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Unconstrained higher spins of mixed symmetry

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Contents

Chapter 1

Introduction

When combined with General Relativity, the Standard Model of elementary particles provides a description of the fundamental forces of Nature that is in astonishing agreement with the experimental data available up to now. However, these theories rely upon two completely different frameworks, since General Relativity describes *classical* gravitational interactions, while the Standard Model describes the other three fundamental forces at the *quantum* level. On the other hand, some of the most spectacular advances in the history of physics originated from the search for a unified description of previously separated phenomena. Trying to overcome this distinction could thus provide a deep insight on both subjects and even on the very notions of space and time. Moreover, gravity couples to matter and, from a fundamental point of view, it is impossible to forego the need for a quantum description of the gravitational interaction, even if it is supposed to play a relevant role in regimes that lie far away from experimental possibilities. In fact, quantum-gravity effects are expected to occur at energies akin to the Plank scale $G_N^{-1/2} \simeq 10^{19}$ GeV, to be compared, for instance, with the Fermi scale $G_F^{-1/2} \simeq 10^2$ GeV that is still under investigation in current accelerator experiments. At any rate, even without direct experimental inputs, one can try to delineate the key features of a consistent quantum theory of gravity that reduces to General Relativity at low energies.

Various approaches were developed over the years to deal with this problem, but up to now String Theory appears most promising. This may sound surprising looking at the origin of the subject, since string models were introduced at the end of the sixties to describe strong interactions [1]. Their most evident signature is indeed a spectrum with infinitely many massive excitations, of increasing mass and spin. These states are in some sense related to "vibrational modes" of the strings, and they immediately recall the trend observed starting from the fifties for hadronic resonances. In the seventies, however, deep inelastic scattering experiments robustly supported the QCD description of strong interactions, where hadronic resonances appear as composite objects. At the same time, Yoneya [2] and independently Scherk and Schwarz [3] pointed out that one can identify the single massless spin 2 particle contained in *all* closed string spectra with the graviton. Furthermore, with a proper tuning of the string

tension, the energy needed to excite massive string states becomes so big to justify the lack of their experimental observation. The ubiquitous presence of a spin 2 mode in closed string spectra and the UV softness displayed by string models, to be compared with the UV divergent behavior of Einstein gravity [4], shortly led to consider String Theory as a paradigm for quantum gravity. This gave rise to an enormous effort aimed at clarifying the properties of this complicated framework[1].

Despite the understanding thus attained of some particular aspects of String Theory, it is worth stressing that a number of fundamental problems are still open. For instance, from the space-time viewpoint String Theory nowadays is still formulated in a *background-dependent* manner, without any analogue of the powerful geometric background-independent presentation of Einstein gravity. Aside from a handful of examples, all solvable string models describe infinitely many fields of increasing mass and spin interacting in a Minkowski background. This view was made more precise starting from the eighties, with the development of the second quantized reformulation of String Theory, called String Field Theory[2]. This framework connects somehow the string proposal to Quantum Field Theory on a flat background, despite the presence of rather unconventional unbounded mass spectra and of higher-derivative interactions. Einstein gravity is thus recovered via the self-interactions of massless spin 2 quanta in a Minkowski background, while the pure-gravity UV divergences [4] are cured via the exchange of infinite towers of massive modes with spin $s > 2$.

In conclusion, String Theory somehow restates the old idea that the framework of Quantum Field Theory could also account for gravitational phenomena, but at the price of considering also *higher-spin excitations*. The new states that appear at the string scale weaken the gravitational potential and restore the high-energy predictivity that is lost in the non-renormalizable spin 2 model. This path qualitatively resembles that leading from the Fermi theory of weak interactions to the Standard Model. It is thus natural to demand whether the string picture is compelling in order to obtain a UV completion of Einstein gravity, or whether the key is rather to be found in the coupling with higher spins[3]. Moreover, a carefully analysis of the properties of these complicated systems of interacting higher-spin fields could also unveil the path toward a potential background-independent framework. The quest for possible generalizations

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[1]For a non-technical review of the present status of String Theory, that contains many references to books and technical reviews on the subject, see for instance [5]. Some books on String Theory are also signaled in [6].

[2]While at the free level the second quantized formulation of all string models was completed in the eighties [7], the situation is slightly different at the interacting level. In fact, a Chern-Simons-like action proposed by Witten [8] nicely describes interacting bosonic open strings, while for closed strings a long term research activity culminated in a proposal by Zwiebach [9], that presents some unusual features with respect to ordinary Quantum Field Theory.

[3]Actually, the answer could be even simpler, since the recent improvement of the computational techniques for loop amplitudes provides some encouraging evidence to the effect that at least $\mathcal{N} = 8$ supergravity [10] might be finite in $D = 4$. Recent results on this subject are reported in [11], that also contains a wide list of references to the previous literature. The possible finiteness of $\mathcal{N} = 8$ supergravity could drastically simplify the present understanding of the features requested by a quantum theory of gravity, but the string UV completion could continue to play an important role in view of its possible phenomenological applications, related to lower values of $\mathcal{N}$.

of the string setup and for a deeper understanding of its peculiar features is indeed one of the strongest motivations for this Thesis, as well as for much of the current literature on higher spins[4].

The study of higher-spin field theories has nevertheless a long history: Majorana proposed Lorentz covariant wave equations for arbitrary spin particles already in 1932 [14], and some years later Dirac independently reexamined the problem [15]. At the end of the thirties Wigner put the subject on a more solid basis, and clearly pointed out that elementary particles can be classified by the unitary irreducible representations of the isometry group of space-time [16]. In the four-dimensional Minkowski case this is the Poincaré group $ISO(1,3)$, whose irreducible representations are labeled by mass squared and spin. These values are not restricted a priori, and the analysis of all corresponding linear covariant wave equations was completed at the end of the forties [17]. In the meantime the first Lagrangians for spin 2 and spin 3/2 particles were obtained by Fierz and Pauli [18, 19] and by Rarita and Schwinger [20]. However, even for the free theory, the search for a Lagrangian formulation for arbitrary higher-spin fields was not completed in this first stage. The problem was reconsidered only starting from the fifties, and it was then solved for spin $\frac{5}{2} \leq s \leq 4$ in [21, 22].

Finally, in the mid seventies Singh and Hagen proposed Lagrangians for free *massive* particles of arbitrary spin in [23, 24]. In a four-dimensional Minkowski background the Lagrangian description of the free dynamics was then completed in 1978 by Fronsdal and Fang [25, 26], who considered the *massless* limit of the Singh-Hagen construction and showed the expected emergence of a gauge symmetry. Shortly thereafter the Fang-Fronsdal results were actually recovered conjecturing the presence of an abelian gauge symmetry [27, 28] and analyzing the consequences of this request. The resulting massless free theory follows the lines of the metric formalism for gravity, and involves the smallest field content giving a covariant description. The basic fields are fully symmetric tensors $\varphi_{\mu_1 \ldots \mu_s}$ and spinor-tensors $\psi^\alpha{}_{\mu_1 \ldots \mu_s}$, that generalize the vector potential A_μ, the linearized metric fluctuation $h_{\mu\nu}$ and the gravitino field $\psi^\alpha{}_\mu$. Their gauge transformations read

$$\delta \varphi_{\mu_1 \ldots \mu_s} = \partial_{(\mu_1} \Lambda_{\mu_2 \ldots \mu_s)}, \qquad \delta \psi^\alpha{}_{\mu_1 \ldots \mu_s} = \partial_{(\mu_1} \epsilon^\alpha{}_{\mu_2 \ldots \mu_s)}, \qquad (1.1)$$

where here and in the following a couple of parentheses denotes a complete symmetrization of the indices it encloses, with the minimum possible number of terms and with *unit* overall normalization. However, differently from the "low-spin" examples with $s \leq 2$, in the general case the fields are subjected to *algebraic (γ-)trace constraints* inherited from the massive free theory,

$$\varphi_{\mu_1 \ldots \mu_{s-4} \lambda\rho}{}^{\lambda\rho} = 0, \qquad \slashed{\psi}_{\mu_1 \ldots \mu_{s-3} \lambda}{}^{\lambda} = 0. \qquad (1.2)$$

Similar constraints also appear for the gauge parameters,

$$\Lambda_{\mu_1 \ldots \mu_{s-3} \lambda}{}^{\lambda} = 0, \qquad \slashed{\epsilon}_{\mu_1 \ldots \mu_{s-2}} = 0. \qquad (1.3)$$

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[4] For reviews on the current status of higher-spin field theories see [12, 13] and references therein.

The role played by these constraints will be discussed in great detail in the following. To conclude this first overview of the subject, let me recall that a frame-like formulation of the free theory [29] was also developed by Vasiliev and others starting from the beginning of the eighties. While the Fang-Fronsdal setup extends to higher-spin fields the linearized Einstein-Hilbert presentation of gravity, the Vasiliev setup extends somehow its Cartan-Weyl form in the McDowell-Mansouri presentation [30].

Looking at this brief summary of the evolution of the *free* theory, the large time span separating Majorana's paper and Fronsdal's work could appear surprising. Aside from technical issues and from the changing fortunes of field theory, this can be reasonably related to the difficulties that were soon encountered when trying to introduce interactions for higher-spin particles. But before commenting further on interactions, let me continue for a while to focus on the free theory, in order to better delineate the content of this Thesis.

In fact, the Singh-Hagen [23, 24] and the Fang-Fronsdal results [25, 26] can be improved in two directions. On one hand one can look for alternative formulations of the free dynamics that are more likely to lead to an interacting theory. The already recalled frame-like formulation of [29] is the first example. One the other hand one can follow the old idea of Kaluza and Klein [31], and consider a space-time with additional dimensions. This is possible if the extra dimensions are compact and have a characteristic scale so small to explain why they have not been detected so far in experiments. Let me start considering this latter possibility.

Models with extra dimensions were originally developed to unify Einstein gravity with electromagnetism, but this idea is crucial in String Theory. Indeed, string models can be quantized without breaking the Lorentz symmetry only in $D = 26$ for bosonic strings and only in $D = 10$ for superstrings[5]. If the dimension of space-time is greater than five the recalled results do not suffice to depict all the representations of the isometry group of a flat background. Further steps are thus needed in order to covariantly describe the propagation of arbitrary free modes. For instance, in the massless case the irreducible representations of the Poincaré group $ISO(1, D-1)$ are built upon the irreducible representations of the little group $SO(D-2)$. These are no longer labeled by a single half-integer, the spin, but rather by Young tableaux with $N \leq \frac{D-2}{2}$ rows. The Fang-Fronsdal results only apply to single-row tableaux in arbitrary dimensions, corresponding to fully symmetric fields $\varphi_{\mu_1 \ldots \mu_s}$ or $\psi^{\alpha}{}_{\mu_1 \ldots \mu_s}$. The other *"mixed-symmetry"* cases, corresponding to arbitrary Young tableaux, are covariantly described by more general (spinor-)tensors $\varphi_{\mu_1 \ldots \mu_{s_1}, \nu_1 \ldots \nu_{s_2}, \ldots}$ and $\psi^{\alpha}{}_{\mu_1 \ldots \mu_{s_1}, \nu_1 \ldots \nu_{s_2}, \ldots}$, that possess several families of fully (anti)symmetric space-time indices and that require a dedicated treatment.

The development of String Field Theory stimulated the study of mixed-symmetry fields in the mid eighties, even if some discussions date back to previous years [32]. In fact, all string spectra contain fields of mixed symmetry, that are associated to products of different types of bosonic or fermionic oscillators. Free String Field Theory thus

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[5]For a clear explanation of this statement see for instance [6], where detailed references to the papers where the existence of critical dimensions was first noticed can be also found.

provides a covariant description of arbitrary massive modes, at least in $D = 10$ or in $D = 26$. A number of authors then worked on how to extend this result also to the massless case. This should not only be considered as an "academic problem", since the dependence of couplings and masses on the same parameter α' has long suggested the idea that String Theory could be a broken phase of a higher-spin *gauge* theory. However, the breaking mechanism at work still lacks a precise formulation, and a proper analysis of this problem requires a full control of mixed-symmetry gauge fields.

After some partial results regarding particular classes of mixed-symmetry fields [33], several covariant formulations for arbitrary massless modes were proposed. In [34, 35] gauge invariant actions for Bose fields were derived using BRST techniques. These covariant formulations are related to the $\alpha' \to \infty$ limit of Free String Field Theory, and thus involve a number of auxiliary fields. Furthermore, they naturally describe reducible representations of the Poincaré group. Another approach was then developed for both Bose and Fermi fields via an extension of the light-cone representations of the Poincaré algebra [36].

Finally, Labastida proposed a set of covariant field equations for arbitrary massless Bose and Fermi fields of mixed symmetry exploiting an abelian gauge symmetry [37, 38]. His formulation extends the Fang-Fronsdal setting of [25, 26]: it involves multi-symmetric tensors $\varphi_{\mu_1 \dots \mu_{s_1}, \nu_1 \dots \nu_{s_2}, \dots}$ and (spinor-)tensors $\psi^{\alpha}{}_{\mu_1 \dots \mu_{s_1}, \nu_1 \dots \nu_{s_2}, \dots}$, possessing several families of symmetrized space-time indices. It thus relies on the minimal field content required by a covariant description[6]. On the other hand, the gauge transformations take a more complicated form in the mixed-symmetry case, since they involve one independent gauge parameter for each index family,

$$\delta \varphi_{\mu_1 \dots \mu_{s_1}, \nu_1 \dots \nu_{s_2}, \dots} = \partial_{(\mu_1} \Lambda^{(1)}{}_{\mu_2 \dots \mu_s), \nu_1 \dots \nu_{s_2}, \dots} + \partial_{(\nu_1 |} \Lambda^{(2)}{}_{\mu_1 \dots \mu_s, | \nu_2 \dots \nu_{s_2}), \dots} + \dots,$$

$$\delta \psi^{\alpha}{}_{\mu_1 \dots \mu_{s_1}, \nu_1 \dots \nu_{s_2}, \dots} = \partial_{(\mu_1} \epsilon^{(1) \alpha}{}_{\mu_2 \dots \mu_s), \nu_1 \dots \nu_{s_2}, \dots} + \dots. \tag{1.4}$$

Qualitative features of the fully symmetric case still persist, and fields and gauge parameters satisfy some (γ-)trace constraints. However, in this case it is possible to consider different (γ-)traces, related to different index families, and only some of their linear combinations are forced to vanish.

The field equations of [37, 38] are non-Lagrangian[7], but in [39] Labastida also built Lagrangians leading to equivalent equations of motion for Bose fields[8]. On the other hand, for Fermi fields the search for a Lagrangian formulation initiated long ago by Fierz and Pauli [19] was completed only seventy years later in [41], within the project that led to the present Thesis. While [39] and [41] contain only Lagrangians for

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[6]Actually, without any symmetry relating the various index families these fields only describe reducible representations of the Lorentz group. At any rate, as we shall see more in detail in the following, performing standard Young projections they can be easily adapted to describe irreducible representations.

[7]In this Thesis the term denotes a set of field equations that were originally obtained not starting from an action principle, but may or may not be eventually connected to a less conventional one.

[8]In some low-dimensional cases the equivalence between the two setups actually involves some subtleties that were discussed in [40] and that will be analyzed in Chapter 4.

massless fields of mixed symmetry, the corresponding massive theories can be obtained by a standard Kaluza-Klein reduction.

The structure of the Lagrangians for mixed-symmetry fields is rather involved, since they contain a number of higher (γ-)traces of the fields that grows proportionally to the number of index families. This is a consequence of the more complicated form of the (γ-)trace constraints. The procedure to build the Lagrangians and their structure will be discussed in detail in Chapter 2, while Chapter 4 will show how this involved form leads to the emergence of Weyl-like symmetries in low enough space-time dimensions. This Thesis is in fact mainly based on the content of [41] and of its companion paper [40], where the Labastida formulation for Bose fields was reconsidered in order to embed it in an *unconstrained* framework.

In fact, the Fang-Fronsdal [25, 26] and Labastida [39, 41] descriptions of the free dynamics are compact and elegant, but they present at least an undesirable feature. It is the unusual presence of off-shell constraints, needed to ensure the propagation of the correct number of degrees of freedom. These constraints are algebraic and thus in principle should not obstruct a non-linear deformation of the theory, but in the tensionless limit $\alpha' \to \infty$ string spectra do not quite result in sums of fields subject to the Labastida constraints. For instance, in this limit the fully symmetric modes of the first Regge trajectory of the open bosonic string are described by interesting systems of fields usually called "bosonic triplets" [35, 42, 43], that propagate reducible representations including, together with a given set of spin-s modes, lower spin chains as well and do not rest on Fronsdal-like trace constraints. In a similar fashion, the fermionic counterparts of these triplets, defined in [42, 43], play a role in orientifolds [44] of the 0B theory [45], although not in superstrings as a result of their GSO projections [46]. In general, it is thus natural to ask how one can eliminate the constraints and, as the string example suggests, whether their removal could be of help when attempting to switch on interactions. The first concrete step in this direction was performed by Pashnev, Tsulaia, Buchbinder and collaborators [47] starting from the late nineties. These authors developed further the BRST techniques adopted in [34, 35], in order to describe the propagation of a single massless spin s mode. They also treated fermions and extended these results to constant curvature backgrounds, but for a spin-s field in their construction the constraints are removed at the price of introducing $\mathcal{O}(s)$ *auxiliary fields*.

An alternative solution to the problem was proposed in 2002 by Francia and Sagnotti for fully symmetric fields [48]. They eliminated the Fang-Fronsdal constraints without introducing any additional fields, via Lagrangians and field equations that are *non local*. On the other hand, the non-localities can be eliminated by a partial gauge fixing involving the (γ-)trace of the gauge parameters, that recovers the Fang-Fronsdal presentation. The non-local terms are thus harmless, and their introduction also gives an additional benefit: in fact, the resulting field equations and Lagrangians find a role for the higher-spin curvatures introduced by de Wit and Freedman [28, 49]. These objects are gauge invariant under unconstrained gauge transformations, and sit at the top of a recursive chain of generalized connections that are characterized by simple

gauge variations. Higher-spin connections and curvatures thus naturally extend the linearized Christoffel symbols and the Riemann tensor of General Relativity. Even if at the free level one can only display these algebraic similarities, they provide hints on a possible geometrical non-linear realization of higher-spin models. Further details and comments on this interesting alternative can be found in [42, 50]. This formulation of the free theory for symmetric fields reached its final form in [51], with the identification of non-local Lagrangians leading to the correct current-current exchanges. Shortly after [48] appeared, the Francia-Sagnotti results were instead extended to arbitrary mixed-symmetry massless Bose fields in [52], and more recently also to the case of massive fully symmetric Bose and Fermi fields [53]. Similarly, unconstrained Lagrangians for both massless and massive mixed-symmetry fields were proposed in the context of the already mentioned BRST approach [54].

The two presentations recalled here eliminate the constraints, but present again some unusual features: in the fully symmetric case, for instance, one has to face field equations containing either $\mathcal{O}(s)$ auxiliary fields or $\mathcal{O}(s)$ derivatives of the basic spin s field. A simpler solution was developed for fully symmetric fields by Francia and Sagnotti in [42], observing that the Fang-Fronsdal field equations

$$
\begin{aligned}
\mathcal{F}_{\mu_1 \ldots \mu_s} &\equiv \Box\, \varphi_{\mu_1 \ldots \mu_s} \,-\, \partial_{(\mu_1} \partial \cdot \varphi_{\mu_2 \ldots \mu_s)} \,+\, \partial_{(\mu_1} \partial_{\mu_2} \varphi_{\mu_3 \ldots \mu_s)\lambda}{}^{\lambda} = 0\,, \\
\mathcal{S}^{\alpha}{}_{\mu_1 \ldots \mu_s} &\equiv i\,\left\{ \partial\!\!\!/\, \psi^{\alpha}{}_{\mu_1 \ldots \mu_s} \,-\, \partial_{(\mu_1} \psi\!\!\!/\,^{\alpha}{}_{\mu_2 \ldots \mu_s)} \right\} = 0\,,
\end{aligned}
\tag{1.5}
$$

vary under the *unconstrained* gauge transformations (1.1) as

$$
\begin{aligned}
\delta\,\mathcal{F}_{\mu_1 \ldots \mu_s} &= 3\,\partial_{(\mu_1} \partial_{\mu_2} \partial_{\mu_3} \Lambda_{\mu_4 \ldots \mu_s)\lambda}{}^{\lambda}\,, \\
\delta\,\mathcal{S}^{\alpha}{}_{\mu_1 \ldots \mu_s} &= -\,2\,i\,\partial_{(\mu_1} \partial_{\mu_2} \epsilon\!\!\!/\,^{\alpha}{}_{\mu_3 \ldots \mu_s)}\,.
\end{aligned}
\tag{1.6}
$$

For both Bose and Fermi fields, introducing a *single* auxiliary field one can thus consider the equivalent field equations[9]

$$
\begin{aligned}
\mathcal{A}_{\mu_1 \ldots \mu_s} &\equiv \mathcal{F}_{\mu_1 \ldots \mu_s} \,-\, 3\,\partial_{(\mu_1} \partial_{\mu_2} \partial_{\mu_3} \alpha_{\mu_4 \ldots \mu_s)} = 0\,, \\
\mathcal{W}^{\alpha}{}_{\mu_1 \ldots \mu_s} &\equiv \mathcal{S}^{\alpha}{}_{\mu_1 \ldots \mu_s} \,+\, 2\,i\,\partial_{(\mu_1} \partial_{\mu_2} \xi^{\alpha}{}_{\mu_3 \ldots \mu_s)} = 0\,,
\end{aligned}
\tag{1.7}
$$

that display the unconstrained gauge symmetry

$$
\left\{
\begin{aligned}
\delta\,\varphi_{\mu_1 \ldots \mu_s} &= \partial_{(\mu_1} \Lambda_{\mu_2 \ldots \mu_s)}\,, \\
\delta\,\alpha_{\mu_1 \ldots \mu_{s-3}} &= \Lambda_{\mu_1 \ldots \mu_{s-3}\lambda}{}^{\lambda}\,,
\end{aligned}
\right.
\qquad
\left\{
\begin{aligned}
\delta\,\psi^{\alpha}{}_{\mu_1 \ldots \mu_s} &= \partial_{(\mu_1} \epsilon^{\alpha}{}_{\mu_2 \ldots \mu_s)}\,, \\
\delta\,\xi^{\alpha}{}_{\mu_1 \ldots \mu_{s-2}} &= \epsilon\!\!\!/\,^{\alpha}{}_{\mu_1 \ldots \mu_{s-2}}\,.
\end{aligned}
\right.
\tag{1.8}
$$

Eqs. (1.7) reduce to the Fang-Fronsdal field equations after a partial gauge-fixing that does not spoil the original constrained gauge symmetry, since it involves the (γ-)trace of the gauge parameter. As those in eq. (1.5), the equations of motion (1.7) are non-Lagrangian, but in [56] Lagrangians leading to equivalent field equations were also presented (for reviews see [57, 51]). For Bose fields they involve a further

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[9]For a spin 3 field an equation of motion of the form (1.7) was actually proposed long ago by Schwinger in [55].

single additional Lagrange multiplier transforming as the triple divergence of the gauge parameter,

$$\delta\,\beta_{\mu_1\dots\mu_{s-4}} \;=\; \partial\cdot\partial\cdot\partial\cdot\Lambda_{\mu_1\dots\mu_{s-4}}\,, \tag{1.9}$$

while for Fermi fields they involve a *single* additional Lagrange multiplier transforming as

$$\delta\,\lambda^{\alpha}{}_{\mu_1\dots\mu_{s-3}} \;=\; \partial\cdot\partial\cdot\epsilon^{\alpha}{}_{\mu_1\dots\mu_{s-3}}\,. \tag{1.10}$$

In this context current-current exchanges were also considered in both Minkowski [51] and constant curvature backgrounds [58].

In conclusion, in the fully symmetric case an unconstrained metric-like formulation can be obtained at the modest price of introducing two auxiliary fields, for both Bose and Fermi fields. With the same logic it is also possible to obtain a "minimal" unconstrained formulation for arbitrary fields of mixed symmetry, introducing at most one compensator for each Labastida constraint on the gauge parameters and at most one Lagrange multiplier for each Labastida constraint on the field. This program was completed in [40, 41], and the first part of Chapter 3 will be devoted to this issue. To conclude, let me stress that from an orthodox point of view the field equations (1.7) also contain higher-derivative terms, and the same is true for the corresponding Lagrangian field equations. However, all higher derivative terms involve the compensators and as such are manifestly pure gauge contributions. They are thus not expected to create any problems, but for fully symmetric Bose fields a really "orthodox solution" to the problem, with only unconstrained fields, a fixed number of auxiliary fields and a two-derivative Lagrangian was proposed in [59]. It was obtained via an elegant off-shell truncation of the triplets of String Field Theory [35], and was followed by another proposal [53] that is more akin to the spirit of the minimal unconstrained formulation. Although these solutions rely upon a wider field content, they represent an interesting alternative to the minimal setup. Such an alternative formulation was also presented for Bose and Fermi fields of mixed symmetry in [40, 41]. It was obtained via a proper reformulation of the approach of [53], and will be discussed in the second part of Chapter 3.

Aside from its simplicity, the minimal formulation of [56, 57, 51, 40, 41] has the advantage of leading easily to the non-local formulation via the elimination of the auxiliary fields in terms of the basic field. On the other hand, it was linked to the triplets systems emerging from the $\alpha' \to \infty$ limit of String Field Theory in [43, 56], and so also to the related BRST-like constructions. It thus somehow bridges the gap between the algebraic "string inspired" framework and the geometrical non-local setup.

I can now conclude this overview of the present status of the free theory recalling that Lagrangians and field equations for mixed-symmetry fields were also recently obtained in a frame-like approach [60] and in its unfolded reformulation [61], both in flat and in constant curvature spaces. In general, higher-spin interactions present technical (and conceptual) difficulties related to the presence of higher derivatives, and the term "unfolding" denotes a systematic procedure aimed at eliminating them via the introduction of additional fields [62, 63, 13]. While this happens also in String Field Theory, at present there is a crucial difference, since in the resulting unfolded higher-

spin systems the field equations are of first order. For fully symmetric fields this line of development led to the Vasiliev equations, that describe non-linear interactions in an (Anti-)de Sitter background [62, 63, 13], and one natural goal is clearly to understand how a similar result can be also attained for mixed-symmetry fields. On the other hand, some steps toward the extension of the Labastida results to constant curvature spaces were made in [64], while these efforts have been accompanied during the years by a world-line first-quantized formulation of the dynamics of higher-spin fields [65]. Finally, mixed-symmetry fields have also been treated in a light-cone formalism [66]. Although not covariant, this approach can in fact efficiently deal with some interaction issues.

This excursus on higher spins was mainly driven by analogies with String Theory. They strongly motivated the study of mixed-symmetry representations of the higher-dimensional Lorentz groups, and also the analysis of the possible unconstrained realizations of the free theory. On the other hand, independent motivations for the development of new tools for higher spins come from the difficulties encountered when dealing with their interactions. Even if this Thesis focuses on some aspects of the free theory, let me conclude this introduction by briefly recalling the present constraints on the features of a would be systematic development of the non-linear theory.

Amusingly, some rough intuitions of possible difficulties appeared about eighty years ago in the first, but almost ignored[10], paper on the subject [14]. The Majorana equations indeed rely on an infinite chain of higher-spin fields, and the author noticed that switching on interactions may forbid the decoupling of a single spin s mode from the infinite chain! Clearer hints on the differences between interacting systems of higher-spin fields and their $s \leq 2$ counterparts came from the work of Fierz and Pauli [19]. Actually, they looked for an action principle in order to overcome the problems displayed by the field equations proposed by Fierz [18] for spin s field,

$$
\begin{aligned}
\left(\Box + m^2 \right) \varphi_{\mu_1 \dots \mu_s} &= 0 \,, \\
\partial \cdot \varphi_{\mu_1 \dots \mu_{s-1}} &= 0 \,, \\
\varphi_{\mu_1 \dots \mu_{s-2} \lambda}{}^{\lambda} &= 0 \,,
\end{aligned}
\tag{1.11}
$$

that are no longer compatible when one tries to couple them to an electromagnetic field via the minimal coupling $\partial_\mu \to \partial_\mu - e\,A_\mu$. Similar complications also emerged in the work of Dirac [15].

As we have already seen in the previous pages, the activity on higher spins was then frozen for some years, probably also due to the lack of clearcut phenomenological motivations. The situation drastically changed with the discovery of higher-spin resonances, and the problem was then reconsidered in more modern terms. At any rate, rather than surpassing the old difficulties the new techniques signaled even more severe drawbacks. For instance, in [68] it was pointed out that the minimal coupling with an external electromagnetic field leads to noncausalities or to the propagation

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[10]A notable exception was given by Wigner, who referred to the Majorana paper in his time-honored 1939 paper [16]. The Majorana approach was also recently reconsidered in [67].

of spurious degrees of freedom even for spin $3/2$ and spin 2 particles[11]. Similar problems were identified in the minimal coupling of spin 2 particles with gravity [71]. Moreover, in the same decade Weinberg realized that, under reasonable assumptions essentially related to analogies with gravity, massless higher-spin particles cannot give rise to long-range interactions in an S-matrix scheme [72]. Coleman and Mandula then proved a famous theorem [73] stating that, again under reasonable assumptions, a Quantum Field Theory leading to a non-trivial S-matrix can only admit a symmetry algebra given by the direct sum of the Poincaré algebra and of an internal algebra, whose generators commute with the Poincaré ones. This rules out conventional gauge theories with a *finite* number of gauge fields of spin $s \geq 3$, that would be associated to forbidden symmetry generators.

More vigorous attempts followed the discovery of supergravity [74], that displayed a new mechanism enabling the coupling of gravity with spin $3/2$ particles. On the other hand, it was soon realized that the supergravity "miracle" cannot be extended to higher spin particles [75, 76]. Schematically, the main difference is that, after the minimal coupling with gravity, the gauge variation of the Rarita-Schwinger action is proportional to the Ricci tensor, and as such can be compensated by the variation of the gravitational action. On the other hand, starting from spin $5/2$ this is no longer true [75], and the whole Riemann tensor appears in the gauge variation of the higher-spin action. The variation of the gravitational action cannot cancel the Weyl tensor, while the loss of gauge invariance would lead at least to the propagation of spurious degrees of freedom. For spin 2 the situation is slightly more subtle, but the conclusions remain the same [76]. Moreover, up to now no consistent modifications of the couplings or of the transformation laws were found, so that a proper generalization of the known gauge symmetry could be recovered. This fact is at the origin of the common lore about the impossibility of consistently coupling higher-spin fields with gravity[12], and all these results were then summarized in the extension of the Coleman-Mandula theorem due to Haag, Lopuszanski and Sohnius [77]. These authors essentially confirmed the conclusions of [73], barring the inclusion of supersymmetry, that can be taken into account considering graded algebras rather than only ordinary ones. Other no-go arguments that confirm and extend the classical results recalled here were also presented more recently in [78].

In spite of these apparently discouraging no-go results, the problem was reconsidered at the beginning of the eighties by the Göteborg group, that identified surprising cubic vertices for self-interacting massless particles of any spin in a light-cone formalism [79]. This result and the advent of String Field Theory gave a new mighty input to the field, and various cubic vertices were found in flat-space in a light-cone formalism [80, 66], in a covariant approach [81, 82, 83] or resorting to BRST techniques [84]. On the other hand, the severe problems encountered at the quartic level led to the realization that consistent interactions for spin $s > 2$ particles should involve *infinitely* many higher-

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[11]Actually, it was later shown that, at least for spin 2 [69] and spin $3/2$ particles [70], one can overcome these problems via a judicious choice of non-minimal interactions.

[12]It is also the origin of the upper limit of 32 supercharges considered in supergravity theories. Crossing the border of $\mathcal{N} = 8$ the graviton multiplet would indeed contain also higher-spin helicities.

spin fields at the same time [82, 85]. This observation perfectly agrees with the most evident signature of string spectra, and also bypasses the Coleman-Mandula theorem, since it contrasts one of its basic assumptions.

After these positive results, also supported and completed by more recent analyses [86], Fradkin and Vasiliev discovered a cubic vertex for higher-spin fields in interaction with gravity [87]. They forewent the problems identified by Aragone and Deser [71, 75] introducing a cosmological constant Λ, which allows the introduction of higher-derivative couplings depending on inverse powers of Λ. Notice that the presence of higher-derivative interactions suffices to bypass the available no-go results, and indeed non-minimal cubic vertices describing the interactions of massless higher-spin fields with gravity were recently singled out even in a Minkowski background [88]. A non-vanishing cosmological constant Λ simply leads to them in a natural way, without the need of introducing a new dimensionful coupling constant. At any rate, the Fradkin-Vasiliev result is a milestone for the modern view of higher-spin theories. In fact, in the following years Vasiliev built consistent non-linear field equations for an infinite set of fully symmetric higher-spin fields interacting on an (Anti-)de Sitter background of arbitrary dimension [62, 63]. To reach this goal he essentially gauged an infinite-dimensional extension of the Lorentz algebra [89] within an unfolded formulation of the dynamics (for a review of the Vasiliev setup see, for instance, [13], while interesting related contributions are due to Sezgin and Sundell [90]). The properties of the resulting non-linear equations have been under active investigation in recent years and, for instance, the first exact solutions were displayed by various groups [91]. On the other hand, up to now the Vasiliev equations still lack an action principle, and they exploit a formalism that is rather remote from that usually adopted to describe low-spin systems. Furthermore, a similar set of non-linear field equations until now was not obtained for mixed-symmetry fields, for which only no-go results are available in literature [92] if one looks for non-abelian deformations of the gauge algebra. Alternative approaches to the problem, as those presented in this Thesis, could thus be of help in overcoming these difficulties. At the same time, it is quite natural to compare the frame-like path followed by Vasiliev and collaborators in studying higher-spin interactions to a metric-like description. While the first approach somehow extends the algebraic Yang-Mills setup, the second could shed light on the geometry underlying these interactions, as the metric formulation does in the spin 2 case.

In conclusion, one might say that higher-spin interactions have presently two vastly different realizations. In the first, provided by the Vasiliev setting, their gauge symmetry is unbroken, while in the second, provided by String Theory, it is somehow spontaneously broken. Unfortunately, as anticipated, we do not understand precisely how to bridge the gap between the two, or even how to attach a precise meaning to the breaking mechanism at work in String Theory. The higher-spin gauge symmetries should be restored in a proper high-energy regime of String Theory, but nowadays this remains a rather obscure corner of this framework. Some important steps forward came from the study of the high-energy limits of the string scattering amplitudes [93],

but a detailed characterization of this regime is still missing[13]. At any rate, String Theory and the Vasiliev model share some neat signatures, to wit spectra containing infinitely many higher-spin modes and higher-derivative interactions. Clearly, these features restate the old border between the $s \leq 2$ and the higher-spin physics, but these two examples at least show that systems of interacting higher-spin fields coupled to gravity exist. Still, the present understanding of their properties is far from being fully satisfactory, and its improvement provides a fascinating task for the future.

Structure and original content of this Thesis

For the reader's benefit let me summarize the structure of this Thesis, recalling again the original results that it contains.

- In Chapter 2 the constrained formulation of the free dynamics for arbitrary mixed-symmetry fields will be discussed in detail. First of all, the identification of the non-Lagrangian Labastida equations will be reviewed with the aid of the compact notation proposed in [40, 41]. Furthermore, it will be shown that they indeed single out the correct physical polarizations, thus filling a gap of their first presentation in [37, 38]. Then, the Labastida Lagrangians for Bose fields [39] will be constructed in an original way, exploiting the consequences of the Bianchi identities satisfied by the constrained fields. The Lagrangians for mixed-symmetry Fermi fields of [41] will be built with the same technique. Chapter 2 closes with some comments on the Lagrangian field equations, and on their reduction to the Labastida form. Finally, I will also describe how to adapt the whole construction to irreducible fields propagating the degrees of freedom of an irreducible representation of the Lorentz group.

- In Chapter 3 the constrained Lagrangian free theory will be reformulated in various unconstrained settings, following [40, 41] and introducing suitable sets of auxiliary fields. I will begin by considering a minimal setup, introducing at most one compensator for each Labastida constraint on the gauge parameters and at most one Lagrange multiplier for each Labastida constraint on the field. Then, at the price of slightly enlarging the field content, I will also present a formulation that is free from higher-derivative terms, but still contains a number of auxiliary fields only depending on the number of index families carried by the basic fields and not on their Lorentz labels.

- In Chapter 4 the Lagrangians for mixed-symmetry fields will be reconsidered, in order to show the emergence of Weyl-like symmetries in backgrounds with low-enough dimensions. Their presence was first recognized in [40, 41], even if a complete classification is still lacking when more than two index families are

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[13]The AdS/CFT correspondence [94] could also provide a natural arena for investigating these relations [95]. A proper tensionless limit in the bulk should correspond to a weak limit on the boundary, and some investigations of this regime were pursued analyzing the higher-spin conserved currents of free super Yang-Mills models on the boundary [96, 97].

present. In the bosonic case I will also identify here some Weyl-invariant models in $D > 4$ that were not discussed in [40].

- Chapter 5 closes this Thesis with a brief summary and a discussion of some possible developments of the work reviewed here.

The original results contained in this Thesis were obtained in collaboration with Dario Francia, Jihad Mourad and Augusto Sagnotti and were presented in [40, 41]. They were also concisely reviewed in [98].

Chapter 2

Constrained theory

In the mid eighties Labastida proposed a set of *covariant* field equations describing the propagation of the free *massless* modes associated to generic representations of the Lorentz group in arbitrary space-time dimensions [37, 38]. As anticipated in the Introduction, in order to achieve this result he considered multi-symmetric *gauge* fields $\varphi_{\mu_1 \dots \mu_{s_1}, \nu_1 \dots \nu_{s_2}, \dots}$ and $\psi^{\alpha}{}_{\mu_1 \dots \mu_{s_1}, \nu_1 \dots \nu_{s_2}, \dots}$, so that his "metric-like" formulation generalizes that obtained by Fronsdal and Fang for fully symmetric fields at the end of the seventies [25, 26]. As in the fully symmetric case, in the Labastida approach fields and gauge parameters are subjected to algebraic *(γ-)trace constraints*.

In the Fang-Fronsdal works the constraints are inherited from the description of massive fields proposed by Singh and Hagen [23, 24]. In fact, Fronsdal and Fang identified the massless Lagrangians considering the limit of the massive ones. On the other hand, Labastida derived his results conjecturing a set of gauge transformations and then looking for gauge invariant field equations, following the reexamination of the Fang-Fronsdal construction proposed by Curtright [27]. He selected his equations of motion without referring to an action principle, but in a successive paper he also derived Lagrangians leading to equivalent field equations for Bose fields [39]. Nevertheless, even at the free level, for Fermi fields the construction of quadratic Lagrangians initiated long ago by Fierz and Pauli [19] was left incomplete until recently. In fact, Lagrangians for Fermi fields of mixed symmetry were presented only in [41].

In this chapter I will review the identification of the gauge transformations, the constraints and the field equations forcing multi-symmetric fields to propagate free massless modes [37, 38]. Following a simple argument presented in [41], I will also verify that these conditions describe indeed the correct physical polarizations. Furthermore, I will present an original derivation of the Labastida bosonic Lagrangians [39, 40], based on the Bianchi identities satisfied by the gauge fields. This technique will also prove useful in the construction of the fermionic Lagrangians [41] that follows, and that represents one of the main original results of this Thesis. Then I will make some comments on the related Lagrangian field equations, and on their link with the non-Lagrangian ones proposed by Labastida. Since in the following I am going to deal mainly with reducible fields, this chapter closes with a description of how the theory

adapts to irreducible fields subjected to suitable Young projections.

While the results presented here refer only to massless fields, I would like to call to the reader's attention that a Stueckelberg-like description of the corresponding massive fields can be obtained via a standard Kaluza-Klein circle reduction [99].

2.1 Non-Lagrangian field equations and constraints

The non-Lagrangian field equations for Bose and Fermi fields of mixed-symmetry can be derived following similar logical steps. The choice of describing them separately is aimed at simplifying the presentation, but the reader is invited to notice the similarities between the two setups.

2.1.1 Bose fields

The well-known example of linearized gravity already exhibits the structure of the covariant field equations for arbitrary free higher-spin fields. In fact, the equation of motion for the linearized metric fluctuation $h_{\mu\nu}$ reads

$$R_{\mu\nu} \equiv \Box\, h_{\mu\nu} - \partial_\mu \partial \cdot h_\nu - \partial_\nu \partial \cdot h_\mu + \partial_\mu \partial_\nu\, h_\lambda{}^\lambda = 0 \,. \tag{2.1}$$

It enforces the vanishing of the linearized Ricci tensor, that provides a gauge invariant completion of $\Box\, h_{\mu\nu}$ under the linearized diffeomorphisms $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. The latter observation can be taken as a guiding principle in order to look for the equations of motion for arbitrary massless fields. For instance, the non-Lagrangian form of the Fronsdal equations [25] for fully symmetric Bose fields reads

$$\mathcal{F}_{\mu_1\ldots\mu_s} \equiv \Box\, \varphi_{\mu_1\ldots\mu_s} - \partial_{(\mu_1} \partial \cdot \varphi_{\mu_2\ldots\mu_s)} + \partial_{(\mu_1} \partial_{\mu_2}\, \varphi_{\mu_3\ldots\mu_s)\lambda}{}^\lambda = 0 \,, \tag{2.2}$$

where I would like to stress again that here and in the following a couple of parentheses denotes a complete symmetrization of the indices it encloses, with the minimum possible number of terms and with unit overall normalization. Eq. (2.2) is invariant under the transformations

$$\delta\, \varphi_{\mu_1\ldots\mu_s} = \partial_{(\mu_1} \Lambda_{\mu_2\ldots\mu_s)} \tag{2.3}$$

with *traceless* gauge parameters. As pointed out by Curtright [27], one can actually identify the field equations (2.2) and the constraints

$$\Lambda_{\mu_1\ldots\mu_{s-3}\lambda}{}^\lambda = 0 \tag{2.4}$$

looking for a *second order* kinetic tensor that contains $\Box\, \varphi_{\mu_1\ldots\mu_s}$ and that is left invariant under the gauge transformations (2.3). In fact, they provide the natural generalization of the abelian linearized diffeomorphisms $\delta h_{\mu\nu} = \partial_{(\mu} \xi_{\nu)}$. The only possibility to achieve such goal is to constrain the gauge parameters according to (2.4), and the result is unique if one avoids to introduce traces of $\Box\, \varphi_{\mu_1\ldots\mu_s}$. This latter hypothesis

is dictated by the quest for the simplest solution, and leads to eq. (2.2). On the other hand, it hampers the construction of Lagrangian field equations, so that it will be relaxed in Section 2.2.1. At any rate, the correctness of the resulting field equations can then be checked a posteriori, via a counting of the degrees of freedom propagated by their solutions. For the Fronsdal equations (2.2) this check was performed by a number of authors with different techniques [27, 28].

For multi-symmetric Bose fields one can follow a similar path. In particular, when dealing with *unprojected* tensors of rank $(s_1, \ldots, s_N)$, one can generalize eq. (2.3) introducing an independent gauge parameter for each index family. These carry $(s_i - 1)$ vector indices in the i-th index family, so that

$$\delta\, \varphi_{\mu_1 \ldots \mu_{s_1}, \nu_1 \ldots \nu_{s_2}, \ldots} \,=\, \partial_{(\mu_1} \Lambda^{(1)}{}_{\mu_2 \ldots \mu_{s_1}), \nu_1 \ldots \nu_{s_2}, \ldots} + \partial_{(\nu_1 |} \Lambda^{(2)}{}_{\mu_1 \ldots \mu_{s_1}, | \nu_2 \ldots \nu_{s_2}), \ldots} + \ldots \, , \quad (2.5)$$

where here and in the following a vertical bar signals that the symmetrization also encompasses the indices lying between the next bar and the closing parenthesis. Looking at eq. (2.5), the need for an efficient and compact notation to deal with mixed-symmetry fields becomes evident, and in the following I will resort to that introduced in [40, 41]. It is based on the removal of all space-time indices, but *"family" indices* are needed in order to select the groups of space-time indices where the operators are acting upon. For instance, in the following a gradient which carries an index symmetrized with those in the i-th family will be denoted by

$$\partial^i \varphi \,\equiv\, \partial_{(\mu_1^i |} \varphi_{\ldots, | \mu_2^i \ldots \mu_{s_i + 1}^i), \ldots} \, , \quad (2.6)$$

while a divergence contracting an index in the i-th family will be denoted by

$$\partial_i \varphi \,\equiv\, \partial^\lambda \varphi_{\ldots, \lambda \mu_1^i \ldots \mu_{s_i - 1}^i, \ldots} \, . \quad (2.7)$$

The position of the family indices respects a useful convention of [40, 41], according to which *lower* family indices are associated to operators removing Lorentz indices, while *upper* family indices are associated to operators adding Lorentz indices, to be symmetrized with all other indices belonging to the group identified by the family label. Consistently with this rule, the gauge parameter lacking one space-time index in the i-th family will be denoted by Λ_i. In conclusion, one can rewrite eq. (2.5) in the synthetic form

$$\delta\, \varphi \,=\, \partial^i \Lambda_i \, . \quad (2.8)$$

This compact notation makes also manifest the existence of gauge for gauge transformations for mixed-symmetry fields. Differential forms already display this behavior, and being fully antisymmetric objects they are but a particular class of mixed-symmetry fields. At any rate, for arbitrary mixed-symmetry fields one can easily recognize that

$$\delta\, \Lambda_i \,=\, \partial^j \Lambda_{[ij]} \quad (2.9)$$

leaves φ invariant provided the $\Lambda_{[ij]}$ are antisymmetric in their family indices. In general the chain would continue, since transformations of the $\Lambda_{[ij]}$ leaving invariant

the Λ_i could also exist. One can proceed in this fashion for a number of steps equal to the number of index families carried by φ.

