a = -1$. Consequence: the energy along the right (resp. left) branch is lower (resp. higher) than the one of the trivial solution.



p large: positive transcritical radial bifurcations

Corollary

The branches consist of positive functions.

SKETCH: If it was not the case, there would be a point solution along the branch with a double root, hence $= 0$. There is no bifurcation from 0. □

p large: positive transcritical radial bifurcations

Corollary

The branches consist of positive functions.

Sketch: If it was not the case, there would be a point solution along the branch with a double root, hence $= 0$. There is no bifurcation from 0. □

Theorem

Radial bifurcations obtained for the $C^{2,\alpha}(\Omega)$-norm are unbounded and do not intersect each other. Moreover, along bifurcations starting from $(1 + \lambda_{i,rad}, 1)$, the solutions always possess the same number of intersections with 1*.*

Sketch: The number of crossings with 1 stays constant because otherwise a non-constant radial solution u s.t. $u - 1$ has a double root would exists. Since the branches do not intersect each other, Rabinowitz's principle says they must be undounded.

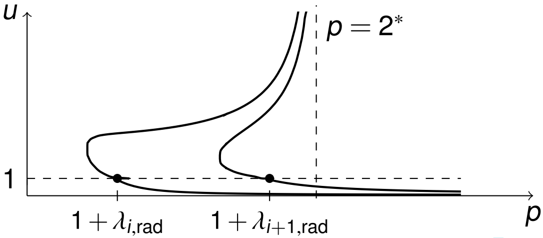
p large: multiplicity results (radial domains)

Theorem

Assume Ω is a ball.

- *In dimension 2, for any $n \in \mathbb{N}_0$, there exists $p_n > 2$ such that, for any $p > p_n$, at least $2n-1$ positive solutions exist*
- *In dimension $\geqslant 3$, for any $2 < p < 2^*$ and $n \in \mathbb{N}_0$, at least $2n-1$ positive solutions exist if the measure of the ball Ω is large enough.*

In both cases, n of these solutions are bounded in L^∞.



p large: degeneracy results (radial domains)

Conjecture

One can choose $p_n = 1 + \lambda_{n,rad}$ and "large enough" as $1 + \lambda_{n,rad}(R) < p$.

The conjecture is proved as soon as $\forall z \in \left]0, +\infty\right[$, $\int_0^z t^{1-\nu} J_\nu^3(t)\,\mathrm{d}t > 0$, with $\nu \in \frac{1}{2}\mathbb{N}$, is established.

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Theorem

On balls, there exists a degenerate positive radial solution for some p provided that the measure of Ω is large enough.

The super-critical case $p \geqslant 2^*$

$p \geqslant 2^*$

Theorem (Adimurthi, Yadava '91)

Let $p = 2^*$ and $\Omega = B_R$. One consider the problem

$$(\mathcal{P}_p) \begin{cases} -\Delta u + u = |u|^{p-2} u, & \text{in } \Omega \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$

1. *If $N \geqslant 3$ and $1 + \lambda_{2,rad}(R) < p$, then $(\mathcal{P}_p)$ admits a positive solution which is radially increasing.*
2. *If $N \in \{4,5,6\}$ and $p < 1 + \lambda_{2,rad}(R)$, then $(\mathcal{P}_p)$ admits a positive solution which is radially decreasing.*
3. *If $N = 3$, there exists an $R^* > 0$ such that for $R \in \,]0, R^*[$, $(\mathcal{P}_p)$ only admits constant positive solutions.*

$p \geqslant 2^*$

Theorem (X-J. Wang, '91)

When $p = 2^*$ and $\Omega = R\Omega_1$ with R large enough, $(\mathcal{P}_p)$ possesses at least one non-constant positive solution.

$p \geqslant 2^*$

Theorem (X-J. Wang, '91)

When $p = 2^*$ *and* $\Omega = R\Omega_1$ *with* R *large enough,* $(\mathcal{P}_p)$ *possesses at least one non-constant positive solution.*

Theorem (E. Serra & P. Tilli, '11)

Assume $a \in L^1(]0, R[)$ *is increasing, not constant and satisfies* $a > 0$ *in* $]0, R[$*, then for any* $p \in]2, +\infty[$, $-\Delta u + u = a(|x|)|u|^{p-2}u$ *with NBC possesses a positive radially increasing solution.*

Trick: work on the space of radially increasing functions.

$p \geqslant 2^*$

Proposition

Assume Ω is a ball of radius R. If u is a radial solution of $(\mathcal{P}_p)$ such that $u(0) < 1$, then $\|u\|_{L^\infty} \leqslant \exp(1/2)$.