Having fixed the notation, we can now look for equations of motion that are invariant under the gauge transformations (2.8). Eqs. (2.1) and (2.2) suggest that it is necessary to consider also traces of the gauge fields in order to obtain them. In the mixed-symmetry case it is thus convenient to introduce the T_{ij} operators acting as

$$ T_{ij}\,\varphi \;\equiv\; \varphi_{\dots,\;{}^{\lambda}{}_{\mu_1^i\dots\mu_{s_i-1}^i,\,\dots,\,\lambda\mu_1^j\dots\mu_{s_j-1}^j,\,\dots}}\,. \tag{2.10} $$

One can then verify that the Fronsdal-Labastida kinetic tensor [37]

$$ \mathcal{F} \;\equiv\; \Box\,\varphi - \partial^i\,\partial_i\,\varphi + \frac{1}{2}\,\partial^i\partial^j\,T_{ij}\,\varphi \tag{2.11} $$

is invariant under the transformations (2.8), provided the gauge parameters satisfy the constraints

$$ T_{(ij}\,\Lambda_{k)} \;=\; 0\,. \tag{2.12} $$

All needed computational rules follow from the algebra of the operators T_{ij}, ∂_i and ∂^i, that is presented in Appendix A. Moreover, this is the unique *two-derivative* gauge-invariant completion of $\Box\,\varphi$ that does not contain its traces[1]. This led Labastida to propose in [37] the field equation

$$ \mathcal{F} \;=\; 0 \tag{2.13} $$

to describe the dynamics of massless mixed-symmetry Bose fields. In [39] he also verified the correctness of eq. (2.13) in a number of simple examples, performing the counting of the propagated degrees of freedom. The first proof of the Labastida conjecture was then provided much later in the third reference of [52], while I will now show that eqs. (2.8), (2.12) and (2.13) lead to the propagations of the correct polarizations for generic mixed-symmetry bosons reviewing the simple argument presented in [41].

In order to proceed, it is convenient to resort momentarily to an explicit oscillator realization of the index-free notation. This is attained folding a generic multi-symmetric gauge field $\varphi_{\mu_1\dots\mu_{s_1},\,\nu_1\dots\nu_{s_2},\,\dots}$ with commuting vectors $u^{i\,\mu}$ according to

$$ \varphi \;\equiv\; \frac{1}{s_1!\,\dots\,s_N!}\,u^{1\,\mu_1}\dots u^{1\,\mu_{s_1}}\,u^{2\,\nu_1}\dots u^{2\,\nu_{s_2}}\dots\;\varphi_{\mu_1\dots\mu_{s_1},\,\nu_1\dots\nu_{s_2},\,\dots}\,. \tag{2.14} $$

The form taken by the various operators introduced until now is described in detail in Appendix A and, for instance, divergences and gradients can be cast in the form

$$ \partial_i \;\equiv\; \partial^\mu\,\frac{\partial}{\partial\,u^{i\,\mu}}\,, \qquad\qquad \partial^i \;\equiv\; \partial_\mu\,u^{i\,\mu}\,. \tag{2.15} $$

In studying the plane-wave solutions of the Labastida equation (2.13), that in momentum space takes the form

$$ p^2\,\varphi - (p\cdot u^i)\left(p\cdot\frac{\partial}{\partial\,u^i}\right)\varphi + \frac{1}{2}\,(p\cdot u^i)\,(p\cdot u^j)\,T_{ij}\,\varphi \;=\; 0\,, \tag{2.16} $$

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[1]As recalled in the Introduction, relaxing the upper bound on the number of derivatives and accepting the presence of non-local terms, one can actually obtain a gauge-invariant completion of $\Box\,\varphi$ without any need for constraints [48, 42, 51, 52].

it is then convenient to decompose φ according to

$$\varphi = (p \cdot u^i)\, \zeta_i + \widehat{\varphi}\,, \tag{2.17}$$

sorting out its portion $\widehat{\varphi}$ that is independent of $(p \cdot u^i)$,

$$\frac{\partial}{\partial\, (p \cdot u^i)}\, \widehat{\varphi} = 0\,. \tag{2.18}$$

Substituting in (2.16) then gives

$$p^2\, \widehat{\varphi} - (p \cdot u^i)\left(p \cdot \frac{\partial}{\partial\, u^i}\right)\widehat{\varphi} + \frac{1}{2}\, (p \cdot u^i)\, (p \cdot u^j)\, T_{ij}\, \widehat{\varphi}$$
$$+ \frac{1}{6}\, (p \cdot u^i)\, (p \cdot u^j)\, (p \cdot u^k)\, T_{(ij}\, \zeta_{k)} = 0\,. \tag{2.19}$$

The four terms appearing in this sum must vanish independently, since they carry different powers of $(p \cdot u^i)$. Therefore, eq. (2.19) is equivalent to the four conditions

$$p^2\, \widehat{\varphi} = 0\,, \tag{2.20}$$

$$\left(p \cdot \frac{\partial}{\partial\, u^i}\right)\widehat{\varphi} = 0\,, \tag{2.21}$$

$$T_{ij}\, \widehat{\varphi} = 0\,, \tag{2.22}$$

$$T_{(ij}\, \zeta_{k)} = 0\,. \tag{2.23}$$

Eq. (2.23) now implies that one can gauge away the ζ_i, since they are subject to the Labastida constraints, and eq. (2.20) then leads to the mass-shell condition. One can thus let $p = p^+$ and the conditions (2.18) and (2.21) imply respectively

$$\frac{\partial}{\partial\, u^{i-}}\, \widehat{\varphi} = 0\,, \qquad\qquad \frac{\partial}{\partial\, u^{i+}}\, \widehat{\varphi} = 0\,. \tag{2.24}$$

All components of $\widehat{\varphi}$ along the two light-cone directions are thus absent on account of eqs. (2.24), while eq. (2.22) forces the remaining transverse components to be traceless. In conclusion, the Labastida equation (2.13) propagates the degrees of freedom of representations of the little group $O(D-2)$, as expected on account of its uniqueness. At this stage these are reducible representations, since I did not enforce any projection on φ, but in Section 2.4 we shall see how the theory can be easily adapted to describe irreducible representations.

Let me stress that in order to reach this result one does not have to impose any condition on the gauge field φ, but eventually all its traces vanish *on-shell*. On the other hand, Labastida verified the correctness of eq. (2.13) in a number of examples adopting the *off-shell* techniques introduced in [100]. In order to match the correct number of degrees of freedom, he thus also identified a set of double trace constraints on the fields, generalizing the one identified by Fronsdal for a given spin s field,

$$\varphi_{\mu_1 \ldots \mu_{s-4}\, \lambda\rho}{}^{\lambda\rho} = 0\,. \tag{2.25}$$

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The origin of the constraints can be better understood recalling that in the fully symmetric case one must impose the constraint (2.25) in order to obtain a gauge invariant Lagrangian leading to a field equation equivalent to (2.2). This was first pointed out in [27, 28], and indeed the double trace of the field appears on the right-hand side of the Bianchi identity

$$\partial \cdot \mathcal{F}_{\mu_1 \dots \mu_{s-1}} - \frac{1}{2} \partial_{(\mu_1} \mathcal{F}_{\mu_2 \dots \mu_{s-1}) \lambda}{}^{\lambda} = -\frac{3}{2} \partial_{(\mu_1} \partial_{\mu_2} \partial_{\mu_3} \varphi_{\mu_4 \dots \mu_{s-1}) \lambda \rho}{}^{\lambda \rho}, \qquad (2.26)$$

that generalizes the contracted Bianchi identity of linearized gravity,

$$\partial \cdot R_\mu - \frac{1}{2} \partial_\mu R_\lambda{}^{\lambda} = 0. \qquad (2.27)$$

As we shall see in Section 2.2.1, eq. (2.26) indeed plays a crucial role in the construction of the Lagrangian for a spin s field, as eq. (2.27) does in the spin 2 case. In a similar fashion the Labastida constraints on the field φ can be identified looking at the Bianchi identities

$$\partial_i \mathcal{F} - \frac{1}{2} \partial^j T_{ij} \mathcal{F} = -\frac{1}{12} \partial^j \partial^k \partial^l T_{(ij} T_{kl)} \varphi, \qquad (2.28)$$

that provide a neat rationale for the Labastida Lagrangians of [39], as showed in [40]. In fact, in the present notation the constraints on the field introduced in [37] read

$$T_{(ij} T_{kl)} \varphi = 0, \qquad (2.29)$$

so that they cancel the right-hand side of eq. (2.28), precisely as the constraint (2.25) does in the fully symmetric setting.

Notice that, quite differently from what Fronsdal's case could naively suggest, *not* all traces of the gauge parameters and *not* all double traces of the field vanish. Only the linear combinations appearing in eqs. (2.12) and (2.29) do, and this simple observation is at the origin of the main differences in the structure of the Lagrangians for fully symmetric and for mixed-symmetry fields that I am about to exhibit. Notice also that, as in Fronsdal's case, those in eq. (2.29) are the strongest gauge invariant constraints available since

$$T_{(ij} T_{kl)} \delta \varphi = \partial_{(i} T_{jk} \Lambda_{l)} + \partial^m \left\{ T_{(ij} T_{kl} \Lambda_{m)} - T_{m(i} T_{jk} \Lambda_{l)} \right\}. \qquad (2.30)$$

The last terms can be cast in this form manifestly related to the constraints (2.12) via the rewriting

$$T_{(ij} T_{kl)} \Lambda_m = T_{(ij} T_{kl} \Lambda_{m)} - T_{m(i} T_{jk} \Lambda_{l)}. \qquad (2.31)$$

The Labastida constraints (2.12) and (2.29) are nicely "covariant" in the family index language, but they are *not* independent. For instance, considering a further trace of $T_{(ij} T_{kl)} \varphi$ one obtains

$$Y_{\{5,1\}} T_{mn} T_{(ij} T_{kl)} \varphi = 0 \qquad (2.32)$$

without any need of enforcing the Labastida constraints. In this expression $Y_{\{5,1\}}$ denotes the Young projector onto the irreducible $\{5, 1\}$ representation of the permutation

group acting on the family indices[2]. Eq. (2.32) reflects a special property of products of identical tensors, that in the two-index T_{ij} case can only build Young diagrams in family-index space with even numbers of boxes in each row. This fact will have important consequences in Chapter 3, where I will describe how one can remove the Labastida constraints adding a suitable set of auxiliary fields. Roughly speaking, one would like to associate an auxiliary field to each constraint, but their linear dependence gives rise to a number of subtleties.

2.1.2 Fermi fields

As anticipated, when dealing with Fermi fields one can follow a similar path. To begin with, let me stress that here and in the following all spinor indices are hidden for simplicity. Then, notice that even for Fermi fields a well-know paradigmatic example already displays some key features of the arbitrary-spin construction. In fact, the Rarita-Schwinger equation for a massless field of spin $3/2$, that is usually presented in the form

$$\gamma^{\mu\nu\rho}\, \partial_\nu\, \psi_\rho \,=\, 0\,, \qquad (2.33)$$

is manifestly invariant under the gauge transformation

$$\delta\, \psi_\mu \,=\, \partial_\mu\, \epsilon\,, \qquad (2.34)$$

simply because two derivatives commute. Moreover, in general it can be combined with its γ-trace in order to obtain the equation

$$\slashed{\partial}\, \psi_\mu \,-\, \partial_\mu\, \slashed{\psi} \,=\, 0\,, \qquad (2.35)$$

that provides a gauge invariant completion of $\slashed{\partial}\psi_\mu$. In complete analogy with the bosonic construction of the previous section, one can thus recover the field equations for arbitrary higher-spin fermions postulating a proper set of gauge transformations and looking for a gauge invariant completion of the "naive" generalization of the Dirac equation. In the fully symmetric setting the natural gauge transformation for a spin $s + \frac{1}{2}$ field is

$$\delta\, \psi_{\mu_1\ldots\mu_s} \,=\, \partial_{(\mu_1}\, \epsilon_{\mu_2\ldots\mu_s)}\,. \qquad (2.36)$$

The corresponding Fang-Fronsdal field equation [26] is

$$\mathcal{S}_{\mu_1\ldots\mu_s} \,\equiv\, i\, \left\{\, \slashed{\partial}\, \psi_{\mu_1\ldots\mu_s} \,-\, \partial_{(\mu_1}\slashed{\psi}_{\mu_2\ldots\mu_s)}\, \right\} \,=\, 0\,, \qquad (2.37)$$

that is invariant under (2.36) provided the parameter satisfies

$$\slashed{\epsilon}_{\mu_1\ldots\mu_{s-2}} \,=\, 0\,. \qquad (2.38)$$

Again, the result is unique up to possible combinations with its γ-traces, that are anyway needed to obtain Lagrangian field equations.

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[2]The conventions for the permutation group and related tools are spelled out at the end of Appendix A. Further details on these matters can be found, for instance, in [101].

In the mixed-symmetry case it is natural to extend eq. (2.36) considering, for *unprojected* spinor-tensors, an independent gauge parameter for each index family. It is clearly convenient to abide by the notation presented in the previous section, that permits to cast the gauge transformations in the form

$$\delta\,\psi\,=\,\partial^{i}\,\epsilon_{i}\,. \tag{2.39}$$

Notice that this expression has the same form as that considered for Bose fields, so that gauge for gauge transformations are generically present also in this setup. However, an ingredient is still missing in the family-index notation: one would also like to compute γ-traces, and this can be done introducing

$$\gamma_{i}\,\psi\,\equiv\,\gamma^{\lambda}\,\psi_{\ldots,\,\lambda\,\mu_{1}^{i}\,\ldots\,\mu_{s_{i}-1}^{i},\,\ldots}\,. \tag{2.40}$$

The Fang-Fronsdal-Labastida kinetic tensor [38]

$$\mathcal{S}\,\equiv\,i\,\left\{\,\slashed{\partial}\,\psi\,-\,\partial^{i}\,\gamma_{i}\,\psi\,\right\} \tag{2.41}$$

thus gives a gauge invariant completion of $\slashed{\partial}\,\psi$, provided one enforces the constraints

$$\gamma_{(i}\,\epsilon_{j)}\,=\,0\,. \tag{2.42}$$

Again, this statement can be verified looking at the algebra of the operators introduced until now, that is presented in Appendix A. The answer is unique if one considers only first order differential operators[3] and avoids to introduce γ-traces of $\slashed{\partial}\,\psi$.

This led Labastida to propose in [38] the field equation

$$\mathcal{S}\,=\,0 \tag{2.43}$$

in order to describe the dynamics of massless mixed-symmetry Fermi fields. Performing the counting of the propagated degrees of freedom, in [39] he also verified the correctness of eq. (2.43) in a number of simple examples. For generic mixed-symmetry fermions this issue was finally clarified in [41], following the lines already depicted for Bose fields in Section 2.1.1.

Let me thus review how the argument adapts to Fermi fields. Even in this setup it is convenient to resort to an oscillator realization of the index-free notation, introducing a commuting vector $u^{i\,\mu}$ for each index family. Eqs. (2.15) then enable to cast the momentum-space Labastida equation in the form

$$\slashed{p}\,\psi\,-\,(p\cdot u^{i})\,\gamma_{i}\,\psi\,=\,0\,. \tag{2.44}$$

As in the bosonic case, one can then decompose the gauge field ψ according to

$$\psi\,=\,(p\cdot u^{i})\,\chi_{i}\,+\,\widehat{\psi}\,, \tag{2.45}$$

---

[3]Even for Fermi fields, relaxing the upper bound on the number of derivatives one can actually obtain gauge invariant completions of $\slashed{\partial}\,\psi$ without any need for constraints [48, 42, 51, 52].

with

$$\frac{\partial}{\partial\,(p\cdot u^i)}\,\widehat{\psi} = 0\,. \tag{2.46}$$

Substituting in eq. (2.44) then gives

$$\not{p}\,\widehat{\psi} - (p\cdot u^i)\,\gamma_i\,\widehat{\psi} - \frac{1}{2}\,(p\cdot u^i)\,(p\cdot u^j)\,\gamma_{(i}\chi_{j)} = 0\,, \tag{2.47}$$

and one is led to the three independent conditions

$$\not{p}\,\widehat{\psi} = 0\,, \tag{2.48}$$

$$\gamma_i\,\widehat{\psi} = 0\,, \tag{2.49}$$

$$\gamma_{(i}\chi_{j)} = 0\,, \tag{2.50}$$

since the terms in the sum carry different powers of $(p\cdot u^i)$. The condition (2.50) then ensures the possibility of gauging away altogether the χ_i, and one is left with the Dirac equation (2.48). It leads again to the mass-shell condition $p^2 = 0$, so that one can consider a momentum p with a single non-vanishing light-cone component as, for instance, p^+. Therefore, the lack of dependence of $\widehat{\psi}$ on $(p\cdot u^i)$ translates into the condition

$$\frac{\partial}{\partial\,u^{i-}}\,\widehat{\psi} = 0\,, \tag{2.51}$$

while the Dirac equation (2.48) turns into the standard projection

$$\gamma_+\,\widehat{\psi} = 0\,, \tag{2.52}$$

that halves the on-shell components of $\widehat{\psi}$. Finally, eq. (2.49) becomes

$$\left(\gamma^+\,\frac{\partial}{\partial\,u^{i+}} + \gamma^{\underline{m}}\,\frac{\partial}{\partial\,u^{i\underline{m}}}\right)\widehat{\psi} = 0\,, \tag{2.53}$$

where I have used $\underline{m}$ to denote the transverse Lorentz index in order to distinguish it from the family indices. Multiplying eq. (2.53) by γ_+ and making use of eq. (2.52) one is finally led to the conditions

$$\frac{\partial}{\partial\,u^{i+}}\,\widehat{\psi} = 0\,, \tag{2.54}$$

$$\gamma^{\underline{m}}\,\frac{\partial}{\partial\,u^{i\underline{m}}}\,\widehat{\psi} = 0\,. \tag{2.55}$$

All components of $\widehat{\psi}$ along the two light-cone directions thus are absent on account of eqs. (2.51) and (2.54), while eq. (2.55) forces the remaining transverse components to be γ-traceless. In conclusion, the Labastida equation (2.43) propagates in general representations of the little group $O(D-2)$, as expected on account of its uniqueness. As already stressed when dealing with the same issue for bosons, at this stage these are reducible representations, since I did not enforce any projection on ψ. On the other hand, it is possible to describe irreducible representations following the path outlined in Section 2.4.

In order to reach this result one does not have to impose any additional condition on the gauge field ψ. However, in order to match the correct number of degrees of freedom via the techniques of [100] Labastida also identified a set of triple γ-trace constraints on the fields. These generalize the one identified by Fang and Fronsdal for a given spin $s + \frac{1}{2}$ field,

$$\psi\!\!\!/_{\mu_1 \ldots \mu_{s-4} \lambda}{}^{\lambda} = 0 \,, \tag{2.56}$$

that, as in the bosonic case, is needed in order to obtain a gauge invariant Lagrangian leading to a field equation equivalent to (2.37). In fact, the triple γ-trace of the field appears on the right-hand side of the Bianchi identity

$$\partial \cdot \mathcal{S}_{\mu_1 \ldots \mu_{s-1}} - \frac{1}{2} \, \partial\!\!\!/ \, \mathcal{S}_{\mu_1 \ldots \mu_{s-1}} - \frac{1}{2} \, \partial_{(\mu_1} \mathcal{S}_{\mu_2 \ldots \mu_{s-1}) \lambda}{}^{\lambda} = i \, \partial_{(\mu_1} \partial_{\mu_2} \psi\!\!\!/_{\mu_3 \ldots \mu_{s-1}) \lambda}{}^{\lambda} \,. \tag{2.57}$$

In a similar fashion the Labastida constraints on mixed-symmetry fields can be identified looking at the Bianchi identities

$$\partial_i \mathcal{S} - \frac{1}{2} \, \partial\!\!\!/ \, \gamma_i \mathcal{S} - \frac{1}{2} \, \partial^j T_{ij} \mathcal{S} - \frac{1}{6} \, \partial^j \gamma_{ij} \mathcal{S} = \frac{i}{6} \, \partial^j \partial^k T_{(ij} \gamma_{k)} \psi \,, \tag{2.58}$$

that in Section 2.2.2 will play a crucial role in the construction of the Lagrangians of [41]. Indeed, in the present notation the constraints on the fields of [38] read

$$T_{(ij} \gamma_{k)} \psi = 0 \,, \tag{2.59}$$

and thus cancel the right-hand side of eq. (2.58), precisely as the constraint (2.56) does in the fully symmetric setting. The new symbol appearing in eq. (2.58) simply denotes an antisymmetric combination of two "family" γ-matrices,

$$\gamma_{ij} = \frac{1}{2} \left(\gamma_i \gamma_j - \gamma_j \gamma_i \right) . \tag{2.60}$$

The idea underlying this choice is to extend to operators carrying family indices the common convention of working with an antisymmetric basis for γ-matrices.

Notice that, again in sharp contrast with the fully symmetric setting, *not* all γ-traces of the gauge parameters and *not* all triple γ-traces of the fields vanish. Only the linear combinations appearing in eqs. (2.42) and (2.59) do, and even for fermions this simple observation is at the origin of the main differences in the structure of the Lagrangians for fully symmetric and for mixed-symmetry fields. Notice also that, as in the Fang-Fronsdal case, those in eq. (2.59) are the strongest gauge invariant constraints available since

$$\begin{aligned} T_{(ij} \gamma_{k)} \delta \psi &= \partial\!\!\!/ \, T_{(ij} \epsilon_{k)} + \partial_{(i} \gamma_j \epsilon_{k)} \\ &\quad + \partial^l \left\{ T_{(ij} \gamma_k \epsilon_{l)} - T_{l(i} \gamma_j \epsilon_{k)} - \gamma_l T_{(ij} \epsilon_{k)} \right\} . \end{aligned} \tag{2.61}$$

The last terms can be cast in this form manifestly related to the constraints (2.42) via the rewriting

$$T_{(ij} \gamma_{k)} \epsilon_l = T_{(ij} \gamma_k \epsilon_{l)} - T_{l(i} \gamma_j \epsilon_{k)} - \gamma_l T_{(ij} \epsilon_{k)} \,. \tag{2.62}$$

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As in the bosonic case, the Labastida constraints (2.42) and (2.59) are nicely "covariant" in the family index language, but they are *not* independent. In the fermionic case the simplest example of this fact is slightly more involved, but it can be again identified resorting to the permutation group acting on family indices. In this respect, the Labastida constraints (2.59) carry a $\{3\}$ representation, since they are fully symmetric in the family indices. On the other hand, a trace T_{ij} carries a $\{2\}$ representation while an antisymmetrized γ-trace γ_{ij} carries a $\{1,1\}$ representation. Thus, the combination $T_{lm} T_{(ij} \gamma_{k)}$ admits the decomposition

$$\{3\} \otimes \{2\} = \{5\} \oplus \{4,1\} \oplus \{3,2\} \,, \tag{2.63}$$

while the combination $\gamma_{lm} T_{(ij} \gamma_{k)}$ admits the decomposition

$$\{3\} \otimes \{1,1\} = \{4,1\} \oplus \{3,1,1\} \,. \tag{2.64}$$

It is thus natural to expect the existence of an identically vanishing combination of $\{4,1\}$-projected traces and antisymmetrized γ-traces of $T_{(ij} \gamma_{k)} \psi$. This is indeed the case, since

$$\begin{aligned}
&\left(\gamma_{mi} T_{(jk} \gamma_{l)} + \gamma_{mj} T_{(ik} \gamma_{l)} \right) \psi - 15 \, Y_{\{4,1\}} \, T_{ij} \, T_{(kl} \gamma_{m)} \, \psi \\
&+ \left[(i,j) \leftrightarrow (k,l) \right] = 0
\end{aligned} \tag{2.65}$$

without any need of imposing the constraints. The first term is automatically $\{4,1\}$-projected on account of the interchange between the couples of indices (i,j) and (k,l) indicated in the last line. Even in this setting, the lack of linear dependence for the constraints will give rise to some subtleties when we will proceed to an unconstrained description of the dynamics in Chapter 3.

2.2 Lagrangians

Moving toward a Lagrangian formulation, Bose and Fermi fields begin to follow different paths. As we shall see, the request of gauge invariance and the Bianchi identities satisfied by the fields provide a clearcut rationale for the construction of the Lagrangians for both types of fields. However, the structure of the results is rather different, and for Fermi fields a number of subtleties related to the presence of γ-matrices emerge.

2.2.1 Bose fields

Even when dealing with the Lagrangians, the example of linearized gravity already displays the logic that will drive the treatment of the arbitrary case. The equation of motion (2.1) is indeed non-Lagrangian, but in general it is equivalent to the one following from the linearized Einstein-Hilbert Lagrangian

$$\mathcal{L} = \frac{1}{2} \, h^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} \, \eta_{\mu\nu} R_\lambda{}^\lambda \right) \,, \tag{2.66}$$

that contains the linearized Ricci tensor appearing in eq. (2.1) and its only available trace. Furthermore, the relative coefficient entering eq. (2.66) is fixed uniquely by the requirement of gauge invariance. In fact, up to total derivatives, the gauge variation of (2.66) under the linearized diffeomorphisms $\delta\, h_{\mu\nu} = \partial_{(\mu}\, \xi_{\nu)}$ is

$$\delta\, \mathcal{L} \,=\, -\, \xi^{\mu} \left(\partial \cdot R_{\mu} \,-\, \frac{1}{2}\, \partial_{\mu}\, R_{\lambda}{}^{\lambda} \right) \,=\, 0\,, \tag{2.67}$$

and vanishes on account of the Bianchi identity (2.27).

The Lagrangians for fully symmetric bosons take a very similar form, since the Fronsdal constraints (2.25) on the fields induce similar constraints on the kinetic tensors,

$$\mathcal{F}_{\mu_1 \dots \mu_{s-4}\, \lambda\rho}{}^{\lambda\rho} \,=\, 0\,. \tag{2.68}$$

Thus, the Lagrangians can only contain $\mathcal{F}_{\mu_1 \dots \mu_s}$ and its trace. The structure of the Bianchi identities (2.26) eventually fixes the relative coefficient to be the same entering the Einstein-Hilbert Lagrangian. The final result,

$$\mathcal{L} \,=\, \frac{1}{2}\, \varphi^{\mu_1 \dots \mu_s} \left(\mathcal{F}_{\mu_1 \dots \mu_s} \,-\, \frac{1}{2}\, \eta_{(\mu_1\mu_2}\, \mathcal{F}_{\mu_3 \dots \mu_s)\lambda}{}^{\lambda} \right)\,, \tag{2.69}$$

is indeed gauge invariant since

$$\delta\, \mathcal{L} \,=\, -\, \frac{s}{2}\, \Lambda^{\mu_1 \dots \mu_{s-1}} \left(\partial \cdot \mathcal{F}_{\mu_1 \dots \mu_{s-1}} \,-\, \frac{1}{2}\, \partial_{(\mu_1} \mathcal{F}_{\mu_2 \dots \mu_{s-1})\lambda}{}^{\lambda} \right)$$
$$+\, \frac{3}{4} \binom{s}{3} \Lambda^{\mu_1 \dots \mu_{s-3}\,\lambda}{}_{\lambda}\, \partial \cdot \mathcal{F}_{\mu_1 \dots \mu_{s-3}\,\lambda}{}^{\lambda}\,. \tag{2.70}$$

The first line builds the Bianchi identity (2.26), whose right-hand side vanishes in the constrained theory, while the second one is proportional to the Fronsdal constraint (2.4) on the gauge parameter. Notice that, even ignoring the condition (2.68), the cancelation of the first two terms in the gauge variation of an ansatz of the form

$$\mathcal{L} \,=\, \frac{1}{2}\, \varphi^{\mu_1 \dots \mu_s} \left(\mathcal{F}_{\mu_1 \dots \mu_s} \,+\, a_1\, \eta_{(\mu_1\mu_2}\, \mathcal{F}_{\mu_3 \dots \mu_s)\lambda}{}^{\lambda} \right.$$
$$\left. +\, a_2\, \eta_{(\mu_1\mu_2}\eta_{\mu_3\mu_4}\, \mathcal{F}_{\mu_5 \dots \mu_s)\lambda\rho}{}^{\lambda\rho} \,+\, \dots \right) \tag{2.71}$$

would require a Bianchi identity free from the "classical anomaly" appearing in the right-hand side of eq. (2.26). This leads to the Fronsdal constraint and eventually to eq. (2.68).

In the mixed-symmetry case the structure of the Lagrangians drastically changes, since *not* all double traces of the kinetic tensor $\mathcal{F}$ are forced to vanish. Hence, the Lagrangians can and indeed do contain additional contributions with multiple traces of $\mathcal{F}$. On the other hand, the correspondence between constraints on the fields and on the kinetic tensors is still true. In fact, the Labastida constraints on the fields (2.29) induce the conditions

$$T_{(ij}\, T_{kl)}\, \mathcal{F} \,=\, 0\,, \tag{2.72}$$

so that only particular linear combinations of traces of $\mathcal{F}$ can enter the Lagrangians. Before identifying the relevant ones, let me briefly comment on eq. (2.72). The commutation rules of Appendix A lead to

$$
\begin{aligned}
T_{(ij}\, T_{kl)}\, \mathcal{F} \;=\;& 3\,\square\, T_{(ij}\, T_{kl)}\, \varphi \;-\; \partial^m \partial_m\, T_{(ij}\, T_{kl)}\, \varphi \;+\; \partial^m \partial_{(i}\, T_{jk}\, T_{l)m}\, \varphi \\
& +\; \frac{1}{2}\, \partial^m \partial^n\, T_{mn}\, T_{(ij}\, T_{kl)}\, \varphi\,.
\end{aligned}
\tag{2.73}
$$

All terms barring the last one in the first line are manifestly annihilated by the Labastida constraints (2.29). However, the remaining one can be also related to the constraints on φ noticing that for a couple of traces a symmetrization over three indices induces a symmetrization over the whole four indices. This is again a consequence of the presence of two identical T_{ij} tensors.

In order to select the non-vanishing contributions that enter the Lagrangians we can now rely upon the *permutation group acting on the family indices*. One can indeed classify the multiple traces of $\mathcal{F}$ of the form

$$
T_{i_1 j_1} \dots T_{i_p j_p}\, \mathcal{F}
\tag{2.74}
$$

according to its irreducible representations. The final result is that only a particular class of representations does not vanish on account of eq. (2.72). To reach this conclusion, let me begin by noticing that a product of identical T_{ij} tensors admits only components labeled by Young tableaux with an even number of boxes in each row. This follows from an already mentioned theorem discussed, for instance, in [102]. Then, those associated to a tableau with at least four boxes in the first row are built upon a sum of contributions containing at least a symmetrization over four indices. These terms can be divided into three sets, according to the different displacements of the four symmetrized indices over the T_{ij} tensors. They can indeed contain a combination of the form $T_{(ij}\, T_{kl)}$, or of the form $T_{(ij}\, T_{k|m}\, T_{|l)n}$ or, finally, of the form $T_{m(i}\, T_{j|n}\, T_{p|k}\, T_{l)q}$. Since they act on $\mathcal{F}$, the first two terms are manifestly annihilated by the conditions (2.72), either directly or via the enlargement of the symmetrizations guaranteed by the presence of identical tensors. The remaining one, with the symmetrized indices spread over four traces, can be related to the previous ones noticing that

$$
\begin{aligned}
T_{m(i}\, T_{j|n}\, T_{p|k}\, T_{l)q} \;=\;& T_{(ij}\, T_{kl}\, T_{mn}\, T_{pq)} \;-\; T_{(ij}\, T_{k|(m|}\, T_{|l)|n}\, T_{pq)} \\
& -\; T_{(ij}\, T_{kl)}\, T_{(mn}\, T_{pq)}\,.
\end{aligned}
\tag{2.75}
$$

Just to clarify the notation, I would like to recall that in the previous expressions the vertical bars are used to stress that the symmetrization also applies to the indices after the next bar and so on, until a parenthesis signals the end of the group.

In conclusion, only *two-column projected* multiple traces of $\mathcal{F}$ can enter the Lagrangians, and it is convenient to denote them concisely as

$$
\mathcal{F}^{[p]}{}_{i_1 j_1, \dots, i_p j_p} \;=\; Y_{\{2^p\}}\, T_{i_1 j_1} \dots T_{i_p j_p}\, \mathcal{F}\,.
\tag{2.76}
$$

Here $Y_{\{2^p\}}$ denotes the Young projector onto the $\{2, \ldots, 2\}$ irreducible representation of the permutation group acting on the family indices. Hence, $\mathcal{F}^{[1]}{}_{ij}$ simply denotes a trace T_{ij} while, for instance,

$$\mathcal{F}^{[2]}{}_{i_1 j_1, \, i_2 j_2} \, = \, \frac{1}{3} \left\{ \, 2 \, T_{i_1 j_1} \, T_{i_2 j_2} \, - \, T_{i_1 (j_1} T_{j_2) i_2} \, \right\} \mathcal{F} \, . \tag{2.77}$$

An ansatz for the bosonic Lagrangians should contain all these terms, contracted with suitable products of invariant tensors. It is thus natural to introduce the new operators

$$\eta^{ii} \, \varphi \, = \, \eta_{(\mu_1^i \mu_2^i \,|\, \varphi \ldots, \,|\, \mu_3^i \ldots \mu_{s_i+2}^i), \ldots} \, ,$$

$$\eta^{ij} \, \varphi \, = \, \frac{1}{2} \sum_{n \, = \, 1}^{s_i + 1} \eta_{\mu_n^i \, (\mu_1^j \,|\, \varphi \ldots, \, \ldots \mu_{n-1}^i \mu_{n+1}^i \ldots, \, \ldots, \,|\, \mu_2^j \ldots \mu_{s_j+1}^j), \ldots} \, , \qquad i \neq j \, , \tag{2.78}$$

where the somehow unconventional factor $1/2$ entering the second line will prove convenient in the presentation of a number of results.

We now have all the ingredients to propose an ansatz displaying the structure of the Lagrangians for arbitrary Bose fields of mixed symmetry [39, 40]. It reads

$$\mathcal{L} \, = \, \frac{1}{2} \, \langle \, \varphi \, , \, \sum_{p \, = \, 0}^{N} k_p \, \eta^{i_1 j_1} \ldots \eta^{i_p j_p} \, \mathcal{F}^{[p]}{}_{i_1 j_1, \, \ldots, \, i_p j_p} \, \rangle \, , \tag{2.79}$$

where I have introduced a convenient scalar product that is described in detail in Appendix A. The upper limit of the sum is the number of index families, simply because it is not possible to antisymmetrize over more than N family indices, as a $\{2^{N+1}\}$ Young projection would require. Barring some unimportant differences in the notation, the ansatz (2.79) is equivalent to that proposed by Labastida in [39]. He did not explicitly consider the two-column projections but, as we have seen, for any given product of traces only a single two-column component exists. Thus, one can also present (2.79) in a redundant fashion, without displaying the projections explicitly. At any rate, Labastida fixed the coefficients k_p looking for a self-adjoint Einstein-like tensor, while in the following we shall directly look for a gauge invariant action. A self-adjoint extension of $\mathcal{F}$ is clearly automatically gauge invariant, but the approach I am reviewing here has the advantage of leading to an unconstrained extension of the theory in a clearcut way. Moreover, it also definitely shows that the Bianchi identities provide the conceptual frame for the Lagrangians.

As we learned from gravity and from the Fronsdal examples, the gauge invariance of the Lagrangians is guaranteed by the Bianchi identities, that in the *constrained* mixed-symmetry case read

$$\partial_i \mathcal{F} \, - \, \frac{1}{2} \, \partial^j \, T_{ij} \, \mathcal{F} \, = \, 0 \, , \tag{2.80}$$

since the "classical anomaly" appearing on the right-hand side of eq. (2.28) cancels imposing the Labastida constraints (2.29). Clearly, eq. (2.80) cannot directly deal with the higher traces contained in the ansatz (2.79), but we shall see in a while that

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suitable combinations of multiple traces of the Bianchi identities provide all the needed information. In fact, the gauge variation of eq. (2.79) reads

$$
\delta \mathcal{L} = - \sum_{p=0}^{N} \frac{1}{2^{p+1}} \langle \, T_{i_1 j_1} \ldots T_{i_p j_p} \Lambda_k \, , \; k_p \, \partial_k \mathcal{F}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p}
$$
$$
+ \, (p + 1) \, k_{p+1} \, \partial^l \, \mathcal{F}^{[p+1]}{}_{i_1 j_1, \ldots, i_p j_p, kl} \rangle \, ,
\tag{2.81}
$$

where the factors $1/2$ come from the definition (2.78) of the η^{ij} tensors. In analogy with the Fronsdal case, part of this gauge variation is proportional to the constraints (2.12) on the gauge parameters. Again, one can identify these contributions with the aid of the permutation group acting on the family indices. The divergence terms in eq. (2.81) indeed admit $\{3, 2^{p-1}\}$ and $\{2^p, 1\}$ Young projections, but the first one would produce terms annihilated by the constraints (2.12) in the left entries of the scalar products. This can be realized resorting to arguments similar to those used to discard from the ansatz (2.79) the tableaux with more than two columns. In conclusion, one can cast the gauge variation (2.81) in the form

$$
\delta \mathcal{L} = - \sum_{p=0}^{N} \frac{1}{2^{p+1}} \langle \, Y_{\{2^p, 1\}} \, T_{i_1 j_1} \ldots T_{i_p j_p} \Lambda_k \, , \; k_p \, Y_{\{2^p, 1\}} \, \partial_k \mathcal{F}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p}
$$
$$
+ \, (p + 1) \, k_{p+1} \, \partial^l \, \mathcal{F}^{[p+1]}{}_{i_1 j_1, \ldots, i_p j_p, kl} \rangle \, ,
\tag{2.82}
$$

since the gradient terms are already $\{2^p, 1\}$-projected in their free family indices. Notice that for $p = 0$ the terms

$$
k_0 \, \partial_k \mathcal{F} + k_1 \, \partial^l \, T_{kl} \, \mathcal{F}
\tag{2.83}
$$

appear in the right entry of the scalar product. Fixing conventionally $k_0 = 1$, the Bianchi identities (2.80) annihilate this sum provided that $k_1 = -\frac{1}{2}$. The other coefficients can be similarly fixed resorting to suitable combinations of traces of the Bianchi identities.

The multiple traces of eq. (2.80) read

$$
\partial_k \, T_{i_1 j_1} \ldots T_{i_p j_p} \, \mathcal{F} - \frac{1}{2} \sum_{n=1}^{p} \partial_{(i_n} \, T_{j_n) k} \prod_{r \neq n}^{p} T_{i_r j_r} \, \mathcal{F}
$$
$$
- \frac{1}{2} \, \partial^l \, T_{i_1 j_1} \ldots T_{i_p j_p} \, T_{kl} \, \mathcal{F} \, = \, 0 \, ,
\tag{2.84}
$$

and in order to compare them with eq. (2.82) one has to compute their $\{2^p, 1\}$ projections. Due to the symmetries of these expressions, the full Young projector can be chosen to coincide with that associated to the single standard Young tableau

i_1	j_1
$\vdots$	$\vdots$
i_p	j_p
k	

$$\tag{2.85}$$

which corresponds to the choice of standard labeling $i_n = 2\,n-1$, $j_n = 2\,n$ for $1 \leq n \leq p$ and $k = 2\,p+1$. Thus, thanks to a standard property of Young tableaux, when acting with $Y_{\{2^p,1\}}$ all symmetrizations in $(\,i_n, j_n, k\,)$ vanish. This means that the terms in the sum can be manipulated according to

$$Y_{\{2^p,1\}}\,\partial_{(\,i_n}\,T_{j_n\,)\,k}\,\prod_{r\neq n}^{p}\,T_{i_r j_r}\,\mathcal{F} \;=\; -\,Y_{\{2^p,1\}}\,\partial_k\,T_{i_1 j_1}\ldots T_{i_p j_p}\,\mathcal{F}\,, \qquad (2.86)$$

while the products of traces can be everywhere replaced with their two-column projections on account of eq. (2.72). The $\{2^p,1\}$ Young projection of eq. (2.84) thus reads

$$(\,p+2\,)\,Y_{\{2^p,1\}}\,\partial_k\,\mathcal{F}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p} \;-\; \partial^l\,\mathcal{F}^{[p+1]}{}_{i_1 j_1, \ldots, i_p j_p, kl} \;=\; 0\,. \qquad (2.87)$$

Notice that this result was obtained using only elementary properties of the involved two-column projections, without any need of knowing their detailed form! At any rate, the identities (2.87) contain the same terms entering eq. (2.82), and one can use them to eliminate the gauge variation of the ansatz (2.79) provided the coefficients satisfy

$$k_{p+1} \;=\; -\,\frac{k_p}{(\,p+1\,)(\,p+2\,)}\,. \qquad (2.88)$$

Since $k_0 = 1$, this recursion relation is solved by

$$k_p \;=\; \frac{(-1)^p}{p!\,(\,p+1\,)!}\,, \qquad (2.89)$$

and indeed the coefficients in eq. (2.89) are those first identified by Labastida in [39], barring a slight change of conventions and a typo in their relative signs.

In order to identify the coefficients (2.89) we only kept track of a particular class of traces of the Bianchi identities (2.80). Notice, however, that in the constrained theory this is the only non-trivial one, since the combination

$$\partial_k\,\mathcal{F}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p} \;-\; \frac{1}{2}\sum_{n=1}^{p}\,\partial_{(\,i_n}\,\mathcal{F}^{[p]}{}_{j_n\,)\,k, \ldots, i_{r\neq n} j_{r\neq n}, \ldots} \qquad (2.90)$$

appearing in eq. (2.84) is actually $\{2^p,1\}$-projected when eq. (2.72) holds. The same is true also for the gradient terms. A simpler manifestation of this fact can be recognized in the fully symmetric case where, for instance, the trace of the Bianchi identity (2.26) simply reduce to the gradient of the double trace of $\mathcal{F}$, that vanishes on account of eq. (2.68).

Before closing this section, let me also review Labastida's derivation of the coefficients (2.89), exploiting the self-adjointness of the Einstein-like tensors that will be useful to identify the field equations following from the Lagrangians (2.94). Starting from

$$\mathcal{E} \;=\; \sum_{p=0}^{N}\,k_p\,\eta^{i_1 j_1}\ldots\eta^{i_p j_p}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,\mathcal{F} \qquad (2.91)$$

and resorting to the expression for the multiple traces of $\mathcal{F}$ that is displayed in eq. (B.22) one finds

$$\langle\,\varphi\,,\,\mathcal{E}\,(\,\phi\,)\,\rangle\,=\,\langle\,\mathcal{E}\,(\,\varphi\,)\,,\,\phi\,\rangle$$

$$-\,\sum_{p=0}^{N}\,\frac{1}{2^{p}}\,\langle\,Y_{\{2^{p}\}}\,T_{i_{1}j_{1}}\ldots T_{i_{p}j_{p}}\,\varphi\,,\,\left[\,(\,p+1\,)\,k_{p}\,+\,\frac{k_{p-1}}{p}\,\right]\,Y_{\{2^{p}\}}\sum_{n=1}^{p}\,\partial_{i_{n}}\partial_{j_{n}}\prod_{r\neq n}^{p}\,T_{i_{r}j_{r}}\,\phi$$

$$-\,\frac{1}{2}\,\left[\,k_{p}\,+\,(\,p+1\,)\,(\,p+2\,)\,k_{p+1}\,\right]\,\partial^{k}\partial^{l}\,Y_{\{2^{p+1}\}}\,T_{i_{1}j_{1}}\ldots T_{i_{p}j_{p}}\,T_{kl}\,\phi\,\rangle\,. \tag{2.92}$$

This rewriting holds up to constraints and total derivatives. It follows from the combination of some of the contributions entering eq. (B.22), that is admitted when the multiple traces of φ or ϕ are substituted with their two-column projections. The relevant manipulations are similar to those adopted in the treatment of multiple traces of the Bianchi identities. In conclusion, if the k_p are those in eq. (2.89), the $\mathcal{E}$ tensors are self-adjoint since eq. (2.92) up to total derivatives reduces to

$$\langle\,\varphi\,,\,\mathcal{E}\,(\,\phi\,)\,\rangle\,=\,\langle\,\mathcal{E}\,(\,\varphi\,)\,,\,\phi\,\rangle\,. \tag{2.93}$$

To recapitulate, the Lagrangian for an arbitrary Bose field of mixed-symmetry is

$$\mathcal{L}\,=\,\frac{1}{2}\,\langle\,\varphi\,,\,\sum_{p=0}^{N}\,\frac{(-1)^{p}}{p!\,(\,p+1\,)!}\,\eta^{i_{1}j_{1}}\ldots\eta^{i_{p}j_{p}}\,\mathcal{F}^{[p]}{}_{i_{1}j_{1},\ldots,\,i_{p}j_{p}}\,\rangle\,. \tag{2.94}$$

Its most evident signature is the presence of higher traces of $\mathcal{F}$. Let me stress that this property is definitely related to the presence of more than one family of symmetrized space-time indices or, equivalently, to the number of rows of the Young tableau describing the representation of the Lorentz group under scrutiny. Therefore, it is independent of the need for constraints, that is dictated by the number of columns in the Young tableau.

A clarifying example is provided by irreducible mixed-symmetry bosons with at most two columns. In the present symmetric convention, these fields carry an arbitrary number of families comprising at most two space-time indices, while all symmetrizations over any group of three indices vanish. They are the simplest class of mixed-symmetry Bose fields and they are actually *unconstrained*. Their Lagrangians can be easily obtained generalizing the expression

$$\mathcal{L}\,=\,-\,\frac{1}{2}\,\partial^{\mu}\,h^{\nu}{}_{[\mu}\,\partial_{\nu}\,h_{\rho]}{}^{\rho}\,, \tag{2.95}$$

where the couple of square brackets denotes a complete antisymmetrization of the indices it encloses, with the minimum possible number of terms and with unit overall normalization. The Lagrangian (2.95) coincides with the linearized Einstein-Hilbert one (2.66), up to total derivatives. The peculiarity of eq. (2.95) is its manifest gauge invariance under linearized diffeomorphisms, simply because two derivatives commute.

In a similar fashion, when written directly in terms of the field, the Lagrangian for an arbitrary two-column boson[4] $\varphi_{\mu_1^1 \mu_2^1, \ldots, \mu_1^p \mu_2^p; \nu_1 \ldots \nu_q}$ must be of the form

$$\mathcal{L} \sim \partial^{\mu_1} \varphi^{\mu_2}{}_{[\mu_1|, \ldots,}{}^{\mu_{p+1}}{}_{|\mu_p; \mu_{p+1} \ldots \mu_{p+q}|}$$
$$\times \partial_{|\mu_{p+q+1}} \varphi_{\mu_{p+q+2}|}{}^{\mu_{p+2}}{}_{, \ldots, |\mu_{2p+1}]}{}^{\mu_{2p+1}; \mu_{2p+2} \ldots \mu_{2p+q+1}}. \tag{2.96}$$

This expression is again manifestly gauge invariant due to the antisymmetrizations it contains and, up to total derivatives, must be proportional to the Lagrangian (2.94), that is uniquely fixed by the request for gauge invariance. Furthermore, in eq. (2.96) the presence of higher traces of the field is manifest.

2.2.2 Fermi fields

In the fully symmetric case Bose and Fermi fields behave in a very similar way. Actually, the Fang-Fronsdal constraint (2.38) on the field induces a similar condition on the kinetic spinor-tensor,

$$\mathcal{\not{S}}_{\mu_1 \ldots \mu_{s-3} \lambda}{}^{\lambda} = 0, \tag{2.97}$$

and the Lagrangians thus take the form

$$\mathcal{L} = \frac{1}{2} \, \bar{\psi}^{\mu_1 \ldots \mu_s} \left(\mathcal{S}_{\mu_1 \ldots \mu_s} - \frac{1}{2} \, \gamma_{(\mu_1} \mathcal{\not{S}}_{\mu_2 \ldots \mu_s)} - \frac{1}{2} \, \eta_{(\mu_1 \mu_2} \mathcal{S}_{\mu_3 \ldots \mu_s) \lambda}{}^{\lambda} \right) + \text{h.c.}. \tag{2.98}$$

The gauge variation of eq. (2.98) does not entail particularly subtleties and reads

$$\delta \mathcal{L} = - \frac{s}{2} \, \bar{\epsilon}^{\mu_1 \ldots \mu_{s-1}} \left(\partial \cdot \mathcal{S}_{\mu_1 \ldots \mu_{s-1}} - \frac{1}{2} \, \not{\partial} \, \mathcal{\not{S}}_{\mu_1 \ldots \mu_{s-1}} - \frac{1}{2} \, \partial_{(\mu_1} \mathcal{S}_{\mu_2 \ldots \mu_{s-1}) \lambda}{}^{\lambda} \right)$$
$$+ \frac{1}{2} \binom{s}{2} \bar{\not{\epsilon}}^{\mu_1 \ldots \mu_{s-2}} \, \partial \cdot \mathcal{\not{S}}_{\mu_1 \ldots \mu_{s-2}} + \frac{3}{4} \binom{s}{3} \bar{\epsilon}^{\mu_1 \ldots \mu_{s-3} \lambda}{}_{\lambda} \, \partial \cdot \mathcal{S}_{\mu_1 \ldots \mu_{s-3} \lambda}{}^{\lambda} + \text{h.c.}. \tag{2.99}$$

The first group of terms cancels on account of the constrained Bianchi identity coming from eq. (2.57), while the others cancel on account of the constraint (2.56) on the gauge parameter.

In the mixed-symmetry case, the structure of the Lagrangians is roughly modified as in the bosonic case, since *not* all triple γ-traces of the kinetic tensor $\mathcal{S}$ are forced to vanish. Hence, the Lagrangians can and indeed do contain additional contributions with multiple γ-traces of $\mathcal{S}$. Still in analogy with the bosonic case, the Labastida constraints on the fields (2.59) induce similar constraints

$$T_{(ij} \gamma_{k)} \mathcal{S} = 0 \tag{2.100}$$

on the kinetic tensors. As a consequence, only particular linear combinations of γ-traces of $\mathcal{S}$ can enter the Lagrangians. Notice that eq. (2.100) can be easily checked via a direct computation:

$$T_{(ij} \gamma_{k)} \mathcal{S} = - i \left(2 \, \not{\partial} \, T_{(ij} \gamma_{k)} \psi + \partial^l T_{(ij} T_{kl)} \psi + \partial^l T_{(ij} \gamma_{k) l} \psi \right). \tag{2.101}$$

---

[4]Notice that the indices coming after the semicolon are actually fully antisymmetrized between each others. As warned in Appendix A, I will often abide on this convention also in the following.

The first two terms in this expression are directly related to the constraints (2.59), while the last one can be linked to them expanding the antisymmetric γ-trace according to

$$T_{(ij}\,\gamma_{k)\,l}\,\psi \;=\; -\,\gamma_l\,T_{(ij}\,\gamma_{k)}\,\psi \;-\; T_{(ij}\,T_{kl)}\,\psi\,. \tag{2.102}$$

In order to select the non-vanishing combinations entering the Lagrangians one can again classify all the candidate terms according to the irreducible representations of the permutation group acting on the family indices. Moreover, as anticipated in Section 2.1.2, it is convenient to work in an antisymmetric basis for γ-matrices, introducing the fully antisymmetrized γ-traces

$$\gamma_{k_1\dots k_q} \;=\; \frac{1}{q!}\,\gamma_{[k_1}\gamma_{k_2}\cdots\gamma_{k_q]}\,. \tag{2.103}$$

The final outcome of this analysis is that, for any given number of traces and antisymmetric γ-traces of the kinetic tensor $\mathcal{S}$, all Young projections in family indices with more than two columns vanish in the constrained theory on account of eq. (2.100). This condition identifies the wide class of terms that can enter fermionic Lagrangians which, as a result, are far more involved than their bosonic counterparts. For N families the Lagrangians are in fact bound to rest on $\mathcal{O}(N^2)$ distinct Young projections, with a first column of length $l_1 \leq N$ and a second column of length $l_2 \leq l_1$, that in general is actually shorter, since antisymmetric γ-traces $\gamma_{k_1\dots k_q}$ of $\mathcal{S}$ can accompany the ordinary T_{ij} traces that already enter the bosonic Lagrangians. Before using this information to present an ansatz for the Lagrangians for mixed-symmetry Fermi fields, let me sketch a proof of this statement. This goal can be reached repeating the analysis performed in the bosonic case. First, one can recognize that all the Young projections associated to tableaux with at least three columns involve a sum of terms containing at least a symmetrization over three family indices. One can then study how the symmetrized indices can be spread over the products of traces and antisymmetrized γ-traces, to eventually recognize that all the different possibilities can be related to the Labastida-like constraints (2.100). This detailed analysis is performed in [41], but here I would like to refer to a simpler argument, that was also introduced in the same paper. Indeed, one can notice that

$$T_{i_1 j_1}\dots T_{i_{p-1}j_{p-1}}T_{(ab}\,\gamma_{c)\,k_1\dots k_{q-1}} \tag{2.104}$$

admits all Young projections allowed by the very general expression

$$T_{i_1 j_1}\dots T_{i_{p-1}j_{p-1}}T_{ab}\,\gamma_{c\,k_1\dots k_{q-1}}\,, \tag{2.105}$$

aside from the two-column one. Furthermore, acting on each irreducible component of (2.104) the permutation group can generate the entire corresponding irreducible subspaces, and in particular the irreducible components of (2.105) that it contains. This suffices to show that all Young projections of the generic expression (2.105) with more than two columns are actually related to terms containing an expression annihilated by the constraints. In fact, the previous observation shows that they can be

expressed as linear combinations of (2.104) with similar quantities obtained permuting indices, while (2.104) is related to the constraints via the identity

$$T_{(ab}\gamma_{c)\,k_1\ldots k_{q-1}} = (-1)^{q+1}\,\gamma_{k_1\ldots k_{q-1}}\,\gamma_{(a}\,T_{bc)} + (-1)^q\,\gamma_{[k_1\ldots k_{q-2}}\,T_{k_{q-1}](a}\,T_{bc)}\,. \quad (2.106)$$

The first term in eq. (2.106) indeed contains manifestly symmetrized triple γ-traces, while the second can be related to them extending the symmetrizations as pertains to a product of identical T tensors.

In conclusion, only *two-column projected* multiple traces of $\mathcal{S}$ can enter the constrained Lagrangians, and it is convenient to denote them as

$$(\gamma^{[q]}\,\mathcal{S}^{[p]})_{i_1j_1,\ldots,i_pj_p;\,k_1\ldots k_q} = Y_{\{2^p,1^q\}}\,T_{i_1j_1}\ldots T_{i_pj_p}\,\gamma_{k_1\ldots k_q}\,\mathcal{S} \quad (2.107)$$

Here $Y_{\{2^p,1^q\}}$ denotes the Young projector onto the $\{2,\ldots,2,1,\ldots,1\}$ irreducible representation of the permutation group acting on the family indices. Furthermore, the reader should appreciate that in eq. (2.107) colons separate again groups of symmetric indices, while a semicolon precedes the left-over group of antisymmetric ones. Hence, barring the substitution $\mathcal{F}\to\mathcal{S}$, for $q=0$ these terms reduce to those introduced in the previous section, while for $p=0$ they simply reduce to fully antisymmetrized γ-traces,

$$(\gamma^{[q]}\,\mathcal{S}^{[0]})_{k_1\ldots k_q} = \gamma_{k_1\ldots k_q}\,\mathcal{S}\,. \quad (2.108)$$

All other terms contain both traces and antisymmetrized γ-traces, as for instance

$$(\gamma^{[1]}\,\mathcal{S}^{[1]})_{ij;\,k} = \frac{1}{3}\left\{\,2\,T_{ij}\,\gamma_k - T_{k(i}\,\gamma_{j)}\,\right\}\mathcal{S}\,. \quad (2.109)$$

An ansatz for the fermionic Lagrangians should contain all these terms, contracted with suitable products of invariant tensors. It is of course natural to consider both the η^{ij} operators introduced in eq. (2.78) and the operators

$$\gamma^i\,\psi \equiv \gamma_{(\mu_1^i|}\,\psi_{\ldots,|\mu_2^i\ldots\mu_{s_i+1}^i),\ldots}\,, \quad (2.110)$$

together with their antisymmetric combinations

$$\gamma^{k_1\ldots k_q} = \frac{1}{q!}\,\gamma^{[k_1}\gamma^{k_2}\ldots\gamma^{k_q]}\,. \quad (2.111)$$

We now have all the ingredients to propose an ansatz for the structure of the Lagrangians for Fermi fields of mixed symmetry [41]. It reads

$$\mathcal{L} = \frac{1}{2}\,\langle\,\bar{\psi}\,,\,\sum_{p,\,q=0}^{N}\,k_{p,q}\,\eta^{i_1j_1}\ldots\eta^{i_pj_p}\,\gamma^{k_1\ldots k_q}\,(\gamma^{[q]}\,\mathcal{S}^{[p]})_{i_1j_1,\ldots,i_pj_p;\,k_1\ldots k_q}\,\rangle + \text{h.c.}\,,$$
$$(2.112)$$

where I am resorting to the scalar product that was already introduced for bosons, and that is described in detail in Appendix A. Again, the upper limit of the sum is the number of index families, simply because it is not possible to antisymmetrize over more than N family indices, as would be required by a Young tableaux with

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$N + 1$ rows. We shall see shortly that the $k_{p,q}$ coefficients in eq. (2.112) are uniquely determined by the condition that $\mathcal{L}$ be gauge invariant, up to the convenient choice $k_{0,0} = 1$.

As in the bosonic case and in the fully symmetric example, the gauge invariance of the Lagrangians is guaranteed by the Bianchi identities, that for *constrained* mixed-symmetry fermions read

$$\partial_i \mathcal{S} - \frac{1}{2}\, \slashed{\partial}\, \gamma_i\, \mathcal{S} - \frac{1}{2}\, \partial^j\, T_{ij}\, \mathcal{S} - \frac{1}{6}\, \partial^j\, \gamma_{ij}\, \mathcal{S} = 0\,. \tag{2.113}$$

In fact, the "classical anomaly" appearing in the right-hand side of eq. (2.58) cancels after imposing the Labastida constraints (2.59) on the field. Again, eq. (2.113) cannot directly deal with the higher traces contained in the ansatz (2.112), but suitable combinations of multiple γ-traces of the Bianchi identities can, barring some new subtleties that I shall dwell upon shortly. In order to proceed, one can begin by noticing that the identities collected in Appendix B make it possible to recast the gauge variation of eq. (2.112), up to total derivatives, in the form

$$\begin{aligned}
\delta \mathcal{L} = -\sum_{p,q=0}^{N} \frac{1}{2^{p+1}} \big\langle\, T_{i_1 j_1} \dots T_{i_p j_p}\, \bar{\epsilon}_l\, \gamma_{k_1 \dots k_q}\,,\;\; & k_{p,q}\, \partial_l\, (\gamma^{[q]} \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p\, ;\, k_1 \dots k_q} \\
+\, (q+1)\, k_{p,q+1}\, \slashed{\partial}\, (\gamma^{[q+1]} \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p\, ;\, k_1 \dots k_q\, l} & \\
+\, (p+1)\, k_{p+1,q}\, \partial^m\, (\gamma^{[q]} \mathcal{S}^{[p+1]})_{i_1 j_1, \dots, i_p j_p,\, lm\, ;\, k_1 \dots k_q} & \\
+\, (q+1)(q+2)\, k_{p,q+2}\, \partial^m\, (\gamma^{[q+2]} \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p\, ;\, k_1 \dots k_q\, lm}\, \big\rangle &\, +\, \text{h.c.}. \quad (2.114)
\end{aligned}$$

All $(\gamma^{[m]} \mathcal{S}^{[n]})$ terms that are right entries of the scalar products are two-column projected by assumption. This statement applies to all free indices carried by the last three terms, while the first is slightly more complicated, since due to the l index carried by the divergence it also allows projections with three boxes in the first row. However, the left entry of the scalar product makes this type of projections proportional to the constraints on the gauge parameters (2.42). As usual, this can be realized resorting to arguments similar to those used to discard from the ansatz (2.112) the tableaux with more than two columns. Thus, up to the Labastida constraints one can effectively restrict the attention to two-column projections of the gauge variation (2.114).

On the other hand, in sharp contrast with the bosonic case, here two types of two-column projections are allowed for the terms of the type

$$T_{i_1 j_1} \dots T_{i_p j_p}\, \bar{\epsilon}_l\, \gamma_{k_1 \dots k_q} \tag{2.115}$$

that are left entries of the scalar products. One can obtain for them a two-column projection acting on $\bar{\epsilon}_l$ with an already two-column projected combination of traces and antisymmetric γ-traces, but the possible outcomes are

$$\{ 2^p,\, 1^q \} \otimes \{ 1 \} = \{ 2^p,\, 1^{q+1} \} \oplus \{ 2^{p+1},\, 1^{q-1} \} \oplus \{ 3,\, 2^{p-1},\, 1^q \}\,, \tag{2.116}$$

where the last term is related to the constraints on the gauge parameters. Thus, after using the constraints (2.42), the gauge variation (2.114) can be cast in the form

$$
\delta\mathcal{L} = -\sum_{p,q=0}^{N} \frac{1}{2^{p+1}} \left\langle\, Y_{\{2^p,1^{q+1}\}}\, T_{i_1 j_1}\ldots T_{i_p j_p}\, \bar{\epsilon}_l\, \gamma_{k_1\ldots k_q}\,,\right.
$$

$$
k_{p,q}\,\partial_l\,(\,\gamma^{[q]}\,\mathcal{S}^{[p]}\,)_{i_1 j_1,\ldots,i_p j_p;k_1\ldots k_q} + (\,q+1\,)\,k_{p,q+1}\,\partial\!\!\!/\,(\,\gamma^{[q+1]}\,\mathcal{S}^{[p]}\,)_{i_1 j_1,\ldots,i_p j_p;k_1\ldots k_q\,l}
$$

$$
+ (\,p+1\,)\,k_{p+1,q}\,\partial^m\,(\,\gamma^{[q]}\,\mathcal{S}^{[p+1]}\,)_{i_1 j_1,\ldots,i_p j_p,lm;k_1\ldots k_q}
$$

$$
\left.+ (\,q+1\,)(\,q+2\,)\,k_{p,q+2}\,\partial^m\,(\,\gamma^{[q+2]}\,\mathcal{S}^{[p]}\,)_{i_1 j_1,\ldots,i_p j_p;k_1\ldots k_q\,lm}\,\right\rangle
$$

$$
-\sum_{p,q=0}^{N} \frac{1}{2^{p+1}} \left\langle\, Y_{\{2^{p+1},1^{q-1}\}}\, T_{i_1 j_1}\ldots T_{i_p j_p}\, \bar{\epsilon}_l\, \gamma_{k_1\ldots k_q}\,,\right.
$$

$$
k_{p,q}\,\partial_l\,(\,\gamma^{[q]}\,\mathcal{S}^{[p]}\,)_{i_1 j_1,\ldots,i_p j_p;k_1\ldots k_q}
$$

$$
\left.+ (\,p+1\,)\,k_{p+1,q}\,\partial^m\,(\,\gamma^{[q]}\,\mathcal{S}^{[p+1]}\,)_{i_1 j_1,\ldots,i_p j_p,lm;k_1\ldots k_q}\,\right\rangle + \text{h.c.}\,. \tag{2.117}
$$

Notice that the first term in eq. (2.114) gives contributions to both groups of projected terms above, while the second term gives a single contribution. The analysis of the gradient terms is slightly more subtle, but the end result is that one of them, the third term in eq. (2.114), contributes to both. Indeed,

$$
\{2^p, 1^{q+1}\} \otimes \{1\} = \{2^p, 1^{q+2}\} \oplus \{2^{p+1}, 1^q\} \oplus \{3, 2^{p-1}, 1^{q+1}\}\,,
$$
$$
\{2^{p+1}, 1^{q-1}\} \otimes \{1\} = \{2^{p+1}, 1^q\} \oplus \{2^{p+2}, 1^{q-2}\} \oplus \{3, 2^p, 1^{q-1}\}\,, \tag{2.118}
$$

so that both types of projections are compatible with the $\{2^{p+1}, 1^q\}$ projection carried by $(\,\gamma^{[q]}\,\mathcal{S}^{[p+1]}\,)$. This is no longer true for the fourth term in eq. (2.114).

At this stage fermionic fields bring about a novel type of complication. Indeed, the γ-traces of the Bianchi identities do *not* suffice to directly set to zero the two groups of terms of eq. (2.117), but these, on the other hand, are not fully independent and can be partly combined using the constraints (2.42) on the gauge parameters. In fact the projections $\{2^p, 1^{q+1}\}$ and $\{2^{p+1}, 1^{q-1}\}$ appearing in eq. (2.116) can actually arise from *pairs* of neighboring contributions of the form (2.115), with values of p differing by one unit and with values of q differing by two units. The key step to proceed is to relate these two types of terms, that are only apparently different, making use of the constraints (2.42) on the gauge parameters and of proper relabelings. To this end, it proves particularly convenient to manipulate the combinations $\bar{\epsilon}_l\, \gamma_{k_1\ldots k_q}$ according to

$$
\bar{\epsilon}_l\, \gamma_{k_1\ldots k_q} = \frac{1}{q+1}\,\bar{\epsilon}_{[l}\, \gamma_{k_1\ldots k_q]} + \frac{1}{q+1}\sum_{n=1}^{q}(-1)^{(n+1)}\,\bar{\epsilon}_{(l}\, \gamma_{k_n)\,k_1\ldots k_{r\neq n}\ldots k_q}\,. \tag{2.119}
$$

The terms entering the sum can then be rewritten more conveniently as

$$
\bar{\epsilon}_{(l}\, \gamma_{k_1)\,k_2\ldots k_q} = \bar{\epsilon}_{(l}\, \gamma_{k_1)}\, \gamma_{k_2\ldots k_q} - T_{[k_2\,|\,(l}\,\bar{\epsilon}_{k_1)}\, \gamma_{|\,k_3\ldots k_q]}\,, \tag{2.120}
$$

where the first contribution on the right-hand side clearly vanishes in the constrained setting, on account of eq. (2.42). Enforcing the constraints thus leads to

$$\sum_{n=1}^{q} (-1)^{(n+1)} \, \bar{\epsilon}_{(l} \, \gamma_{k_n) \, k_1 \dots k_{r \neq n} \dots k_q} \; \rightarrow \; T_{l \, [k_1} \bar{\epsilon}_{k_2} \, \gamma_{k_3 \dots k_q]} \,, \tag{2.121}$$

since the sum with oscillating signs induces a complete antisymmetry in the k_r indices, that as such forbids the simultaneous presence of two of them on the trace T. As a result, in the constrained theory eq. (2.119) can be finally cast in the form

$$T_{i_1 j_1} \dots T_{i_p j_p} \, \bar{\epsilon}_l \, \gamma_{k_1 \dots k_q} \; = \; \frac{1}{q+1} \, T_{i_1 j_1} \dots T_{i_p j_p} \left(\bar{\epsilon}_{[l} \, \gamma_{k_1 \dots k_q]} \, + \, T_{l \, [k_1} \bar{\epsilon}_{k_2} \, \gamma_{k_3 \dots k_q]} \right), \tag{2.122}$$

where the first term is annihilated by the $\{2^{p+1}, 1^{q-1}\}$ projection, while the second is similarly annihilated by the $\{2^p, 1^{q+1}\}$ projection. In conclusion one is led to

$$Y_{\{2^{p+1}, 1^{q-1}\}} \, T_{i_1 j_1} \dots T_{i_p j_p} \, \bar{\epsilon}_l \, \gamma_{k_1 \dots k_q} \; = \; \frac{1}{q+1} \, Y_{\{2^{p+1}, 1^{q-1}\}} \, T_{i_1 j_1} \dots T_{i_p j_p} \, T_{l \, [k_1} \bar{\epsilon}_{k_2} \, \gamma_{k_3 \dots k_q]} \,,$$

$$Y_{\{2^p, 1^{q+1}\}} \, T_{i_1 j_1} \dots T_{i_p j_p} \, \bar{\epsilon}_l \, \gamma_{k_1 \dots k_q} \; = \; \frac{1}{q+1} \, Y_{\{2^p, 1^{q+1}\}} \, T_{i_1 j_1} \dots T_{i_p j_p} \, \bar{\epsilon}_{[l} \, \gamma_{k_1 \dots k_q]} \,. \tag{2.123}$$

While the second of these relations is rather trivial, the first is quite interesting, since it connects a relatively complicated projection of p traces and a single q-fold antisymmetric γ-trace to a simpler one involving $p+1$ traces and a $(q-2)$-fold antisymmetric γ-trace with the maximum possible number of antisymmetrizations. Let me stress that one can also proceed in the opposite direction, decreasing the value of p. The relevant identity, that can be proved with similar techniques is then

$$Y_{\{2^p, 1^{q+1}\}} \, T_{i_1 j_1} \dots T_{i_p j_p} \, \bar{\epsilon}_l \, \gamma_{k_1 \dots k_q}$$
$$= \frac{(-1)^{q+1}}{p \, (q+1)} \, Y_{\{2^p, 1^{q+1}\}} \sum_{n=1}^{p} \left(\prod_{r \neq n}^{p} T_{i_r j_r} \right) \bar{\epsilon}_{(i_n} \, \gamma_{j_n) \, k_1 \dots k_q \, l} \,, \tag{2.124}$$

but it is simpler to work with terms whose projections involve the maximum number of antisymmetrizations compatible with their tensorial character.

One can now manipulate the second group of terms in eq. (2.117) using the key relation appearing in the first line of eq. (2.123). As a result, after a proper relabeling eq. (2.117) can be finally turned into

$$\delta \mathcal{L} = - \sum_{p, q = 0}^{N} \frac{1}{2^{p+1}} \, \big\langle \, Y_{\{2^p, 1^{q+1}\}} \, T_{i_1 j_1} \dots T_{i_p j_p} \, \bar{\epsilon}_l \, \gamma_{k_1 \dots k_q} \,,$$

$$k_{p,q} \, \partial_l \, (\, \gamma^{[q]} \, \mathcal{S}^{[p]} \,)_{i_1 j_1, \dots, i_p j_p \,; \, k_1 \dots k_q}$$

$$+ \frac{(q+1)(q+2)}{p(q+3)} \, k_{p-1, q+2} \sum_{n=1}^{p} \partial_{(i_n |} \, (\, \gamma^{[q+2]} \, \mathcal{S}^{[p-1]} \,)_{\dots, \, i_{r \neq n} j_{r \neq n}, \dots \,; \, | j_n) \, l \, k_1 \dots k_q}$$

$$+ (q+1) \, k_{p, q+1} \, \slashed{\partial} \, (\, \gamma^{[q+1]} \, \mathcal{S}^{[p]} \,)_{i_1 j_1, \dots, i_p j_p \,; \, k_1 \dots k_q \, l}$$

$$+ (p+1) k_{p+1,q} \, \partial^m \, (\gamma^{[q]} \, \mathcal{S}^{[p+1]})_{i_1 j_1, \dots, i_p j_p, lm; k_1 \dots k_q}$$

$$+ \frac{(q+1)(q+2)}{q+3} k_{p,q+2} \sum_{n=1}^{p} \partial^m \, (\gamma^{[q+2]} \, \mathcal{S}^{[p]})_{\dots, i_r \neq n j_r \neq n, \dots, m (i_n; j_n) l k_1 \dots k_q}$$

$$+ (q+1)(q+2) k_{p,q+2} \, \partial^m \, (\gamma^{[q+2]} \, \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q lm} \rangle + \text{h.c.}, \quad (2.125)$$

where I enforced the symmetry under interchanges of the (i_n, j_n) pairs brought about by the products of identical T tensors present on the right-hand sides of the scalar products. Furthermore, the contributions in the last two terms are proportional, since

$$(\gamma^{[q+2]} \, \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q lm} = (\gamma^{[q+2]} \, \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_{p-1} j_{p-1}, m (i_p; j_p) l k_1 \dots k_q}, \quad (2.126)$$

because a symmetrization in (i_p, j_p, m) vanishes on account of the two-column projection. Therefore, on account of the constraints (2.42), eq. (2.114) finally reduces to

$$\delta \mathcal{L} = - \sum_{p,q=0}^{N} \frac{1}{2^{p+1}} \langle \, Y_{\{2^p, 1^{q+1}\}} \, T_{i_1 j_1} \dots T_{i_p j_p} \, \bar{\epsilon}_l \, \gamma_{k_1 \dots k_q} ,$$

$$k_{p,q} \, \partial_l \, (\gamma^{[q]} \, \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q}$$

$$+ \frac{(q+1)(q+2)}{p(q+3)} k_{p-1,q+2} \sum_{n=1}^{p} \partial_{(i_n|} \, (\gamma^{[q+2]} \, \mathcal{S}^{[p-1]})_{\dots, i_r \neq n j_r \neq n, \dots; |j_n) l k_1 \dots k_q}$$

$$+ \frac{(q+1)(q+2)(q+p+3)}{q+3} k_{p,q+2} \, \partial^m \, (\gamma^{[q+2]} \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q lm}$$

$$+ (q+1) k_{p,q+1} \, \slashed{\partial} \, (\gamma^{[q+1]} \, \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q l}$$

$$+ (p+1) k_{p+1,q} \, \partial^m \, (\gamma^{[q]} \, \mathcal{S}^{[p+1]})_{i_1 j_1, \dots, i_p j_p, lm; k_1 \dots k_q} \rangle + \text{h.c.}. \quad (2.127)$$

In the fermionic case, the γ-traces of the Bianchi identities take the rather involved form presented in eq. (B.28), but it is possible to extract from them the far simpler chain of $\{2^p, 1^{q+1}\}$-projected identities

$$(p+q+2) \, Y_{\{2^p, 1^{q+1}\}} \, \partial_l \, (\gamma^{[q]} \, \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q}$$

$$+ \frac{1}{q+3} \, Y_{\{2^p, 1^{q+1}\}} \sum_{n=1}^{p} \partial_{(i_n|} \, (\gamma^{[q+2]} \, \mathcal{S}^{[p-1]})_{\dots, i_r \neq n j_r \neq n, \dots; |j_n) l k_1 \dots k_q}$$

$$+ (-1)^{q+1} \, \slashed{\partial} \, (\gamma^{[q+1]} \, \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q l} - \partial^m \, (\gamma^{[q]} \, \mathcal{S}^{[p+1]})_{i_1 j_1, \dots, i_p j_p, lm; k_1 \dots k_q}$$

$$- \frac{1}{q+3} \, \partial^m \, (\gamma^{[q+2]} \, \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q lm} = 0. \quad (2.128)$$

This can be done resorting to techniques similar to those applied to manipulate eq. (2.84) in the bosonic case, and more details are presented in Appendix C. Comparing eqs. (2.127) and (2.128) one can now recognize that, in order to obtain a gauge invariant Lagrangian, the coefficients should satisfy the two recursion relations

$$k_{p+1,q} = - \frac{1}{(p+1)(p+q+2)} k_{p,q},$$

$$k_{p,q+1} = \frac{(-1)^{q+1}}{(q+1)(p+q+2)} k_{p,q}, \quad (2.129)$$

whose solution is

$$k_{p,q} = \frac{(-1)^{p+\frac{q(q+1)}{2}}}{p!\,q!\,(p+q+1)!} \tag{2.130}$$

if $k_{0,0} = 1$. This result is also fully consistent with the vanishing of the other terms in eq. (2.127). Notice that this expression includes as special cases the coefficients first obtained by Labastida for Bose fields: in fact, letting $q = 0$ in eq. (2.130) recovers the result presented in eq. (2.89).

The coefficients given in eq. (2.130) determine completely the constrained Lagrangians (2.112) for mixed-symmetry Fermi fields. Together with the unconstrained Lagrangians that we shall come across in Chapter 3, these are the main results of this Thesis.

Interestingly, eq. (2.130) can be also obtained demanding that the kinetic operator be self-adjoint, along the lines of what was done by Labastida for Bose fields in [39]. The idea is to build a Rarita-Schwinger-like tensor such that

$$\langle \bar{\psi}, \mathcal{E} \rangle + \text{h.c.} = 2\langle \bar{\psi}, \mathcal{E} \rangle, \tag{2.131}$$

up to constraints and total derivatives, starting from a general ansatz of the type

$$\mathcal{E} = \sum_{p,q=0}^{N} k_{p,q}\, \eta^{i_1 j_1} \ldots \eta^{i_p j_p}\, \gamma^{k_1 \ldots k_q}\, T_{i_1 j_1} \ldots T_{i_p j_p}\, \gamma_{k_1 \ldots k_q}\, \mathcal{S}\,. \tag{2.132}$$

Whereas this sum apparently differs from the one contained in eq. (2.112), in the constrained case they actually coincide, since each term gives rise to a *unique* two-column projection. Hence, making use of the result for the general γ-trace of the $\mathcal{S}$ tensor given in eq. (B.26), up to total derivatives and constraints one is led to the condition

$$\langle \bar{\psi}, \mathcal{E} \rangle + \text{h.c.} = 2\langle \bar{\psi}, \mathcal{E} \rangle + \sum_{p,q=0}^{N} \frac{i}{2^{p+1}} \langle\, Y_{\{2^p,1^q\}}\, T_{i_1 j_1} \ldots T_{i_p j_p}\, \bar{\psi}\, \gamma_{k_1 \ldots k_q}\,,$$

$$-\left[\,(p+q+1)\,k_{p,q} - \frac{(-1)^q}{q}\,k_{p,q-1}\,\right] Y_{\{2^p,1^q\}}\, \gamma_{[k_1 \ldots k_{q-1}}\, \partial_{k_q]}\prod_{r=1}^{p} T_{i_r j_r}\,\psi$$

$$+\left[\,k_{p,q} - (-1)^q\,\frac{q+1}{p}\,k_{p-1,q+1}\,\right] Y_{\{2^p,1^q\}} \sum_{n=1}^{p} \gamma_{k_1 \ldots k_q\,(i_n}\, \partial_{j_n)}\prod_{r\neq n}^{p} T_{i_r j_r}\,\psi$$

$$+\left[\,k_{p,q} + (-1)^q\,(q+1)\,(p+q+2)\,k_{p,q+1}\,\right] \partial^l\, Y_{\{2^p,1^{q+1}\}}\, \gamma_{k_1 \ldots k_q l}\prod_{r=1}^{p} T_{i_r j_r}\,\psi$$

$$+\left[\,k_{p,q} + (-1)^q\,\frac{p+1}{q}\,k_{p+1,q-1}\,\right] \partial^l\, Y_{\{2^{p+1},1^{q-1}\}}\, \gamma_{[k_1 \ldots k_{q-1}}\, T_{k_q]l}\prod_{r=1}^{p} T_{i_r j_r}\,\psi\,\rangle, \tag{2.133}$$

so that the remainder vanishes precisely if the coefficients $k_{p,q}$ are those of eq. (2.130). To reiterate, the coefficients for the general fermionic Lagrangians can be derived in two distinct ways: enforcing the gauge symmetry via the Bianchi identities as in [41],

or alternatively requiring that the kinetic operator be self adjoint, as was done only for Bose fields in [39]. In Section 3.1.2 we shall see that the first method appears more natural in the unconstrained case. Moreover, as in the bosonic case, it also definitely shows that the Bianchi identities provide the frame of the far more involved Lagrangians (2.112).

In order to summarize the result of this rather long discussion, let me briefly comment on the structure of the Lagrangian for an arbitrary Fermi field of mixed-symmetry, that reads

$$\mathcal{L} = \frac{1}{2} \langle \, \bar{\psi} \, , \, \sum_{p,\,q=0}^{N} \frac{(-1)^{\,p+\frac{q\,(q+1)}{2}}}{p!\,q!\,(\,p+q+1\,)!} \, \eta^{\,p}\,\gamma^{\,q}\,(\,\gamma^{\,[q]}\,\mathcal{S}^{\,[p]}\,) \, \rangle \, + \, \text{h.c.} \, , \qquad (2.134)$$

where for brevity I have introduced the shorthand

$$\eta^{\,p}\,\gamma^{\,q}\,(\,\gamma^{\,[q]}\,\mathcal{S}^{\,[p]}\,) \, \equiv \, \eta^{\,i_1 j_1} \dots \eta^{\,i_p j_p}\,\gamma^{\,k_1 \dots k_q}\,(\,\gamma^{\,[q]}\,\mathcal{S}^{\,[p]}\,)_{i_1 j_1,\,\dots,\,i_p j_p\,;\,k_1 \dots k_q} \, . \qquad (2.135)$$

As in the bosonic case, the most evident signature of the Lagrangian (2.134) is the presence of higher γ-traces of $\mathcal{S}$. Again, this property is definitely related to the presence of more than one family of symmetrized space-time indices or, equivalently, to the number of rows of the Young tableau describing the representation of the Lorentz group under scrutiny. On the contrary, the need for constraints is dictated by the number of columns in the Young tableau.

A clarifying example is provided by fully antisymmetric, "single-column" fermions. They are the simplest class of mixed-symmetry Fermi fields and they are actually *unconstrained*. Their Lagrangians can be easily obtained generalizing the usual presentation of the Rarita-Schwinger Lagrangian, that takes the form

$$\mathcal{L} = \frac{i}{2} \, \bar{\psi}_\mu \, \gamma^{\mu\nu\rho} \, \partial_\nu \, \psi_\rho \, + \, \text{h.c.} \, . \qquad (2.136)$$

Up to total derivatives, it actually coincides with the Lagrangian that one can extract from eq. (2.98) for a spin $3/2$ fermion. Eq. (2.136) is manifestly left invariant under the gauge transformation $\delta \psi_\mu = \partial_\mu \epsilon$, simply because two derivatives commute. In a similar fashion, when written directly in terms of the field, the Lagrangian for an arbitrary single-column fermion[5] $\psi_{\nu_1 \dots \nu_q}$ can be cast in the form

$$\mathcal{L} = i \, \frac{(-1)^{\frac{q\,(q-1)}{2}}}{2\,q!} \, \bar{\psi}_{\mu_1 \dots \mu_q} \, \gamma^{\mu_1 \dots \mu_q \lambda \nu_1 \dots \nu_q} \, \partial_\lambda \, \psi_{\nu_1 \dots \nu_q} \, + \, \text{h.c.} \, . \qquad (2.137)$$

This expression is again manifestly gauge invariant due to the presence of the fully antisymmetrized γ-matrix and, up to total derivatives, must be proportional to the Lagrangian (2.134), that is uniquely fixed by the request for gauge invariance. Furthermore, in eq. (2.96) the presence of higher γ-traces of the field is manifest.

---

[5]Notice the slight abuse of notation: here the indices carried by the field are fully antisymmetrized, rather than fully symmetrized as, for instance, in eq. (2.98).

2.3 Lagrangian field equations

In the previous sections, together with a set of gauge transformations and a set of constraints to be enforced on fields and gauge parameters, we first presented gauge invariant differential operators and then gauge invariant quadratic Lagrangians built upon them. We also saw that the null eigenfunctions of these differential operators propagate free massless modes. The next step is to ask whether the quadratic Lagrangians (2.94) and (2.134) give rise to field equations equivalent to the non-Lagrangian ones of eqs. (2.13) and (2.43), that set to zero these differential operators.

First of all one has to vary the Lagrangians in order to extract the corresponding field equations. In eq. (2.92) I showed that the Einstein-like tensor

$$
\mathcal{E}_{\mathrm{Bose}} \;=\; \frac{(-1)^{p}}{p\,!\,(\,p+1\,)\,!}\;\eta^{i_1 j_1}\ldots\eta^{i_p j_p}\,\mathcal{F}^{\,[\,p\,]}_{i_1 j_1,\,\ldots,\,i_p j_p} \tag{2.138}
$$

entering eq. (2.94) is self-adjoint, as first pointed out by Labastida [39]. As a consequence, up to total derivatives, the $\delta\,\varphi$ variation of the Lagrangians (2.94) can be cast in the form

$$
\delta\,\mathcal{L}_{\mathrm{Bose}} \;=\; \langle\;\delta\,\varphi\;,\;\mathcal{E}_{\mathrm{Bose}}\;\rangle\,. \tag{2.139}
$$

Furthermore, in eq. (2.133) I proved that similar considerations apply to the Rarita-Schwinger-like spinor-tensor

$$
\mathcal{E}_{\mathrm{Fermi}} \;=\; \frac{(-1)^{p+\frac{q\,(q+1)}{2}}}{p\,!\,q\,!\,(\,p+q+1\,)\,!}\;\eta^{i_1 j_1}\ldots\eta^{i_p j_p}\,\gamma^{\,k_1\ldots k_q}\,(\,\gamma^{\,[\,q\,]}\,\mathcal{S}^{\,[\,p\,]}\,)_{i_1 j_1,\,\ldots,\,i_p j_p\,;\,k_1\ldots k_q}\,. \tag{2.140}
$$

Again up to total derivatives, the $\delta\,\bar\psi$ variation of the Lagrangians (2.134) can thus be cast in the form

$$
\delta\,\mathcal{L}_{\mathrm{Fermi}} \;=\; \langle\;\delta\,\bar\psi\;,\;\mathcal{E}_{\mathrm{Fermi}}\;\rangle\,, \tag{2.141}
$$

where the $\delta\,\bar\psi$ dependence of $\delta\,\mathcal{L}$ is now manifest.

At this point one would be tempted to declare that the Lagrangian field equations are given by $\mathcal{E}_{\mathrm{Bose}} = 0$ and $\mathcal{E}_{\mathrm{Fermi}} = 0$. However, a bit of caution is needed, since we are dealing with *constrained* variations. In the Fang-Fronsdal case the free variation actually coincides with the constrained one, since the double traces of the Einstein-like tensors and the triple γ-traces of the Rarita-Schwinger-like tensors vanish. This is almost obvious recalling eqs. (2.68) and (2.97). On the other hand, as first[6] noticed in [40, 41], in the mixed-symmetry case some subtleties arise, since

$$
T_{(\,ij}\,T_{kl\,)}\,\mathcal{E}_{\mathrm{Bose}} \;\neq\; 0\,, \qquad\qquad T_{(\,ij}\,\gamma_{k\,)}\,\mathcal{E}_{\mathrm{Fermi}} \;\neq\; 0\,, \tag{2.142}
$$

even if

$$
T_{(\,ij}\,T_{kl\,)}\,\mathcal{F} \;=\; 0\,, \qquad\qquad T_{(\,ij}\,\gamma_{k\,)}\,\mathcal{S} \;=\; 0\,. \tag{2.143}
$$

---

[6]Actually, in [39] Labastida briefly commented on this issue, but he did not seem to fully realize the subtleties that I am about to describe.

At first sight this result could look strange. Nevertheless, in the mixed-symmetry case a new class of operators enters the (γ-)traces of expressions containing a bunch of invariant tensors. In fact, as shown in Appendix A, the operators

$$S^i{}_j \, \varphi \; \equiv \; \varphi_{\ldots, \, (\mu^i_1 \ldots \mu^i_{s_i} |, \, \ldots, \, | \mu^i_{s_i+1}) \, \mu^j_1 \ldots \mu^j_{s_j-1}, \, \ldots} \, . \tag{2.144}$$

are needed to close the algebra of the operators T_{ij} and η^{ij} or γ_i and γ^i. Looking at eq. (2.144) one can see that, for $i \neq j$, the net effect of the $S^i{}_j$ operators is to displace indices from one family to another. Their ubiquitous presence in the (γ-)traces of the $\mathcal{E}$ tensors can thus be manifested looking at very simple examples, such as

$$\eta^{\lambda\rho} \, \eta_{\lambda(\mu} \, \varphi_{\nu),\rho} \; = \; \varphi_{(\mu,\nu)} \, . \tag{2.145}$$

These new operators play a crucial role in leading to eqs. (2.142). In fact, they permit to spread the symmetrized indices also over the $S^i{}_j$, rather than only over products of (γ-)traces. This avoids the reconstruction of $T_{(ij} \, T_{kl)} \, \mathcal{F}$ or $T_{(ij} \, \gamma_{k)} \, \mathcal{S}$ in some terms, that as such are *not* annihilated by eqs. (2.143). To be more concrete, let me dwell further upon these subtleties in a simplified setting. For instance, Bose fields with only two index families already display this behavior, but their Einstein-like tensors take the relatively simple form

$$\mathcal{E} \; = \; \mathcal{F} \, - \, \frac{1}{2} \, \eta^{ij} \, T_{ij} \, \mathcal{F} \, + \, \frac{1}{36} \, \eta^{ij} \, \eta^{kl} \left(2 \, T_{ij} \, T_{kl} \, - \, T_{i\,(k} \, T_{l)\,j} \right) \mathcal{F} \, . \tag{2.146}$$

Enforcing the Labastida constraints on the field, the symmetrized double traces of $\mathcal{E}$ read

$$T_{(ij} \, T_{kl)} \, \mathcal{E} \; = \; - \, \frac{1}{36} \, S^m{}_{(i} \, S^n{}_{j|} \left(2 \, T_{|kl)} \, T_{mn} \, - \, T_{m|k} \, T_{l)n} \right) \mathcal{F} \, . \tag{2.147}$$

The combination of double traces of $\mathcal{F}$ entering this expression is actually $\{2,2\}$-projected in its family indices, so that it is orthogonal to the $\{4\}$-projected constraints (2.72). This explains why the result does not vanish even in the constrained theory. As a consequence, in order to obtain a field equation satisfying the same constraints as the gauge field a projection is needed. However, the projected field equations would take a very involved form, so that it is more convenient to reach them via the addition of some independent contributions to the naive equations, coming from suitable Lagrange multiplier terms to be added to the Lagrangians. I will postpone this discussion to Chapter 3, where these terms will naturally emerge in the unconstrained framework that I will present there.

Independently of the precise form taken by the Lagrangian field equations, some very general comments regarding their equivalence to the non-Lagrangian Labastida ones can be already made. Even in this respect, the paradigmatic example of linearized gravity provides important hints. In fact, in all space-time dimensions $D \neq 2$ the Lagrangian field equation

$$R_{\mu\nu} \, - \, \frac{1}{2} \, \eta_{\mu\nu} \, R_\lambda{}^\lambda \; = \; 0 \tag{2.148}$$

can be reduced to the non-Lagrangian one (2.1) combining it with its trace. On the other hand, in two dimensions a couple of pathologies appear: the Einstein-Hilbert

action actually vanishes, even if the linearized Ricci tensor $R_{\mu\nu}$ does not, while the formal expression (2.148) is traceless. The first statement becomes manifest writing the Lagrangian directly in terms of the fields as in eq. (2.95), where an antisymmetrization over three indices is present. Therefore, in two dimensions we should compare a non-Lagrangian setup where the field is bounded to satisfy the non-trivial equation of motion

$$R_{\mu\nu} = 0 \,, \tag{2.149}$$

and a Lagrangian setup, where no equations of motion are present. On the other hand, in $D = 2$ the little group is degenerate and no local degrees of freedom are present. A three-component symmetric tensor $h_{\mu\nu}$ can thus be forced to propagate no degrees of freedom in two different ways, either imposing a gauge invariant field equation, or stating directly that all its components are pure gauge. Indeed, in $D = 2$ an arbitrary shift of the field is a symmetry of the action! More precisely, the usual gauge transformations in $D = 2$ are accompanied by Weyl transformations, that in their linearized form read

$$\delta \, h_{\mu\nu} = \eta_{\mu\nu} \, \Omega \,. \tag{2.150}$$

When combined, these two sets of gauge transformations suffice to rebuild an arbitrary shift of the field. Even without resorting to eq. (2.95), the presence of a pathological behavior can be recognized looking at eq. (2.148) and considering $R_{\mu\nu}$ as an independent tensor. Since the combination in eq. (2.148) is traceless in $D = 2$, it is evident that the Lagrangian field equation is at least weaker than the non-Lagrangian one, because it displays an additional symmetry. In $D = 2$ the traceless part of the linearized Ricci tensor $R_{\mu\nu}$ then disappears altogether, and this brings together the two observations we made.