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Assume Ω is a ball of radius R. If u is a radial solution of $(\mathcal{P}_p)$ such that $u(0) < 1$, then $\|u\|_{L^\infty} \leqslant \exp(1/2)$.

Proof. In radial coordinates, the equation writes

$$-u'' - \frac{N-1}{r} u' + u = u^{p-1}.$$

Multiplying by u', we get

$$\frac{\mathrm{d}}{\mathrm{d}r} h(r) = -\frac{N-1}{r} u'^2(r) \leqslant 0,$$

where

$$h(r) := \frac{u'^2(r)}{2} + \frac{u^p(r)}{p} - \frac{u^2(r)}{2}.$$

In particular, this means that $h(r) \leqslant h(0)$ for any r.

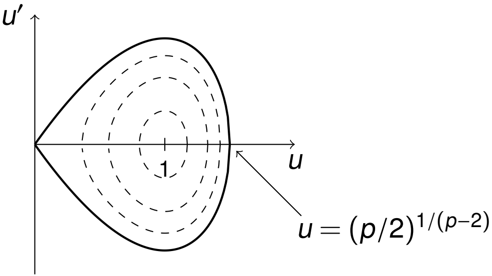
$p \geqslant 2^*$

PROOF (CONT'D). The assumption $u(0) < 1$ implies

$$h(0) = \frac{u^p(0)}{p} - \frac{u^2(0)}{2} = u^2(0)\Big(\frac{u^{p-2}(0)}{p} - \frac{1}{2}\Big) \leqslant 0.$$

Thus

$$\|u\|_{L^\infty} \leqslant \Big(\frac{p}{2}\Big)^{1/(p-2)} \leqslant \exp(1/2). \qquad \square$$



$p \geqslant 2^*$

Theorem

Assume Ω is a ball. Then, for any $n \in \mathbb{N}_0$, there exists p_n s.t., for any $p \in [p_n, +\infty[$, $(\mathcal{P}_p)$ has at least n positive radially symmetric solutions.

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SKETCH: As we saw, radial bifurcations are transcritical and along the right branch (starting with $p > 1 + \lambda_{i,\text{rad}}$) $u_p(0) < 1$. Thus all u belonging to that branch must satisfy $\|u\|_{L^\infty} \leqslant \exp(1/2)$. Since 1 is the only solution for $p \approx 2$, the branch must exist for all p large. □

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Sketch: As we saw, radial bifurcations are transcritical and along the right branch (starting with $p > 1 + \lambda_{i,\text{rad}}$) $u_p(0) < 1$. Thus all u belonging to that branch must satisfy $\|u\|_{L^\infty} \leqslant \exp(1/2)$. Since 1 is the only solution for $p \approx 2$, the branch must exist for all p large. □

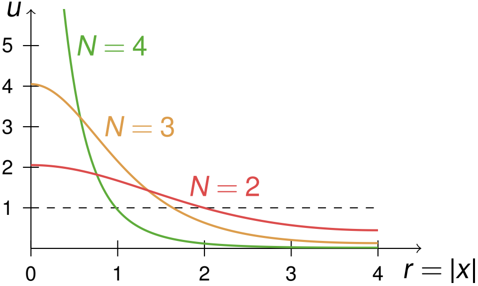
Conjecture

$p_n = 1 + \lambda_{n,rad}$.

Some numerical computations

Radial ground state for $p = 0.95 + \lambda_{2,\mathrm{rad}} < 2^*$ on B_4

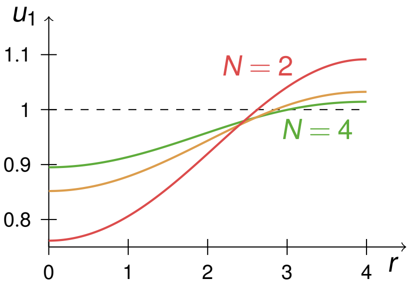
Using the Mountain Pass Algorithm in the space of radial functions:

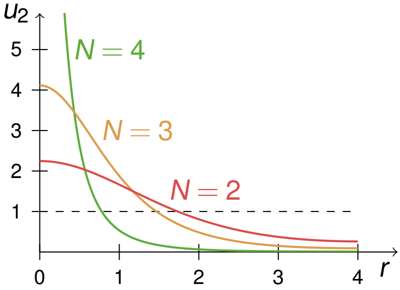


N	2^*	$1+\lambda_{2,\mathrm{rad}}$	$\mathcal{E}(1)$	$\min u$	$\max u$	$\mathcal{E}(u)$
2	∞	2.92	7.60	0.447	2.05	7.45
3	6	3.26	50.58	0.130	4.05	34.85
4	4	3.65	280.58	0.016	13.31	66.39

Radial ground state for $p = 1.1 + \lambda_{2,\text{rad}} < 2^*$ on B_4

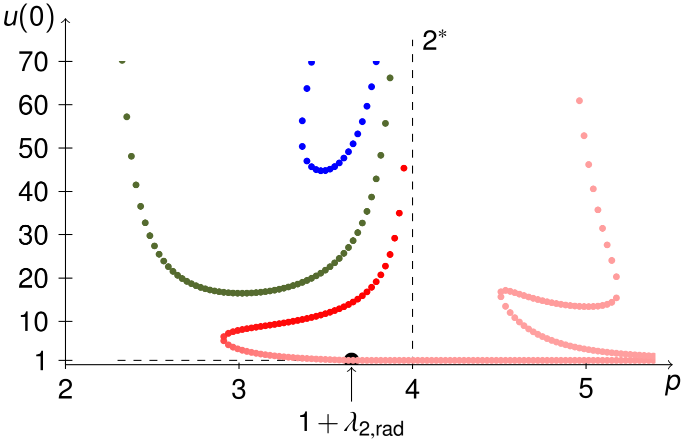
Using the Mountain Pass Algorithm in the space of radial functions with initial functions $x \mapsto 1 \pm 0.2|x|$.





N	$1+\lambda_{2,\text{rad}}$	$\mathcal{E}(1)$	$\min u_1$	$\max u_1$	$\mathcal{E}(u_1)$	$\min u_2$	$\max u_2$	$\mathcal{E}(u_2)$
2	2.92	8.48	0.76	1.09	8.47	0.261	2.25	7.39
3	3.26	54.30	0.85	1.03	54.29	0.092	4.12	30.74
4	3.65	294.63	0.90	1.01	294.62	0.008	17.25	49.61

Bifurcation diagram $N = 4$, $R = 4$



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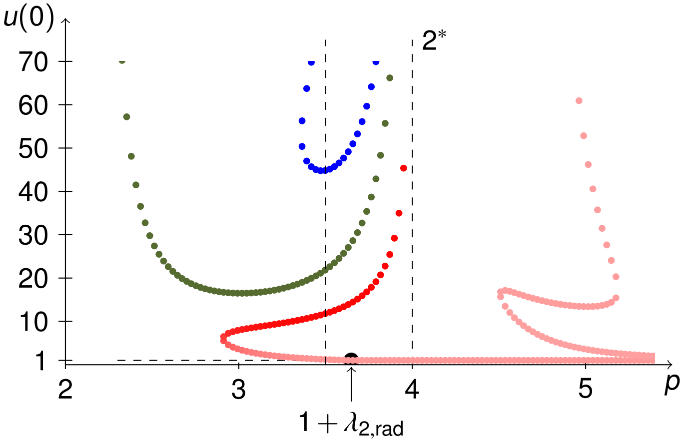
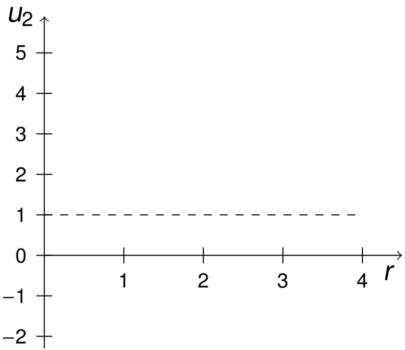


Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

Shape of the solutions for $p = 3.5 < 1 + \lambda_{2,\text{rad}}$.



$\mathcal{E}(1) \approx 270.709$

Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

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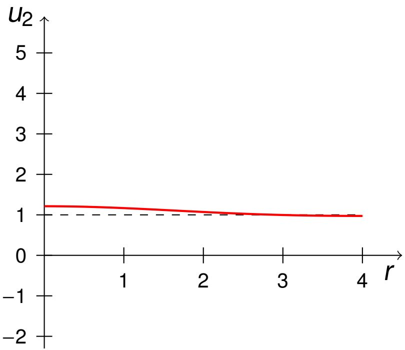
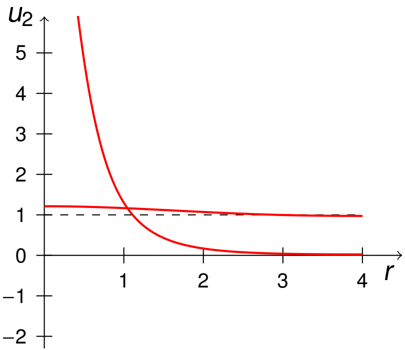


$$\mathcal{E}(1) \approx 270.709$$

$$u(0) \approx 1.213 \quad \Rightarrow \mathcal{E} \approx 270.753$$

Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

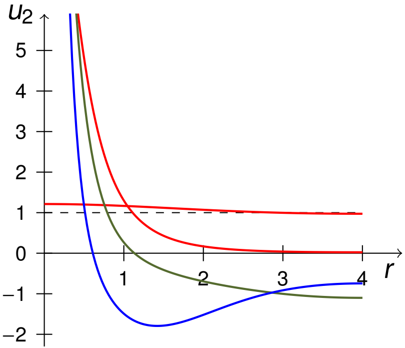
Shape of the solutions for $p = 3.5 < 1 + \lambda_{2,\text{rad}}$.