In conclusion, in some degenerate cases the Lagrangian field equations can not be directly equivalent to the non-Lagrangian ones, but at the same time new Weyl-like symmetries emerge. The rich structure of two-dimensional gravity naturally leads to wonder whether it is the only example presenting this pathological behavior. Actually, another example should be rather familiar, since the Rarita-Schwinger Lagrangian (2.136) manifestly vanishes in $D = 2$, even if $\mathcal{S}_\mu$ does not. However, these two-dimensional examples are the only pathological cases in the fully symmetric setting, for both Bose and Fermi fields.

Moving to the mixed-symmetry setup, the far more involved form of the Lagrangians makes it plausible that in particular space-time dimensions the tensors (2.138) and (2.140) could coincide by chance with some particular $o\,(D)$ components of the tensors $\mathcal{F}$ or $\mathcal{S}$. This is indeed the only way to obtain a new symmetry with respect to the non-Lagrangian field equations. The resulting $o\,(D)$ component could then also vanish when written in terms of the fields. A detailed discussion of these issues leading to the identification of a rich set of pathological cases will be presented in Chapter 4. On the other hand, we can already identify some mixed-symmetry fields behaving exactly like spin 2 or spin 3/2 fields in two space-time dimensions. Indeed, the actions (2.96) for $\{2^p, 1^q\}$-projected Bose fields manifestly vanish in $D \leq 2\,p + q$, while these tensors are available for $D \geq p + q$. In a similar fashion, the actions (2.137) for fully antisym-

metric Fermi fields with q vector indices manifestly vanish in $q \leq D \leq 2\,q$ even if the corresponding $\mathcal{S}$ spinor-tensors do not vanish for $D \geq q + 1$. In all these pathological cases the fields should not propagate any degrees of freedom[7], and this is compatible with the vanishing of their actions. Since any shift of the fields leaves invariant the actions, the usual gauge transformations must be accompanied by additional Weyl-like symmetries as in the case of two-dimensional gravity. Indeed, let me stress that the formal expressions (2.138) and (2.140) for the Einstein-like tensors can vanish when they are written in terms of the fields only if they coincide with a vanishing $o\,(D)$ component of the non-trivial kinetic tensors.

2.4 Irreducible fields

The gauge fields considered in the previous sections were in general only symmetric under the interchange of pairs of vector indices belonging to the same set, but did not possess any symmetry relating different sets. In other words, they were *reducible* $gl(D)$ tensors or spinor-tensors. Fields of this type naturally emerge in String Theory, where they are associated with products of bosonic oscillators, together with more general reducible $gl(D)$ tensors, where some sets of indices are actually antisymmetrized rather than symmetrized. In [40, 41] we also described how the previous results can be rewritten in an antisymmetric framework, while in the following I will briefly concentrate on Young projected $gl(D)$ spinor-tensors, still with manifestly symmetric index sets. They propagate the degrees of freedom of *irreducible* representations of the Lorentz group, up to possible (Majorana)Weyl projections in the fermionic case, and in this respect they are more along the lines of the usual low-spin examples. Since Young projections only affect in a non-trivial fashion the vector indices carried by the gauge fields, the changes occurring in this case were essentially the same for both Bose and Fermi fields.

The condition of irreducibility states that the symmetrization of a given line of the corresponding Young tableaux with any additional index belonging to one of the lower lines gives a vanishing result. This condition can be neatly formulated using the $S^i{}_j$ operators introduced in Appendix A as

$$S^i{}_j\,\varphi = 0\,, \qquad S^i{}_j\,\psi = 0\,, \qquad \forall\ i < j\,, \quad 2 \leq j \leq N\,, \qquad (2.151)$$

where N is the number of index families. The Fronsdal-Labastida kinetic operator (2.11) and the Fang-Fronsdal-Labastida kinetic operator (2.41) *commute with the $S^i{}_j$ operators.* In a similar fashion, the operators building the Einstein-like tensors (2.138) and the Rarita-Schwinger-like tensors (2.140) also commute with the $S^i{}_j$ operators,

---

[7]The gauge transformations, the field equations and, possibly, the constraints introduced until now force $gl(D)$ tensors and spinor-tensors to propagate the degrees of freedom of the $o\,(D-2)$ representation associated to the same Young tableau identifying them. On the other hand, a general result of representation theory (see, for instance, the first reference in [101], §10-6) implies that, for $O(n)$ groups, if the total number of boxes in the first two columns of a tableau exceeds n, the corresponding traceless tensor vanishes. A similar result also applies to spinor-tensor representations.

on account of the relations

$$[\, S^m{}_n \,, \, \eta^{i_1 j_1} \ldots \eta^{i_p j_p} \, \gamma^{k_1 \ldots k_q} \, T_{i_1 j_1} \ldots T_{i_p j_p} \, \gamma_{k_1 \ldots k_q} \,] \, = \, 0 \,, \qquad (2.152)$$

or more simply because they are "family" scalars. As a consequence, in the constrained theory Lagrangians and field equations take exactly the same form for reducible and irreducible gauge fields. On the other hand, for irreducible fields not all traces of φ or $\mathcal{F}$ and not all γ-traces of ψ or $\mathcal{S}$ remain independent, and therefore it is possible to reduce the number of those appearing in the field equations and in the Lagrangians, although the redundant description remains correct.

Moreover, in general irreducible fields involve fewer gauge parameters than their reducible counterparts, and how this number is reduced was already stated in the second reference of [33]: the relevant irreducible gauge parameters can be associated to all admissible Young diagrams obtained stripping one box from the diagram corresponding to the gauge field. The difference could be sizeable: if an irreducible gauge field is characterized by a Young diagram containing a number of identical rows, only a single gauge parameter is associated to all of them.

It is instructive to recover these results in an algebraic fashion, that will also prove very useful in Chapter 4. They follow in fact rather directly from the structure of the $S^i{}_j$ operators that implement the irreducibility condition. For instance, let me begin by considering how to select independent gauge parameters in the bosonic case, since no relevant changes occur in the fermionic case. First of all, eq. (2.151) must be preserved under gauge transformations, and this implies that

$$\partial^k \left(S^i{}_j \Lambda_k + \delta^i{}_k \Lambda_j \right) = 0 \,, \qquad i < j \,, \qquad (2.153)$$

which forces the terms between parentheses to define a gauge-for-gauge transformation:

$$S^i{}_j \Lambda_k + \delta^i{}_k \Lambda_j = \partial^l \Lambda^i{}_{[kl]j} \,, \qquad i < j \,. \qquad (2.154)$$

Up to the last irrelevant term, one thus obtains a set of constraints that are conveniently analyzed starting from the highest available value of k. In this case, in fact, i is necessarily less than k, being constrained to be less than j, so that eq. (2.154) reduces to the more familiar condition that the last gauge parameter in the chain, Λ_k, be irreducible,

$$S^i{}_j \Lambda_k = 0 \,. \qquad (2.155)$$

The remaining contributions can then be used to determine the other parameters corresponding to lower values of k in terms of this solution and of additional independent ones that emerge, one for each step, from the homogeneous parts of eqs. (2.155). Let me stress that the independent parameters solve further irreducibility conditions, and therefore can only exist if the corresponding Young diagrams are admissible. In conclusion, one thus obtains an irreducible gauge parameter for each admissible Young diagram built stripping one box from the original diagram for the gauge field, precisely as anticipated. Furthermore, the independent parameters lead by construction to irreducible gauge transformations where no explicit Young projectors are needed.

Let us see how this works, referring for example to an irreducible $\{4, 2\}$ gauge field, for which the conditions (2.154) read

$$S^1{}_2 \Lambda_1 + \Lambda_2 = 0 \, ,$$
$$S^1{}_2 \Lambda_2 = 0 \, . \qquad (2.156)$$

To begin with, Λ_1 could have three irreducible components, a $\{3, 2\}$, a $\{4, 1\}$ and a $\{5\}$, while Λ_2 could have a $\{4, 1\}$ and a $\{5\}$. Now the second of eqs. (2.156) eliminates directly the $\{5\}$ component of Λ_2, and then the first eliminates the $\{5\}$ component of Λ_1. The two $\{4, 1\}$ components are then connected by the first equation, while the $\{3, 2\}$ component is a zero mode of the first equation, and as such is not constrained. In space-time notation the conclusion is that, when adapted to an irreducible $\{4, 2\}$ field, the original reducible gauge transformation of the form (2.5) can be recast as

$$
\begin{aligned}
\delta \, \varphi_{\mu_1\mu_2\mu_3\mu_4 \, , \, \nu_1\nu_2} \; = \; & \partial_{(\mu_1} \Lambda^{(1)\{3,2\}}{}_{\mu_2\mu_3\mu_4) \, , \, \nu_1\nu_2} \\
& + \left(\partial_{(\mu_1} \Lambda^{(1)\{4,1\}}{}_{\mu_2\mu_3\mu_4) \, , \, \nu_1\nu_2} \; - \; \partial_{(\nu_1|} \Lambda^{(1)\{4,1\}}{}_{(\mu_1\mu_2\mu_3 \, , \, \mu_4) \, | \nu_2)} \right) \, . \quad (2.157)
\end{aligned}
$$

Let me stress again that no explicit Young projector is needed here, since both combinations are already $\{4, 2\}$ projected, as one can see enlarging the cycle that contains the μ indices to comprise also a ν index.

In full analogy, one can relate to each other the irreducible components of the (γ-)traces of φ and ψ, or of $\mathcal{F}$ and $\mathcal{S}$, simply computing the (γ-)traces of eq. (2.151) and moving to the left the $S^i{}_j$ operators. For instance, for Bose fields the relevant conditions are

$$S^m{}_n \, T_{i_1 j_1} \ldots T_{i_p j_p} \mathcal{F} + \sum_{i=1}^{p} \delta^m{}_{(i_n} T_{j_n)n} \prod_{r \neq n}^{p} T_{i_r j_r} \mathcal{F} \; = \; 0 \, , \qquad (2.158)$$

that can be solved resorting to the arguments presented for eq. (2.154). These identities should not be regarded as a mere curiosity, since they will play a crucial role in simplifying some expressions that we shall come across in Chapter 4.

Chapter 3

Unconstrained theory

In the previous chapter I have presented second order field equations and Lagrangians for massless Bose fields of mixed symmetry and first order field equations and Lagrangians for massless Fermi fields of mixed symmetry. The field content was the minimal one leading to a covariant description, since the representations of the Lorentz group associated with Young tableaux $\{s_1, s_2, \ldots\}$ were described via single Young-projected multi-symmetric tensors $\varphi_{\mu_1 \ldots \mu_{s_1}, \nu_1 \ldots \nu_{s_2}, \ldots}$ or spinor-tensors $\psi^{\alpha}{}_{\mu_1 \ldots \mu_{s_1}, \nu_1 \ldots \nu_{s_2}, \ldots}$. As repeatedly stressed, however, in the formulation presented in Chapter 2 fields and gauge parameters must satisfy some algebraic (γ-)trace *constraints*. In this chapter I will describe how one can recover a more conventional description, removing the constraints at the price of introducing a number of *auxiliary fields*. Aside from bringing these systems closer to their "low-spin" counterparts, this operation could well remove some complications in the treatment of these complicated models at the interacting level. With this aim in mind, I shall present a way to forego the need for constraints preserving as much as possible the simplicity of the construction developed in the previous chapter. Thus, I will try to contain the enhancement of the field content, introducing at most one auxiliary field for each constraint. To do that, I shall follow the main lines of the strategy adopted by Francia and Sagnotti to build their local unconstrained formulation for fully symmetric fields [42, 56, 57, 51]. This "minimal" procedure will lead to higher-derivative terms involving some of the new auxiliary fields, but this does not create any problem since the extra derivatives appear in pure-gauge contributions. At any rate, I will also show how to remove them enlarging a bit the field content, via additional auxiliary fields whose number only depends on the number of index families, and not on the total number of space-time indices. These unconstrained formulations will be obtained reexamining in a critical fashion the consequences of the principle of gauge invariance underlying the constrained theory.

Let me close these cursory remarks by recalling that the introduction of these auxiliary fields has another important benefit. As anticipated in the Introduction, eliminating them in terms of the basic fields φ or ψ one indeed ends up with the geometrical non-local formulation of the dynamics [48, 52, 42, 50]. While I will not deal with this point, I would like to stress that the formulation I am going to present is in this sense the minimal local one allowing a contact with the *higher-spin geometry*.

3.1 Minimal Lagrangian formulation

In the Lagrangian theory of Chapter 2, the Labastida constraints can be eliminated in a "minimal" way in two steps:

- the constraints (2.12) or (2.42) on the gauge parameters can be eliminated via the introduction of at most one compensator field for each constraint;

- the constraints (2.29) or (2.59) on the gauge fields can be eliminated via the introduction of at most one Lagrange multiplier for each constraint.

This program can be pursued following similar steps for both Bose and Fermi fields. Nevertheless, the two types of fields will be treated separately as in the previous chapter, also because Fermi fields involve again some additional subtleties related to the presence of γ-matrices. At any rate, the reader is invited to notice the similarities between the two setups.

3.1.1 Bose fields

In the previous chapter I showed that the Labastida constraints

$$T_{(ij}\Lambda_{k)} = 0 \tag{3.1}$$

lead to second order gauge invariant field equations. Indeed, computing the gauge variation of the Fronsdal-Labastida tensor (2.11) one obtains

$$\delta\,\mathcal{F} = \frac{1}{6}\,\partial^i\partial^j\partial^k\,T_{(ij}\Lambda_{k)}\,. \tag{3.2}$$

I also showed that the Labastida constraints

$$T_{(ij}\,T_{kl)}\,\varphi = 0 \tag{3.3}$$

lead to gauge invariant Lagrangians, since

$$\partial_i\,\mathcal{F} - \frac{1}{2}\,\partial^j\,T_{ij}\,\mathcal{F} = -\frac{1}{12}\,\partial^j\partial^k\partial^l\,T_{(ij}\,T_{kl)}\,\varphi\,, \tag{3.4}$$

and Bianchi identities free of the "classical anomaly" appearing on the right hand-side of eq. (3.4) guarantee the gauge invariance of the Lagrangians (2.94).

Looking at eq. (3.2), it seems that the simplest solution to overcome the constraints (3.1) would be to introduce for each of them one auxiliary field transforming as

$$\delta\,\alpha_{ijk} = \frac{1}{3}\,T_{(ij}\Lambda_{k)}\,. \tag{3.5}$$

This was indeed the strategy followed in [42, 56, 57, 51] in order to obtain an unconstrained local formulation for fully symmetric Bose fields. Consistently with the

notation adopted for the gauge parameters Λ_i, the auxiliary fields α_{ijk} would carry one less space-time index in the i-th, j-th and k-th families with respect to the gauge field φ. Introducing these auxiliary fields, unconstrained gauge invariant field equations could be obtained defining the new kinetic tensors

$$\mathcal{A} \equiv \mathcal{F} - \frac{1}{2} \partial^i \partial^j \partial^k \alpha_{ijk} \,, \tag{3.6}$$

and forcing them to vanish[1]. Furthermore, $\mathcal{A}$ would satisfy the Bianchi identities

$$\partial_i \mathcal{A} - \frac{1}{2} \partial^j T_{ij} \mathcal{A} = -\frac{1}{4} \partial^j \partial^k \partial^l \mathcal{C}_{ijkl} \,, \tag{3.7}$$

where the tensors

$$\mathcal{C}_{ijkl} = \frac{1}{3} \left\{ T_{(ij} T_{kl)} \varphi - 3 \partial_{(i} \alpha_{jkl)} - \frac{3}{2} \partial^m \left(T_{(ij} \alpha_{kl)m} - T_{m(i} \alpha_{jkl)} \right) \right\} \tag{3.8}$$

provide a gauge invariant completion of the Labastida constraints (3.3).

On the other hand, as anticipated in Chapter 2, the Labastida constraints (3.1) or (3.3) are *not* independent. As a consequence, gauge invariant combinations of the α_{ijk} exist. This means that, if regarded as independent, they would *not* be pure-gauge fields as the single compensator introduced for a given spin s field in [42, 56, 57, 51]. This is however a crucial requirement, since it enables one to eliminate the auxiliary fields by a partial gauge fixing, while still maintaining the constrained gauge symmetry needed to force the gauge fields to propagate the proper massless modes. Before showing how it is possible to overcome the problem, let me support these arguments by displaying explicitly a gauge invariant combination of the α_{ijk}. In order to do that, it is convenient to first exploit the linear dependence of the Labastida constraints on the fields. In Section 2.1.1 we have already seen an example of this fact, and in the present setup the result shown in eq. (2.32) implies that the φ part of a $\{5, 1\}$-projected combination of traces of the $\mathcal{C}_{ijkl}$ tensors vanishes. On the contrary, there are no group theoretical arguments forcing its α_{ijk} portion to vanish as well. Moreover, if it does not vanish, it must define a gauge invariant combination of these fields, since the $\mathcal{C}_{ijkl}$ are gauge invariant. A direct computation finally shows that the divergence terms in $Y_{\{5,1\}} T_{mn} \mathcal{C}_{ijkl}$ actually vanish, while the gradient ones do not. In conclusion,

$$Y_{\{5,1\}} T_{mn} \mathcal{C}_{ijkl} = -\frac{3}{2} Y_{\{5,1\}} \partial^p T_{mn} \left(T_{(ij} \alpha_{kl)p} - T_{p(i} \alpha_{jkl)} \right) , \tag{3.9}$$

where the $\{5, 1\}$ projection is applied only on the free indices and not on the dumb index p. The combination of double traces of the α_{ijk} fields appearing in eq. (3.9) is

---

[1]As already recalled in the Introduction, for a spin 3 field an equation of motion of this form was first proposed by Schwinger in [55], and independently rediscovered by Francia and Sagnotti in [48]. This result was then extended to arbitrary symmetric fields in [42]. The other papers already recalled [56, 57, 51] present a Lagrangian extension of the non-Lagrangian field equations of the form (3.6) in the fully symmetric case.

available in general from two-families onward and it is gauge invariant, as the reader can verify using its explicit expression

$$
\begin{aligned}
Y_{\{5,1\}} \, T_{mn} \, \mathcal{C}_{ijkl} \; \sim \; \partial^p \Big\{ & \left(T_{n(i} \, T_{jk} \, \alpha_{lm)p} + T_{m(i} \, T_{jk} \, \alpha_{ln)p} \right) \\
- \, 2 & \left(T_{p(i|} \, T_{n|j} \, \alpha_{klm)} + T_{p(i|} \, T_{m|j} \, \alpha_{kln)} \right) + \left(T_{pn} \, T_{(ij} \, \alpha_{klm)} + T_{pm} \, T_{(ij} \, \alpha_{kln)} \right) \\
- \, 2 & \left(T_{(ij} \, T_{kl} \, \alpha_{m)np} + T_{(ij} \, T_{kl} \, \alpha_{n)mp} \right) + \left(T_{p(i} \, T_{jk} \, \alpha_{lm)n} + T_{p(i} \, T_{jk} \, \alpha_{ln)m} \right) \Big\} .
\end{aligned} \tag{3.10}
$$

Let me close this digression by noticing that the linear dependence of the Labastida constraints does not manifest itself in simple mixed-symmetry models with small numbers of space-time indices. In fact, with fewer traces of the α_{ijk} it is not possible to obtain gauge invariant combinations.

As I have argued before, the presence of gauge invariant combinations of the α_{ijk} fields definitely shows that one cannot consider them independent. The most economical solution to the problem consists in regarding them as "composite" objects, introducing a set of fundamental compensators that transform as

$$
\delta \, \Phi_i \; = \; \Lambda_i \, . \tag{3.11}
$$

On the other hand, the new compensators enter field equations and Lagrangians only through their symmetrized traces, so that they can be gauged away without spoiling the Labastida constrained gauge symmetry. For this reason, for brevity in the following I will often abide by the notation

$$
\alpha_{ijk} \; \equiv \; \frac{1}{3} \, T_{(ij} \, \Phi_{k)} \, , \tag{3.12}
$$

explicitly displaying the Φ dependence of the "composite compensators" α_{ijk} only where it is particularly relevant.

Furthermore, the Φ_i compensators naturally emerge when trying to enlarge the set of gauge transformations via a Stueckelberg-like realization of the missing unconstrained ones. In fact, this can be done shifting the gauge field as

$$
\varphi \; \rightarrow \; \varphi \, - \, \partial^i \, \Phi_i \, . \tag{3.13}
$$

Notice that the gradients present in eq. (3.13) imply that the compensators cannot have the same dimension of the gauge field. This fact is at the origin of the (harmless) *higher-derivative* term that appears in the new kinetic tensor of the unconstrained theory. Either directly or via the shift (3.13), one can indeed cast it in the form

$$
\mathcal{A} \; \equiv \; \mathcal{F} \, - \, \frac{1}{6} \, \partial^i \partial^j \partial^k \, T_{(ij} \, \Phi_{k)} \, , \tag{3.14}
$$

and define the equation of motion

$$
\mathcal{A} \; = \; 0 \, , \tag{3.15}
$$

that is left invariant under the *unconstrained* gauge transformations

$$
\begin{aligned}
\delta \, \varphi &= \partial^i \, \Lambda_i \, , \\
\delta \, \Phi_i &= \Lambda_i \, .
\end{aligned} \tag{3.16}
$$

I will comment more on the issue of higher derivatives in Section 3.2.1, where I will show how to obtain a two-derivative unconstrained Lagrangian formulation for Bose fields, joining the Φ_i compensators with other auxiliary fields of the same dimension as the gauge field φ. At any rate, when working with the fundamental compensators Φ_i, the $\mathcal{C}_{ijkl}$ tensors appearing on the right-hand sides of the Bianchi identities (3.7) take the far simpler form

$$\mathcal{C}_{ijkl} = \frac{1}{3} \, T_{(ij} \, T_{kl)} \, (\, \varphi \, - \, \partial^m \, \Phi_m \,) \, , \tag{3.17}$$

as should be clear looking at the Stueckelberg method to introduce the Φ_i. Furthermore, eq. (3.12) was introduced to eliminate all gauge invariant combinations of the α_{ijk} fields, so that it implies

$$Y_{\{5,1\}} \, T_{mn} \, \mathcal{C}_{ijkl} \, (\, \varphi \, , \Phi_i \,) \, = \, 0 \, . \tag{3.18}$$

We have thus seen how to eliminate the constraints (3.1) on the gauge parameters and we can now move on and analyze the role of the constraints (3.3) on the fields. Since they are related to the Lagrangians, whose complexity increases with the number of index families, let me return for a while to the fully symmetric case in order to better delineate the strategy that I will follow. Moreover, I will begin by keeping the constraint on the gauge parameter in order to focus only on the elimination of the double trace constraint on φ. Using the Bianchi identity (2.26) in eq. (2.70), in this framework the gauge variation of the Fronsdal Lagrangian can be cast in the form

$$\delta \, \mathcal{L} \, = \, - \, 3 \, \binom{s}{4} \, \partial \cdot \partial \cdot \partial \cdot \Lambda^{\mu_1 \dots \mu_{s-4}} \, \varphi_{\mu_1 \dots \mu_{s-4} \lambda \rho}{}^{\lambda \rho} \, . \tag{3.19}$$

Instead of imposing the constraint (2.25) on the gauge field, one can thus cancel it introducing a Lagrange multiplier with $(s-4)$ space time indices that transforms as

$$\delta \, \beta_{\mu_1 \dots \mu_{s-4}} \, = \, \partial \cdot \partial \cdot \partial \cdot \Lambda_{\mu_1 \dots \mu_{s-4}} \, . \tag{3.20}$$

In fact, one can couple it to the double trace of the gauge field and add the new term to the Fronsdal Lagrangian:

$$\mathcal{L} \, = \, \frac{1}{2} \, \varphi^{\mu_1 \dots \mu_s} \left(\mathcal{F}_{\mu_1 \dots \mu_s} \, - \, \frac{1}{2} \, \eta_{(\mu_1 \mu_2} \, \mathcal{F}_{\mu_3 \dots \mu_s) \lambda}{}^{\lambda} \right)$$
$$+ \, 3 \, \binom{s}{4} \, \beta^{\mu_1 \dots \mu_{s-4}} \, \varphi_{\mu_1 \dots \mu_{s-4} \lambda \rho}{}^{\lambda \rho} \, . \tag{3.21}$$

The resulting Lagrangian contains unconstrained fields, but is still gauge invariant only under constrained gauge transformations. One can also relax this requirement introducing suitable compensator terms that lead to the Lagrangian[2] of [56, 57, 51].

---

[2]Actually the Lagrangians of [56, 57, 51] were presented in the far more effective index-free convention which inspired the one that I am using to describe mixed-symmetry fields. On the other hand, the space-time indices are here displayed in order to keep this cursory example immediately accessible.

Thus, in the fully symmetric setting a couple of auxiliary fields suffices to eliminate the constraints for any value of the spin s.

In the mixed-symmetry case I will now attack the problem following the same strategy. To begin with, let me continue for a while to discard the terms proportional to the constraints on the gauge parameters (3.1). As we have seen in eq. (2.82), the gauge variation of the Labastida Lagrangian (2.94) thus reads

$$
\delta \mathcal{L} = \sum_{p=0}^{N} \frac{(p+1)\, k_{p+1}}{2^{p+1}} \langle\, Y_{\{2^p,1\}}\, T_{i_1 j_1} \ldots T_{i_p j_p}\, \Lambda_k\, ,
$$
$$
(p+2)\, Y_{\{2^p,1\}}\, \partial_k\, \mathcal{F}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p} \,-\, \partial^l\, \mathcal{F}^{[p+1]}{}_{i_1 j_1, \ldots, i_p j_p, kl}\, \rangle\, , \qquad (3.22)
$$

where the k_p are the coefficients of eq. (2.89). In order to continue one has to identify the unconstrained analogues of the identities (2.87) that we used at this stage in the constrained setting. As a starting point one can consider again the multiple traces of the Bianchi identities of eq. (2.84), but some of the manipulations that lead from them to eq. (2.87) were based on the constraints on $\mathcal{F}$ of eq. (2.72). These are clearly not available here, since they were induced by the constraints on φ that we are eliminating. Some care is thus in order, but it is still true that the divergence terms of eq. (2.84) can be combined when computing a two-column projection. Thus, the $\{2^p, 1\}$ projection of a product of p traces of the Bianchi identities reads

$$
(p+2)\, Y_{\{2^p,1\}}\, \partial_k\, T_{i_1 j_1} \ldots T_{i_p j_p}\, \mathcal{F} \,-\, Y_{\{2^p,1\}}\, \partial^l\, T_{i_1 j_1} \ldots T_{i_p j_p}\, T_{kl}\, \mathcal{F}
$$
$$
= -\frac{1}{6}\, Y_{\{2^p,1\}}\, T_{i_1 j_1} \ldots T_{i_p j_p}\, \partial^l \partial^m \partial^n\, T_{(kl}\, T_{mn)}\, \varphi \qquad (3.23)
$$

To compare it with the gauge variation, one must now reconstruct the two-column projections of the traces of $\mathcal{F}$. This is the point where in Section 2.2.1 we resorted for brevity to the constraints (2.72). On the other hand, one can notice that

$$
Y_{\{2^p,1\}}\, \partial_k\, T_{i_1 j_1} \ldots T_{i_p j_p}\, \mathcal{F} \,=\, Y_{\{2^p,1\}}\, \partial_k\, \mathcal{F}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p}\, , \qquad (3.24)
$$

since all other terms in the expansion

$$
T_{i_1 j_1} \ldots T_{i_p j_p}\, \mathcal{F} \,=\, Y_{\{2^p\}}\, T_{i_1 j_1} \ldots T_{i_p j_p}\, \mathcal{F} \,+\, Y_{\{4,2^{p-1}\}}\, T_{i_1 j_1} \ldots T_{i_p j_p}\, \mathcal{F} \,+\, \ldots \qquad (3.25)
$$

would be annihilated by the $\{2^p, 1\}$ projection. Thus, the result that was obtained via the constraints (2.72) is still correct. In a similar fashion, the presence of identical tensors promotes the $\{2^p, 1\}$ projection in the gradient term to a $\{2^{p+1}\}$ projection comprising also the lower index l. We can thus conclude that

$$
(p+2)\, Y_{\{2^p,1\}}\, \partial_k\, \mathcal{F}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p} \,-\, \partial^l\, \mathcal{F}^{[p+1]}{}_{i_1 j_1, \ldots, i_p j_p, kl}
$$
$$
= -\frac{1}{6}\, Y_{\{2^p,1\}}\, T_{i_1 j_1} \ldots T_{i_p j_p}\, \partial^l \partial^m \partial^n\, T_{(kl}\, T_{mn)}\, \varphi\, , \qquad (3.26)
$$

where for brevity I avoided to normal order the terms on the right-hand side. Notice that simple considerations fix again the form of the result, even without knowing

explicitly the form taken by the projections involved. Using these identities the gauge variation (3.22) can be eventually cast in the form

$$\delta\mathcal{L} = \sum_{p=0}^{N} \frac{(p+1)\,k_{p+1}}{48} \times$$
$$\times \langle\, \partial_{(k}\partial_l\partial_{m|}\,\eta^{i_1j_1}\dots\eta^{i_pj_p}\,Y_{\{2^p,1\}}\,T_{i_1j_1}\dots T_{i_pj_p}\,\Lambda_{|n)}\,,\,T_{(kl}\,T_{mn)}\,\varphi\,\rangle\,. \quad (3.27)$$

In analogy with the fully symmetric case, it is thus possible to deal with *unconstrained fields* introducing the Lagrangians

$$\mathcal{L} = \frac{1}{2}\,\langle\,\varphi\,,\,\sum_{p=0}^{N}\frac{(-1)^p}{p!\,(p+1)!}\,\eta^{i_1j_1}\dots\eta^{i_pj_p}\,\mathcal{F}^{[p]}{}_{i_1j_1;\dots;i_pj_p}\,\rangle$$
$$+\frac{1}{24}\,\langle\,\beta_{ijkl}\,,\,T_{(ij}\,T_{kl)}\,\varphi\,\rangle\,, \quad (3.28)$$

that are left invariant under *constrained gauge transformations* provided the Lagrange multipliers β_{ijkl} transform as

$$\delta\beta_{ijkl} = \frac{1}{2}\sum_{p=0}^{N}\frac{(-1)^p}{p!\,(p+2)!}\,\partial_{(i}\partial_j\partial_{k|}\,\eta^{i_1j_1}\dots\eta^{i_pj_p}\,Y_{\{2^p,1\}}\,T_{i_1j_1}\dots T_{i_pj_p}\,\Lambda_{|l)}\,. \quad (3.29)$$

Indeed, let me recall that in eq. (2.30) we also saw that $T_{(ij}\,T_{kl)}\,\varphi$ is left invariant by these transformations. Notice that the new term added to the Lagrangian (2.94) enforces on-shell the Labastida constraints (3.3) via the equations of motion of the Lagrange multipliers. Moreover, the result is consistent with that in eq. (3.20), since in the single-family case the $\{2^p,1\}$ projections are clearly not available while the $p=0$ term in the sum reproduces eq. (3.20).

The linear dependence of the Labastida constraints leads again to some subtleties: here it induces a symmetry of the Lagrangian (3.28) under shifts of the Lagrange multipliers of the form

$$\delta\beta_{ijkl} = \eta^{m_1n_1}\dots\eta^{m_kn_k}\,Y_{\text{odd}}\,L_{ijkl,\,m_1n_1,\dots,\,m_kn_k}\,. \quad (3.30)$$

These transformations are characterized by Y_{odd}, that must be a Young projector associated to a tableau having an odd number of boxes in some rows. To be more concrete, let me consider the simplest example

$$\delta\beta_{ijkl} = \eta^{mn}\,Y_{\{5,1\}}\,L_{ijkl,\,mn} \quad (3.31)$$

that is already visible for two-family fields, and from which all other transformations can be extracted as particular cases. The variation of the Lagrangian (3.28) under the shifts (3.31) is

$$\delta\mathcal{L} = \frac{1}{48}\,\langle\,L_{ijkl,\,mn}\,,\,Y_{\{5,1\}}\,T_{mn}\,T_{(ij}\,T_{kl)}\,\varphi\,\rangle\,, \quad (3.32)$$

and vanishes on account of the repeatedly stressed properties of products of identical T_{ij} tensors (see for instance eq. (2.32)). A similar cancelation clearly accompanies a

generic transformation of the form (3.30). As a consequence, some combinations of traces of the β_{ijkl} Lagrange multipliers are left undetermined by the field equations, but they can be anyhow gauged away exploiting the new symmetry. Moreover, this observation also implies that part of the gauge transformations (3.29) is actually ineffective in the Lagrangian (3.28). One could indeed normal-order eq. (3.29) commuting the η^{ij} operators through the divergences, and a generic term of the sum would still start with a bunch of η^{ij} operators. When considering the gauge variation of the Lagrangian, these act as trace operators on $T_{(ij}T_{kl)}\varphi$. As we have seen, their Young projections with odd numbers of boxes in some rows vanish. On the other hand, projections of this kind are admitted by the terms appearing in eq. (3.29). Indeed, in general their lower family indices carry either components that can couple to the resulting products of traces or components that annihilate them. This can be recognized looking at the manifest symmetries of these terms, and selecting the compatible structures according to

$$\{2^p, 1\} \otimes \{3\} = \{5, 2^{p-1}, 1\} \oplus \{4, 2^p\} \oplus \dots. \tag{3.33}$$

This observation will play a role in the definition of a proper Stueckelberg-like shift for the Lagrangian theory, and to this effect more details and examples can be found in [40].

To build the Lagrangians (3.28), I revisited the gauge variation of the constrained Lagrangians (2.94). As a consequence, the Einstein-like tensors still only contain the projected traces of $\mathcal{F}$ that are not *fully* proportional to symmetrized double traces of φ. On the other hand, if one relaxes the constraints some terms proportional to the double traces of φ pop up in eq. (2.94). These are at the origin of the "classical anomalies" that in Section 2.2.1 we canceled imposing the constraints. This observation suggests an alternative approach to the elimination of the constraints that leads to a different presentation of the Lagrangians for unconstrained fields. In fact, one can directly extract the terms proportional to $T_{(ij}T_{kl)}\varphi$ in the Lagrangians (2.94). This can be done introducing the quantities

$$\widehat{\mathcal{F}}^{[p]}{}_{i_1 j_1, \dots, i_p j_p} = \mathcal{F}^{[p]}{}_{i_1 j_1, \dots, i_p j_p} - \frac{1}{2}\, \partial^k \partial^l\, Y_{\{4, 2^{p-1}\}}\, T_{i_1 j_1} \dots T_{i_p j_p} T_{kl}\, \varphi\,, \tag{3.34}$$

that coincide with the two-column projected traces of $\mathcal{F}$ up to the terms proportional to the symmetrized double traces of φ that the latter contain (see for instance eq. (B.22)). One can eventually use them to obtain the gauge invariant Lagrangians

$$\mathcal{L} = \frac{1}{2}\, \langle\, \varphi\,,\, \sum_{p=0}^{N} \frac{(-1)^p}{p!\,(p+1)!}\, \eta^{i_1 j_1} \dots \eta^{i_p j_p}\, \widehat{\mathcal{F}}^{[p]}{}_{i_1 j_1; \dots; i_p j_p}\, \rangle$$

$$+ \frac{1}{24}\, \langle\, \mathcal{B}_{ijkl}\,,\, T_{(ij}T_{kl)}\,\varphi\,\rangle\,, \tag{3.35}$$

where the $\mathcal{B}_{ijkl}$ are *gauge invariant* Lagrange multipliers enforcing on-shell the double trace constraints. These Lagrangians are not written in terms of the multiple traces of $\mathcal{F}$ but their Einstein-like tensors are manifestly self-adjoint, since they contain *only* the non-trivial terms that survive in eq. (2.92) when one imposes the constraints.

Since the bare φ terms in eq. (3.34) can be cast in a form manifestly proportional to $T_{(ij} T_{kl)} \varphi$, the new Lagrangians are clearly related to those presented in eq. (3.28) via a field redefinition of the Lagrange multipliers. In this respect they do not introduce any real novelty, but this different viewpoint can be of some interest nonetheless. For instance, it manifestly shows that the complicated gauge transformation of eq. (3.29) are not an intrinsic feature, since they can be eliminated via a judicious choice of the terms quadratic in φ. Finally, notice that the new tensors entering the Lagrangians satisfy the "anomaly-free" Bianchi identities

$$(p+2)\, Y_{\{2^p,1\}}\, \partial_k\, \widehat{\mathcal{F}}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p} \; - \; \partial^l\, \widehat{\mathcal{F}}^{[p+1]}{}_{i_1 j_1, \ldots, i_p j_p, kl} \; = \; 0\,, \qquad (3.36)$$

as the gauge invariance of eq. (3.35) suggests.

To summarize, in eq. (3.15) I have presented non-Lagrangian field equations that are left invariant by the unconstrained gauge transformations (3.16), while in eqs. (3.28) and (3.35) I have presented Lagrangians for unconstrained gauge fields that are still invariant only under constrained gauge transformations. We can now combine these two improvements and build a fully unconstrained Lagrangian theory. As a matter of principle, this can be done simply refining the Stueckelberg-like shift of eq. (3.13). For instance, performing the shift

$$\varphi \;\to\; \varphi \,-\, \partial^i\, \Phi_i\,,$$

$$\beta_{ijkl} \;\to\; \beta_{ijkl} \,-\, \frac{1}{2} \sum_{p=0}^{N} \frac{k_p}{p+2}\, \partial_{(i} \partial_j \partial_{k|}\, \eta^{i_1 j_1} \ldots \eta^{i_p j_p}\, Y_{\{2^p,1\}}\, T_{i_1 j_1} \ldots T_{i_p j_p} \Phi_{|l)} \quad (3.37)$$

in eq. (3.28), one ends up with a fully unconstrained Lagrangian, where the compensators Φ_i only appear through their symmetrized traces. The shift of the Lagrange multipliers is crucial in order to fulfill this important condition, and it maps the β_{ijkl} into their gauge invariant completions. Since the Stueckelberg-like shift of φ has the same structure as a gauge transformation, the Φ_i portion of the β_{ijkl} shifts cancels the terms that in the gauge variation would be annihilated via the Bianchi identities. In fact, in the present context they would not be expressible in terms of $T_{(ij} \Phi_{k)}$. This approach makes it clear that a fully unconstrained Lagrangian exists, and will prove useful when dealing with its field equations. On the other hand, it does not provide a clear hint on the structure of the compensator terms, and further computations are needed. Since the steps involved are exactly the same, rather than dealing with the Stueckelberg-like shifts (3.37) I will now fix the precise form of the compensator terms reconsidering the gauge variation of the Lagrangian (3.28). I will thus compensate the terms proportional to $T_{(ij} \Lambda_{k)}$ adding new contributions to the Lagrangian, rather than taking advantage of the constraints (3.1).

To fix the structure of the fully unconstrained Lagrangians one can thus start from

$$\mathcal{L}_0 \;=\; \frac{1}{2} \left\langle\, \varphi\,,\; \sum_{p=0}^{N} \frac{(-1)^p}{p!\,(p+1)!}\, \eta^{i_1 j_1} \ldots \eta^{i_p j_p}\, \mathcal{A}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p}\, \right\rangle$$

$$+\, \frac{1}{8} \left\langle\, \beta_{ijkl}\,,\; \mathcal{C}_{ijkl}\, \right\rangle\,, \qquad (3.38)$$

that correspond to eq. (3.28) up to the substitution of $\mathcal{F}$ with $\mathcal{A}$ and of $T_{(ij}T_{kl)}\varphi$ with $\mathcal{C}_{ijkl}$. Since the tensors $\mathcal{A}$ and $\mathcal{C}_{ijkl}$ of eqs. (3.14) and (3.17) are gauge invariant under unconstrained gauge transformations, the gauge variation of the trial Lagrangian (3.38) can be cast in the form

$$\delta\mathcal{L} = -\sum_{p=0}^{N}\frac{k_p}{2^{p+1}}\langle\, Y_{\{3,2^{p-1}\}}\,T_{i_1 j_1}\dots T_{i_p j_p}\,\Lambda_k\,,\, Y_{\{3,2^{p-1}\}}\,\partial_k\,\mathcal{A}^{[p]}{}_{i_1 j_1,\dots,i_p j_p}\,\rangle\,. \quad (3.39)$$

These are indeed the terms that were discarded in eq. (3.22). As I have already stressed in Section 2.2.1, the left entries of the scalar products are proportional to $T_{(ij}\Lambda_{k)}$ on account of the $\{3,2^{p-1}\}$ projection that they carry. However, we now have to refine this argument, since we must identify the precise relation to compensate them adding new terms to the Lagrangians. The needed techniques are similar to those adopted to manipulate the two-column projections of the traces of the Bianchi identities. On the other hand, the chain of steps leading to the result is more involved, and in the following I will only quote the final result. The interested reader can find more details in Appendix C. At any rate, these manipulations recast the gauge variation (3.39) in the form

$$\delta\mathcal{L} = -\sum_{p=0}^{N}\frac{p\,k_p}{2^{p+1}(p+2)}\langle\, T_{i_1 j_1}\dots T_{i_p j_p}\,\Lambda_k\,,\, \partial_{(k}\,\mathcal{A}^{[p]}{}_{i_1 j_1),i_2 j_2,\dots,i_p j_p}\,\rangle\,, \quad (3.40)$$

where now only symmetrized traces of the gauge parameters appear. This suffices to identify the structure of the fully unconstrained Lagrangians

$$\mathcal{L} = \frac{1}{2}\langle\, \varphi\,,\, \sum_{p=0}^{N}\frac{(-1)^p}{p!\,(p+1)!}\,\eta^{i_1 j_1}\dots\eta^{i_p j_p}\,\mathcal{A}^{[p]}{}_{i_1 j_1,\dots,i_p j_p}\,\rangle$$

$$-\frac{1}{4}\langle\, \alpha_{ijk}\,,\, \sum_{p=0}^{N-1}\frac{(-1)^p}{p!\,(p+3)!}\,\eta^{i_1 j_1}\dots\eta^{i_p j_p}\,\partial_{(i}\,\mathcal{A}^{[p+1]}{}_{jk),i_1 j_1,\dots,i_p j_p}\,\rangle$$

$$+\frac{1}{8}\langle\, \beta_{ijkl}\,,\, \mathcal{C}_{ijkl}\,\rangle\,. \quad (3.41)$$

Notice that these Lagrangians possess the same symmetry under shifts of the Lagrange multipliers as their counterparts in eq. (3.28). Actually, in eq. (3.18) we have seen that, when working in terms of the compensators Φ_i, the $\mathcal{C}_{ijkl}$ tensors satisfy the same properties as the combinations $T_{(ij}T_{kl)}\varphi$.

In the previous pages I have extended the constrained Labastida framework proposing the Lagrangians (3.28) and (3.35) for unconstrained fields still subjected to constrained gauge transformations, and the Lagrangians (3.41) for fully unconstrained fields. I can now close this section presenting their field equations. Those coming from the Lagrangians (3.35) can be obtained almost by inspection, since I stressed that their Einstein-like tensors are self-adjoint as in the constrained theory. On the other hand, the subtleties discussed in Section 2.3 do not arise in this framework since the fields are *unconstrained*. The equations of motion following from the Lagrangian

(3.35) thus read

$$E_\varphi \; : \; \sum_{p=0}^{N} \frac{(-1)^p}{p!\,(p+1)!} \, \eta^{i_1 j_1} \cdots \eta^{i_p j_p} \, \mathcal{F}^{[p]}{}_{i_1 j_1,\,\ldots,\,i_p j_p} + \frac{1}{2}\, \eta^{ij}\, \eta^{kl}\, \mathcal{B}_{ijkl} = 0 \,, \quad (3.42)$$

$$E_\mathcal{B} \; : \; T_{(ij}\, T_{kl)}\, \varphi = 0 \,, \tag{3.43}$$

where the projections that I discussed in Section 2.3 can be recovered in E_φ via the elimination of the $\mathcal{B}_{ijkl}$ tensors. This is the first example where the removal of the constraints simplify, technically at least, the discussion of mixed-symmetry fields.

The equations of motion following from the Lagrangians (3.28) take a similar form, that reads

$$E_\varphi \; : \; \sum_{p=0}^{N} \frac{(-1)^p}{p!\,(p+1)!} \, \eta^{i_1 j_1} \cdots \eta^{i_p j_p} \, \mathcal{F}^{[p]}{}_{i_1 j_1,\,\ldots,\,i_p j_p} + \frac{1}{2}\, \eta^{ij}\, \eta^{kl}\, \mathcal{B}_{ijkl}$$

$$- \frac{1}{4} \sum_{p=0}^{N} \frac{(-1)^p}{p!\,(p+1)!} \, \eta^{i_1 j_1} \cdots \eta^{i_p j_p}\, \partial^k \partial^l\, Y_{\{4,2p-1\}}\, T_{kl}\, \varphi^{[p]}{}_{i_1 j_1,\,\ldots,\,i_p j_p} = 0 \,, \quad (3.44)$$

$$E_\beta \; : \; T_{(ij}\, T_{kl)}\, \varphi = 0 \,, \tag{3.45}$$

where the $\mathcal{B}_{ijkl}$ are now the combinations

$$\mathcal{B}_{ijkl} = \beta_{ijkl} - \frac{1}{2} \sum_{p=0}^{N-1} \frac{(-1)^p}{p!\,(p+3)!} \, \eta^{i_1 j_1} \cdots \eta^{j_p j_p}\, \partial_{(i}\partial_j\, \varphi^{[p+1]}{}_{kl),\,i_1 j_1,\,\ldots,\,i_p j_p} \,, \quad (3.46)$$

that are invariant under constrained gauge transformations. Notice that in these expressions I have extended the notation adopted for $\mathcal{F}$ in order to denote the two-column projected traces of φ. Moreover, when combined with eq. (3.45), eq. (3.44) can be cast in the form (3.42), so that in both cases we can effectively refer to these equations of motion.

Finally, the field equations following from the Lagrangians (3.41) can be also obtained from eqs. (3.42) and (3.43). In fact, these Lagrangians result from the action of the Stueckelberg-like shifts (3.37) in eq. (3.28). One can thus identify the corresponding field equations for φ and the β_{ijkl} as

$$E_\varphi \; : \; \sum_{p=0}^{N} \frac{(-1)^p}{p!\,(p+1)!} \, \eta^{i_1 j_1} \cdots \eta^{i_p j_p} \, \mathcal{A}^{[p]}{}_{i_1 j_1,\,\ldots,\,i_p j_p} + \frac{1}{2}\, \eta^{ij}\, \eta^{kl}\, \mathcal{B}_{ijkl} = 0, \quad (3.47)$$

$$E_\beta \; : \; \mathcal{C}_{ijkl} = 0 \,, \tag{3.48}$$

where, barring some subtleties that I will dwell upon in a while, the $\mathcal{B}_{ijkl}$ tensors are obtained shifting the corresponding tensors in eq. (3.46) according to eq. (3.37). Furthermore, the equations of motion for the compensators Φ_i must provide the conservation conditions for external unconstrained currents that are missing in the theory

with constrained gauge invariance. In the general mixed symmetry case they thus read

$$\partial_i E_\varphi + \sum_{p=0}^{N} \frac{2}{p!\,(p+2)!}\, \eta^{m_1 n_1} \dots \eta^{m_p n_p}\, Y_{\{2^p,1\}}\, T_{m_1 n_1} \dots T_{m_p n_p}\, \partial^j \partial^k \partial^l \, (E_\beta)_{ijkl} \, = \, 0 \,.$$

(3.49)

The equations of motion (3.47) and (3.48) can then be reduced to eqs. (3.42) and (3.43) undoing the Stueckelberg-like shift or, more precisely, performing a partial gauge fixing that does not spoil the Labastida constrained gauge symmetry. As a consequence, when in Section 4.1.1 I will discuss the reduction of these three sets of field equations to the Labastida form $\mathcal{F} = 0$ I will only deal with eqs. (3.42) and (3.43). These indeed suffice to capture the relevant features of all equations of motion following from the various approaches.

A last comment is in order, since the linear dependence of the Labastida constraints plays a role at this stage. In fact, the Stueckelberg-like shifts (3.37) map the β_{ijkl} into their gauge invariant completions, but we have seen that the gauge transformations (3.29) can be redefined eliminating all terms that do not affect the Lagrangians. Only the shifts built upon these refined transformations will lead to $\mathcal{B}_{ijkl}$ tensors expressible in terms of the combinations $\alpha_{ijk}(\Phi)$, while the "naive" form presented in eq. (3.37) does not. A more detailed discussion supported by explicit two-family examples can be found in [40].

3.1.2   Fermi fields

In the previous chapter I showed that the Labastida constraints

$$\gamma_{(i}\epsilon_{j)} \, = \, 0$$

(3.50)

lead to first order gauge invariant field equations. Indeed, computing the gauge variation of the Fronsdal-Labastida tensor (2.11) one obtains

$$\delta\mathcal{S} \, = \, -\frac{i}{2}\, \partial^i \partial^j\, \gamma_{(i}\epsilon_{j)} \,.$$

(3.51)

I also showed that the Labastida constraints

$$T_{(ij}\gamma_{k)}\, \psi \, = \, 0$$

(3.52)

lead to gauge invariant Lagrangians, since

$$\partial_i \mathcal{S} - \frac{1}{2}\,\slashed{\partial}\,\gamma_i\,\mathcal{S} - \frac{1}{2}\,\partial^j\, T_{ij}\,\mathcal{S} - \frac{1}{6}\,\partial^j\,\gamma_{ij}\,\mathcal{S} \, = \, \frac{i}{6}\,\partial^j \partial^k\, T_{(ij}\gamma_{k)}\,\psi \,,$$

(3.53)

and Bianchi identities free from the "classical anomaly" appearing on the right hand-side of eq. (3.53) guarantee the gauge invariance of the Lagrangians (2.134).

The simplest solution to eliminate the constraints (3.50) on the gauge parameters would be again to introduce for each of them one auxiliary field transforming as

$$\delta\,\xi_{ij} \, = \, \frac{1}{2}\,\gamma_{(i}\epsilon_{j)} \,.$$

(3.54)

However, even in this setup the linear dependence of the Labastida constraints (3.50) would imply the existence of gauge invariant combinations of the ξ_{ij} fields. At any rate, in analogy with the bosonic case, this problem can be overcame resorting to a set of fundamental compensators transforming as

$$\delta \Psi_i = \epsilon_i \, . \tag{3.55}$$

The constrained gauge symmetry then guarantees that the Ψ_i enter field equations and Lagrangians only through their symmetrized γ-traces. For this reason, in the following I will often abide by the notation

$$\xi_{ij} \equiv \frac{1}{2} \, \gamma_{(i} \epsilon_{j)} \, . \tag{3.56}$$

Before going on along the lines followed for bosons, let me support further these arguments by displaying a gauge invariant combination of the ξ_{ij}. A convenient starting point is the combination (2.65), that displays a vanishing linear combination of γ-traces of the constraints (3.52) on the fields ψ. Considering the gauge invariant completions of these constraints,

$$\mathcal{Z}_{ijk} = \frac{i}{3} \left\{ T_{(ij} \, \psi_{k)} - 2 \, \partial_{(i} \xi_{jk)} - \not\partial \gamma_{(i} \xi_{jk)} \right.$$
$$\left. - \, \partial^l \left(2 \, T_{(ij} \, \xi_{k)l} - T_{l(i} \, \xi_{jk)} - \gamma_{l(i} \xi_{jk)} \right) \right\} \, , \tag{3.57}$$

one can built the same combination of γ-traces acting on the $\mathcal{Z}_{ijk}$ tensors. Yet, there are no group theoretical arguments forcing its ξ_{ij} part to vanish, and indeed

$$\left(\gamma_{m(i} \, \mathcal{Z}_{j)kl} + \gamma_{m(k} \, \mathcal{Z}_{l)ij} \right) - 15 \, Y_{\{4,1\}} \left(T_{ij} \, \mathcal{Z}_{klm} + T_{kl} \, \mathcal{Z}_{ijm} \right)$$
$$= \frac{i}{3} \, \partial^n \left\{ 8 \, T_{(ij} \, T_{kl)} \, \xi_{mn} - 4 \, T_{mn} \, T_{(ij} \, \xi_{kl)} + 4 \, T_{m(i|} \, T_{n|j} \, \xi_{kl)} - 2 \, T_{(m|(i} \, T_{jk} \, \xi_{l)|n)} \right\}$$
$$+ \frac{i}{3} \, \partial^n \left\{ 2 \, T_{(i|(m} \, \gamma_{n)|j} \, \xi_{kl)} - 2 \, \gamma_{(m|(i} \, T_{jk} \, \xi_{l)|n)} \right\} \, . \tag{3.58}$$

The terms on the right-hand side of this expression are available from two families onward and define a gauge invariant combination of the ξ_{ij} fields. In analogy with the bosonic case, linear combinations involving a smaller number of γ-traces cannot lead to gauge invariant combinations of the ξ_{ij}, so that this problem can be avoided in simple mixed-symmetry models as in the fully symmetric case. Moreover, when the $\mathcal{Z}_{ijk}$ are expressed in terms of the independent compensators Ψ_i, they clearly satisfy the same set of linear relations as the combinations $T_{(ij} \gamma_{k)} \psi$: for instance

$$\left(\gamma_{m(i} \, \mathcal{Z}_{j)kl} + \gamma_{m(k} \, \mathcal{Z}_{l)ij} \right) - 15 \, Y_{\{4,1\}} \left(T_{ij} \, \mathcal{Z}_{klm} + T_{kl} \, \mathcal{Z}_{ijm} \right) = 0 \, . \tag{3.59}$$

Now that we have identified the needed auxiliary fields, we can introduce the new kinetic tensors

$$\mathcal{W} = \mathcal{S} + \frac{i}{2} \, \partial^i \partial^j \gamma_{(i} \Psi_{j)} \, , \tag{3.60}$$

that are left invariant under the *unconstrained* gauge transformations

$$\delta \psi = \partial^i \epsilon_i \, ,$$
$$\delta \Psi_i = \epsilon_i \, . \tag{3.61}$$

One can then obtain gauge invariant field equations imposing

$$\mathcal{W} = 0 \,. \tag{3.62}$$

This "minimal" recipe to recover an unconstrained gauge symmetry leads again to a higher-derivative kinetic tensor. Indeed, $\mathcal{W}$ contains two gradients acting on the Ψ_i, but the corresponding term is harmless since it can be gauged away without spoiling the constrained Labastida gauge symmetry. At any rate, ordinary single derivative field equations will be presented in Section 3.2.2 at the price of increasing a bit the number of auxiliary fields.

Notice that the new kinetic tensors satisfy the Bianchi identities

$$\partial_i \,\mathcal{W} - \frac{1}{2} \, \partial\!\!\!/ \, \gamma_i \,\mathcal{W} - \frac{1}{2} \, \partial^j \, T_{ij} \,\mathcal{W} - \frac{1}{6} \, \partial^j \, \gamma_{ij} \,\mathcal{W} = \frac{i}{2} \, \partial^j \partial^k \, \mathcal{Z}_{ijk} \,, \tag{3.63}$$

where the tensors

$$\mathcal{Z}_{ijk} = \frac{1}{3} \, T_{(ij} \, \gamma_{k)} \left(\psi - \partial^l \, \Psi_l \right) \tag{3.64}$$

coincide with those previously introduced in eq. (3.57), as can be seen expressing the ξ_{ij} in terms of the Ψ_i. Moreover, as suggested by the form taken by the $\mathcal{Z}_{ijk}$, it is also possible to introduce the Ψ_i compensators performing a Stueckelberg-like shift

$$\psi \to \psi - \partial^i \, \Psi_i \,. \tag{3.65}$$

Having seen how to eliminate the constraints (3.50) on the gauge parameters, we can move on and analyze the constraints (3.52) on the field. The strategy will be that already delineated in the bosonic case. Relaxing the constraints (3.52), the gauge variation of the Lagrangian (2.134) no longer vanishes, but using the Bianchi identities (3.53) one can control the remaining terms proportional to $T_{(ij} \, \gamma_{k)} \, \psi$. Rather than imposing the Labastida constraints (3.52), these can then be canceled via the gauge variation of some extra terms containing a proper set of Lagrange multipliers. To identify them, let me continue to discard the contributions that are proportional to $\gamma_{(i} \epsilon_{j)}$ as in Section 2.2.2. In this fashion we can focus only on the portion of the gauge variation that is really linked to the Labastida constraints on the fields ψ. Under this hypothesis it is again possible to combine the two kinds of contributions appearing in eq. (2.117), as we did in the constrained theory. Thus, we can start to reconsider the gauge variation of the Lagrangian (2.134) from eq. (2.127), that I write again here for clarity, denoting by $k_{p,q}$ the coefficients of eq. (2.130):

$$\delta \mathcal{L} = - \sum_{p,q=0}^{N} \frac{1}{2^{p+1}} \frac{k_{p,q}}{(q+3)(p+q+2)} \, \langle \, T_{i_1 j_1} \ldots T_{i_p j_p} \, \bar{\epsilon}_l \, \gamma_{k_1 \ldots k_q} \,,$$

$$(q+3)(p+q+2) \, Y_{\{2^p, 1^{q+1}\}} \, \partial_l \left(\gamma^{[q]} \mathcal{S}^{[p]} \right)_{i_1 j_1, \ldots, i_p j_p \,; k_1 \ldots k_q}$$

$$+ \, Y_{\{2^p, 1^{q+1}\}} \sum_{n=1}^{p} \partial_{(i_n |} \left(\gamma^{[q+2]} \mathcal{S}^{[p-1]} \right)_{\ldots, i_r \neq n j_r \neq n, \ldots \,; | j_n) l \, k_1 \ldots k_q}$$

$$+ \, (-1)^{q+1} \, (q+3) \, \partial\!\!\!/ \left(\gamma^{[q+1]} \mathcal{S}^{[p]} \right)_{i_1 j_1, \ldots, i_p j_p \,; k_1 \ldots k_q l}$$

$$- \left(q + 3 \right) \partial^m \left(\gamma^{[q]} \mathcal{S}^{[p+1]} \right)_{i_1 j_1, \dots, i_p j_p, lm; k_1 \dots k_q}$$

$$- \partial^m \left(\gamma^{[q+2]} \mathcal{S}^{[p]} \right)_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q lm} \rangle + \text{h.c.} . \tag{3.66}$$

In order to proceed it is then necessary to identify the unconstrained analogue of eq. (2.128). In the fermionic case this step is more subtle than its bosonic counterpart. Indeed, a generic γ-trace $T_{i_1 j_1} \dots T_{i_p j_p} \gamma_{k_1 \dots k_q}$ of the Bianchi identities (3.53) admits a couple of two-column projections since

$$\{2^p, 1^q\} \otimes \{1\} = \{2^p, 1^{q+1}\} \oplus \{2^{p+1}, 1^{q-1}\} \oplus \{3, 2^{p-1}, 1^q\} . \tag{3.67}$$

While in the constrained case the constraints on $\mathcal{S}$ of eq. (2.100) allow to simply consider the first of them to reach eq. (2.128), in the unconstrained case one has to combine different consequences of the Bianchi identities in order to reproduce the terms entering eq. (3.66). This rather involved procedure is sketched in Appendix C, but the final outcome is

$$\left(p + q + 2 \right) Y_{\{2^p, 1^{q+1}\}} \partial_l \left(\gamma^{[q]} \mathcal{S}^{[p]} \right)_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q}$$

$$+ \frac{1}{q+3} Y_{\{2^p, 1^{q+1}\}} \sum_{n=1}^{p} \partial_{(i_n |} \left(\gamma^{[q+2]} \mathcal{S}^{[p-1]} \right)_{\dots, i_r \neq n j_r \neq n, \dots; | j_n) l k_1 \dots k_q}$$

$$+ (-1)^{q+1} \slashed{\partial} \left(\gamma^{[q+1]} \mathcal{S}^{[p]} \right)_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q l} - \partial^m \left(\gamma^{[q]} \mathcal{S}^{[p+1]} \right)_{i_1 j_1, \dots, i_p j_p, lm; k_1 \dots k_q}$$

$$- \frac{1}{q+3} \partial^m \left(\gamma^{[q+2]} \mathcal{S}^{[p]} \right)_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q lm}$$

$$= \frac{i}{p+q+3} Y_{\{2^p, 1^{q+1}\}} \Bigg\{ \left(p + 2 \right) T_{i_1 j_1} \dots T_{i_p j_p} \gamma_{k_1 \dots k_q} \partial^m \partial^n T_{(lm} \gamma_{n)} \psi$$

$$- \frac{1}{q+3} \sum_{n=1}^{p} \prod_{r \neq n}^{p} T_{i_r j_r} \gamma_{k_1 \dots k_q l (i_n |} \partial^m \partial^n \left(T_{| j_n)(m} \gamma_{n)} + T_{mn} \gamma_{| j_n)} \right) \psi \Bigg\} . \tag{3.68}$$

The complicated structure of the constraint terms on the right-hand side actually comes from the combination of different γ-traces of the Bianchi identities. At any rate, substituting eq. (3.68) in eq. (3.66) one can identify the gauge variation of the set of Lagrange multipliers to be added to the Lagrangian (2.134). Removing the Labastida constraints on the fields thus leads to consider the Lagrangians

$$\mathcal{L} = \frac{1}{2} \langle \bar{\psi} , \sum_{p, q=0}^{N} \frac{(-1)^{p + \frac{q(q+1)}{2}}}{p! \, q! \, (p+q+1)!} \eta^p \gamma^q \left(\gamma^{[q]} \mathcal{S}^{[p]} \right) \rangle$$

$$+ \frac{i}{12} \langle \bar{\lambda}_{ijk} , T_{(ij} \gamma_{k)} \psi \rangle + \text{h.c.} , \tag{3.69}$$

where I resorted to the shorthand notation of eq. (2.135). These Lagrangians are gauge invariant under constrained gauge transformations provided that the λ_{ijk} transform

as

$$
\delta\,\lambda_{ijk} \;=\; \sum_{p,\,q=0}^{N}\, \frac{1}{p+q+2}\; \eta^{m_1 n_1}\ldots \eta^{m_p n_p}\, \gamma^{\,l_1\ldots l_q} \times
$$

$$
\times\; \partial_{(i}\,\partial_{j|}\left\{ \frac{(p+2)\,k_{p,q}}{(p+q+3)}\; Y_{\{2^p,1^{q+1}\}}\, T_{m_1 n_1}\ldots T_{m_p n_p}\, \gamma_{l_1\ldots l_q}\, \epsilon_{|k)} \right.
$$

$$
\left. +\; \frac{(-1)^{q+1}\,(p+1)\,k_{p+1,\,q-2}}{q\,(q+1)\,(p+q+1)}\; Y_{\{2^{p+1},1^{q-1}\}}\, T_{m_1 n_1}\ldots T_{m_p n_p}\, T_{|k)\,[\,l_1}\gamma_{l_2\ldots l_{q-1}}\,\epsilon_{l_q]} \right\}. \quad (3.70)
$$

In analogy with the bosonic case, the linear dependence of the Labastida constraints displayed in eq. (2.65) implies the existence of a symmetry of the Lagrangians (3.69) under shifts of the Lagrange multipliers of the type

$$
\delta\lambda_{ijk} \;=\; \eta^{lm}\, M_{ijk,\,lm} \;+\; \gamma^{lm}\, N_{ijk;\,lm}\,. \quad (3.71)
$$

The symmetry holds provided that the $M_{ijk,\,lm}$ and $N_{ijk;\,lm}$ parameters are $\{4,1\}$ projected in their family indices and are related according to

$$
N_{ij\,(k\,;\,l)\,m} \;+\; N_{kl\,(i\,;\,j)\,m} \;=\; -\frac{1}{2}\,\big(\, M_{ijm,\,kl} \,+\, M_{klm,\,ij}\,\big)\,. \quad (3.72)
$$

In fact, the shift (3.71) gives rise to

$$
\delta\,\mathcal{L} \;=\; \frac{i}{24}\,\big\langle\, \bar{M}_{ijk,\,mn}\,,\; T_{mn}\,T_{(ij}\,\gamma_{k)}\,\psi\,\big\rangle \quad (3.73)
$$

$$
-\;\frac{i}{12}\,\big\langle\, \bar{N}_{ijk;\,mn}\,,\; \big(\gamma_{mn\,(i}\,T_{jk)} \,-\, T_{(ij}\,T_{k)\,[\,m}\,\gamma_{n]}\big)\,\psi\,\big\rangle \;+\; \text{h.c.}\,. \quad (3.74)
$$

The first scalar product can give contributions in the $\{5\}$, $\{4,1\}$ and $\{3,2\}$ representations of the permutation group acting on the family indices. On the other hand, the first term in the second scalar product can give contributions in the $\{3,1,1\}$ and $\{2,1,1,1\}$, and finally the last can give contributions in the $\{4,1\}$ and $\{3,1,1\}$. Therefore, if the M and N parameters are $\{4,1\}$ projected and are related to one another as in eq. (3.72), an interesting symmetry of the Lagrangians (2.134) emerges. It manifests itself rather clearly when one tries to reduce the Lagrangian field equations to the Labastida form (2.43), since all contributions that can be shifted away by (3.71) are left undetermined. Furthermore, as in the bosonic case, an important lesson to be drawn from this symmetry is that only part of the gauge transformation (3.70) of the Lagrange multipliers λ_{ijk} is effective in the variation of the Lagrangian. As we shall see in a while, this is important when trying to extend at the Lagrangian level the Stueckelberg-like shift (3.65).

For Bose fields we also saw an alternative presentation of the Lagrangians for unconstrained fields. One can follow the same approach even for Fermi fields. In particular, looking at eq. (B.26) one can recognize that the combinations

$$
\widehat{\mathcal{S}}^{[p,q]}{}_{i_1 j_1,\,\ldots,\,i_p j_p;\,k_1\ldots k_q} \;=\; Y_{\{2^p,1^q\}}\, T_{i_1 j_1}\ldots T_{i_p j_p}\,\gamma_{k_1\ldots k_q}\,\mathcal{S}
$$

$$
+\; i\,\partial^l\, Y_{\{3,2^{p-1},1^q\}}\, T_{i_1 j_1}\ldots T_{i_p j_p}\,\gamma_{k_1\ldots k_q l}\,\psi\,, \quad (3.75)
$$

allow to define the Lagrangians

$$\mathcal{L} = \frac{1}{2} \langle \, \bar{\psi} \, , \, \sum_{p,\,q=0}^{N} \frac{(-1)^{p+\frac{q\,(q+1)}{2}}}{p\,!\,q\,!\,(\,p+q+1\,)\,!} \, \eta^{p} \, \gamma^{q} \, (\, \gamma^{[q]} \, \widehat{\mathcal{S}}^{[p]} \,) \, \rangle$$

$$+ \, \frac{i}{12} \, \langle \, \bar{\mathcal{Y}}_{ijk} \, , \, T_{(ij}\,\gamma_{k)}\,\psi \, \rangle \, + \, \text{h.c.} \, , \tag{3.76}$$

that are gauge invariant under constrained transformations even with gauge invariant Lagrange multipliers $\mathcal{Y}_{ijk}$. Notice that the portion of the new Lagrangians containing both $\bar{\psi}$ and ψ is no more written in terms of the projected traces of $\mathcal{S}$, but rather directly in terms of the fields. As in the bosonic case, these Lagrangians are related to those in eq. (3.69) via a field redefinition of the Lagrange multipliers, but this rewriting will prove useful when dealing with the field equations. Furthermore, the new tensors entering the Lagrangians satisfy "anomaly-free" Bianchi identities with the same structure as those in eq. (2.128).

To summarize, in eq. (3.62) I have presented non-Lagrangian field equations that are left invariant under the full unconstrained gauge transformations (3.61), while in eqs. (3.69) and (3.76) I have presented Lagrangians for unconstrained gauge fields that are still only invariant under constrained gauge transformations. As we did for Bose fields, we can now combine these two results and build a fully unconstrained Lagrangian theory. Again, in principle this can be done simply refining the Stueckelberg-like shift of eq. (3.65). This can be done defining also a proper shift of the Lagrange multipliers according to

$$\psi \, \to \, \psi \, - \, \partial^{i} \, \Psi_{i} \, ,$$
$$\zeta_{ijk} \, \to \, \lambda_{ijk} \, - \, \Delta_{ijk} \, (\, \Psi \,) \, . \tag{3.77}$$

The $\Delta_{ijk}(\Psi)$ that appear in eq. (3.77) can be formally obtained replacing ϵ_i with Ψ_i in the gauge variation of the Lagrange multipliers λ_{ijk}, so that the ζ_{ijk} are mapped into gauge invariant combinations. Performing these shifts in eq. (3.69), one ends up with a fully unconstrained Lagrangian, where the compensators Ψ_i only appear through their symmetrized γ-traces. As for Bose fields, the shift of the Lagrange multipliers is crucial in order to satisfy this important condition, since it cancels the terms that would not be expressible in terms of $\gamma_{(i}\,\Psi_{j)}$. Again, this approach makes it clear that a fully unconstrained Lagrangian exists but, at least at this level, it does not provide a clear hint on the structure of the compensator terms. More work is needed to display these terms, and to this end I will now follow the path outlined in Section 3.1.1 for Bose fields. Rather than dealing directly with the shifts (3.77), I will in fact reconsider the gauge variation of a suitable modification of the Lagrangian (3.69), compensating the terms proportional to $\gamma_{(i}\,\epsilon_{k)}$ via the addition of new compensator terms.

A convenient starting point is thus provided by the trial Lagrangian obtained sub-

stituting $\mathcal{S}$ with $\mathcal{W}$ and $T_{(ij}\gamma_{k)}\psi$ with $\mathcal{Z}_{ijk}$ in eq. (3.69):

$$\mathcal{L}_0 = \frac{1}{2}\left\langle \bar{\psi}\,,\, \sum_{p,\,q=0}^{N} \frac{(-1)^{p+\frac{q(q+1)}{2}}}{p!\,q!\,(p+q+1)!}\,\eta^p\,\gamma^q\,(\,\gamma^{[q]}\,\mathcal{W}^{[p]}\,)\,\right\rangle$$

$$+ \frac{1}{4}\left\langle\,\bar{\lambda}_{ijk}\,,\,\mathcal{Z}_{ijk}\,\right\rangle + \text{h.c.} \ . \tag{3.78}$$

The gauge variation of the first line of eq. (3.78) takes exactly the form of eq. (2.134), barring the substitution of $\mathcal{S}$ with $\mathcal{W}$. On the other hand, since we are relaxing the constraints (3.50) on the gauge parameters, we now have to keep the terms that in Section 2.2.2 we discarded in going from eq. (2.114) to eq. (2.127). These can eventually be canceled by the gauge variation of some compensator terms to be added to $\mathcal{L}_0$. However, differently from the bosonic case, here the compensator terms originate from two types of remainders: in fact, in the constrained theory we first eliminated the $\{3, 2^{p-1}, 1^q\}$-projected terms in eq. (2.114) and then we used again these constraints in order to combine the two classes of two-column projections appearing in eq. (2.117). In the unconstrained theory the first type of remainders can be compensated exploiting the precise dependence on the constraints of the $\{3, 2^{p-1}, 1^q\}$ components of the $T_{i_1 j_1}\ldots T_{i_p j_p}\,\bar{\epsilon}_l\,\gamma_{k_1\ldots k_q}$ combinations appearing on the left entries of the scalar products in eq. (2.114). The precise relation is rather involved and it is derived in Appendix C, but the end result is

$$\left\langle Y_{\{3,2^{p-1},1^q\}}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,\bar{\epsilon}_l\,\gamma_{k_1\ldots k_q}\,,\,\partial_l\,(\,\gamma^{[q]}\,\mathcal{W}^{[p]}\,)_{i_1 j_1,\,\ldots,\,i_p j_p;\,k_1\ldots k_q}\,\right\rangle$$

$$= \left\langle\,(p+1)\,T_{i_2 j_2}\ldots T_{i_p j_p}\,T_{(i_1 j_1}\,\bar{\epsilon}_{l)}\,\gamma_{k_1\ldots k_q} - T_{i_2 j_2}\ldots T_{i_p j_p}\,T_{[k_1|(l}\,\bar{\epsilon}_{i_1}\,\gamma_{j_1)}\,\gamma_{|k_2\ldots k_q]}\,,\right.$$

$$\left.\frac{p}{3\,(p+1)(p+q+2)}\,\partial_{(l|}\,(\,\gamma^{[q]}\,\mathcal{W}^{[p]}\,)_{|i_1 j_1),\,\ldots,\,i_p j_p;\,k_1\ldots k_q}\,\right\rangle. \tag{3.79}$$

Hence, the three-column projected terms appearing in the gauge variation of the trial Lagrangians (3.78) can be compensated adding to them

$$\mathcal{L}_1 = -\frac{1}{4}\left\langle\,\bar{\xi}_{ij}\,,\,\sum_{p,\,q=0}^{N} \eta^{i_1 j_1}\ldots\eta^{i_p j_p}\,\gamma^{k_1\ldots k_q}\times\right.$$

$$\times\left\{\,\frac{(p+1)\,k_{p+1,q-1}}{p+q+2}\,\partial_{(k_1|}(\,\gamma^{[q-1]}\,\mathcal{W}^{[p+1]}\,)_{|ij),\,i_1 j_1,\,\ldots,\,i_p j_p;\,k_2\ldots k_q}\right.$$

$$\left.\left.+ \frac{2\,p\,(p-1)\,q\,k_{p,q+1}}{(p+1)(p+q+3)}\,\partial_{(i_1|}(\gamma^{[q+1]}\,\mathcal{W}^{[p]}\,)_{|ij),\,i_2 j_2,\,\ldots,\,i_p j_p;\,j_1 k_1\ldots k_q}\,\right\}\,\right\rangle + \text{h.c.}\,, \tag{3.80}$$

In the constrained theory we combined the remaining two-column projected terms in eq. (2.117) using the first identity in eq. (2.123), whose unconstrained analogue is

$$Y_{\{2^{p+1},1^{q-1}\}}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,\bar{\epsilon}_l\,\gamma_{k_1\ldots k_q} = \frac{1}{q+1}\,Y_{\{2^{p+1},1^{q-1}\}}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,T_{l[k_1}\bar{\epsilon}_{k_2}\,\gamma_{k_3\ldots k_q]}$$

$$- \frac{2}{q+1}\,Y_{\{2^{p+1},1^{q-1}\}}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,\delta\,\bar{\xi}_{l[k_1}\,\gamma_{k_2\ldots k_q]}\ . \tag{3.81}$$

The gauge variation of $(\mathcal{L}_0 + \mathcal{L}_1)$ thus vanishes up to the terms proportional to $\delta\xi_{ij}$ coming from eq. (3.81), that can be canceled by the gauge transformation of

$$\mathcal{L}_2 = -\frac{1}{4}\,\langle\,\bar{\xi}_{ij}\,,\,\sum_{p,q=0}^{N}\,\eta^{i_1 j_1}\ldots\eta^{i_p j_p}\,\gamma^{k_1\ldots k_q}\,\times$$

$$\times\,\frac{2\,(q+1)}{q+2}\,\Big\{\,k_{p,q+1}\,Y_{\{2^{p+1},1^q\}}\,\partial_{(i|}\big(\,\gamma^{[q+1]}\,\mathcal{W}^{[p]}\,\big)_{i_1 j_1,\ldots,i_p j_p\,;\,|j)\,k_1\ldots k_q}$$

$$-\,(p+1)\,k_{p+1,q+1}\,\partial^m\,(\gamma^{[q+1]}\,\mathcal{W}^{[p+1]})_{i_1 j_1,\ldots,i_p j_p,\,ij\,;\,m\,k_1\ldots k_q}\,\Big\}\,\rangle\,+\,\text{h.c.}\,. \qquad (3.82)$$

As a consequence, our fully unconstrained Lagrangians for mixed-symmetry Fermi fields take the form

$$\mathcal{L} = \frac{1}{2}\,\langle\,\bar{\psi}\,,\,\sum_{p,q=0}^{N}\,\frac{(-1)^{p+\frac{q\,(q+1)}{2}}}{p!\,q!\,(p+q+1)!}\,\eta^{p}\,\gamma^{q}\,(\,\gamma^{[q]}\,\mathcal{W}^{[p]}\,)\,\rangle\,+\,\frac{1}{4}\,\langle\,\bar{\xi}_{ij}\,,\,\Xi_{ij}\,\rangle$$

$$+\,\frac{1}{4}\,\langle\,\bar{\lambda}_{ijk}\,,\,\mathcal{Z}_{ijk}\,\rangle\,+\,\text{h.c.}\,, \qquad (3.83)$$

where

$$\frac{1}{4}\,\langle\,\bar{\xi}_{ij}\,,\,\Xi_{ij}\,\rangle\,=\,\mathcal{L}_1\,+\,\mathcal{L}_2\,, \qquad (3.84)$$

and where the gauge transformations of the Lagrange multipliers are given in eq. (3.70). Notice that the fully unconstrained Lagrangians also display a symmetry under shifts of the Lagrange multipliers of the form (3.71). This essentially follows from eq. (3.59), that holds when one works in terms of the independent compensators Ψ_i.

Following the lines already depicted in the bosonic case, in the previous pages I have extended the constrained framework proposing the Lagrangians (3.69) and (3.76) for unconstrained fields that still allow only constrained gauge transformations, and the Lagrangians (3.83) for fully unconstrained fields. I can now close this section by presenting their field equations. As for Bose fields, those coming from the Lagrangians (3.76) can be obtained almost by inspection, since these Lagrangians satisfy the identity (2.141) as in the constrained theory. Furthermore, in this framework the fields are *unconstrained*, so that the subtleties discussed in Section 2.3 are absent. The equations of motion thus read

$$E_{\bar{\psi}}\,:\,\sum_{p,q=0}^{N}\,\frac{(-1)^{p+\frac{q\,(q+1)}{2}}}{p!\,q!\,(p+q+1)!}\,\eta^{p}\,\gamma^{q}\,(\,\gamma^{[q]}\,\mathcal{S}^{[p]}\,)\,-\,\frac{1}{2}\,\eta^{ij}\,\gamma^{k}\,\mathcal{Y}_{ijk}\,=\,0\,, \qquad (3.85)$$

$$E_{\bar{y}}\,:\,T_{(ij}\,\gamma_{k)}\,\mathcal{S}\,=\,0\,, \qquad (3.86)$$

where the projections of Section 2.3 can be recovered via the eliminations of the $\mathcal{Y}_{ijk}$ tensors from eq. (3.85). This is another example of the technical simplifications following from the elimination of the constraints (3.52). Clearly, the equation of motion coming from the Lagrangians (3.69) can be cast in the same form if one cancels from $E_{\bar{\psi}}$

the terms proportional to $E_{\bar{\lambda}}$. The only difference is that now the $\mathcal{Y}_{ijk}$ are expressed in terms of the basic fields via the combinations

$$
\mathcal{Y}_{ijk} \equiv i\,\zeta_{ijk} + i\sum_{p,q=0}^{N} \frac{(-1)^{p+\frac{q(q+1)}{2}+1}}{p!\,q!\,(p+q+3)!}\,\eta^{m_1 n_1}\cdots\eta^{m_p n_p}\,\gamma^{l_1\ldots l_q}\times
$$
$$
\times\,\partial_{(i|}\big(\gamma^{[q]}\,\psi^{[p+1]}\big)_{|jk),\,m_1 n_1,\ldots,\,m_p n_p;\,l_1\ldots l_q}\,,
$$
(3.87)

that are left invariant under constrained gauge transformations. Notice that in these expressions I have extended to ψ the notation adopted in order to denote the two-column projected traces of $\mathcal{S}$.

Finally, even the field equations coming from the Lagrangians (3.83) can be obtained from eqs. (3.85) and (3.86). In fact, these Lagrangians result from the action of the Stueckelberg-like shifts (3.37) in eq. (3.69). One can thus identify the corresponding field equations for $\bar{\psi}$ and the $\bar{\lambda}_{ijk}$ shifting eqs. (3.85) and (3.86) into

$$
E_{\bar{\psi}} \;:\; \sum_{p,q=0}^{N} \frac{(-1)^{p+\frac{q(q+1)}{2}}}{p!\,q!\,(p+q+1)!}\,\eta^{p}\,\gamma^{q}\,\big(\gamma^{[q]}\,\mathcal{W}^{[p]}\big) - \frac{1}{2}\,\eta^{ij}\,\gamma^{k}\,\mathcal{Y}_{ijk} = 0\,,
$$
(3.88)

$$
E_{\bar{\lambda}} \;:\; \mathcal{Z}_{ijk} = 0\,,
$$
(3.89)

where, barring some subtleties that I shall dwell upon in a while, the $\mathcal{Y}_{ijk}$ tensors are obtained shifting the corresponding tensors in eq. (3.87) according to eq. (3.77). Furthermore, the equations of motion for the compensators $\bar{\Psi}_i$ must provide the conservation conditions for external unconstrained currents that are missing in the theory with constrained gauge invariance. In the general mixed symmetry case they thus read

$$
E_{\bar{\Psi}} \;:\; \partial_l\,E_{\bar{\psi}} \;-\; \sum_{p,q=0}^{N} \frac{3\,(q+1)\,(q+2)\,(-1)^{p+\frac{q(q+1)}{2}}}{p!\,(q+3)!\,(p+q+3)!}\,\eta^{i_1 j_1}\ldots\eta^{i_p j_p}\,\gamma^{k_1\ldots k_q}\,Y_{\{2^p,1^{q+1}\}}\times
$$
$$
\times\,\Big\{\,(p+2)\,(q+3)\,T_{i_1 j_1}\ldots T_{i_p j_p}\,\gamma_{k_1\ldots k_q}\partial^m\partial^n\,(E_{\bar{\lambda}})_{lmn}
$$
$$
-\sum_{n=1}^{p}\prod_{r\neq n}^{p} T_{i_r j_r}\,\gamma_{k_1\ldots k_q\,l\,(i_n|}\partial^m\partial^n\,(E_{\bar{\lambda}})_{|j_n)\,mn}\,\Big\} = 0\,.
$$
(3.90)

The equations of motion (3.88) and (3.89) can then be reduced to eqs. (3.85) and (3.86) undoing the Stueckelberg-like shift or, more precisely, performing a partial gauge fixing that does not spoil the Labastida constrained gauge symmetry. As a consequence, when I shall discuss the reduction of these three sets of field equations to the Labastida form $\mathcal{S} = 0$ I shall only deal with eqs. (3.85) and (3.86). These equations indeed suffice to capture the relevant features of all those coming from the various approaches.

A last comment is in order, since the linear dependence of the Labastida constraints plays a role at this stage. In fact, the Stueckelberg-like shifts (3.77) map the λ_{ijk} into their gauge invariant completions, but we have seen that the gauge transformations (3.70) can be redefined eliminating all terms that do not affect the Lagrangians. Only the shifts built upon these refined gauge transformations will lead to $\mathcal{Y}_{ijk}$ tensors

expressible in terms of the combinations $\xi_{ij}(\Psi)$, while the "naive" form presented in eq. (3.77) does not. A more detailed discussion supported by explicit two-family examples can be found in [41].

3.1.3 Irreducible fields

As in most of Chapter 2, the fields considered in the previous sections were in general only symmetric under the interchange of pairs of vector indices belonging to the same set, but did not possess any symmetry relating different sets. They were thus *reducible* $gl(D)$ tensors or spinor-tensors. In Section 2.4 we have already seen the kinds of modifications that emerge when dealing with *irreducible* fields, that are subjected to suitable Young projections. The relations (2.154) involving the gauge parameters are still valid in the unconstrained framework, since we did not exploit the constraints (3.50) to derive them. In a similar fashion, the Einstein-like tensors appearing in the Lagrangians still commute with the $S^i{}_j$ operators as in the constrained setup. Furthermore, the (γ-)traces of the unconstrained kinetic tensors $\mathcal{A}$ or $\mathcal{W}$ still satisfy the same relations as the traces of the constrained kinetic tensors $\mathcal{F}$ of $\mathcal{S}$. For instance, in the bosonic case the identity (2.158) simply becomes

$$S^m{}_n T_{i_1 j_1} \dots T_{i_p j_p} \mathcal{A} + \sum_{i=1}^{p} \delta^m{}_{(i_n} T_{j_n)n} \prod_{r \neq n}^{p} T_{i_r j_r} \mathcal{A} = 0\,. \tag{3.91}$$

On the other hand, in general the irreducibility conditions (2.151) also restrict the extra field content introduced in the previous sections. For instance, eq. (2.154) induces an identical set of conditions on the Φ_i compensators,

$$S^m{}_n \Phi_i + \delta^m{}_i \Phi_n = 0\,. \tag{3.92}$$

As a consequence, the "composite" compensators α_{ijk} satisfy the relations

$$S^m{}_n \alpha_{ijk} + \delta^m{}_{(i} \alpha_{jk)n} = 0\,, \tag{3.93}$$

while computing the symmetrized double traces of (2.151) and moving to the left the $S^m{}_n$ operator one can obtain the relations satisfied by the $\mathcal{C}_{ijkl}$ tensors. These induce identical conditions on the Lagrange multipliers, so that

$$S^m{}_n \beta_{ijkl} + \delta^m{}_{(i} \beta_{jkl)n} = 0\,. \tag{3.94}$$

Similar conditions restrict the field content of the unconstrained fermionic models, so that the Ψ_i are related according to

$$S^m{}_n \Psi_i + \delta^m{}_i \Psi_n = 0\,, \tag{3.95}$$

while their symmetrized γ-traces satisfy

$$S^m{}_n \xi_{ij} + \delta^m{}_{(i} \xi_{j)n} = 0\,. \tag{3.96}$$

Finally, in the fermionic case the Lagrange multipliers are related according to

$$S^m{}_n \lambda_{ijk} + \delta^m{}_{(i} \lambda_{jk)n} = 0\,. \tag{3.97}$$

3.2 Low-derivative Lagrangian formulation

In Section 3.1 we have seen how the constraints can be eliminated introducing a "minimal" number of auxiliary fields [56, 57, 51, 40, 41], that only depends on the number of index families and not on the total number of space-time indices carried by the gauge fields as in the BRST-like constructions of [47, 54]. However, while the constraints on the gauge fields were eliminated adding Lagrange multiplier terms to the Lagrangians, the unconstrained gauge symmetry was recovered adding higher-derivative terms involving extra compensator fields. As repeatedly stressed in the previous pages, these higher-derivative contributions do not represent a real problem since they can be gauged away without spoiling the Labastida constrained gauge symmetry, but they still represent an unconventional ingredient that is absent in lower-spin examples. On the contrary, the BRST-like constructions contain a number of auxiliary fields that grows with the "spin", but associate only the conventional number of derivatives even to the auxiliary fields. In this section I would like to show that the presence of one or more of these peculiarities is not an essential feature of higher-spin dynamics. In fact, in the following I will explain how to combine the good properties of both unconstrained setups, presenting a more conventional description of the free dynamics for massless mixed-symmetry fields. It contains again only unconstrained $gl(D)$ (spinor-)tensors, but they have now second order field equations and Lagrangians in the case of Bose fields and first order field equations and Lagrangians in the case of Fermi fields. Furthermore, the field content is a bit larger with respect to the "minimal" construction, but the number of auxiliary fields still depends only on the number of index families. This means, for instance, that in the fully symmetric framework four auxiliary fields suffice to obtain an ordinary Lagrangian formulation for arbitrary higher-spin bosons and fermions.

Before moving to the details of the construction, let me recall that for fully symmetric Bose and Fermi fields an unconstrained formulation of the free dynamics without higher-derivative terms and with a fixed number of extra fields was first attained in [59]. It is an interesting off-shell variant of the on-shell truncation of the "triplets" [35] of String Field Theory [7] obtained in [42, 43] (see also [103] for some recent developments in the "frame-like" formalism). For fully symmetric Bose fields it was then followed by the construction of [53], that is really tailored to the compensator constructions of [56, 51]. This also stimulated the development of the formulation that was presented in [40, 41], for both Bose and Fermi fields of mixed-symmetry, and that I am going to review.

3.2.1 Bose fields

The idea underlying the construction of the new unconstrained formulation is the replacement of the Φ_i compensators of eq. (3.11) with other fields whose dimensions are at least as high as that of the gauge potentials φ. Let me stress that such fields can not be pure gauge for this reason but, as first proposed in [53], they can be forced

to vanish on shell adding a suitable set of Lagrange multipliers to the Lagrangians.

Whether one relaxes the request for pure-gauge compensators, it is simple to construct gauge invariant kinetic tensors similar to $\mathcal{A}$ but free of higher derivatives, and indeed several options are available. However, it is less straightforward to make sure that the resulting Lagrangians do not propagate additional degrees of freedom, and in [40] the potential problems were discussed in some examples. On the other hand, a systematic way to forego the constraints without introducing higher-derivative terms can be attained refining the Stueckelberg way to enhance the gauge symmetry [40]. This procedure permits to follow closely the steps outlined in the previous section, and in this fashion it also simplifies the check of its reduction to the constrained formulation. In Section 3.1.1 we have indeed seen that the kinetic tensors $\mathcal{A}$ of eq. (3.14) can be obtained via the Stueckelberg-like shift

$$\varphi \rightarrow \varphi - \partial^i \Phi_i, \qquad \text{with} \qquad \delta \Phi_i = \Lambda_i. \tag{3.98}$$

As I have already pointed out, this operation makes it manifest that the Φ_i compensators have higher (mass) dimensions than the gauge field, and this leads directly to the higher-derivative term in eq. (3.14). Furthermore, the shift (3.98) maps φ into a gauge invariant combination, so that arbitrary functions of φ become gauge invariant under its action. However, this condition looks somehow unnatural, since our goal should be only to enhance the constrained gauge symmetry that is already present. Indeed, only the symmetrized traces of the Φ_i compensators play a role in the minimal unconstrained construction. Thus, even if the solution provided by the Stueckelberg-like shift (3.98) is technically very simple, it would be more natural to map φ into a combination that is gauge invariant only under the transformations that are forbidden in the constrained theory.