$$\mathcal{E}(1) \approx 270.709$$
$$u(0) \approx 1.213 \Rightarrow \mathcal{E} \approx 270.753$$
$$u(0) \approx 11.803 \Rightarrow \mathcal{E} \approx 79.730$$

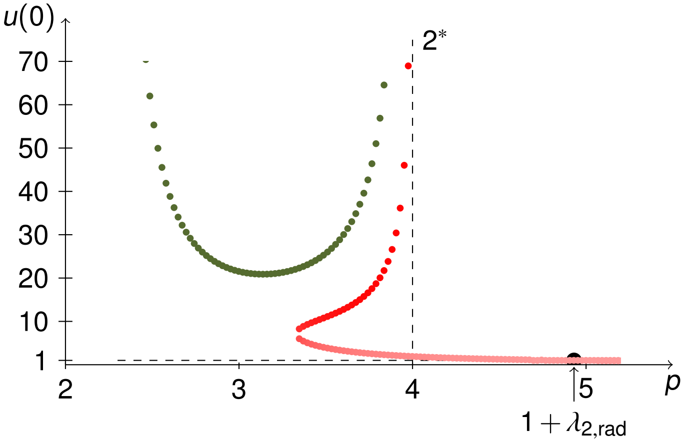
Bifurcation diagram $N = 4$, $R = 4$ (cont'd)

Shape of the solutions for $p = 3.5 < 1 + \lambda_{2,\text{rad}}$.

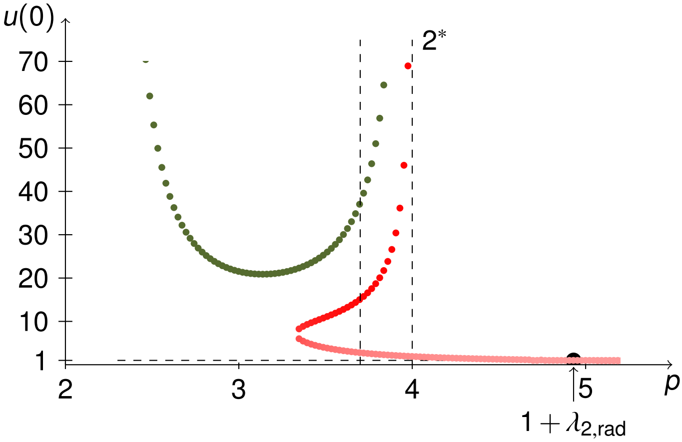


$$
\begin{aligned}
&& \mathcal{E}(1) &\approx 270.709 \\
u(0) &\approx 1.213 & \Rightarrow \mathcal{E} &\approx 270.753 \\
u(0) &\approx 11.803 & \Rightarrow \mathcal{E} &\approx 79.730 \\
u(0) &\approx 21.887 & \Rightarrow \mathcal{E} &\approx 390.387 \\
u(0) &\approx 44.830 & \Rightarrow \mathcal{E} &\approx 436.267
\end{aligned}
$$

Bifurcation diagram $N = 4$, $R = 3$

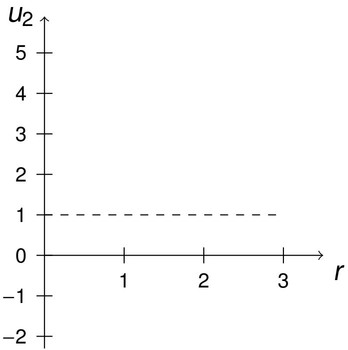


Bifurcation diagram $N = 4$, $R = 3$



Bifurcation diagram $N = 4$, $R = 3$ (cont'd)

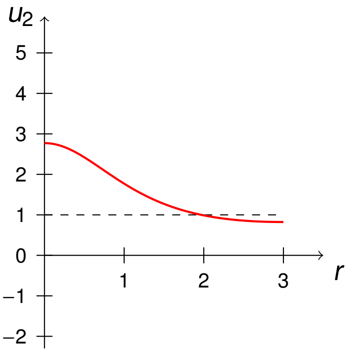
Shape of the solutions for $p = 3.7 < 2^* < 1 + \lambda_{2,\text{rad}}$.



$\mathcal{E}(1) \approx 91.8273$

Bifurcation diagram $N = 4$, $R = 3$ (cont'd)

Shape of the solutions for $p = 3.7 < 2^* < 1 + \lambda_{2,\text{rad}}$.

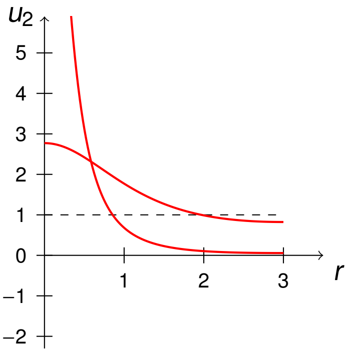


$$\mathcal{E}(1) \approx 91.8273$$

$$u(0) \approx 2.77189 \Rightarrow \mathcal{E} \approx 95.7796$$

Bifurcation diagram $N = 4$, $R = 3$ (cont'd)

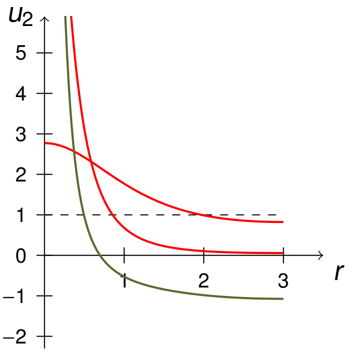
Shape of the solutions for $p = 3.7 < 2^* < 1 + \lambda_{2,\mathrm{rad}}$.



$$\mathcal{E}(1) \approx 91.8273$$
$$u(0) \approx 2.77189 \Rightarrow \mathcal{E} \approx 95.7796$$
$$u(0) \approx 15.1307 \Rightarrow \mathcal{E} \approx 54.283$$

Bifurcation diagram $N = 4$, $R = 3$ (cont'd)

Shape of the solutions for $p = 3.7 < 2^* < 1 + \lambda_{2,\mathrm{rad}}$.



$$\mathcal{E}(1) \approx 91.8273$$

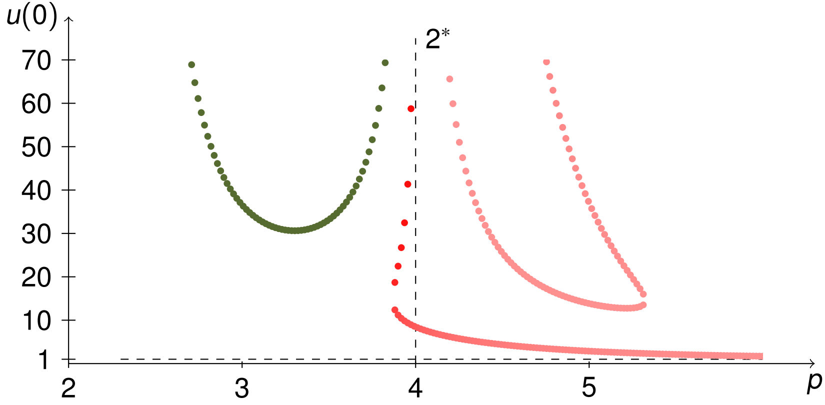
$$u(0) \approx 2.77189 \Rightarrow \mathcal{E} \approx 95.7796$$