Actually, these two comments are related, and a proper redefinition of the shift (3.98) naturally outlines a safe way of introducing a set of compensators with the same dimensions as the gauge field. Indeed, the gauge parameters can be decomposed according to

$$\Lambda_i = \Lambda^{(t)}{}_i + \eta^{jk} \Lambda^{(p)}{}_{i,jk}, \tag{3.99}$$

where the $\Lambda^{(t)}{}_i$ satisfy the conditions

$$T_{(ij} \Lambda^{(t)}{}_{k)} \equiv 0 \tag{3.100}$$

while the $\Lambda^{(p)}{}_{i,jk}$ carry the full amount of gauge symmetry that one would like to add, and are such that

$$T_{(ij|} \eta^{lm} \Lambda^{(p)}{}_{|k),lm} \equiv T_{(ij} \Lambda_{k)}. \tag{3.101}$$

Hence, the Labastida constrained gauge transformations can be cast in the form

$$\delta \varphi = \partial^i \Lambda^{(t)}{}_i, \tag{3.102}$$

while the gauge transformations of the form

$$\delta \varphi = \eta^{ij} \partial^k \Lambda^{(p)}{}_{k,ij} \tag{3.103}$$

are those we would like to allow via the introduction of a proper set of auxiliary fields.

If one accepts compensators that are not pure gauge this goal can be achieved with a set of fields transforming as

$$\delta\,\theta_{ij} \,=\, \partial^k\,\Lambda^{(p)}{}_{k,\,ij}\,. \tag{3.104}$$

They manifestly have the same dimension as the gauge potential, and one can use them to define the Stueckelberg-like shift

$$\varphi \,\to\, \varphi \,-\, \eta^{ij}\,\theta_{ij}\,. \tag{3.105}$$

The combination on the right-hand side of this expression transforms exactly like the gauge potential under (3.102), while it is gauge invariant under (3.103). Performing the shift (3.105) in the Fronsdal-Labastida tensor (2.11) one thus obtains the new kinetic tensor

$$\mathcal{A}_{\theta} \,=\, \mathcal{F} \,-\, \frac{1}{2}\,\left[\,(\,D-2\,)\,\partial^i\partial^j \,+\, \partial^k\partial^{(\,i}\,S^{j\,)}{}_k\,\right]\,\theta_{ij} \,-\, \eta^{ij}\,\mathcal{F}\,(\,\theta_{ij}\,)\,, \tag{3.106}$$

that is gauge invariant under the unconstrained gauge transformations

$$\delta\,\varphi \,=\, \partial^i\,\Lambda_i\,, \tag{3.107}$$

$$\delta\,\theta_{ij} \,=\, \partial^k\,\Lambda^{(p)}{}_{ijk}\,, \tag{3.108}$$

and where $\mathcal{F}\,(\,\theta_{ij}\,)$ is the Fronsdal-Labastida tensor for the θ_{ij}. Moreover, $\mathcal{A}_{\theta}$ satisfies the Bianchi identities

$$\partial_i\,\mathcal{A}_{\theta} \,-\, \frac{1}{2}\,\partial^j\,T_{ij}\,\mathcal{A}_{\theta} \,=\, -\,\frac{1}{12}\,\partial^j\partial^k\partial^l\,T_{(\,ij}\,T_{kl\,)}\,(\,\varphi - \eta^{mn}\,\theta_{mn}\,)\,, \tag{3.109}$$

that have the same form as those emerging in the minimal formulation. As a consequence, a gauge invariant Lagrangian for fully unconstrained fields obtains performing the shift (3.105) in eq. (3.28), or in eq. (3.35).

This step does not conclude our analysis: one cannot gauge away the θ_{ij} exploiting the new gauge symmetry because their transformations (3.104) are not algebraic in the gauge parameters. However, a possible way out of this problem was proposed for the fully symmetric case in [53]. In that framework, a spin s boson is described in an unconstrained fashion adding a compensator transforming as

$$\delta\,\theta_{\mu_1\ldots\mu_{s-2}} \,=\, \partial_{(\,\mu_1}\,\Lambda_{\mu_2\ldots\mu_{s-2}\,)\lambda}{}^{\lambda}\,. \tag{3.110}$$

In analogy with the θ_{ij} of the present setup, $\theta_{\mu_1\ldots\mu_{s-2}}$ is not pure gauge but it can be forced to vanish on shell via the addition of a Lagrange multiplier term of the form

$$\chi^{\mu_1\ldots\mu_{s-2}}\,\left(\,\theta_{\mu_1\ldots\mu_{s-2}} \,-\, \partial_{(\,\mu_1}\,\alpha_{\mu_2\ldots\mu_{s-2}\,)}\,\right) \tag{3.111}$$

The $\alpha_{\mu_1\ldots\mu_{s-3}}$ field entering eq. (3.111) is nothing but the pure-gauge compensator of the minimal construction, that transforms as

$$\delta\,\alpha_{\mu_1\ldots\mu_{s-3}} \,=\, \Lambda_{\mu_2\ldots\mu_s\lambda}{}^{\lambda} \tag{3.112}$$

and can be gauged away without spoiling the Labastida gauge symmetry. This finally permits to effectively gauge away the $\theta_{\mu_1 \dots \mu_{s-2}}$ field via the equation of motion of the gauge invariant Lagrange multiplier $\chi_{\mu_1 \dots \mu_{s-2}}$.

The same strategy leads to consider in the present framework the Lagrangians

$$\mathcal{L} = \frac{1}{2} \langle \, \varphi - \eta^{mn} \theta_{mn} \, , \, \sum_{p=0}^{N} \frac{(-1)^p}{p! \, (p+1)!} \, \eta^{i_1 j_1} \dots \eta^{i_p j_p} \, \mathcal{A}_{\theta}^{[p]}{}_{i_1 j_1, \dots, i_p j_p} \, \rangle$$

$$+ \frac{1}{8} \langle \, \beta_{ijkl} \, , \, T_{(ij} T_{kl)} \, (\, \varphi - \eta^{mn} \theta_{mn} \,) \, \rangle + \langle \, \chi_{ij} \, , \, \theta_{ij} - \partial^k \Pi_{ijk}^{lmn} \alpha_{lmn}(\Phi) \, \rangle \, , \quad (3.113)$$

that can be obtained shifting the φ fields in eq. (3.28) according to eq. (3.105) and adding a new set of gauge invariant Lagrange multipliers χ_{ij}. They play the same role as the $\chi_{\mu_1 \dots \mu_{s-2}}$ multiplier introduced in the previous example and relate each θ_{ij} compensator to the combination of the pure-gauge compensators Ψ_i displaying the same gauge transformation. As a consequence, this Lagrangian is invariant under the gauge transformations that I already defined, and that I collect again here for the sake of clarity:

$$\delta \varphi \;\; = \; \partial^i \Lambda_i \, ,$$
$$\delta \theta_{ij} \;\; = \; \partial^k \Lambda^{(p)}{}_{k, ij} \, ,$$
$$\delta \alpha_{ijk} \;\; = \; \frac{1}{3} \, T_{(ij} \Lambda_{k)} \, ,$$
$$\delta \beta_{ijkl} \;\; = \; \frac{1}{2} \sum_{p=0}^{N} \frac{(-1)^p}{p! \, (p+2)!} \, \partial_{(i} \partial_j \partial_{k|} \eta^{i_1 j_1} \dots \eta^{i_p j_p} \, Y_{\{2^p, 1\}} \, T_{i_1 j_1} \dots T_{i_p j_p} \Lambda^{(t)}{}_{|l)} \, ,$$
$$\delta \chi_{ij} \;\; = \; 0 \, . \qquad (3.114)$$

Notice that the Π_{ijk}^{lmn} appearing in eq. (3.113) is the projector defining the solution of eq. (3.101) for $\Lambda^{(p)}{}_{ijk}$ in the form

$$\Lambda^{(p)}{}_{ijk} \; = \; \Pi_{ijk}^{lmn} \, T_{(lm} \Lambda_{n)} \, . \qquad (3.115)$$

Computing the projector Π_{ijk}^{lmn} represents the main technical difficulty of this construction, and indeed I cannot display an explicit closed form for it in the general case, although it is rather straightforward, if lengthy, to compute it explicitly in specific cases of interest. Thus, for instance, in the fully symmetric case the explicit relation between $\Lambda^{(p)}$ and the full gauge parameter Λ is

$$\Lambda^{(p)} \; = \; \sum_{k=0}^{[\frac{s-3}{2}]} \frac{(-1)^k \, k \, !}{\prod_{i=0}^{k} \sum_{j=0}^{i} [D + 2 \, (s - 2 \, j - 3)]} \, \eta^k \, \Lambda^{[k+1]} \, , \qquad (3.116)$$

where $\Lambda^{[k+1]}$ denotes the $(k + 1)$-th trace of Λ and η^k is a product of k Minkowski metric tensors written with unit overall normalization and with the minimal number of terms needed to be totally symmetric.

Finally, the reduction of the equations of motion to the Fronsdal-Labastida form $\mathcal{F} = 0$ would result from steps similar to those needed for the minimal higher-derivative

Lagrangians. Indeed, the fields θ_{ij} and Φ_i can be gauged away simultaneously, while the Lagrange multipliers β_{ijkl} and χ_{ij} can be expressed in terms of the physical field φ by the equations for φ and θ_{ij}, respectively. Clearly this last statement holds up to the already discussed symmetries of the Lagrangians (3.113) under shifts of the Lagrange multipliers β_{ijkl}. Furthermore, this construction can be adapted to describe irreducible fields following the same lines sketched for the minimal case in Section 3.1.3.

3.2.2 Fermi fields

The extension of the previous results to mixed-symmetry fermions proceeds exactly along the same lines: to begin with, in the gauge variation (2.39) of the field ψ,

$$\delta\,\psi \,=\, \partial^i\,\epsilon_i\,, \tag{3.117}$$

one should isolate the contribution to the gauge parameters that is to vanish in the constrained theory. To this end, let me consider the decomposition

$$\epsilon_i \,=\, \epsilon^{(t)}{}_i \,+\, \gamma^k\,\epsilon^{(g)}{}_{i\,;\,k}\,, \tag{3.118}$$

where

$$\begin{aligned}
\gamma_{(i}\,\epsilon^{(t)}{}_{j)} &\equiv 0\,, \\
\gamma_{(i}\,\gamma^k\,\epsilon^{(g)}{}_{j)\,;\,k} &\equiv \gamma_{(i}\,\epsilon_{j)}\,.
\end{aligned} \tag{3.119}$$

In this fashion, the parameters $\epsilon^{(g)}{}_{i\,;\,k}$ embody precisely the gauge freedom that is absent in the Labastida formulation, and that we would like to allow here without introducing new fields whose dimensions differ from that of ψ. Guided by the solution of the problem for Bose fields presented in the previous section, we are thus led to introduce new compensator fields σ_k such that

$$\delta\,\sigma_k \,=\, \partial^l\,\epsilon^{(g)}{}_{l\,;\,k}\,. \tag{3.120}$$

Notice that we have to give up again pure gauge compensators in order to do that, but this allows the redefinition

$$\psi \,\rightarrow\, \psi \,-\, \gamma^k\,\sigma_k \tag{3.121}$$

of the field ψ, that has the virtue of identifying a combination of fields possessing the same gauge transformation as the *constrained* Labastida field under variations involving *unconstrained* gauge parameters. As we already observed, this is to be contrasted with the Stueckelberg shift (3.65). It is then possible to verify that, after the substitution (3.121), the Labastida tensor (2.41) takes the form

$$\mathcal{W}_\sigma \,=\, \mathcal{S} \,+\, i\,\big[\,(D-2)\,\partial^j \,+\, 2\,\partial^i\,S^j{}_i\,\big]\,\sigma_j \,+\, \gamma^k\,\mathcal{S}\,(\sigma_k)\,, \tag{3.122}$$

where

$$\mathcal{S}\,(\sigma_k) \,=\, i\,(\,\slashed{\partial}\,\sigma_k \,-\, \partial^j\,\gamma_j\,\sigma_k\,) \tag{3.123}$$

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is the Fang-Fronsdal-Labastida tensor for the σ_k, and where, let me reiterate, no higher derivatives appear, precisely because the compensators σ_k have the same physical dimension as ψ.

The Bianchi identities for $\mathcal{W}_\sigma$,

$$\partial_i \, \mathcal{W}_\sigma \, - \, \frac{1}{2} \, \slashed{\partial} \, \gamma_i \, \mathcal{W}_\sigma \, - \, \frac{1}{2} \, \partial^j \, T_{ij} \, \mathcal{W}_\sigma \, - \, \frac{1}{6} \, \partial^j \, \gamma_{ij} \, \mathcal{W}_\sigma$$
$$= \, \frac{i}{6} \, \partial^j \partial^k \, T_{(ij} \, \gamma_{k)} \, (\, \psi \, - \, \gamma^l \, \sigma_l \,) \, , \tag{3.124}$$

take a particularly nice form, as should be clear recalling eq. (3.121).

One can thus follow a relatively straightforward route to a Lagrangian based on the tensor (3.122), starting from the Lagrangian defined by eq. (3.69), where the constraints on the symmetrized triple γ-trace of ψ are enforced by Lagrange multipliers. Since (3.69) is gauge invariant under *constrained* gauge transformations, by construction it will become gauge invariant under unconstrained ones if one performs everywhere the substitution (3.121), that leads to

$$\mathcal{L} = \frac{1}{2} \, \langle \, \bar{\psi} + \bar{\sigma}_k \, \gamma^k \, , \, \sum_{p,q=0}^{N} \frac{(-1)^{\frac{q(q+1)}{2}}}{p! \, q! \, (p+q+1)!} \, \eta^p \, \gamma^q \, (\, \gamma^{[q]} \, \mathcal{W}_\sigma^{[p]} \,) \, \rangle$$
$$+ \, \frac{i}{12} \, \langle \, \bar{\lambda}_{ijk} \, , \, T_{(ij} \, \gamma_{k)} \, (\, \psi \, - \, \gamma^l \, \sigma_l \,) \rangle + \text{h.c.} \, . \tag{3.125}$$

Finally, in order to guarantee that the new compensators σ_k be effectively pure gauge, eq. (3.125) should be supplemented with further constraints, meant to enforce a linear relation between the σ_k and the compensators $\xi_{ij}(\Psi)$ defined in (3.56), so that the complete Lagrangian eventually takes the form

$$\mathcal{L} = \frac{1}{2} \, \langle \, \bar{\psi} + \bar{\sigma}_k \, \gamma^k \, , \, \sum_{p,q=0}^{N} \frac{(-1)^{\frac{q(q+1)}{2}}}{p! \, q! \, (p+q+1)!} \, \eta^p \, \gamma^q \, (\, \gamma^{[q]} \, \mathcal{W}_\sigma^{[p]} \,) \, \rangle$$
$$+ \, \frac{i}{12} \, \langle \, \bar{\lambda}_{ijk} \, , \, T_{(ij} \, \gamma_{k)} \, (\, \psi \, - \, \gamma^l \, \sigma_l \,) \rangle \, + \, \langle \, \bar{\chi}_k \, , \, \sigma_k \, - \, \partial^l \, \Pi_{kl}^{ij} \, \xi_{ij}(\Psi) \rangle + \text{h.c.} \, . \tag{3.126}$$

Π_{kl}^{ij} is the projector allowing to express the solution of eq. (3.118) as

$$\epsilon^{(g)}{}_{k\,;\,l} \, = \, \Pi_{kl}^{ij} \, \gamma_{(i} \, \epsilon_{j)} \, , \tag{3.127}$$

while the transformations that leave eq. (3.126) invariant are

$$\delta \, \psi \, = \, \partial^i \, \left(\, \epsilon^{(t)}{}_i \, + \, \gamma^k \, \epsilon^{(g)}{}_{i\,;\,k} \, \right) \, ,$$
$$\delta \, \sigma_k \, = \, \partial^l \, \epsilon^{(g)}{}_{l\,;\,k} \, ,$$
$$\delta \, \xi_{ij} \, = \, \gamma_{(i} \, \epsilon_{j)} \, ,$$
$$\delta \, \chi_k \, = \, 0 \, , \tag{3.128}$$

with $\delta \, \lambda_{ijk}$ given in (3.70) after replacing ϵ with $\epsilon^{(t)}$.

As in the bosonic case, constructing explicitly the projector Π^{ij}_{kl} represents the main technical difficulty of this construction, and I am again unable to display an explicit closed form for it in the general case. On the other hand, it is still rather straightforward, if lengthy, to compute it explicitly in specific cases of interest. For instance, in the fully symmetric case the explicit relation between $\epsilon^{(g)}$ and the full gauge parameter ϵ is

$$\epsilon^{(g)} = \sum_{n=1}^{[\frac{s}{2}]} \rho_n(D, s)\, \eta^{n-1} \left\{ \not{\epsilon}^{[n-1]} + \frac{1}{2\,n}\, \gamma\, \epsilon^{[n]} \right\} , \qquad (3.129)$$

with

$$\rho_n(D, s) = (-1)^{n+1} \prod_{k=1}^{n} \frac{1}{D + 2\,(s - k - 1)} . \qquad (3.130)$$

Here s is the tensorial rank of ψ and $\epsilon^{[n]}$ denotes the n-th trace of ϵ, while η^n denotes a combination of products of n Minkowski metric tensors defined with unit overall normalization and with the minimal number of terms needed in order to obtain a totally symmetric expression.

Let me close the present Section by noticing that this derivation of the Lagrangian (3.126) makes it also manifest that the equations of motion reduce generically to $\mathcal{S} = 0$, since the fields σ_i and Ψ_i can be gauged away simultaneously, while the Lagrange multipliers λ_{ijk} and χ_k can be expressed in terms of the physical field ψ by the equations for $\bar{\psi}$ and $\bar{\sigma}_k$, respectively. Clearly this last statement holds up to the already discussed symmetries of the Lagrangians (3.126) under shifts of the Lagrange multipliers λ_{ijk}. Again, this formulation can be also adapted to irreducible fields following steps similar to those performed in the minimal case in Section 3.1.3.

Chapter 4

Weyl-like symmetries

Inspired by the example of two-dimensional gravity, in Section 2.3 I identified a class of mixed-symmetry Bose and Fermi fields with vanishing actions. This is a manifestation of a more general phenomenon, since in low space-time dimensions field equations and Lagrangians can display additional accidental symmetries under Weyl-like shifts of the fields. The classification of these "pathological" cases is very interesting, since in most of them the Lagrangian field equations display extra symmetries with respect to the non-Lagrangian ones. Thus, the two sets of field equations are *not* directly equivalent, and in the Lagrangian setup the correct description of the free dynamics involves in a crucial way the extra Weyl-like transformations. The rich structure displayed by two-dimensional gravity also stimulates an effort aimed at classifying the "pathological" examples. Even if at the free level we can only display these algebraic similarities, the analogy could well signal some peculiar features of possible non-linear completions of these models.

Following [40, 41], in Section 4.1 I will thus derive the conditions that identify the mixed-symmetry models with additional Weyl-like symmetries, for both Bose and Fermi fields. In Section 4.2 I will then display some classes of solutions of these conditions in the general N-family case. The results of this analysis show that the two-column Bose fields and the single-column Fermi fields discussed in Section 2.3 are simple examples of a rather rich set of Weyl-invariant mixed-symmetry models. Let me also anticipate that some of these models have non-trivial actions, so that the existence of Weyl-like symmetries and the vanishing of the action do not necessarily go in hands in the mixed-symmetry case as in the fully symmetric setup. Indeed, in Section 4.2 I will exhibit some simple counterexamples to this effect.

4.1 Reduction of the Lagrangian field equations to the Labastida form

In the fully symmetric framework, the minimal local unconstrained Lagrangian field equations of Francia and Sagnotti were shown to be generally equivalent to the Fang-

Fronsdal ones in [56]. This result was achieved computing successive traces of the equations of motion and using them to express the Lagrange multipliers in terms of the gauge fields. This information then enables one to set zero also the $\mathcal{A}$ or $\mathcal{W}$ tensors and their (γ-)traces that are not directly proportional to the constraints. The other (γ-)traces are in fact directly annihilated by the field equations of the Lagrange multipliers. The elimination of the compensators via a partial gauge fixing eventually recovers the Fang-Fronsdal form of eqs. (2.2) or (2.37). As anticipated in Section 2.3, this procedure reduces the Lagrangian field equations to the non-Lagrangian ones in all space-time dimensions and for all spins, with two exceptions: the spin 2 and the spin 3/2 cases in two space-time dimensions.

As I repeatedly pointed out, the elimination of the compensators via a partial gauge fixing does not create any problem even in the mixed-symmetry framework, but the procedure just delineated would become rather complicated for mixed-symmetry fields. Indeed, eq. (2.142) implies that the kinetic tensors and the gauge invariant completions of the multipliers cannot be simply decoupled as in the fully symmetric setup, while the rather involved structure of the quadratic portion of the Lagrangians complicates their simultaneous treatment. On the other hand, the existence of the symmetries (3.31) and (3.71) under shifts of the Lagrange multipliers already shows that for generic mixed-symmetry fields some quantities are left undetermined by the Lagrangian field equations. At the same time it suggests that the classification of the symmetries of the equations of motions could be a more efficient way to treat the problem, and in the following I will proceed along this direction. The logic behind this approach will be exactly the same for both Bose and Fermi fields, but in the fermionic framework the proofs of some statements in Section 4.1.2 are still missing due to the harder technical steps that they involve.

4.1.1 Bose fields

At the end of Section 3.1.1 we have seen that all field equations following from the unconstrained Lagrangians that I presented in eqs. (3.28), (3.35) and (3.41) can be cast in the form

$$E_\varphi \,:\; \sum_{p=0}^{N} \frac{(-1)^p}{p!\,(p+1)!}\, \eta^{i_1 j_1} \ldots \eta^{i_p j_p}\, \mathcal{F}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p} + \frac{1}{2}\, \eta^{ij}\, \eta^{kl}\, \mathcal{B}_{ijkl} \,=\, 0\,, \quad (4.1)$$

$$E_\mathcal{B} \,:\; T_{(ij}\, T_{kl)}\, \varphi \,=\, 0 \quad \Longrightarrow \quad T_{(ij}\, T_{kl)}\, \mathcal{F} \,=\, 0\,, \tag{4.2}$$

possibly after a partial gauge fixing that eliminates the compensators Φ_i. Moreover, these conditions can also conveniently describe the field equations following from the constrained Lagrangians (2.94). In that case the second condition holds by definition even off-shell, while a field equation for φ alone is obtained eliminating the $\mathcal{B}_{ijkl}$ tensors from eq. (4.1). Let me stress that, in this respect, the equivalence between the Lagrangian field equations and the non-Lagrangian Labastida ones

$$\mathcal{F} \,=\, 0 \tag{4.3}$$

must be verified even at the constrained level, as first pointed out in [39]. Rather than adding complications, the unconstrained formulation simplifies the treatment of this problem providing the more manageable form of the field equations displayed in eq. (4.1), that allowed the detailed discussion of [40].

To better describe the framework, let me focus for a while on the fully symmetric case. There the field equations (4.1) and (4.2) take the simpler form

$$E_\varphi \; : \; \mathcal{F}_{\mu_1\dots\mu_s} \; - \; \frac{1}{2}\,\eta_{(\mu_1\mu_2}\,\mathcal{F}_{\mu_3\dots\mu_s)\lambda}{}^{\lambda} \; + \; \eta_{(\mu_1\mu_2}\,\eta_{\mu_3\mu_4}\,\mathcal{B}_{\mu_5\dots\mu_s)} \; = \; 0\,, \qquad (4.4)$$

$$E_\mathcal{B} \; : \; \varphi_{\mu_1\dots\mu_{s-4}\lambda\rho}{}^{\lambda\rho} \; = \; 0 \quad \Longrightarrow \quad \mathcal{F}_{\mu_1\dots\mu_{s-4}\lambda\rho}{}^{\lambda\rho} \; = \; 0\,, \qquad (4.5)$$

and they imply $\mathcal{F}_{\mu_1\dots\mu_s} \; = \; 0$ and $\mathcal{B}_{\mu_1\dots\mu_{s-4}} \; = \; 0$ unless $s = 2$ and $D = 2$. Even if the field equations take a universal form in any space-time dimension, an explicit D dependence indeed emerges when computing their traces. One can better understand its role recalling that in the unique pathological case the combination of the kinetic tensor and its trace appearing in eq. (4.4) becomes traceless. In the mixed-symmetry case a similar behavior is displayed by more complicated models so that, in general, eq. (4.1) *does not* reduce directly to the conditions

$$\mathcal{F} = 0\,, \qquad\qquad \mathcal{B}_{ijkl} = 0\,. \qquad (4.6)$$

The existence of non-trivial solutions of this homogeneous equation calls for the existence of additional symmetries, and let me stress again that we already know an example of this kind. In fact, the symmetry of the Lagrangians under the shift (3.31) of the Lagrange multipliers induces a symmetry of the field equations under shifts

$$\delta\,\mathcal{B}_{ijkl} \; = \; \eta^{mn}\,Y_{\{5,1\}}\,L_{ijkl,mn} \qquad (4.7)$$

even when the $\mathcal{B}_{ijkl}$ are composite objects. For all mixed-symmetry fields carrying a large enough number of space-time indices the field equations thus do not set to zero all trace components of the $\mathcal{B}_{ijkl}$ tensors. On the other hand, those left undetermined do not really enter the linear combinations that are present in the equations of motion or, equivalently, they can be gauged away exploiting the symmetry (4.7). As a result, the leftover components are immaterial insofar as the reduction to the Labastida form (4.3) is concerned. More explicit examples of this statement can be found in [40], where the field equations of some simple models were reduced to the Labastida form dealing with all their traces.

We can now classify the more interesting models where a symmetry involving also the $\mathcal{F}$ portion of eq. (4.1) is present. In these cases some $o(D)$ components of the kinetic tensors are not forced to vanish on-shell by eq. (4.1), as the trace of the linearized Ricci tensor is left undetermined by eq. (4.4) in $D = 2$. Therefore, Lagrangian and non-Lagrangian field equations are *not* directly equivalent. In order to proceed toward the identification of the pathological cases, let me remark that in these preliminary considerations I am treating $\mathcal{F}$ as if it were a generic tensor. This is however *not* true, since it is the specific combination of differential operators acting on the

gauge field φ defined in eq. (2.11). This fact has a couple of notable implications: first of all in some cases the non-Lagrangian field equations could also display additional symmetries, and in highly pathological models the $\mathcal{F}$ tensors even vanish identically[1]. In these examples, rather than discovering a lack of direct equivalence of the different field equations, one can thus only identify a set of symmetries that are common to both setups. Moreover, analyzing the symmetries of the equations of motion we are really interested only in those that can be realized in terms of the fields. This means that we can perform our analysis dealing directly with the expression (4.1), but we must consider only the shift symmetries of $\mathcal{F}$ that can be realized shifting φ. Other possible alternatives would be associated to tensors that cannot be written in the form (2.11), and that are thus of no interest. This information can be codified requesting that the $\delta \mathcal{F}$ transformations also preserve the Bianchi identities, that directly follow from eq. (2.11). When enforcing the field equations (4.2) these take the form

$$\mathscr{B}_i \; : \; \partial_i \mathcal{F} \, - \, \frac{1}{2} \, \partial^j \, T_{ij} \, \mathcal{F} \, = \, 0 \,. \tag{4.8}$$

Furthermore, we must clearly consider only the transformations that preserve also the set of field equations (4.2), or equivalently that preserve

$$T_{(ij} \, T_{kl)} \, \mathcal{F} \, = \, 0 \,. \tag{4.9}$$

Thus, in the following I will first classify the conditions that allow non-trivial shifts of $\mathcal{F}$ preserving the Bianchi identities (4.8) and the constraints (4.9), and then I will show that all these transformations also leave invariant the field equation (4.1). I will finally close the circle, showing explicitly that all these $\mathcal{F}$ shifts can be induced by Weyl-like transformations of the gauge field φ. These thus enable one to complete the reduction of the Lagrangian field equations to the Labastida form via a partial gauge fixing, and thus complete the proof of the equivalence of the two setups in this roundabout way. The conditions that I am going to identify will be then discussed in Section 4.2.1, but I can anticipate that all the pathological models that I am going to present "live" in backgrounds with dimensions $D < \frac{N-2}{2}$, where N is the number of index families. This means that they do not really propagate the degrees of freedom of a mixed-symmetry representation of the Lorentz group, and actually they do not propagate any local degrees of freedom. At any rate, this is true also for two-dimensional gravity, so that the appearance of Weyl-invariant models in arbitrary space-time dimensions could be of some interest in the future.

Looking at eq. (4.1) one can then recognize that, if $\mathcal{F}$ does not vanish identically, in principle it can admit shift symmetries of the form[2]

$$\delta \mathcal{F} \, = \, \eta^{ij} \, \Omega_{ij} \,, \tag{4.10}$$

---

[1]A very simple example, even if highly degenerate, is given by a spin 1 field in one dimension, for which the Fronsdal tensor vanishes identically.

[2]Since we are not considering the cases where $\mathcal{F}$ is identically zero, we are thus going to identify only the conditions that select the models for which the Lagrangian field equations are not directly equivalent to the non-Lagrangian ones. An exhaustive classification of the symmetries of the actions should also take care of the highly degenerate cases where $\mathcal{F}$ itself vanishes.

and we now want to classify those that also preserve the Bianchi identities. In general under shifts of the form (4.10) these vary according to[3]

$$\delta \mathscr{B}_i = -\frac{1}{2}\, \partial^j \left[\, (D-2)\, \Omega_{ij} + S^k{}_{(i}\, \Omega_{j)\,k}\,\right] + \eta^{jk}\, \mathscr{B}_i \left(\Omega_{jk}\right), \qquad (4.11)$$

where the $\mathscr{B}_i \left(\Omega_{jk}\right)$ obtain acting with the differential operator $\partial_i - \frac{1}{2}\, \partial^j\, T_{ij}$ on the Ω_{jk} parameters. Thus, if the conditions

$$(D-2)\, \Omega_{ij} + S^k{}_{(i}\, \Omega_{j)\,k} = 0, \qquad (4.12)$$

are satisfied by non-trivial parameters Ω_{ij} that are themselves subject to the Bianchi identities, the corresponding shift (4.10) preserve indeed eq. (4.8). Notice that the $S^i{}_j$ operators already encountered in Section 2.3 and defined in detail in Appendix A play a crucial role at this stage. Indeed, the conditions (4.12) are nothing but eigenvalue problems for the $S^i{}_j$ operators, parametrized by the space-time dimension D. This outcome restates in a more precise algebraic language that, while field equations and Lagrangians take a universal form that does not depend on the space-time dimension and on the detailed Lorentz structure of the fields under scrutiny, their traces do not. They depend explicitly on D, while we shall see that the behavior of the $S^i{}_j$ operators is directly related to the Lorentz structure of the tensors on which they act. The eigenvalue problems (4.12) will be discussed in Section 4.2.1, but let me stress that in general these conditions are only *sufficient* to identify a symmetry of the Bianchi identities. In fact, the contributions that appear in eq. (4.11) could in principle balance each other. However, one can proceed to look for alternative solutions in an iterative manner, analyzing directly more complicated shifts of $\mathcal{F}$ that involve more than one naked η^{ij} tensor.

The most general shift transformation that one can consider is of the form

$$\delta \mathcal{F} = \eta^{i_1 j_1} \ldots \eta^{i_n j_n}\, \Omega^{(n)}{}_{i_1 j_1, \ldots, i_n j_n}, \qquad (4.13)$$

and varies the Bianchi identities according to

$$\begin{aligned}
\delta \mathscr{B}_k = &-\frac{n}{2}\, \eta^{i_1 j_1} \ldots \eta^{i_{n-1} j_{n-1}}\, \partial^l \Big[\, (D-2)\, \Omega^{(n)}{}_{kl,\, i_1 j_1, \ldots,\, i_{n-1} j_{n-1}} \\
&+ \sum_{r=1}^{n-1} \Omega^{(n)}{}_{i_r\,(k,\,l)\,j_r,\,\ldots,\,i_{s\neq r} j_{s\neq r},\,\ldots} + S^m{}_{(k}\, \Omega^{(n)}{}_{l)\,m,\,i_1 j_1,\,\ldots,\,i_{n-1} j_{n-1}} \Big] \\
&+ \eta^{i_1 j_1} \ldots \eta^{i_n j_n}\, \mathscr{B}_k \left(\Omega^{(n)}{}_{i_1 j_1, \ldots, i_n j_n}\right).
\end{aligned} \qquad (4.14)$$

Following this recursive approach, in order to identify the models that can display additional Weyl-like symmetries one is thus led to consider all the conditions of the form

$$\begin{aligned}
(D-2)\, \Omega^{(n)}{}_{kl,\,i_1 j_1,\,\ldots,\,i_{n-1} j_{n-1}} &+ \sum_{r=1}^{n-1} \Omega^{(n)}{}_{i_r\,(k,\,l)\,j_r,\,\ldots,\,i_{s\neq r} j_{s\neq r},\,\ldots} \\
&+ S^m{}_{(k}\, \Omega^{(n)}{}_{l)\,m,\,i_1 j_1,\,\ldots,\,i_{n-1} j_{n-1}} = 0.
\end{aligned} \qquad (4.15)$$

---

[3] In this section I will often introduce a slight abuse of notation, identifying with $\mathscr{B}_i$ and with "Bianchi identities" only the left-hand side of eq. (4.8).

Their solutions depend again on the space-time dimension and on the Lorentz structure of the involved tensors. Notice also that these conditions are all inequivalent, since the one identified by a given value of n cannot be obtained via the redefinition

$$\Omega_{i_1 j_1, \dots, i_{n-1} j_{n-1}} \rightarrow \eta^{i_n j_n} \Omega_{i_1 j_1, \dots, i_n j_n}, \qquad (4.16)$$

or extracting in a similar way η^{ij} tensors from the parameters characterized by lower values of n. This confirms the initial motivation to come across eqs. (4.15), and one is thus forced to deal with all these conditions in order to classify Weyl-invariant models[4]. On the other hand, whenever one fixes the number of index families N, not all the shifts (4.13) compatible with the Lorentz structure of the field are expected to play a role, simply because the constraints (4.9) imply that all combinations of $N+1$ traces of $\mathcal{F}$ vanish.

I can now show that the same set of conditions emerges even when one considers the variation of the constraints (4.9) under these transformations. Let me start with the simplest example: under the shift (4.10) the symmetrized double traces of $\mathcal{F}$ vary as

$$T_{(ij}\, T_{kl)}\, \delta\, \mathcal{F} \;=\; T_{(ij|}\left[\,(\,D-2\,)\,\Omega_{|kl)} \;+\; S^{m}{}_{|k}\,\Omega_{l)m}\,\right] \;+\; \eta^{mn}\, T_{(ij}\, T_{kl)}\, \Omega_{mn}\,, \qquad (4.17)$$

so that they vanish provided that eq. (4.12) holds and that the parameters satisfy

$$T_{(ij}\, T_{kl)}\, \Omega_{mn} \;=\; 0\,. \qquad (4.18)$$

These last conditions will not play any role when we will look for the shift symmetries of the equation of motion for φ, but they will be crucial in the identification of the pathological models for which, let me reiterate, *all* field equations must possess a symmetry. Otherwise, the quantities left undetermined by eq. (4.1) will be simply forced to vanish by eq. (4.2).

For generic shifts of the form (4.13) it is more convenient to proceed in steps. The traces of the $\mathcal{F}$ variation indeed read

$$T_{ab}\,\delta\,\mathcal{F} \;=\; n\,\eta^{i_1 j_1}\dots\eta^{i_{n-1}j_{n-1}}\Big[\,D\,\Omega^{(n)}{}_{ab,\,i_1 j_1,\dots,\,i_{n-1}j_{n-1}}$$
$$+ \sum_{r=1}^{n-1}\Omega^{(n)}{}_{i_r\,(a,b)\,j_r,\dots,\,i_{s\neq r}j_{r\neq n},\dots} \;+\; S^{k}{}_{(a}\,\Omega^{(n)}{}_{b)k,\,i_1 j_1,\dots,\,i_{n-1}j_{n-1}}\Big]$$
$$+ \eta^{i_1 j_1}\dots\eta^{i_n j_n}\, T_{ab}\,\Omega^{(n)}{}_{i_1 j_1,\dots,\,i_n j_n}\,, \qquad (4.19)$$

and we can simplify them using the relations (4.15). In fact, the aim is to check whether the constraints (4.9) imply *additional* conditions, so that we can restrict the attention only to the cases where the conditions (4.15) are satisfied. When eqs. (4.15) hold the traces of $\delta\,\mathcal{F}$ simply reduce to

$$T_{ab}\,\delta\mathcal{F} \;=\; \eta^{i_1 j_1}\dots\eta^{i_{n-1}j_{n-1}}\Big[\,2\,n\,\Omega^{(n)}{}_{ab,\,i_1 j_1,\dots,\,i_{n-1}j_{n-1}} + \eta^{i_n j_n}\, T_{ab}\,\Omega^{(n)}{}_{i_1 j_1,\dots,\,i_n j_n}\,\Big]. \qquad (4.20)$$

---

[4]As in the first step of this analysis, the conditions (4.15) actually select only the cases where Lagrangian and non-Lagrangian field equations are not directly equivalent, while an exhaustive classification of the symmetries of the actions should also identify all the vanishing $o\,(D)$ components of the $\mathcal{F}$ tensors.

Computing a further trace and using again the relations (4.15) leads eventually to

$$
T_{ab}\,T_{cd}\,\delta\,\mathcal{F} \;=\; 2\,n\,(\,n-1\,)\,\eta^{i_1 j_1}\ldots\eta^{i_{n-2}j_{n-2}}\times
$$

$$
\times\Big[\,2\,\Omega^{(n)}{}_{ab,\,cd,\,i_1 j_1,\,\ldots,\,i_{n-2}j_{n-2}} \;-\; \Omega^{(n)}{}_{a\,(\,c,\,d\,)\,b,\,i_1 j_1,\,\ldots,\,i_{n-2}j_{n-2}}\,\Big]
$$

$$
+\; n\,\eta^{i_1 j_1}\ldots\eta^{i_{n-1}j_{n-1}}\Big[\,D\,T_{ab}\,\Omega^{(n)}{}_{cd,\,i_1 j_1,\,\ldots,\,i_{n-1}j_{n-1}} \;+\; 2\,T_{cd}\,\Omega^{(n)}{}_{ab,\,i_1 j_1,\,\ldots,\,i_{n-1}j_{n-1}}
$$

$$
-\; T_{(\,a\,|\,(\,c}\,\Omega^{(n)}{}_{d\,)\,|\,b\,),\,i_1 j_1,\,\ldots,\,i_{n-1}j_{n-1}} \;+\; \sum_{r=1}^{n-1}\,T_{ab}\,\Omega^{(n)}{}_{i_r\,(\,c,\,d\,)\,j_r,\,\ldots,\,i_{s\neq r}j_{r\neq n},\,\ldots}
$$

$$
+\; T_{ab}\,S^{k}{}_{(\,c}\,\Omega^{(n)}{}_{d\,)\,k,\,i_1 j_1,\,\ldots,\,i_{n-1}j_{n-1}}\,\Big] \;+\; \eta^{i_1 j_1}\ldots\eta^{i_n j_n}\,T_{ab}\,T_{cd}\,\Omega^{(n)}{}_{i_1 j_1,\,\ldots,\,i_n j_n}\,. \qquad (4.21)
$$

After symmetrizing in the four family indices $(\,a,b,c,d\,)$ the first group of terms vanishes while the others combine to give

$$
T_{(\,ab}\,T_{cd\,)}\,\delta\,\mathcal{F} \;=\; \frac{n}{2}\,\eta^{i_1 j_1}\ldots\eta^{i_{n-1}j_{n-1}}\,T_{(\,ab\,|}\Big[\,(\,D-2\,)\,\Omega^{(n)}{}_{|\,cd\,),\,i_1 j_1,\,\ldots,\,i_{n-1}j_{n-1}}
$$

$$
+\; \sum_{r=1}^{n-1}\,\Omega^{(n)}{}_{i_r\,|\,c,\,d\,)\,j_r,\,\ldots,\,i_{s\neq r}j_{r\neq n},\,\ldots} \;+\; S^{k}{}_{|\,c}\,\Omega^{(n)}{}_{d\,)\,k,\,i_1 j_1,\,\ldots,\,i_{n-1}j_{n-1}}\,\Big]
$$

$$
+\; \eta^{i_1 j_1}\ldots\eta^{i_n j_n}\,T_{(\,ab}\,T_{cd\,)}\,\Omega^{(n)}{}_{i_1 j_1,\,\ldots,\,i_n j_n}\,. \qquad (4.22)
$$

In conclusion, when eqs. (4.15) hold even the constraints (4.9) are preserved by a generic shift of the form (4.13) provided the parameters also satisfy the conditions

$$
T_{(\,ab}\,T_{cd\,)}\,\Omega^{(n)}{}_{i_1 j_1,\,\ldots,\,i_n j_n} \;=\; 0\,. \qquad (4.23)
$$

I can now move on to prove that, as expected, all shifts satisfying the conditions (4.15) also leave invariant the field equation (4.1). Let me start again from the simplest example, in order to clarify the strategy that I am going to conform to. The shift (4.10) varies the Einstein-like tensors (2.138) as

$$
\delta\,\mathcal{E} \;=\; \sum_{p=1}^{N}\frac{(-1)^p}{p!\,(\,p+1\,)!}\,\eta^{i_1 j_1}\ldots\eta^{i_p j_p}\,[\,Y_{\{2^p\}}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,,\,\eta^{kl}\,]\,\Omega_{kl}
$$

$$
+\; \sum_{p=0}^{N}\frac{(-1)^p}{p!\,(\,p+1\,)!}\,\eta^{i_1 j_1}\ldots\eta^{i_p j_p}\,\eta^{kl}\,\big(\,Y_{\{2^p\}}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,\big)\,\Omega_{kl}\,. \qquad (4.24)
$$

Taking into account the $\{2^p\}$ Young projection in the family indices, the commutator appearing in eq. (4.24) reduces to

$$
[\,Y_{\{2^p\}}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,,\,\eta^{kl}\,]\,\Omega_{kl}
$$

$$
=\; Y_{\{2^p\}}\,\sum_{n=1}^{p}\,\prod_{r\neq n}^{p}\,T_{i_r j_r}\,\big\{\,(\,D+p-1\,)\,\Omega_{i_n j_n} \;+\; S^{k}{}_{(\,i_n}\,\Omega_{j_n\,)\,k}\,\big\}\,, \qquad (4.25)
$$

as can be seen resorting to the techniques repeatedly used to manipulate the traces of the Bianchi identities in Sections 2.2.1 and 3.1.1 and to the computational rules collected in Appendix B. Once the two-column projection of the second line of eq. (4.24)

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is combined with the first, one is left with

$$\delta \mathcal{E} = \sum_{p=1}^{N} \frac{(-1)^p}{p!\,(p+1)!}\, \eta^{i_1 j_1} \dots \eta^{i_p j_p}\, Y_{\{2^p\}} \sum_{n=1}^{p} \prod_{r \neq n} T_{i_r j_r} \left\{ (D-2)\,\Omega_{i_n j_n} + S^k{}_{(i_n}\Omega_{j_n)k} \right\}$$
$$+ \sum_{p=2}^{N} \frac{(-1)^p}{(p!)^2}\, \eta^{i_1 j_1} \dots \eta^{i_p j_p}\, Y_{\{4,2^{p-2}\}} \sum_{n=1}^{p} \left(Y_{\{2^{p-1}\}} \prod_{r \neq n} T_{i_r j_r} \right) \Omega_{i_n j_n}\,. \qquad (4.26)$$

The arguments used to identify the ansatz for the constrained Lagrangians in Section 2.2.1 now guarantee that the $\{4, 2^{p-2}\}$ projection present in eq. (4.26) can be recast in a form where the family indices carried by a pair of η tensors are fully symmetrized. Hence, the terms in the second line can be compensated by a shift of the $\mathcal{B}_{ijkl}$, in sharp contrast with the whole first line, that vanishes if one tries to symmetrize three or more indices. Therefore, these terms can not be canceled redefining the $\mathcal{B}_{ijkl}$ and must vanish identically, as they indeed do if the conditions (4.12) hold. As in the previous analyses, alternative solutions exist if the contributions appearing in the first line actually embody at least an additional η tensor, and this leads again to consider the other shifts of the form (4.13).

To begin with, one can notice that the conditions (4.15) suffice to eliminate all the $S^i{}_j$ operators that would appear in the variation of the Einstein like-tensors. As a result, when eqs. (4.15) hold all the $\Omega^{(n)}$ parameters that carry a Young projection with more than two columns induce a variation of $\mathcal{E}$ that can be canceled by suitable $\mathcal{B}_{ijkl}$ shifts. On the other hand, for two-column projected Weyl-parameters the invariance of the field equations under shifts satisfying eqs. (4.15) is less evident, and it rests on the precise form of these relations rather than on the mere possibility of eliminating all $S^i{}_j$ operators. To treat this problem in full generality it is convenient to introduce the notation

$$\mathcal{O}(\lambda)^{kl}{}_{ij} = \frac{\lambda}{2}\,\delta_i{}^{(k}\delta^{l)}{}_j + S^{(k}{}_{(i}\delta^{l)}{}_{j)}\,. \qquad (4.27)$$

Restricting the attention to two-column projected parameters the conditions (4.15) then take the form

$$\mathcal{O}(D-n-1)^{ab}{}_{ij}\,\Omega^{(n)}{}_{ab,\,k_1 l_1,\dots,\,k_{n-1}l_{n-1}} = 0\,, \qquad (4.28)$$

simply because any symmetrization over four family indices vanishes. Moreover, starting from eq. (4.26) one can prove via a recursive argument that a shift of the form (4.13) induces the variation

$$\delta \mathcal{E} = \sum_{p=n}^{N} n!\,k_p\, \eta^{i_1 j_1} \dots \eta^{i_p j_p}\, Y_{\{2^p\}} \sum_{r_1 < \dots < r_n} \prod_{s \neq r_q} T_{i_s j_s}\, \mathcal{O}(D-n-1)^{k_1 l_1}{}_{i_{r_1} j_{r_1}} \times$$
$$\times\, \mathcal{O}(D-n-2)^{k_2 l_2}{}_{i_{r_2} j_{r_2}} \dots \mathcal{O}(D-2n)^{k_n l_n}{}_{i_{r_n} j_{r_n}}\, \Omega^{(n)}{}_{k_1 l_1,\dots,\,k_n l_n}\,, \qquad (4.29)$$

up to terms that carry Young projections with more than two columns. Indeed, we have already seen that these can be compensated by suitable $\mathcal{B}_{ijkl}$ shifts and thus do

not have to vanish identically. All crucial contributions are thus displayed in eq. (4.29), whose recursive proof can be obtained via the redefinitions

$$\Omega^{(n)}{}_{k_1 l_1, \ldots, k_n l_n} \ \rightarrow \ \eta^{ab} \, \Omega^{(n+1)}{}_{ab, k_1 l_1, \ldots, k_n l_n} \,, \tag{4.30}$$

and noticing that they imply

$$\mathcal{O} \, (\, \lambda \,)^{ab}{}_{ij} \, \Omega^{(n)}{}_{ab, k_1 l_1, \ldots, k_{n-1} l_{n-1}} \ \rightarrow \ \eta^{ab} \, \mathcal{O} \, (\, \lambda - 2 \,)^{cd}{}_{ij} \, \Omega^{(n+1)}{}_{ab, cd, k_1 l_1, \ldots, k_{n-1} l_{n-1}} \,. \tag{4.31}$$

Therefore, one can move the η^{ab} tensor to the left of the $\mathcal{O}$ operators using (4.31), and then repeat the steps that led to eq. (4.26) to conclude the proof.

The expression appearing in eq. (4.29) is not normal ordered, but it is rather convenient since it contains the same operator defining the conditions (4.28) that we identified analyzing the Bianchi identities. $\mathcal{O} \, (\, D - n - 1 \,)$ does not appear immediately on the left of the Weyl parameter as in eq. (4.28), but one can notice that eqs. (4.28) also imply

$$S^m{}_{(i} \, \Omega^{(n)}{}_{j) m, kl, \ldots} \ - \ S^m{}_{(k} \, \Omega^{(n)}{}_{l) m, ij, \ldots} \ = \ 0 \tag{4.32}$$

via proper reshuffling of their indices. As a consequence, when eqs. (4.28) hold the $\mathcal{O}$ operators commute since

$$[\, \mathcal{O} \, (\, \lambda_1 \,)^{mn}{}_{ij} \, , \, \mathcal{O} \, (\, \lambda_2 \,)^{pq}{}_{kl} \,] \, \Omega_{mn, pq, \ldots} \ = \ S^m{}_{(i} \, \Omega_{j) m, kl, \ldots} \ - \ S^m{}_{(k} \, \Omega_{l) m, ij, \ldots} \,. \tag{4.33}$$

This suffices to show that the equation of motion (4.1) is left invariant by shift transformations of $\mathcal{F}$ of the form (4.13) provided the parameters satisfy the conditions (4.15).

In eqs. (4.15) and (4.23) we have thus identified the set of conditions that select the "physical" shift symmetries of the field equations (4.1) and (4.2), and we can now verify that they indeed follow from suitable Weyl-like shifts of the gauge fields. One can begin by considering a Weyl-like transformation of the form

$$\delta \, \varphi \ = \ \eta^{ij} \, \Theta_{ij} \,, \tag{4.34}$$

that shift the Fronsdal-Labastida tensor (2.11) as

$$\delta \, \mathcal{F} \ = \ \frac{1}{2} \, \partial^i \partial^j \left[\, (\, D - 2 \,) \, \Theta_{ij} \ + \ S^k{}_{(i} \, \Theta_{j) k} \, \right] \ + \ \eta^{ij} \, \mathcal{F} \, (\, \Theta_{ij} \,) \,. \tag{4.35}$$

The $\mathcal{F} \, (\, \Omega_{ij} \,)$ are simply Fronsdal-Labastida tensors for the Ω_{ij} parameters, while the conditions that emerge commuting η^{ij} with the derivatives are again those appearing in eqs. (4.11), (4.17) and (4.24). As a consequence, if the equations of motion admit a shift symmetry subject to the conditions (4.12) and (4.18), it is *always* possible to realize it in the form

$$\delta \, \mathcal{F} \ = \ \eta^{ij} \, \mathcal{F} (\, \Theta_{ij} \,) \tag{4.36}$$

starting from the shift (4.34) of the gauge field. Indeed, the $S^i{}_j$ operators commute with the Fronsdal-Labastida ones, so that

$$(\, D - 2 \,) \, \mathcal{F} (\, \Theta_{ij} \,) \ + \ S^k{}_{(i|} \, \mathcal{F} \, (\, \Theta_{|j) k} \,) \ = \ 0 \tag{4.37}$$

if the Θ_{ij} satisfy the conditions (4.12). Furthermore, an admissible shift of the field should clearly satisfy

$$T_{(ij} T_{kl)} \delta \varphi = 0 \,. \tag{4.38}$$

When eq. (4.12) holds this implies that

$$T_{(ij} T_{kl)} \Theta_{mn} = 0 \,, \tag{4.39}$$

and, as a consequence of eq. (2.73), this leads to

$$T_{(ij} T_{kl)} \mathcal{F}(\Theta_{mn}) = 0 \,. \tag{4.40}$$

The conditions (4.39) then imply that the Θ_{ij} satisfy the same properties as gauge fields, so that the $\mathcal{F}(\Theta_{ij})$ are also subject to the Bianchi identities. In conclusion, the induced shifts (4.36) fulfill all the conditions we selected when working directly in terms of $\mathcal{F}$ shifts, and they thus leave the field equations (4.1) and (4.2) invariant.

Even in this context the terms appearing in eq. (4.35) could well cancel each other, and one is thus led to consider in a recursive fashion also shifts of the form

$$\delta \varphi = \eta^{i_1 j_1} \ldots \eta^{i_n j_n} \Theta^{(n)}{}_{i_1 j_1, \ldots, i_n j_n} \tag{4.41}$$

in order to classify the Weyl-like transformations that leave invariant the traceless part of $\mathcal{F}$. These affect the Fronsdal-Labastida tensor as

$$\delta \mathcal{F} = \frac{n}{2} \, \eta^{i_1 j_1} \ldots \eta^{i_{n-1} j_{n-1}} \partial^l \Big[(D-2) \Theta^{(n)}{}_{kl, i_1 j_1, \ldots, i_{n-1} j_{n-1}}$$
$$+ \sum_{r=1}^{n-1} \Theta^{(n)}{}_{i_r (k, l) j_r, \ldots, i_{s \neq r} j_{s \neq r}, \ldots} + S^m{}_{(k} \Omega^{(n)}{}_{l) m, i_1 j_1, \ldots, i_{n-1} j_{n-1}} \Big]$$
$$+ \eta^{i_1 j_1} \ldots \eta^{i_n j_n} \mathcal{F}\big(\Theta^{(n)}{}_{i_1 j_1, \ldots, i_n j_n}\big) \,, \tag{4.42}$$

so that, whenever the conditions (4.15) hold, the shifts (4.13) leaving invariant the field equations can be realized in the form

$$\delta \mathcal{F} = \eta^{i_1 j_1} \ldots \eta^{i_n j_n} \mathcal{F}\big(\Theta^{(n)}{}_{i_1 j_1, \ldots, i_n j_n}\big) \,. \tag{4.43}$$

Notice that in the bosonic framework the analogy between shifts of $\mathcal{F}$ that preserve the Bianchi identities and shifts of φ that act on $\mathcal{F}$ as in eq. (4.43) is easily seen. Indeed, barring an irrelevant overall gradient, the terms of $\mathcal{F}$ that do not contain $\Box \varphi$ build the same combination of differential operators that enter the Bianchi identities. On the other hand, in the fermionic case this analogy is not so direct and these different viewpoints will be of help in the identification of the relevant Weyl-like transformations.

4.1.2 Fermi fields

In analogy with the bosonic case, all the unconstrained Lagrangians that I presented in eqs. (3.69), (3.76) and (3.83) give rise to field equations that can be cast in the form

$$E_{\bar{\psi}} : \sum_{p,q=0}^{N} \frac{(-1)^{p + \frac{q(q+1)}{2}}}{p! \, q! \, (p+q+1)!} \, \eta^p \gamma^q \big(\gamma^{[q]} \mathcal{S}^{[p]}\big) - \frac{1}{2} \, \eta^{ij} \gamma^k \mathcal{Y}_{ijk} = 0 \,, \tag{4.44}$$

$$E_{\bar{y}} : T_{(ij} \gamma_{k)} \psi = 0 \quad \Longrightarrow \quad T_{(ij} \gamma_{k)} \mathcal{S} = 0 \,, \tag{4.45}$$

possibly after a partial gauge fixing that eliminates the compensators Ψ_i. Moreover, we have seen that these conditions can also describe the field equations following from the constrained Lagrangians. In that case the second condition holds by definition even off-shell, while the field equation for $\bar\psi$ is obtained eliminating the $\mathcal{Y}_{ijk}$ tensors from eq. (4.44). As for bosons, the equivalence of the Lagrangian field equations with the non-Lagrangian Labastida ones

$$\mathcal{S} = 0 \tag{4.46}$$

must thus be verified even at the constrained level. Rather than adding complications, the unconstrained formulation again simplifies the analysis, providing the more manageable form of the field equations displayed in eq. (4.44).

In this respect Bose and Fermi fields display a similar behavior, since in general eq. (4.44) *does not* reduce directly to the conditions

$$\mathcal{S} = 0\,, \qquad\qquad \mathcal{Y}_{ijk} = 0\,. \tag{4.47}$$

The existence of non-trivial solutions of the homogeneous equation (4.44) is again equivalent to the existence of additional symmetries, and even in this setting we already know an example of this kind. Indeed, the shift symmetry (3.71) of the Lagrange multipliers induces a symmetry of the field equations under[5]

$$\delta\,\mathcal{Y}_{ijk} = \eta^{lm}\, M_{ijk,\,lm} + \gamma^{lm}\, N_{ijk;\,lm} \tag{4.48}$$

even when the $\mathcal{Y}_{ijk}$ are composite objects. Therefore, in general the fields equations do not set to zero all γ-trace components of the $\mathcal{Y}_{ijk}$ tensors. On the other hand, the undetermined components do not really enter the linear combinations that are present in the equations of motion or, equivalently, can be gauged away exploiting the symmetry (4.48). As a consequence they are immaterial insofar as the reduction to the Labastida form (4.46) is concerned. More explicit examples of this statement can be found in [41], where the field equations of some simple models were reduced to the Labastida form dealing with all their γ-traces.

As in the bosonic case, one can then move on to consider more interesting shift symmetries that involve also the $\mathcal{S}$ portion of eq. (4.44). In these cases some $o\,(D)$ components of the kinetic tensors are not forced to vanish on-shell by eq. (4.44), and thus Lagrangian and non-Lagrangian field equations are *not* directly equivalent. However, even in the fermionic framework one has to take into account that the $\mathcal{S}$ tensor identifies the combination of differential operators acting on the gauge field ψ defined in eq. (2.11). As a result, in some cases Lagrangian and non-Lagrangian field equations could share additional symmetries and, most importantly, one has to select only the symmetries of the formal expression (4.44) that can be realized in terms of the fields. This information can be again codified in the request for $\delta\,\mathcal{S}$ transformations that also preserve the Bianchi identities, that take the form

$$\mathscr{B}_i \,:\, \partial_i\,\mathcal{S} - \frac{1}{2}\,\slashed{\partial}\,\gamma_i\,\mathcal{S} - \frac{1}{2}\,\partial^j\,T_{ij}\,\mathcal{S} - \frac{1}{6}\,\partial^j\,\gamma_{ij}\,\mathcal{S} = 0\,, \tag{4.49}$$

---

[5]In order to lead to a symmetry of the equations of motion, the $M_{ijk,\,lm}$ and $N_{ijk;\,lm}$ parameters must be $\{4, 1\}$ projected and must be related between each other as in eq. (3.72).

when eqs. (4.45) hold. Furthermore, the relevant $\mathcal{S}$ shifts must also preserve the constraints

$$T_{(ij}\gamma_{k)}\,\mathcal{S} = 0\,. \tag{4.50}$$

In principle one should then analyze these conditions following an iterative procedure similar to that developed for Bose fields. On the other hand, in the present framework this would involve a number of technical difficulties. For this reason, I will present here a detailed analysis of this kind only for the first level of the iteration. For the successive levels I am going to discuss only the variations of the $\mathcal{S}$ tensors induced by the Weyl-like transformation of the fields. These identify a set of conditions that should naturally lead to symmetries of the actions, and for some partial checks of this conjecture I refer the reader to [41]. There the two-family case is discussed in full detail, and for N-family fields some additional checks are presented in Appendix D. In spite of the sharp differences at the technical level, let me stress that no substantial differences with respect to the bosonic setup are expected to be met in the fermionic one. For instance, in Section 4.2.2 I will show that the conditions I am going to treat admit solutions very similar to those emerging for Bose fields. In particular, let me anticipate that I will present a rather rich set of Weyl-invariant models, that all "live" in $D < \frac{N-2}{2}$ space-time dimensions and do not propagate any local degrees of freedom.

The discussion of Section 4.1.1 thus suggests to begin by considering Weyl-like transformations

$$\delta\,\psi = \gamma^{i}\,\Theta_{i}\,, \tag{4.51}$$

that give rise to a variation of the Fang-Fronsdal-Labastida tensor (2.41) of the form

$$\delta\,\mathcal{S} = -\,i\,\partial^{k}\left[\,(D-2)\,\Theta_{k} + 2\,S^{l}{}_{k}\,\Theta_{l}\,\right] - \gamma^{k}\,\mathcal{S}\,(\Theta_{k})\,. \tag{4.52}$$

As in the bosonic case, the formal expression (4.44) in principle can admit shift symmetries of the type[6]

$$\delta\,\mathcal{S} = \gamma^{i}\,\Omega_{i}\,, \tag{4.53}$$

so that one is led to consider the conditions

$$(D-2)\,\Theta_{i} + 2\,S^{j}{}_{i}\,\Theta_{j} = 0 \tag{4.54}$$

in order to select fields that are good candidates to display an extra symmetry of the action. In these cases one can indeed identify the Ω_{i} of eq. (4.53) with $-\,\mathcal{S}\,(\Theta_{i})$ and, since the $S^{i}{}_{j}$ operators commute with $\mathcal{S}$, the Ω_{i} also satisfy eq. (4.54). In order to fully characterize the ansatz one must also take into account that the Weyl-like gauge transformations must preserve the triple γ-trace constraints, so that

$$T_{(ij}\gamma_{k)}\,\delta\,\psi = T_{(ij|}\left[\,(D-2)\,\Theta_{|k)} + 2\,S^{l}{}_{|k)}\,\Theta_{l}\,\right] - \gamma^{l}\,T_{(ij}\gamma_{k)}\,\Theta_{l} \tag{4.55}$$

must vanish. When eq. (4.54) holds this only implies the additional conditions

$$T_{(ij}\gamma_{k)}\,\Theta_{l} = 0\,, \tag{4.56}$$

---

[6]As in the bosonic case, I am discarding here the cases in which $\mathcal{S}$ vanishes identically. This analysis is thus aimed to select the cases where Lagrangian and non-Lagrangian field equations are not directly equivalent.

that in analogy with the bosonic case induce similar conditions on the Ω_i via eq. (2.101).

We can now verify that Weyl-like transformations of the form (4.51) satisfying the conditions (4.54) and (4.56) leave invariant the equations of motion. In the $E_{\bar{y}}$ case eq. (4.55) already suffices to conclude that this is true, while in the $E_{\bar{\psi}}$ case more work is needed. On the other hand, we are dealing with $\delta\,\mathcal{S}$ transformations that can be expressed as $\mathcal{S}\,(\,\gamma^i\,\Theta_i\,)$ on account of eq. (4.52), and that preserve the constraints (4.50) on account of eq. (4.55). As a result, they must preserve the Bianchi identities as well, and a direct check indeed gives

$$
\begin{aligned}
\delta\,\mathscr{B}_i \;=\; &- \,\frac{1}{2}\, \slashed{\partial}\Big[\,(\,D-2\,)\,\Omega_i \;+\; 2\,S^j{}_i\,\Omega_j\,\Big] \\
&- \,\frac{1}{6}\, \partial^j\,\gamma_{[i|}\Big[\,(\,D-2\,)\,\Omega_{|j]} \;+\; 2\,S^k{}_{|j]}\,\Omega_k\,\Big] \;+\; \gamma^j\,\mathscr{B}_i\,(\,\Omega_j\,). \qquad (4.57)
\end{aligned}
$$

As usual, the $\mathscr{B}_i\,(\,\Omega_j\,)$ denote the Bianchi identities for the parameters Ω_j, that clearly vanish when $\Omega_i = -\,\mathcal{S}\,(\,\Theta_i\,)$ and the constraints (4.56) are satisfied. Since the Bianchi identities provide the frame for the structure of the Rarita-Schwinger-like tensors $\mathcal{E}$ of eq. (2.140), the existence of a symmetry of the field equation (4.44) is thus really expected. In order to finally prove this statement one can verify that the transformation (4.51) shifts $\mathcal{E}$ as

$$
\begin{aligned}
\delta\,\mathcal{E} \;=\; &\sum_{p,\,q=0}^{N} k_{p,\,q}\,\eta^{i_1 j_1}\!\ldots\eta^{i_p j_p}\,\gamma^{k_1\ldots k_q}\,[\,Y_{\{2^p,1^q\}}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,\gamma_{k_1\ldots k_q}\,,\,\gamma^l\,]_{(-1)^{q+1}}\,\Omega_l \\
&+\sum_{p,\,q=0}^{N} \frac{(-1)^q\,q}{p+1}\,k_{p,\,q}\,\eta^{i_1 j_1}\!\ldots\eta^{i_{p+1} j_{p+1}}\gamma^{k_1\ldots k_{q-1}}Y_{\{2^{p+1},1^{q-1}\}}\sum_{n=1}^{p+1}\prod_{r\neq n}^{p}T_{i_r j_r}\gamma_{k_1\ldots k_{q-1}\,(i_n}\,\Omega_{j_n)} \\
&+\sum_{p,\,q=0}^{N} (-1)^q\,k_{p,\,q}\,\eta^{i_1 j_1}\!\ldots\eta^{i_p j_p}\,\gamma^{k_1\ldots k_q}\,\gamma^l\,Y_{\{3,2^{p-1},1^q\}}\big(\,Y_{\{2^p,1^q\}}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,\gamma_{k_1\ldots k_q}\,\big)\,\Omega_l \\
&+\sum_{p,\,q=0}^{N} \frac{(-1)^q}{q+1}\,k_{p,\,q}\,\eta^{i_1 j_1}\!\ldots\eta^{i_p j_p}\,\gamma^{k_1\ldots k_{q+1}}\,Y_{\{2^p,1^{q+1}\}}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,\gamma_{[k_1\ldots k_q}\,\Omega_{k_{q+1}]}. \quad (4.58)
\end{aligned}
$$

When one considers the variation of the full equation of motion (4.44), the third line in eq. (4.58) can be compensated by Ω-dependent shifts of the $\mathcal{Y}_{ijk}$ tensors, on account of its three-column projection. On the contrary, the other three lines should vanish directly, as they indeed do provided they combine to rebuild the analogue of the conditions (4.54). To proceed one can prove that, taking into account the two-column projection in the family indices, the (anti-)commutators reduce to

$$
\begin{aligned}
[\,Y_{\{2^p,1^q\}}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,\gamma_{k_1\ldots k_q}\,,\,\gamma^l\,]_{(-1)^{q+1}}\,\Omega_l \;=\; &Y_{\{2^p,1^q\}}\sum_{n=1}^{p}\prod_{r\neq n}^{p}T_{i_r j_r}\,\gamma_{k_1\ldots k_q\,(i_n}\,\Omega_{j_n)} \\
&+ \,Y_{\{2^p,1^q\}}\,T_{i_1 j_1}\ldots T_{i_p j_p}\,\gamma_{[k_1\ldots k_{q-1}|}\big\{\,(\,D+p+q-1\,)\,\Omega_{|k_q]} \;+\; 2\,S^l{}_{|k_q]}\,\Omega_l\,\big\}. \quad (4.59)
\end{aligned}
$$

Substituting this identity in eq. (4.58) and combining the terms containing the same

invariant tensors finally gives

$$\delta \mathcal{E} = \sum_{p,q=0}^{N} k_{p,q}\, \eta^{i_1 j_1} \ldots \eta^{i_p j_p} \gamma^{k_1 \ldots k_q} \times$$

$$\times\, Y_{\{2^p, 1^q\}}\, T_{i_1 j_1} \ldots T_{i_p j_p}\, \gamma_{[k_1 \ldots k_{q-1}|} \left\{ (D-2)\,\Omega_{|k_q]} + 2\, S^l{}_{|k_q]}\,\Omega_l \right\} + \ldots, \tag{4.60}$$

where I have omitted the third line of eq. (4.58), since I have already stressed that proper shifts of the $\mathcal{Y}_{ijk}$ tensors can cancel it. This concludes the proof that all the Weyl-like transformations (4.51) satisfying the conditions (4.54) and (4.56) leave invariant the equations of motion (4.44) and (4.45).

On the other hand, in analogy with the bosonic framework, the contributions appearing in eq. (4.52) could in principle cancel each other, rather than vanish independently. As anticipated, one should then recursively consider all the Weyl-like transformations

$$\delta \psi = \eta^{i_1 j_1} \ldots \eta^{i_n j_n} \gamma^{k_1 \ldots k_m}\, \Theta^{(n,m)}{}_{i_1 j_1, \ldots, i_n j_n; k_1 \ldots k_m}, \tag{4.61}$$

together with the conditions that lead to corresponding $\mathcal{F}$ shifts of the form

$$\delta \mathcal{S} = \eta^{i_1 j_1} \ldots \eta^{i_n j_n} \gamma^{k_1 \ldots k_m}\, \mathcal{S}\left(\Theta^{(n,m)}{}_{i_1 j_1, \ldots, i_n j_n; k_1 \ldots k_m} \right). \tag{4.62}$$

However, the explicit expression

$$\delta \mathcal{S} = -\, i\, m\, \eta^{i_1 j_1} \ldots \eta^{i_n j_n} \gamma^{k_1 \ldots k_{m-1}} \partial^l \left\{ (D-m-1)\, \Theta^{(n,m)}{}_{i_1 j_1, \ldots, i_n j_n; l\, k_1 \ldots k_{m-1}} \right.$$

$$+ \sum_{r=1}^{n} \Theta^{(n,m)}{}_{\ldots, i_s \neq r j_s \neq r, \ldots, l\, (i_r; j_r)\, k_1 \ldots k_{m-1}} + 2\, S^q{}_l\, \Theta^{(n,m)}{}_{i_1 j_1, \ldots, i_n j_n; q\, k_1 \ldots k_{m-1}} \bigg\}$$

$$-\, i\, \frac{n}{m+1}\, \eta^{i_1 j_1} \ldots \eta^{i_{n-1} j_{n-1}} \gamma^{k_1 \ldots k_{m+1}} \partial^l\, \Theta^{(n,m)}{}_{i_1 j_1, \ldots, i_{n-1} j_{n-1}, l\, [k_1; k_2 \ldots k_{m+1}]}$$

$$+\, (-1)^m\, \eta^{i_1 j_1} \ldots \eta^{i_n j_n} \gamma^{k_1 \ldots k_m}\, \mathcal{S}\left(\Theta^{(n,m)}{}_{i_1 j_1, \ldots, i_n j_n; k_1 \ldots k_m} \right) \tag{4.63}$$

shows that parameters carrying the same number of family indices but having different symmetries can enter eq. (4.63) via terms that are saturated with the same combination of invariant tensors. As a consequence, one can obtain less restrictive conditions allowing Weyl-like transformations of the form

$$\delta \psi = \sum_{n=0}^{\left[\frac{m}{2}\right]} \eta^{i_1 j_1} \ldots \eta^{i_n j_n} \gamma^{k_1 \ldots k_{m-2n}}\, \Theta^{(n,m-2n)}{}_{i_1 j_1, \ldots, i_n j_n; k_1 \ldots k_{m-2n}}, \tag{4.64}$$

that affect $\mathcal{S}$ according to

$$\delta \mathcal{S} = \sum_{n=0}^{\left[\frac{m}{2}\right]} \eta^{i_1 j_1} \ldots \eta^{i_n j_n} \gamma^{k_1 \ldots k_{m-2n}}\, \mathcal{S}\left(\Theta^{(n,m-2n)}{}_{i_1 j_1, \ldots, i_n j_n; k_1 \ldots k_{m-2n}} \right), \tag{4.65}$$

provided the chains of n-dependent conditions

$$\frac{n+1}{(m-2n-1)(m-2n)} \, \Theta^{(n+1, m-2(n+1))}{}_{i_1 j_1, \ldots, i_n j_n, l \, [k_1 ; k_2 \ldots k_{m-2n-1}]}$$

$$+ \, (D - m + 2n - 1) \, \Theta^{(n, m-2n)}{}_{i_1 j_1, \ldots, i_n j_n; l \, k_1 \ldots k_{m-2n-1}}$$

$$+ \sum_{r=1}^{n} \Theta^{(n, m-2n)}{}_{\ldots, i_{s \neq r} j_{s \neq r}, \ldots, l \, (i_r ; j_r) \, k_1 \ldots k_{m-2n-1}}$$

$$+ \, 2 \, S^q{}_l \, \Theta^{(n, m-2n)}{}_{i_1 j_1, \ldots, i_n j_n; q \, k_1 \ldots k_{m-2n-1}} \; = \; 0 \qquad (4.66)$$

hold for the allowed values of m. It is thus natural to conjecture that the conditions (4.66) select models whose actions display Weyl-like symmetries when these are supplemented by the conditions

$$T_{(ab} \gamma_{c)} \, \Theta^{(n, m)}{}_{i_1 j_1, \ldots, i_n j_n; k_1 \ldots k_m} \; = \; 0 \, . \qquad (4.67)$$

Nevertheless, the explicit check that these transformations actually leave invariant the equations of motion (4.44) and (4.45) is a non-trivial technical task, that will not be treated here. In Section 4.2.2 I will only show that for fully antisymmetric fermions one of the m-dependent relations (4.66) admits solutions in the space-time dimensions where the action vanishes. This is a good check because in these cases the $\mathcal{S}$ tensor does not vanish, so that the action can do that only if it coincides with a vanishing $o(D)$ component of the kinetic tensor. This implies the existence of a shift symmetry of the field equations, that should be recovered by this analysis. Moreover, a detailed treatment of all transformations of the form (4.64) with up to four family indices can be found in [41]. They suffice to classify all the Weyl-like symmetries emerging in the two-family case, and this fixed the bound on the number of family indices. Further steps toward a proof of this conjecture are also presented in the Appendix D of [41], but the final answer is still missing.

4.2 Pathological cases

We can now take a closer look at the conditions identified in Section 4.1 that select the models with additional Weyl-like symmetries. For both Bose and Fermi fields one is facing eigenvalue problems for the $S^i{}_j$ operators, but whose solutions are to be subjected to further conditions arising from the constraints (4.9) or (4.50). For the reader's convenience, let me thus recall here the definition of the $S^i{}_j$ operators already introduced in Section 2.3:

$$S^i{}_j \, \varphi \; \equiv \; \varphi_{\ldots, (\mu_1^i \ldots \mu_{s_i}^i |, \ldots, | \mu_{s_i+1}^i) \mu_1^j \ldots \mu_{s_j-1}^j, \ldots} \, . \qquad (4.68)$$

As shown in Appendix A, these operators generate a $gl(N)$ algebra if N index families are present, and introducing a proper dictionary this enables to treat the eigenvalue problems resorting to rather standard techniques. In particular, we shall see that a substantial progress in the classification of the solutions can be attained only exploiting

the properties of the $gl(2)$ subalgebras generated by the operators $S^i{}_j$, $S^j{}_i$, $S^i{}_i$ and $S^j{}_j$ with *fixed* family indices i and j. As a consequence, in this section no summations over the indices are left implicit. In dealing with these $gl(2)$ subalgebras it is convenient to turn to the notation

$$L_+^{(ij)} = S^i{}_j \, , \quad L_-^{(ij)} = S^j{}_i \, , \quad L_3^{(ij)} = \frac{1}{2} \left(S^i{}_i - S^j{}_j \right) , \qquad i < j \, , \qquad (4.69)$$

that readily recovers the angular momentum algebra

$$[\, L_3^{(ij)} , L_\pm^{(ij)} \,] = \pm \, L_\pm^{(ij)} \, , \qquad [\, L_+^{(ij)} , L_-^{(ij)} \,] = 2 \, L_3^{(ij)} \, . \qquad (4.70)$$

Moreover, looking at eqs. (4.68) or (A.6), one can recognize that $S^i{}_i$ and $S^j{}_j$ act diagonally on any given tensor or spinor-tensor of rank $(s_1, \ldots, s_N)$. For instance,

$$S^i{}_i \, \varphi = s_i \, \varphi \, , \qquad\qquad S^j{}_j \, \psi = s_j \, \psi \, . \qquad (4.71)$$

Hence, also $L_3^{(ij)}$ acts diagonally on arbitrary tensors or spinor-tensors, with eigenvalues

$$m_{ij} = \frac{s_i - s_j}{2} \, . \qquad (4.72)$$

Notice that the spinor index does not play any role in these relations. Actually, all the techniques that I am going to present only deal with the vector indices carried by the fields, so that in the following I will often explicitly refer only to φ. However, in Section 4.2.2 all the results will be transferred verbatim to the fermionic framework.

In order to proceed, it is convenient to split the problems into combinations of simpler ones, expanding in bases on which the "total angular momentum" $\mathbf{L}_{ij}^2$ acts diagonally. For instance, a gauge field φ of rank $(s_1, \ldots, s_N)$ can be decomposed according to

$$\varphi = \sum_{n=0}^{s_j} \varphi^{\{s_i + s_j - n, \, n\}} \, , \qquad (4.73)$$

where the various components are characterized by the *same* $L_3^{(ij)}$ eigenvalue, that is fixed by the rank of φ to be that displayed in eq. (4.72). On the other hand, the terms $\varphi^{\{s_i + s_j - n, \, n\}}$ can be related to null eigenvectors of $L_+^{(ij)}$ via the standard descent relations

$$\varphi^{\{s_i + s_j - n, \, n\}} = (L_-^{(ij)})^{s_j - n} \, \widehat{\varphi}^{\{s_i + s_j - n, \, n\}} \, , \qquad L_+^{(ij)} \, \widehat{\varphi}^{\{s_i + s_j - n, \, n\}} = 0 \, , \qquad (4.74)$$

so that they are all eigenvectors of $\mathbf{L}_{ij}^2$, but in general with *different* eigenvalues. The $\widehat{\varphi}$ that lie at the tips of the chains are *irreducible* when considering the permutation group acting *only* on the i-th or j-th index families since they satisfy eq. (2.151) with fixed i and j. Indeed, they are annihilated by $L_+^{(ij)}$, so that a vanishing result obtains if one tries to extend the symmetrization beyond the i-th family. However, the lower members of the chains, that are built acting on the $\widehat{\varphi}$ with powers of $L_-^{(ij)}$, are still eigenvectors of the corresponding Young projectors acting on the couple of index families under scrutiny. In fact, they commute with $L_-^{(ij)}$ simply because the

permutation group acts irreducibly within individual $Y^{(ij)}$ eigenspaces. The terms appearing in the decomposition (4.73) can thus be directly identified as

$$\varphi^{\{s_i+s_j-n,\,n\}} \;\equiv\; Y^{(ij)}_{\{s_i+s_j-n,\,n\}}\,\varphi\,, \qquad\qquad (4.75)$$

where $Y^{(ij)}_{\{s_i+s_j-n,\,n\}}$ projects φ on the $\{s_i + s_j - n,\, n\}$ irreducible representation of the permutation group acting only on the i-th and j-th index families. All in all, the operator

$$L^{(ij)}_- L^{(ij)}_+ \;=\; \mathbf{L}^2_{ij} - (L^{(ij)}_3)^2 - L^{(ij)}_3 \qquad\qquad (4.76)$$

is diagonal when acting on this basis, whose members $\varphi^{\{s_i+s_j-n,\,n\}}$ are characterized by the *fixed* value (4.72) for m_{ij} and by a *range* of values of ℓ_{ij},

$$\ell_{ij}(\,n\,) \;=\; \frac{s_i + s_j}{2} - n\,, \qquad\qquad \frac{s_i - s_j}{2} \;\leq\; \ell_{ij} \;\leq\; \frac{s_i + s_j}{2}\,, \qquad (4.77)$$

so that

$$L^{(ij)}_- L^{(ij)}_+ \, \varphi^{\{s_i+s_j-n,\,n\}} \;=\; (\,n - s_i - 1\,)\,(\,n - s_j\,)\,\varphi^{\{s_i+s_j-n,\,n\}}\,. \qquad (4.78)$$

Let me conclude by remarking that the decomposition (4.73) corresponds to a true decomposition in irreducible representations of the permutation group acting on the whole set of space-time indices only in the two-family case. This reason, together with the other technical simplifications that emerge in the two-family case, allowed the exhaustive treatment of this class of fields, in both the bosonic [40] and the fermionic cases [41]. On the other hand, to get a clue on the space of solutions here I will follow an alternative path already outlined in [41] for Fermi fields. When combined with the techniques described in Section 2.4, the information provided by the various $su(2)$ subalgebras (4.70) indeed suffices to solve in the general N-family case various eigenvalue problems. Rather than performing an exhaustive analysis, in the following I will thus present some general classes of solutions, in order to show the richness of the set of Weyl-invariant models in arbitrary dimensions.

4.2.1   Bose fields

In most of this Thesis I worked with reducible fields. The presence of more than one family of symmetrized indices is indeed the only feature that shapes the structure of constraints, field equations and Lagrangians. On the other hand, one can directly look for *irreducible* solutions of the eigenvalue problems of Section 4.1.1, since one can clearly decompose a reducible field in irreducible components and analyze each of them independently. Furthermore, this approach is suggested by a simple observation. In Section 2.3 we have seen that in any space-time dimension irreducible two-column Bose fields with Weyl-like symmetries exist, but we also know that fully symmetric bosons admit Weyl-invariant actions only for $s = 2$ and in $D = 2$. Since all multi-symmetric fields carry a single-row irreducible component, it is evident that the presence of Weyl-like symmetries is intimately related to the detailed irreducible structure of the fields.

When dealing with irreducible fields, one can also take advantage of the irreducibility conditions of eq. (2.151). In fact, they relate the parameters that enter the shifts

of Section 4.1.1, and one can use them to simplify the corresponding eigenvalue problems. In order to display this mechanism, let me consider the simplest class of Weyl-like transformations, those of the form (4.10). As we have seen, they leave invariant the equation of motion (4.1) provided the conditions (4.12) hold. For a field φ of rank $(s_1, \ldots, s_N)$ these give rise to two groups of equations of the form

$$(D + 2\, s_1 - 6)\, \Omega_{ii} + 2 \sum_{k=1}^{i-1} S^k{}_i\, \Omega_{ik} + 2 \sum_{k=i+1}^{N} S^k{}_i\, \Omega_{ik} = 0\,, \qquad 1 \leq i \leq N\,,$$

$$(D + s_i + s_j - 4)\, \Omega_{ij} + \sum_{k=1}^{i-1} S^k{}_i\, \Omega_{jk} + \sum_{k \neq i}^{j-1} S^k{}_j\, \Omega_{ik} + S^i{}_j\, \Omega_{ii}$$

$$+ \sum_{k=i+1}^{j-1} S^k{}_i\, \Omega_{jk} + S^j{}_i\, \Omega_{jj} + \sum_{k=j+1}^{N} S^k{}_{(i}\, \Omega_{j)k} = 0\,, \qquad i < j\,. \qquad (4.79)$$

The $S^i{}_j$ operators modify the number of space-time indices contained in the various families, and this allows the simultaneous presence of more than one parameter in each condition, even if they have a different rank. On the other hand, in order to preserve the irreducibility conditions (2.151) that project φ on its $\{s_1, \ldots, s_N\}$ component, the parameters must also satisfy the relations

$$S^k{}_l\, \Omega_{ij} + \delta^k{}_{(i}\, \Omega_{j)l} = 0\,, \qquad k < l\,, \qquad (4.80)$$

that actually leave only one independent parameter for each symmetry type. This can be seen resorting to the procedure outlined in Section 2.4 in the discussion of the structure of the gauge transformations of irreducible fields. At any rate, let me anticipate that these conditions suffice to "diagonalize" eqs. (4.79), reducing the identification of their solutions to simple algebraic equations involving the space-time dimension and the rank of the fields.

Indeed, one can directly use the relations (4.80) to simplify some of the terms in eqs. (4.79) since for proper index choices they read

$$\begin{aligned} S^k{}_i\, \Omega_{jk} &= - \Omega_{ij}\,, & k < i\,, \\ S^i{}_j\, \Omega_{ii} &= - 2\, \Omega_{ij}\,, & i < j\,. \end{aligned} \qquad (4.81)$$

As a result the conditions (4.79) can be cast in the form

$$[\, D + 2\, s_1 - 2\, (i+2) \,]\, \Omega_{ii} + 2 \sum_{k=i+1}^{N} S^k{}_i\, \Omega_{ik} = 0\,, \qquad 1 \leq i \leq N\,,$$

$$[\, D + s_i + s_j - (i+j+3) \,]\, \Omega_{ij} + \sum_{k=i+1}^{j-1} S^k{}_i\, \Omega_{jk} + S^j{}_i\, \Omega_{jj}$$

$$+ \sum_{k=j+1}^{N} S^k{}_{(i}\, \Omega_{j)k} = 0\,, \qquad i < j\,. \qquad (4.82)$$

If all parameters are related by eq. (4.80), even the remaining terms in the first group of equations can be related to Ω_{ii}, while in the second group of equations all terms can be related to Ω_{ij}. To this end one can act with the $S^k{}_i$ or $S^k{}_j$ operators appearing in eq. (4.82) on some of the consequences of eq. (4.80), obtaining

$$
\begin{aligned}
S^k{}_i S^i{}_k \Omega_{ij} &= - S^k{}_i \Omega_{jk}, & k &> i, \\
S^k{}_i S^i{}_k \Omega_{ii} &= -2 S^k{}_i \Omega_{ik}, & k &> i,
\end{aligned}
\tag{4.83}
$$

and similar conditions with i exchanged with j. The operators on the left-hand sides are of the form $L_-^{(ik)} L_+^{(ik)}$ and, as we have already seen, standard "angular momentum" techniques imply that they act diagonally on eigenvectors of $\mathbf{L}_{ik}^2$. Furthermore, the relations (4.80) imply

$$
S^k{}_l \Omega_{ij} = 0, \qquad k < l, \quad k \neq i, j,
\tag{4.84}
$$

so that the parameters appearing on the right-hand sides of eqs. (4.83) are actually eigenvectors of $\mathbf{L}_{ik}^2$ or can be simply related to highest-weight eigenvectors. Acting with $L_-^{(ik)}$ one then remains inside the same ℓ_{ik} multiplet. As a result, even the left-hand sides are eigenvectors of $\mathbf{L}_{ik}^2$ with the same value of ℓ_{ik}. In conclusion, the $L_-^{(ik)} L_+^{(ik)}$ operators appearing in eqs. (4.83) indeed act diagonally, and their eigenvalues can be easily computed exploiting these considerations. For instance, for the first group of conditions (4.83),

$$
\ell_{ik}(\Omega_{ij}) = \ell_{ik}(\Omega_{jk}) = m_{ik}(\Omega_{jk}) = \frac{s_i - s_k + 1}{2},
\tag{4.85}
$$

while

$$
m_{ik}(\Omega_{ij}) = \frac{s_i - s_k - 1}{2}.
\tag{4.86}
$$

One can eventually use this information to cast eqs. (4.83) in the form

$$
\begin{aligned}
(s_i - s_k + 1)\,\Omega_{ij} &= - S^k{}_i \Omega_{jk}, & i &< k,\ k \neq j, \\
(s_i - s_k + 2)\,\Omega_{ij} &= - S^j{}_i \Omega_{jj}, & i &< j, \\
(s_i - s_k + 1)\,\Omega_{ii} &= - S^k{}_i \Omega_{ik}, & i &< k.
\end{aligned}
\tag{4.87}
$$

When dealing with irreducible fields, the conditions (4.79) are thus equivalent to

$$
\left[D - 2(N+2) + 2\left(\sum_{k=i+1}^{N} s_k - (N-i-1)\,s_i \right) \right] \Omega_{ii} = 0, \qquad 1 \leq i \leq N,
$$

$$
\left[D - 2(N+2) + \sum_{k=i+1}^{N} s_k - (N-i-1)\,s_i \right.
$$

$$
\left. + \sum_{k=j+1}^{N} s_k - (N-j-1)\,s_j \right] \Omega_{ij} = 0, \qquad\qquad i < j, \quad (4.88)
$$

so that they reduce to a system of algebraic equations for the coefficients that multiply the parameters. In conclusion, one can identify Weyl-invariant models simply checking

whether this homogeneous system admits solutions for proper values of the space-time dimension D and of the s_i that describe the rank of the field φ. Before dealing with this problem, let me recall that this analysis does not suffice to classify all Weyl-invariant models for two reasons. First of all, the manipulations that I performed to reach eqs. (4.88) are only valid when all the Ω_{ij} are available, and then only when $s_i \geq 2$ for $1 \leq i \leq N$. The other degenerate cases should be treated separately, and one should also discuss the other eigenvalues problems of the form (4.15). In the following I will briefly discuss some simple solutions of these further conditions, but the scrutiny of eqs. (4.88) already suffices to display a rich set of Weyl-invariant models in arbitrary space-time dimensions.

In the study of eqs. (4.88), one cannot forget that the irreducibility conditions (4.80) played a crucial role in reaching this rewriting of eqs. (4.79). Therefore one has to look for solutions that are compatible with them. For instance, the irreducibility of φ implies that if a parameter Ω_{ij} vanishes, all other parameters Ω_{kl} with $k \geq i$ and $l \geq j$ must vanish as well. Indeed, in principle Ω_{ij} carries all the irreducible components that are admitted by the Ω_{kl} with $k \geq i$ and $l \geq j$, and eqs. (4.80) relate all the contributions associated to the same Young tableau. Thus, a Weyl-like transformation with a non-trivial Ω_{NN} parameter must contain all the Ω_{ij}. We can then begin by looking for Weyl-like symmetries of this kind, since the condition for Ω_{NN} in (4.88) takes the simple form

$$D + 2\,s_N - 2\,(N+2) = 0\,, \tag{4.89}$$

and admits solutions for $D \leq 2\,N$. In these cases the other equations reduce to

$$\left(\sum_{k=i+1}^{N-1} s_k - (N-i-1)\,s_i \right) \Omega_{ii} = 0\,, \qquad 1 \leq i \leq N-2\,,$$

$$\left(\sum_{k=i+1}^{N} s_k - (N-i-1)\,s_i + \sum_{k=j+1}^{N} s_k - (N-j-1)\,s_j \right) \Omega_{ij} = 0\,, \tag{4.90}$$

where the latter hold for all $i < j < N$. These conditions iteratively fix the s_i with $i < N$ to be identical, so that a non-trivial solution of eqs. (4.79) exists in $D = 2\,(N - s_N + 2)$ for all $\{s, \ldots, s, s_N\}$-projected fields with $2 \leq s_N \leq N+1$. This means that these fields admit a shift transformation of $\mathcal{F}$ that leaves invariant the equation of motion (4.1) in the corresponding space-time dimension. The irreducibility conditions (4.80) also suffice to fix the detailed structure of this shift, since they can be inverted to relate all the parameters to Ω_{NN} via

$$\Omega_{ij} = \frac{1}{(s_i - s_j + 1)\,(s_j - s_N + 2)}\,S^N{}_i\,S^N{}_j\,\Omega_{NN}\,, \qquad i < j < N\,,$$

$$\Omega_{iN} = -\frac{1}{s_i - s_N + 2}\,S^N{}_i\,\Omega_{NN}\,, \qquad i < N\,. \tag{4.91}$$

The Lorentz structure of Ω_{NN} can be easily extracted from eqs. (4.80), since they imply

$$S^k{}_l\,\Omega_{NN} = 0\,, \qquad 1 \leq k < l \leq N\,, \tag{4.92}$$

so that Ω_{NN} is a $\{s, \ldots, s, s_N - 2\}$-projected tensor. In the two-family case these results agree with those obtained in [40] on the complete classification of the Weyl-like symmetries emerging in that setup. The analysis of eq. (4.1) indeed selected the candidate Weyl-invariant models gathered in Table 4.1. The first column collects the models where eq. (4.1) is left invariant under the shifts of the form (4.10) that we are considering, and contains $\{s, 2\}$ bosons in $D = 4$ and $\{s, 3\}$ bosons in $D = 2$. The second column then collects the models where eq. (4.1) is left invariant under shifts of the form (4.13) involving parameters with four family indices.