$$u(0) \approx 15.1307 \Rightarrow \mathcal{E} \approx 54.283$$

$$u(0) \approx 37.412 \Rightarrow \mathcal{E} \approx 168.972$$

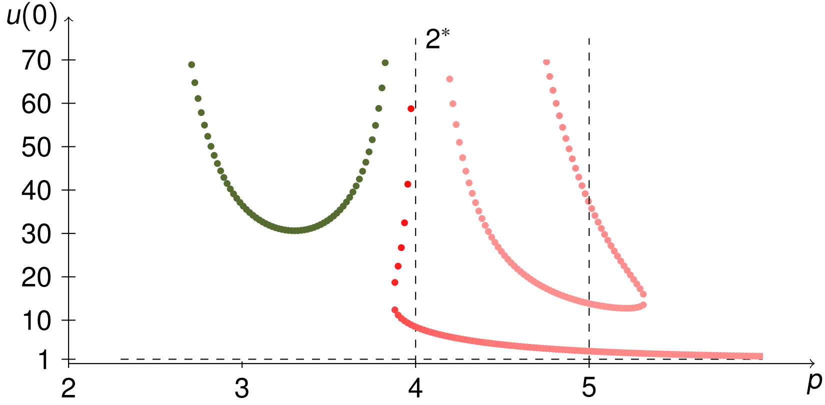
Bifurcation diagram $N = 4$, $R = 2$

$1 + \lambda_2 \approx 8.59365$



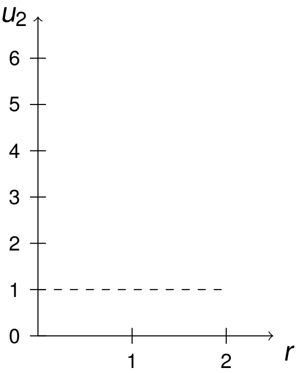
Bifurcation diagram $N = 4$, $R = 2$

$1 + \lambda_2 \approx 8.59365$



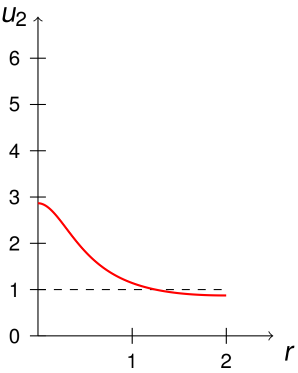
Bifurcation diagram $N = 4$, $R = 2$ (cont'd)

Shape of the solutions for $2^* < p = 5 < 1 + \lambda_{2,\mathrm{rad}}$.



Bifurcation diagram $N = 4$, $R = 2$ (cont'd)

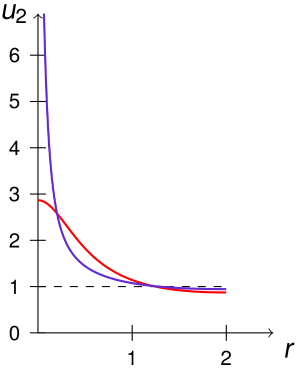
Shape of the solutions for $2^* < p = 5 < 1 + \lambda_{2,\text{rad}}$.



$u(0) \approx 2.86611$

Bifurcation diagram $N = 4$, $R = 2$ (cont'd)

Shape of the solutions for $2^* < p = 5 < 1 + \lambda_{2,\text{rad}}$.

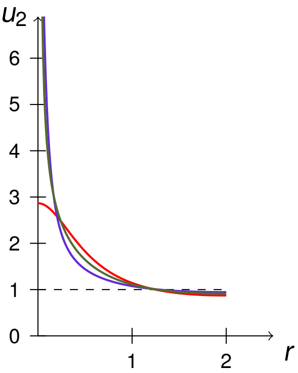


$u(0) \approx 2.86611$

$u(0) \approx 13.8393$

Bifurcation diagram $N = 4$, $R = 2$ (cont'd)

Shape of the solutions for $2^* < p = 5 < 1 + \lambda_{2,\text{rad}}$.



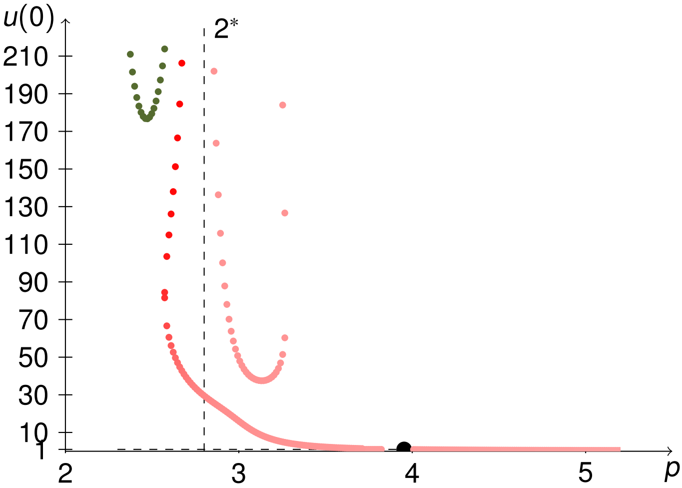
$u(0) \approx 2.86611$

$u(0) \approx 13.8393$

$u(0) \approx 37.0332$

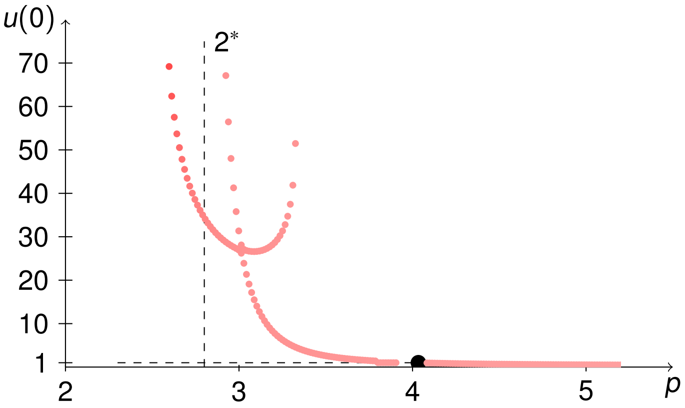
Bifurcation diagram $N = 7$, $R = 5$

$1 + \lambda_{2,\text{rad}} \approx 3.95325$



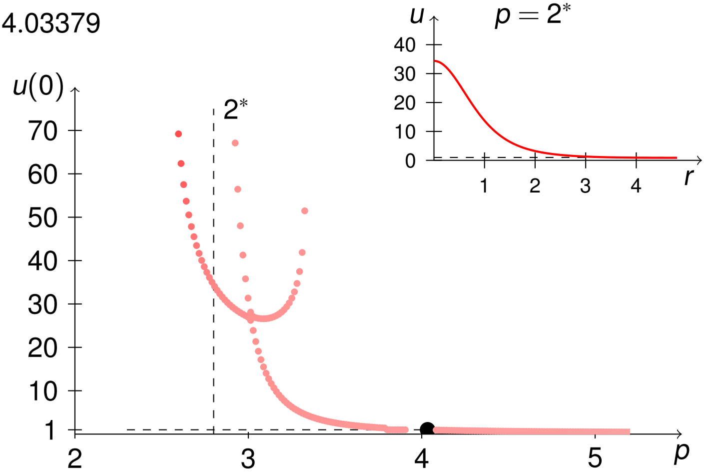
Bifurcation diagram $N = 7$, $R = 4.9$

$1 + \lambda_2 \approx 4.03379$



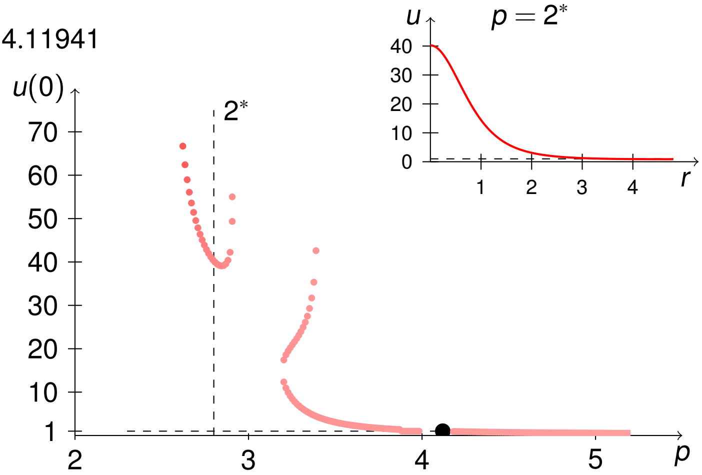
Bifurcation diagram $N = 7$, $R = 4.9$

$1 + \lambda_2 \approx 4.03379$



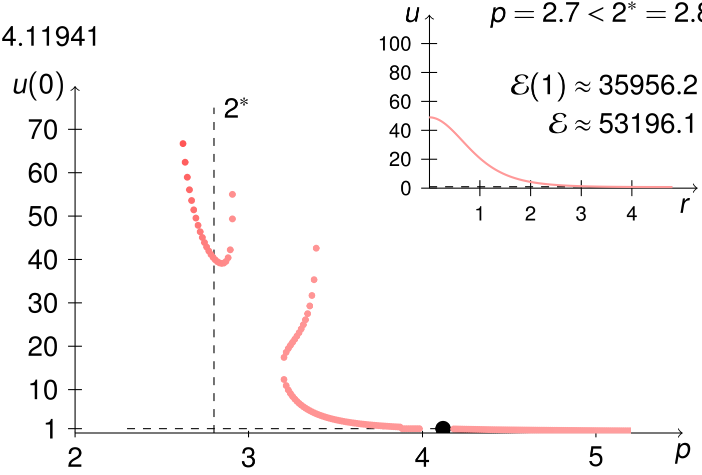
Bifurcation diagram $N = 7$, $R = 4.8$

$1 + \lambda_2 \approx 4.11941$



Bifurcation diagram $N = 7$, $R = 4.8$

$1 + \lambda_2 \approx 4.11941$



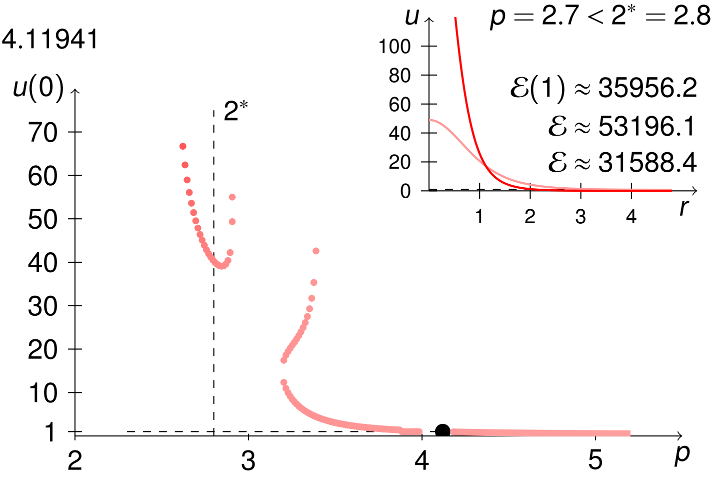
$p = 2.7 < 2^* = 2.8$

$\mathcal{E}(1) \approx 35956.2$

$\mathcal{E} \approx 53196.1$

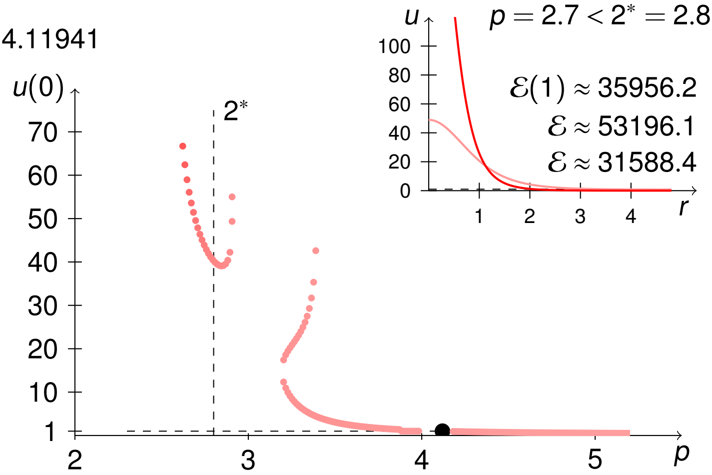
Bifurcation diagram $N = 7$, $R = 4.8$

$1 + \lambda_2 \approx 4.11941$



Bifurcation diagram $N = 7$, $R = 4.8$

$1 + \lambda_2 \approx 4.11941$



It is known [Adimurthi & S. L. Yadava '97] that if $N \geqslant 7$ and R is small enough, positive solutions for $p = 2^*$ must be constant.

Thank you for your attention.

Krasnoselskii-Boehme-Marino theorem (1/2)

Theorem (Krasnoselskii-Boehme-Marino)

Let $F : I \times H \to K : (t,u) \mapsto F(t,u)$ be a continuous function, where $I \subseteq \mathbb{R}$ is an interval, and H and K are Banach spaces, such that $F(\lambda,0) = 0$ for any $\lambda \in I$.

- *If F is of class $\mathcal{C}^1$ in a neighborhood of $(\lambda,0)$ and $(\lambda,0)$ is a bifurcation point of F then $\partial_u F(\lambda,0)$ is not invertible.*
- *Let assume that for each $(\lambda,u) \in I \times H$,*

  $$F(\lambda,u) = L(\lambda,u) - N(\lambda,u), \quad L(\lambda,\cdot) = \lambda\mathbb{1} - T \quad \text{and} \quad N(\lambda,u) = o(\|u\|),$$

  with T linear, T and N compact, and the last equality being uniform on each compact set of λ.
  If λ_* is an eigenvalue of T with odd multiplicity, then $(\lambda_*,0)$ is a global bifurcation point for $F(t,u) = 0$.

Krasnoselskii-Boehme-Marino theorem (2/2)

Theorem (Krasnoselskii-Boehme-Marino (cont'd))

- *Let assume that* H *is a Hilbert space and that for each* $(\lambda, u) \in I \times \mathbb{R}$, $F(\lambda, u) = \nabla_u h(\lambda, u)$ *where*

$$h(\lambda, u) = \tfrac{1}{2}\langle L(\lambda, u), u\rangle - g(\lambda, u),$$
$$L(\lambda, \cdot) = \lambda \mathbb{1} - T, \quad \text{and} \quad \nabla g(\lambda, u) = o(\|u\|),$$

with T *linear and symmetric,* $g(\lambda, \cdot) \in \mathcal{C}^2$ *for all* λ*, and the last equality being uniform on each compact set of* λ*.*
If λ_* *is an eigenvalue of* T *with finite multiplicity and* $h(\lambda, \cdot)$ *verifies the Palais-Smale condition for each* λ*, then* $(\lambda_*, 0)$ *is a bifurcation point for* $F(t, u) = 0$*.*

Return