	$\mathcal{F}'$		$\mathcal{F}''$	
D	s_1	s_2	s_1	s_2
2	2	0		
2	2	1		
2	3	1		
2	s	3		
3	2	1	2	2
4	s	2		

Table 4.1: Two-family irreducible bosons with Weyl-like symmetries of eq. (4.1)

On the other hand, in order to identify a Weyl-like symmetry of the action one must also check whether the corresponding transformations leave invariant the field equations (4.2). As we have seen in eq. (4.17), this means that the Ω_{ij} parameters must also satisfy the constraints

$$T_{(ij} T_{kl)} \Omega_{mn} = 0 \,. \tag{4.93}$$

In general these conditions would simply reduce the number of independent components of the parameters entering the shift (4.10), but in some cases they could even trivialize it. Rather than identifying a true Weyl-like symmetry of the action, the corresponding transformation thus only signals an accidental symmetry of the formal expression (4.1). In order to clarify this statement let me consider a $\{4, 3\}$ field in $D = 2$, that provides the simplest example to this effect. Displaying explicitly the space-time indices, the candidate Weyl-like transformation identified in the previous analysis reads

$$\delta \, \varphi_{\mu_1 \ldots \mu_4 , \, \nu_1 \nu_2 \nu_3} = \frac{1}{3} \, \eta_{(\mu_1 \mu_2} \Omega_{\mu_3 \mu_4) (\nu_1 \nu_2 , \, \nu_3)} - \frac{1}{3} \, \eta_{(n_1 | (\mu_1} \Omega_{\mu_2 \mu_3 \mu_4) | \nu_2 , \, \nu_3)}$$
$$+ \, \eta_{(\nu_1 \nu_2 |} \Omega_{\mu_1 \ldots \mu_4 , \, | \nu_3)} \,, \tag{4.94}$$

with a single independent $\{4, 1\}$-projected Weyl parameter. On the other hand, in $D = 2$ an irreducible $\{4, 3\}$ field actually admits only two independent components. Denoting the two space-time directions as 0 and 1, one can then choose them to be $\varphi_{0000 , \, 111}$ and $\varphi_{1111 , \, 000}$, and their Weyl-like shifts (4.94) explicitly read

$$\delta \, \varphi_{0000 , \, 111} = 3 \left(\Omega_{0000 , \, 1} - 2 \, \Omega_{0011 , \, 1} \right) ,$$
$$\delta \, \varphi_{1111 , \, 000} = -3 \left(\Omega_{1111 , \, 0} - 2 \, \Omega_{1100 , \, 0} \right) . \tag{4.95}$$

Furthermore, the conditions (4.93) simply reduce to the double trace constraint

$$\eta^{\lambda\rho}\,\eta^{\delta\sigma}\,\Omega_{\lambda\rho\delta\sigma\,;\,\nu} \,=\, 0\,, \tag{4.96}$$

that in components reads

$$-\,2\,\Omega_{1100\,,\,0} + \Omega_{1111\,,\,0} \,=\, 0\,, \qquad\qquad \Omega_{0000\,,\,1} - 2\,\Omega_{0011\,,\,1} \,=\, 0\,. \tag{4.97}$$

These are exactly the combinations that appear in eq. (4.95) and, as a result, a non-trivial shift of the form (4.94) cannot preserve the constraints in $D = 2$. A similar behavior is common to all $\{s, 3\}$ models in two space-time dimensions, with the exception of the $\{3, 3\}$ case, where the fields are actually unconstrained. With the same strategy one can also analyze the $\{s, 2\}$ models in $D = 4$, but in these cases the constraints (4.93) do not suffice to trivialize the φ or $\mathcal{F}$ shifts. Thus, for these fields the Weyl-like transformations we have identified leave the actions invariant. In the two-family case all pathological models can be studied with this direct approach and the reader can find in Table 4.2 the full set of two-family Weyl-invariant models that were identified in [40].

	$\mathcal{F}'$		$\mathcal{F}''$	
D	s_1	s_2	s_1	s_2
2	2	0		
2	2	1		
2	3	1		
2	3	3		
3	2	1	2	2
4	s	2		

Table 4.2: Two-family irreducible bosons with Weyl-invariant actions

In conclusion, not all solutions of eqs. (4.79) that I have presented effectively identify models that admit Weyl-like symmetries. On the other hand, one can at least select a subclass which does, and which extends the example of two-column Bose fields of Section 2.3. In fact, a general result of representation theory (see, for instance, the first reference in [101], §10-6) implies that, for $O(n)$ groups, if the total number of boxes in the first two columns of a tableau exceeds n, the corresponding traceless tensor vanishes. Thus, in $D = 2$ the $\{4, 1\}$-projected parameter of eq. (4.94) does not carry a traceless component. As a result, the constraint (4.93) can and indeed do annihilate all its components. On the other hand, the $\{s\}$-projected parameters of the Weyl transformations of $\{s, 2\}$ fields in $D = 4$ carry also a traceless component, that as such cannot be affected by the constraints (4.93). This pattern extends to the general N-family case, where the $\{s, \ldots, s, s_N - 2\}$-projected Weyl parameters emerging for $\{s, \ldots, s, s_N\}$ fields in $D = 2(N - s_N + 2)$ in general do not admit a traceless component. The only exception is provided by the class of $\{s, \ldots, s, 2\}$-projected fields in $D = 2N$. Therefore, their Weyl-like parameters contain some

components that cannot be affected by the constraints (4.93), and eqs. (4.2) can be preserved by non-trivial shifts. We can thus conclude that the presence of Weyl-like symmetries is not confined to unconstrained two-column bosons, and the paradigmatic examples of $\{s, \ldots, s, 2\}$-projected fields in $D = 2N$ display all the features of generic higher-spin mixed-symmetry fields.

In this fashion I have shown that the conditions (4.12) admit a rather rich set of solutions in arbitrary dimensions, and I can now show that all shifts of the form (4.13) actually play a role. In fact, all conditions (4.15) admit at least one solution. Rather than proposing a detailed treatment similar to that presented in the previous pages, I will identify simple solutions for all of them confining the attention to two-column Bose fields. For instance, irreducible bosons of the form $\{2^N\}$ must display Weyl-like symmetries in $N \leq D \leq 2N$ since their actions vanish in these space-time dimensions. Indeed, let me recall that, since $\mathcal{F}$ does not vanish identically, the formal expression (2.138) for the Einstein-like tensor can vanish only if it coincides with a vanishing $o(D)$ component of the kinetic tensor. Hence, the other components can be arbitrarily shifted and this is the origin of the Weyl-like symmetries that we have to recover from the conditions (4.10) or (4.13). In the previous analysis we have identified for this class of fields only Weyl-like symmetries in $D = 2N$, and I can now show that in the other space-time dimensions the relevant Weyl-like transformations are of the form (4.13).

Let me begin by noticing that in an algebraic language the Weyl-like transformations are to satisfy

$$S^k{}_1 \, \Omega_{1k} \, = \, - \, \Omega_{11} \, , \qquad \text{for } k \text{ fixed and } k \neq 1 \, , \tag{4.98}$$

ore more generally

$$S^k{}_1 \, \Omega_{1k, \, 22, \, \ldots, \, pp} \, = \, - \, \Omega_{11, \, 22, \, \ldots, \, pp} \, , \qquad \text{for } k \text{ fixed and } k > p \, , \tag{4.99}$$

for the higher shifts of eq. (4.13). These are but simple consequences of the vanishing of all symmetrizations over three space-time indices. As a result, the $\frac{N(N+1)}{2}$ conditions of eq. (4.79) effectively reduce to the single equation

$$(D - 2) \, \Omega_{11} \, + \, 2 \sum_{k \, = \, 2}^{N} S^k{}_1 \, \Omega_{1k} \, = \, 0 \, , \tag{4.100}$$

and finally to

$$(D - 2N) \, \Omega_{11} \, = \, 0 \, , \tag{4.101}$$

that recovers the non-trivial solution for $D = 2N$ that we already identified. In a similar fashion, one can recognize that all equations of the form (4.15) are equivalent to the conditions

$$(D - 2N + p - 1) \, \Omega_{11, \, 22, \, \ldots, \, pp} \, = \, 0 \, , \tag{4.102}$$

so that the p-th trace of a $\{2^N\}$-projected field $\varphi_{\mu_1^1 \mu_2^1; \, \ldots; \, \mu_1^N \mu_2^N}$ is not determined by the Lagrangian field equation in $D = 2N - p + 1$. We have thus recovered the expected Weyl-like symmetries, but in order to do that we had to consider all the available

conditions of the form (4.15). Let me stress that the presence of shift symmetries with more "naked" invariant tensors could be expected, since lowering the number of space-time dimensions one has to lower the rank of the Weyl-like parameter in order to allow a traceless component for it. Nevertheless, the fact that these solutions are identified by the conditions (4.15) is not at all obvious, since in principle they could emerge as particular solutions of the conditions (4.12).

To summarize, even if a full classification of Weyl-invariant models in arbitrary dimensions is deferred to a future work, I have identified two paradigmatic classes of examples and I would like to briefly comment on their properties. First of all, $\{s^{N-1}, 2\}$ fields in $D = 2N$ and $\{2^p, 1^q\}$ fields in $D \leq 2p + q$ do *not* propagate any local degrees of freedom. This can be seen resorting to the standard theorem that I applied to select the $\{s^{N-1}, 2\}$ cases out of the $\{s^{N-1}, s_N\}$ class. The same is true even for the other models collected in Tables 4.1 and 4.2, that complete the full classification when only two index families are present. It is thus reasonable to expect that we have identified a key feature of the pathological models, and that the Lagrangian field equations (4.1) are *directly* equivalent to the Labastida ones (4.3) in all space-time dimensions $D > \frac{N-2}{2}$. In fact, above this bound N-family fields propagate the degrees of freedom of a proper N-row representation of the Lorentz group. Moreover, looking at eq. (A.12) or at the form taken by the conditions (4.15), one can recognize that for large D the $S^i{}_j$ can be discarded, so that the field equations reduce rather directly to the Labastida form. In addition, the Labastida counting of the degrees of freedom [39] should provide the correct answer of eq. (A.39) independently of the value of D, so that the modifications related to the emergence of new Weyl-like gauge transformations are expected to occur only outside of the range of validity of eq. (A.39).

On the other hand, aside from this common signature the pathological models can display different properties. In eq. (4.102) we have indeed seen that two-column fields have vanishing actions in all cases where Weyl-like symmetries exist, but in general this is not necessary by any means. Indeed, one can verify that the action of a $\{3, 2\}$ field does not vanish in $D = 4$. This result can be also recovered noticing that if an action vanishes in a given space-time dimension it must also vanish in all lower ones. No additional subtleties are involved in this example since $\{3, 2\}$ fields are actually unconstrained. Therefore, in the mixed-symmetry case the existence of Weyl-like symmetries does not imply in general that the Lagrangian is a total derivative.

4.2.2 Fermi fields

Even when dealing with Fermi fields it is convenient to study the eigenvalue problems of Section 4.1.1 directly in the irreducible case. As for Bose fields, the irreducibility conditions (2.151) greatly simplify the search for Weyl-invariant actions. On the other hand, I will again present only some paradigmatic classes of solutions, deferring to a future work a more detailed analysis. In analogy with the bosonic case, I will thus discuss in some detail the conditions (4.54), that identify Weyl-like transformations of the form (4.53). For the more involved conditions (4.66), that identify Weyl-like

transformations of the form (4.65), I will only present simple solutions associated to fully antisymmetric fields, just to show that they all play a role, rather than looking for new Weyl-invariant models.

Let me thus begin by considering Weyl-like transformations of the form

$$\delta \psi = \gamma^i \Theta_i \,, \tag{4.103}$$

for which the irreducibility of ψ translates into the conditions

$$\delta^i{}_k \Theta_j + S^i{}_j \Theta_k = 0 \,, \qquad i < j \tag{4.104}$$

that again leave only one independent Θ_i parameter for each symmetry type. These are to be combined with the conditions allowing (4.103) to define a Weyl symmetry, that were given in eq. (4.54) and that one can rewrite in the convenient form

$$(D + 2\, s_i - 4)\, \Theta_i + 2 \sum_{j=1}^{i-1} S^j{}_i \Theta_j + 2 \sum_{j=i+1}^{N} S^j{}_i \Theta_j = 0 \,, \qquad 1 \le N \,. \tag{4.105}$$

As in the preceding section s_i denotes the number of Lorentz labels in the i-th family for the spinor-tensor ψ. Eqs. (4.104) then reduce (4.105) to

$$[D + 2\, s_i - 2\,(i+1)]\, \Theta_i - 2 \sum_{j=i+1}^{N} S^j{}_i\, S^i{}_j\, \Theta_i = 0 \,, \qquad 1 \le N \,. \tag{4.106}$$

The sum is absent in the last of these equations, that is simply

$$[D + 2\, s_N - 2\,(N+1)]\, \Theta_N = 0 \,, \tag{4.107}$$

and admits non-trivial solutions provided the condition

$$D + 2\, s_N = 2\,(N+1) \tag{4.108}$$

holds. On the other hand, the irreducibility conditions (4.104) again imply that if Θ_N does not vanish then all the Θ_i cannot vanish as well. As a consequence, one has to discuss also the other conditions with $i < N$. To this end one can notice that, in analogy with the bosonic case, the irreducibility conditions (4.104) ensure that the operators $S^j{}_i\, S^i{}_j$ act *diagonally* on Θ_i. In fact, $S^j{}_i$ and $S^i{}_j$ are the $L_-^{(ij)}$ and $L_+^{(ij)}$ operators for the $su(2)$ subalgebra of $gl(N)$ connecting the two rows i and j, and eq. (4.104) implies that

$$L_+^{(ij)} \Theta_j = 0 \,, \qquad \Theta_j = - L_+^{(ij)} \Theta_i \,. \tag{4.109}$$

As a result, Θ_i and Θ_j have the same "total angular momentum" quantum number

$$\ell_{ij}(\Theta_i) = \ell_{ij}(\Theta_j) = \frac{s_i - s_j + 1}{2} \,, \tag{4.110}$$

that is also the "magnetic" quantum number of Θ_j, that lies at the tip of the $su(2)$ chain and lacks one space-time index of the j-th row when compared to the gauge field

ψ. At the same time, the "magnetic" quantum number of Θ_i, that lacks an index in the i-th row compared to ψ, is

$$m_{ij}(\Theta_i) = \frac{s_i - s_j - 1}{2}, \qquad (4.111)$$

so that finally

$$S^j{}_i S^i{}_j \Theta_i = (s_i - s_j + 1)\Theta_i. \qquad (4.112)$$

In conclusion, when the condition (4.108) holds the remaining equations (4.106) with $i < N$ reduce to the chain of conditions

$$\left[(N - i - 1)s_i - \sum_{j=i+1}^{N-1} s_j\right]\Theta_i = 0, \qquad 1 \leq i \leq N - 2, \qquad (4.113)$$

while the $(N-1)$-th condition is also identically satisfied on account of eq. (4.108). Since under these conditions all Θ_i must be non-vanishing, a solution obtains only if *all* the coefficients in (4.113) vanish, which is the case for $\{s, \ldots, s, s_N\}$ irreducible fields with s arbitrary and $s_N \leq N$ in the space-time dimensions determined by eq. (4.108). As for Bose fields, the irreducibility conditions also suffice to identify the detailed form of the Weyl-like transformations because they relate the Θ_i as

$$\Theta_i = -\frac{1}{s_i - s_j + 1} S^j{}_i \Theta_j, \qquad i < j. \qquad (4.114)$$

These results agree with those obtained in the exhaustive classification of the shift symmetries of the equation of motion (4.44) performed for two-family fields in [41]. The outcome of that analysis is presented in Table 4.3, where the reader can recognize the presence of $\{s, 2\}$ fields in $D = 2$ and of $\{s, 1\}$ fields in $D = 4$. They appear in the first column, that collects the fields admitting shift symmetries of the form (4.103). The second column then collects the models that admit shift symmetries of the form

$$\delta\psi = \eta^{ij}\Theta_{ij} + \gamma^{ij}\widetilde{\Theta}_{ij}, \qquad (4.115)$$

while the third column collects finally the models that admit shift symmetries of the form

$$\delta\psi = \eta^{ij}\gamma^k\Theta_{ij;k}. \qquad (4.116)$$

As in the bosonic case, this first step does not suffice to identify Weyl-like symmetries of the actions: the solutions we identified are also to preserve the Labastida constraints (3.52). In eq. (4.55) we have seen that, whenever eqs. (4.105) hold, for the Weyl-like shifts that we are considering this is guaranteed by the additional conditions

$$T_{(ij}\gamma_{k)}\Theta_l = 0. \qquad (4.117)$$

On the other hand, the conditions (4.114) imply that all Θ_l parameters have the same $gl(D)$ structure as the irreducible Θ_N, that in this sense represents the most convenient choice for the single independent quantity left over in eq. (4.105). Therefore,

	$\mathcal{S}$		$\mathcal{S}'$		$\hat{\mathcal{S}}'$	
D	s_1	s_2	s_1	s_2	s_1	s_2
2	1	0	2	1	s	3
2	1	1				
2	s	2				
3			1	1		
4	s	1				

Table 4.3: Two-family irreducible fermions with Weyl-like symmetries of eq. (4.44)

the transformations of the $\{s,\dots,s,1\}$ fields in $D = 2\,N$ rest on a $\{s,\dots,s\}$ conformal Weyl parameter, that in this space-time dimension admits a γ-traceless component that as such cannot be annihilated by the triple γ-trace constraints (4.117). This is consistent with the fact that for two-family fields one can check explicitly that $\{s,1\}$ fields admit Weyl-like symmetries of the type (4.103) compatible with the Labastida constraints. On the contrary, for higher values of s_N the standard theorem already recalled in Section 4.2.1 states that a γ-traceless component does not exist for $\{s,\dots,s,s_N-1\}$-projected parameters in $D = 2\,(\,N - s_N + 1\,)$. As a consequence, the previous considerations protecting this type of shift are no more available and indeed in the $\{s,2\}$ case in $D = 2$ one can verify that the shift transformations are trivialized by the constraints (4.117). As first recognized in [41], at two-families a cancelation of this kind also concerns the $\{s,3\}$ case in $D = 2$, so that the true Weyl-invariant models are only those displayed in Table 4.4. While I do not have a complete argument to this effect, the explicit analysis of the two-family case thus suggests that the triple γ-trace conditions could eliminate at least some of the solutions (4.108) in low enough space-time dimensions. Nevertheless, as in the bosonic case we have at least identified a rather general class of solutions that extends the simple example of fully antisymmetric Fermi fields, that are actually unconstrained fields and thus do not display all the features of generic mixed-symmetry fermions.

	$\mathcal{S}$		$\mathcal{S}'$	
D	s_1	s_2	s_1	s_2
2	1	0		
2	2	2		
3			1	1
4	s	1		

Table 4.4: Two-family irreducible fermions with Weyl-invariant actions

Let me support these statements by displaying the simplest case were the constraints (4.117) obstruct the promotion of the symmetries of eq. (4.44) to symmetries of the action. It is provided by a $\{3,2\}$ field, whose $E_{\bar{\psi}}$ field equation is left invariant by the

shift transformation

$$\delta\psi_{\mu_1\mu_2\mu_3,\nu_1\nu_2} = -\frac{1}{2}\,\gamma_{(\mu_1}\Theta_{\mu_2\mu_3)(\nu_1,\nu_2)} + \gamma_{(\nu_1|}\Theta_{\mu_1\mu_2\mu_3,|\nu_2)}\,, \qquad (4.118)$$

that involves a single $\{3,1\}$-projected parameter. Even in this example the irreducibility conditions leave only two independents components in $D=2$. Let me denote the two space-time directions as 0 and 1, and let me choose them to be $\psi_{000,11}$ and $\psi_{111,00}$. Their Weyl-like transformations then read

$$\begin{aligned}\delta\psi_{000,11} &= 2\,\gamma_1\,\Theta_{000,1} - 3\,\gamma_0\,\Theta_{001,1}\,, \\ \delta\psi_{111,00} &= 2\,\gamma_0\,\Theta_{111,0} + 3\,\gamma_1\,\Theta_{001,1}\,,\end{aligned} \qquad (4.119)$$

where I have chosen to identify the three independent components of the parameter with $\Theta_{000,1}$, $\Theta_{111,0}$ and $\Theta_{001,1}$. Furthermore, in this context the conditions (4.117) reduce to the constraint

$$\eta^{\lambda\rho}\,\gamma^\sigma\,\Theta_{\lambda\rho\sigma,\nu} = 0\,, \qquad (4.120)$$

that implies the relations

$$\gamma_1\left(\Theta_{111,0} + \frac{1}{3}\Theta_{000,1}\right) + \gamma_0\Theta_{001,1} = 0\,,$$

$$\gamma_0\left(\Theta_{000,1} + \frac{1}{3}\Theta_{111,0}\right) - \gamma_1\Theta_{001,1} = 0\,. \qquad (4.121)$$

Resorting to a standard representations of the two-dimensional γ-matrices of the form

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad (4.122)$$

one can then recognize that the conditions (4.121) imply

$$\Theta_{000,1} = \begin{pmatrix} f_1(t,x) \\ f_2(t,x) \end{pmatrix} \Rightarrow \Theta_{111,0} = -\begin{pmatrix} f_1(t,x) \\ f_2(t,x) \end{pmatrix},\ \Theta_{001,1} = \frac{2}{3}\begin{pmatrix} -f_1(t,x) \\ f_2(t,x) \end{pmatrix}. \ (4.123)$$

Finally, substituting this result in eqs. (4.119) one can notice that it eliminates the combinations of the parameter components that they contain.

I can now close this section by showing that the conditions (4.66) admit a solution. Indeed, in Section 2.3 I pointed out that the actions for fully antisymmetric Fermi fields vanish manifestly for $N \le D \le 2N$, and I noticed that the formal expressions for their field equations in terms of $\mathcal{S}$ must possess Weyl-like symmetries. On the other hand, with the previous analysis I have identified a Weyl-like symmetry for these fields only in $D = 2N$, and I can now recover the missing symmetries solving the conditions (4.66) for spinor-tensor forms. First of all, in this case the η^{ij} tensors do not enter Lagrangians and field equations, so that the chains of conditions (4.66) reduce to the far simpler independent equations

$$(D-q-1)\,\Theta^{(q)}{}_{k_1\dots k_{q-1}l} + 2\,S^m{}_l\,\Theta^{(q)}{}_{k_1\dots k_{q-1}m} = 0\,, \qquad (4.124)$$

whose solutions identify Weyl-like shifts of the form

$$\delta\,\psi \,=\, \gamma^{\,k_1\ldots k_q}\,\Theta^{\,(q)}{}_{k_1\ldots k_q}\,. \tag{4.125}$$

Let me also stress that no other conditions are to be imposed on these parameters, because all spinor-tensor forms are *unconstrained*. Furthermore, for this type of fields $S^m{}_l\,\psi = 0$ for all $l \neq m$, since it is clearly impossible to symmetrize their space-time indices. As a result for all $l \neq m$

$$S^m{}_l\,\delta\,\psi \,=\, \gamma^{\,k_1\ldots k_q}\left(\,S^m{}_l\,\Theta^{\,(q)}{}_{k_1\ldots k_q} + (-1)^{\,q+1}\,\delta^m{}_{[\,k_1}\Theta^{\,(q)}{}_{k_2\ldots k_q\,]\,l}\,\right) \,=\, 0\,, \tag{4.126}$$

so that eq. (4.124) becomes

$$(\,D - 2\,N + q - 1\,)\,\Theta^{\,(q)}{}_{k_1\ldots k_{q-1}l} \,=\, 0\,. \tag{4.127}$$

Fully antisymmetric fields can admit a single independent q-th γ-trace, that as we now see is left undetermined in $D = 2\,N - q + 1$, since it can be shifted by non-vanishing $\Theta^{\,(q)}$ parameters. Therefore, as expected, we are recovering the existence of Weyl-like symmetries for these fields for $N + 1 \leq D \leq 2\,N$. Notice, finally, that the Lagrangians (2.137) are total derivatives also for $D = N$, but in these cases the $\mathcal{S}$ spinor-tensors vanish as well, as one can see computing explicitly their only available component. This explains why these special space-time dimensions do not emerge from the previous discussion.

The final considerations of the previous section can be repeated almost verbatim also in the fermionic case. Even in this framework we cannot present a full classification of the Weyl-invariant models, but we have identified two paradigmatic classes of examples: $\{s^{N-1},1\}$ fields in $D = 2\,N$ and fully antisymmetric $\{1^N\}$ fields in $N \leq D \leq 2\,N$. In both cases and in all the additional examples contained in Tables 4.3 and 4.4 no local degrees of freedom are involved. Furthermore, looking at eq. (A.24) and at the structure of the conditions (4.66), one can realize that for large D the $S^i{}_j$ operators cannot obstruct the reduction of the Lagrangian field equations to the Labastida form (4.46). As a result, it is natural to expect that the lack of propagating degrees of freedom be a key signature of the pathological models, and that for $D > \frac{N-2}{2}$ the two forms of the field equations are directly equivalent. Even if no degrees of freedom are involved, the two classes of examples that I have selected can present rather different behaviors. In fact, the Lagrangians of fully antisymmetric fermions are total derivatives when they admit Weyl-like symmetries but, for instance, this is not the case for $\{3,1\}$ fermions in $D = 4$.

Chapter 5

Conclusions

This Thesis presents three alternative local and covariant Lagrangian formulations for field transforming in arbitrary representations of the Lorentz group in Minkowski backgrounds of generic dimension. The various options detailed here can be regarded as natural extensions of the metric formulation of linearized gravity, since they all reduce to it in the spin-2 case. In the metric formalism the graviton polarizations are described in arbitrary space-time dimensions via a symmetric tensor $h_{\mu\nu}$ that varies under linearized diffeomorphisms as

$$\delta h_{\mu\nu} = \partial_{(\mu} \xi_{\nu)} \,. \tag{5.1}$$

However, moving to generic representations of the Lorentz group the natural extension of this setup, involving multi-symmetric tensors $\varphi_{\mu_1\dots\mu_{s_1},\nu_1\dots\nu_{s_2},\dots}$ subjected to the gauge transformations

$$\delta \varphi_{\mu_1\dots\mu_{s_1},\nu_1\dots\nu_{s_2},\dots} = \partial_{(\mu_1} \Lambda^{(1)}{}_{\mu_2\dots\mu_s),\nu_1\dots\nu_{s_2},\dots} + \partial_{(\nu_1|} \Lambda^{(2)}{}_{\mu_1\dots\mu_s,|\nu_2\dots\nu_{s_2}),\dots} + \dots \tag{5.2}$$

or their fermionic analogues, does not lead to a *local* description of the dynamics. This is the very origin of the three complementary approaches reviewed here, since a local metric-like Lagrangian formulation can be achieved either constraining fields and gauge parameters or adding auxiliary fields.

The constrained formulation reviewed in Chapter 2 for Bose fields took its final form in the late eighties, with the identification of the Lagrangians of eq. (2.94) by Labastida [39]. In the compact notation adopted in this Thesis, it involves multi-symmetric tensors φ transforming under gauge transformations as

$$\delta \varphi = \partial^i \Lambda_i \,, \tag{5.3}$$

but fields and gauge parameters are subjected to the constraints

$$T_{(ij} T_{kl)} \varphi = 0 \,, \qquad T_{(ij} \Lambda_{k)} = 0 \,. \tag{5.4}$$

In the same years Labastida also proposed field equations forcing multi-symmetric spinor-tensors ψ to describe free massless modes [38, 39]. To this end, he identified the gauge transformations

$$\delta \psi = \partial^i \epsilon_i \tag{5.5}$$

and the constraints

$$T_{(ij}\gamma_{k)}\psi = 0\,, \qquad\qquad \gamma_{(i}\epsilon_{j)} = 0\,, \qquad\qquad (5.6)$$

but did not relate his field equations to an action principle. The constrained formulation of the dynamics was then completed twenty years later in [41], with the identification of the Lagrangians (2.134). The choice of working with multi-symmetric (spinor-)tensors, that was adopted also in this Thesis, is merely conventional. Indeed, in the third-reference of [52] and in [40, 41] it was pointed out how one can easily adapt the formalism to describe fields carrying also antisymmetric sets of indices.

The constrained formulation involves the minimal field content leading to a *local* covariant description, but does not link directly to the *non-local* geometrical description [48, 52, 42, 51, 53] involving the curvatures of de Wit and Freedman [28] or their mixed-symmetry generalizations [52]. In order to fill this gap, in [40, 41] we also completed the alternative program initiated by Francia and Sagnotti in [56]. Its goal is the construction of local Lagrangians that do not rest upon the constraints (5.4) or (5.6), via the introduction of a minimal number of auxiliary fields. As reviewed in Section 3.1, the treatment of fully-symmetric fields developed in [56, 57, 51] can be extended to arbitrary mixed-symmetry Bose fields introducing a set of "composite" compensators $\alpha_{ijk}(\Phi)$ and a set of Lagrange multipliers β_{ijkl}. For Fermi fields the same result can be achieved introducing a set of "composite" compensators $\xi_{ij}(\Psi)$ and a set of Lagrange multipliers λ_{ijk}. The α_{ijk} and the ξ_{ij} cannot be regarded as independent fields on account of the linear dependence of the Labastida constraints on the gauge parameters, and this leads to the introduction of the "fundamental" compensators Φ_i and Ψ_i. At any rate, the labeling of the auxiliary fields with the family indices used throughout this Thesis makes it manifest that their number depends only on the number of index families carried by the fields and not on their detailed Lorentz structure. Moreover, the "composite" compensators and the Lagrange multipliers share the index structure of the constraints on gauge parameters and fields, so that this field content is expected to be the minimal one leading to an unconstrained covariant formulation of the dynamics.

However, the minimal formulation of Section 3.1 contains higher-derivative terms involving the compensators. These can be gauged away via a partial gauge fixing that does not spoil the constrained gauge symmetry, and therefore are not expected to create any problem. Nevertheless, their presence stimulated the development of the third approach reviewed in this Thesis, and first presented in [40, 41] further elaborating the treatment of fully symmetric bosons presented in [53]. In fact, in Section 3.2 it is shown how it is possible to obtain a truly conventional description of the free theory, without resorting to higher-derivative terms but still involving a number of auxiliary fields only depending on the number of index families. Clearly, in principle one can deal even with alternative setups, adding more auxiliary fields with suitable Stueckelberg-like gauge symmetries. Various examples are provided by [47, 54, 59]. Even if at the free level it is not possible to select a "best" solution among all these equivalent formulations, let me stress again that the unconstrained frameworks reviewed here have a minimal field content among those that turn into

the geometrical non-local formulation after the auxiliary fields are eliminated. On the contrary, the field content of the constrained formulation is too limited to allow a direct link with higher-spin geometry.

Aside from describing the construction of field equations and Lagrangians in these three setups, this Thesis also illustrates some peculiar properties of their structure. The most striking one is the emergence, in particular low space-time dimensions, of Weyl-like symmetries of the form

$$\delta\,\varphi \,=\, \eta^{ij}\,\Omega_{ij}\,, \qquad\qquad \delta\,\psi \,=\, \gamma^{i}\,\Omega_{i}\,. \qquad\qquad (5.7)$$

Qualitatively they are allowed by the rather complicated form taken by the Lagrangians, that contain a number of (γ-)traces of the gauge fields that grows proportionally to the number of index families that they carry. A full classification of these pathological cases, that somehow generalize the example of two-dimensional gravity, is still missing. Yet, the various classes of examples presented in Chapter 4 and the complete classification for two-family fields presented in [40, 41] vindicate the expectation that Weyl-like symmetries are confined to space-time dimensions were the corresponding fields do not propagate any degrees of freedom. At any rate, the analogy with two-dimensional gravity suggests that these models could present nonetheless a rich structure, that it would be worth investigating in the future.

Looking for a better characterization and understanding of Weyl-invariant models is only one of the directions along which the work presented in this Thesis could be further developed. First of all, it would be interesting to adapt the results reviewed here to backgrounds of constant curvature. This could provide a way to make contact with the Vasiliev construction and with its possible mixed-symmetry extensions. As repeatedly stressed in this Thesis, all string spectra contain mixed-symmetry excitations, and this line of development could then be crucial in order to understand the interplay between the Vasiliev setup, its possible generalizations involving mixed-symmetry fields and String Theory. Furthermore, the completion of the free theory for both Bose and Fermi fields naturally calls for a study of the properties of supermultiplets involving even mixed-symmetry fields. This requires the crossing of the $\mathcal{N}=8$ border either via the introduction of higher-spin supermultiplets or even, possibly, via the introduction of higher-spin supercharges.

The real challenge is eventually the study of the interactions of systems of mixed-symmetry fields. Progress in this direction is expected to be slower and more problematic, but the definite hope is that the metric-formulation of the dynamics presented in this Thesis can provide useful insights to this end.

Appendix A

Notation and conventions

The gauge fields treated in this Thesis are multi-symmetric tensors $\varphi_{\mu_1\ldots\mu_{s_1},\,\nu_1\ldots\nu_{s_2},\ldots}$ or spinor tensors $\psi^{\alpha}{}_{\mu_1\ldots\mu_{s_1},\,\nu_1\ldots\nu_{s_2},\ldots}$, whose different groups of indices are fully *symmetric* under interchanges of pairs belonging to the same set, but have no prescribed symmetry relating different sets. They are thus reducible $gl(D)$ tensors or spinor-tensors. As a result, they are perhaps less familiar than Young projected (spinor-)tensors, but are most convenient for the present discussion and play a natural role in String Theory. For instance, for bosonic strings or for the five superstring models these reducible tensors accompany products of bosonic oscillators of the type $\alpha_{-1}^{\mu_1}\ldots\alpha_{-1}^{\mu_{s_1}}\alpha_{-2}^{\nu_1}\ldots\alpha_{-2}^{\nu_{s_2}}\ldots$, that are only symmetric under interchanges of pairs of identical oscillators. Let me stress that this is only a conventional choice, and one can work as conveniently in an antisymmetric basis, dealing with multi-forms [104] rather than with multi-symmetric tensors. Both conventions can be used to describe arbitrary representations of the Lorentz group, and in general massive superstring spectra contain spinor-tensors carrying both symmetric and antisymmetric index sets. At any rate, in this Thesis I only deal with the symmetric convention, while a description of how the various formulations presented here adapt to multi-forms can be found in [40, 41]. On the other hand, in Sections 2.4 and 3.1.3 the modifications that one has to take into account when dealing with irreducible (spinor-)tensors are described in some detail.

Turning to the details of the notation, the reader should be aware that the "mostly plus" convention for the space-time signature is used throughout this Thesis, as in [40, 41, 98]. I also resort to the compact notation introduced in these papers, that extends the index-free notation developed for fully symmetric fields in [48, 42, 43, 56, 57, 51, 53]. It eliminates all space-time indices from tensor relations, so that the gauge fields $\varphi_{\mu_1\ldots\mu_{s_1},\,\nu_1\ldots\nu_{s_2},\ldots}$ and $\psi^{\alpha}{}_{\mu_1\ldots\mu_{s_1},\,\nu_1\ldots\nu_{s_2},\ldots}$ are simply denoted by φ and ψ. When needed, in the reducible case their rank is denoted in general by the list of integer numbers $(s_1, s_2, \ldots)$, both in the bosonic and in the fermionic case, the only exception being the example of fully symmetric fields. For them I resort to the standard convention, and the single label denoting the spin takes as usual both integer and half-integer values. Aside from these basic fields, Lagrangians and field equations involve gradients, divergences and traces of φ or ψ, as well as γ-traces of ψ in the fermionic case. Furthermore, the Lagrangians also contain Minkowski metric tensors related to

one (or two) of the previous index sets and γ-matrices. As a result, *"family indices"* are often needed in order to specify the sets to which some tensor indices belong. As in [40, 41, 98], these family indices are here denoted by small-case Latin letters, and the Einstein convention for summing over pairs of them is used throughout. It actually proves helpful to be slightly more precise: *upper* family indices are thus reserved for operators, like a gradient, which *add* space-time indices, while *lower* family indices are used for operators, like a divergence, which *remove* them. For instance gradients, divergences and traces of a field φ are denoted concisely by $\partial^i \varphi$, $\partial_i \varphi$ and $T_{ij}\varphi$. This shorthand notation suffices to identify the detailed meaning of these symbols, so that

$$
\begin{aligned}
\partial^i \varphi &\equiv \partial_{(\mu_1^i|}\varphi_{\ldots,|\mu_2^i \ldots \mu_{s_i+1}^i),\ldots}\,, \\
\partial_i \varphi &\equiv \partial^\lambda \varphi_{\ldots,\lambda\mu_1^i \ldots \mu_{s_i-1}^i,\ldots}\,, \\
T_{ij}\varphi &\equiv \varphi_{\ldots,}{}^\lambda{}_{\mu_1^i \ldots \mu_{s_i-1}^i,\ldots,\lambda\mu_1^j \ldots \mu_{s_j-1}^j,\ldots}\,.
\end{aligned}
\tag{A.1}
$$

The couple of parentheses that appears in the first line of eq. (A.1) denotes a symmetrization of the indices it encloses. Let me stress that in this Thesis the symmetrizations are *not* of unit strength, but involve nonetheless the minimum possible number of terms. Thus, for instance, the product $T_{(ij}T_{kl)}$ here stands for $T_{ij}T_{kl}+T_{ik}T_{jl}+T_{il}T_{jk}$. Moreover, I also use square brackets to denote antisymmetrizations.

An explicit description of the remaining operations similar to that in eq. (A.1) can be found in the main body of this Thesis, where they are introduced for the first time. On the other hand, *all computational rules follow from the algebra of these operators*, and I will thus momentarily resort to an oscillator realization that it is convenient to derive it. In the present symmetric convention one can obtain an oscillator description introducing commuting vectors $u^{i\mu}$ to cast generic gauge fields in the form

$$
\begin{aligned}
\varphi &\equiv \frac{1}{s_1!\ldots s_N!}\, u^{1\mu_1}\ldots u^{1\mu_{s_1}} u^{2\nu_1}\ldots u^{2\nu_{s_2}}\ldots \varphi_{\mu_1\ldots\mu_{s_1},\nu_1\ldots\nu_{s_2},\ldots}\,, \\
\psi &\equiv \frac{1}{s_1!\ldots s_N!}\, u^{1\mu_1}\ldots u^{1\mu_{s_1}} u^{2\nu_1}\ldots u^{2\nu_{s_2}}\ldots \psi_{\mu_1\ldots\mu_{s_1},\nu_1\ldots\nu_{s_2},\ldots}\,.
\end{aligned}
\tag{A.2}
$$

Divergences, gradients, γ-traces and "γ-matrices" can thus be described via

$$
\begin{aligned}
\partial_i &\equiv \partial^\mu \frac{\partial}{\partial u^{i\mu}}\,, & \partial^i &\equiv \partial_\mu u^{i\mu}\,, \\
\gamma_i &\equiv \gamma^\mu \frac{\partial}{\partial u^{i\mu}}\,, & \gamma^i &\equiv \gamma_\mu u^{i\mu}\,,
\end{aligned}
\tag{A.3}
$$

while traces and "metric tensors" can be introduced either via the commutation relations of the family γ-matrices

$$
\{\gamma_i,\gamma_j\} = 2\,T_{ij}\,, \qquad \{\gamma^i,\gamma^j\} = 4\,\eta^{ij}\,,
\tag{A.4}
$$

or directly as

$$
T_{ij} \equiv \eta^{\mu\nu}\frac{\partial}{\partial u^{i\mu}}\frac{\partial}{\partial u^{j\nu}}\,, \qquad \eta^{ij} \equiv \frac{1}{2}\,\eta_{\mu\nu}u^{i\mu}u^{j\nu}\,.
\tag{A.5}
$$

Interestingly, the algebra of the T_{ij} and η^{ij} operators or, equivalently, that of the γ_i and γ^i operators close only introducing the new operators

$$S^i{}_j \equiv \eta_\mu{}^\nu \, u^{i\mu} \, \frac{\partial}{\partial \, u^{j\nu}} \, . \tag{A.6}$$

For $i = j$ these act as number operators and count the indices in the i-th family,

$$S^i{}_i \, \varphi \, = \, s_i \, \varphi \, , \qquad\qquad S^i{}_i \, \psi \, = \, s_i \, \psi \, , \tag{A.7}$$

while for $i \neq j$ their net effect is to displace indices from the i-th to the j-th index families. Using the natural convention

$$\left[\frac{\partial}{\partial \, u^{i\mu}} \, , \, u^{j\nu} \right] \, = \, \eta_\mu{}^\nu \, \delta_i{}^j \tag{A.8}$$

one can finally compute the algebra closed by the various operators used in this Thesis. Focusing for a while on those that do not involve γ-matrices it reads

$$[\, \partial_i \, , \, \partial^j \,] \; = \; \square \, \delta_i{}^j \, , \tag{A.9}$$

$$[\, T_{ij} \, , \, \partial^k \,] \; = \; \partial_{(i} \delta_{j)}{}^k \, , \tag{A.10}$$

$$[\, \partial_k \, , \, \eta^{ij} \,] \; = \; \frac{1}{2} \, \delta_k{}^{(i} \partial^{i)} \, , \tag{A.11}$$

$$[\, T_{ij} \, , \, \eta^{kl} \,] \; = \; \frac{D}{2} \, \delta_i{}^{(k} \delta_j{}^{l)} + \frac{1}{2} \left(\delta_i{}^{(k} S^{l)}{}_j + \delta_j{}^{(k} S^{l)}{}_i \right) , \tag{A.12}$$

$$[\, S^i{}_j \, , \, \eta^{kl} \,] \; = \; \delta_j{}^{(k} \eta^{l)i} \, , \tag{A.13}$$

$$[\, T_{ij} \, , \, S^k{}_l \,] \; = \; T_{l(i} \delta_{j)}{}^k \, , \tag{A.14}$$

$$[\, S^i{}_j \, , \, \partial^k \,] \; = \; \partial^i \delta_j{}^k \, , \tag{A.15}$$

$$[\, \partial_k \, , \, S^i{}_j \,] \; = \; \delta_k{}^i \partial_j \, , \tag{A.16}$$

$$[\, S^i{}_j \, , \, S^k{}_l \,] \; = \; \delta_j{}^k S^i{}_l - \delta_l{}^i S^k{}_j \, . \tag{A.17}$$

Notice that the commutators of the $S^i{}_j$ operators, the key novelty of the mixed-symmetry case, build the $gl(N)$ Lie algebra if N index families are present. The other relevant commutation relations are

$$[\, \gamma_i \, , \, \partial^j \,] \; = \; \slashed{\partial} \, \delta_i{}^j \, , \tag{A.18}$$

$$[\, \partial_i \, , \, \gamma^j \,] \; = \; \slashed{\partial} \, \delta_i{}^j \, , \tag{A.19}$$

$$[\, \gamma_i \, , \, \eta^{jk} \,] \; = \; \frac{1}{2} \, \delta_i{}^{(j} \gamma^{k)} \, , \tag{A.20}$$

$$[\, T_{ij} \, , \, \gamma^k \,] \; = \; \gamma_{(i} \delta_{j)}{}^k \, , \tag{A.21}$$

$$[\, S^i{}_j \, , \, \gamma^k \,] \; = \; \gamma^i \delta_j{}^k \, , \tag{A.22}$$

$$[\, \gamma_k \, , \, S^i{}_j \,] \; = \; \delta_k{}^i \gamma_j \, , \tag{A.23}$$

while the basic anticommutators are

$$\{ \, \gamma_i \, , \, \gamma^j \, \} \; = \; D \, \delta_i{}^j + 2 \, S^j{}_i \, , \tag{A.24}$$

$$\{ \, \gamma_i \, , \, \slashed{\partial} \, \} \; = \; 2 \, \partial_i \, , \tag{A.25}$$

$$\{ \, \slashed{\partial} \, , \, \gamma^i \, \} \; = \; 2 \, \partial^i \, . \tag{A.26}$$

Altogether these relations give rise to the more involved computational rules of Appendix B, that describes the (anti)commutators involving different copies of these objects that are needed to derive the results presented in this Thesis. In this respect, let me stress that in these cases I always use an antisymmetric basis for "family" γ-matrices, that can be obtained joining the T_{ij} operators with the fully antisymmetric combinations

$$\gamma_{i_1 \ldots i_n} \equiv \frac{1}{n!} \, \gamma_{[i_1} \gamma_{i_2} \ldots \gamma_{i_n]} \, , \qquad (A.27)$$

and joining the η^{ij} operators with the corresponding objects with raised indices,

$$\gamma^{i_1 \ldots i_n} \equiv \frac{1}{n!} \, \gamma^{[i_1} \gamma^{i_2} \ldots \gamma^{i_n]} \, . \qquad (A.28)$$

Furthermore, different groups of *symmetrized* indices borne by a given quantity are separated by colons, while a semicolon signals the beginning of a group of *antisymmetrized* indices.

In order to further simplify the combinatorics, as in [40, 41] it also proves convenient to introduce the scalar product

$$\langle \, \varphi \, , \, \phi \, \rangle \equiv \frac{1}{s_1! \ldots s_n!} \, \varphi_{\mu_1^1 \ldots \mu_{s_1}^1 \, , \, \ldots \, , \, \mu_1^n \ldots \mu_{s_n}^n} \, \phi^{\, \mu_1^1 \ldots \mu_{s_1}^1 \, , \, \ldots \, , \, \mu_1^n \ldots \mu_{s_n}^n} \equiv \frac{1}{s_1! \ldots s_n!} \, \varphi \, \phi \, , \; (A.29)$$

that can be adopted even in the fermionic case in the form

$$\langle \, \bar{\psi} \, , \, \chi \, \rangle \equiv \frac{1}{s_1! \ldots s_n!} \, \bar{\psi}_{\mu_1^1 \ldots \mu_{s_1}^1 \, , \, \ldots \, , \, \mu_1^n \ldots \mu_{s_n}^n} \, \chi^{\, \mu_1^1 \ldots \mu_{s_1}^1 \, , \, \ldots \, , \, \mu_1^n \ldots \mu_{s_n}^n} \equiv \frac{1}{s_1! \ldots s_n!} \, \bar{\psi} \, \chi \, . \; (A.30)$$

Inside the brackets it is then possible to integrate by parts and to turn η's into traces without introducing any s_i-dependent combinatoric factors, since for instance

$$\langle \, \varphi \, , \, \partial^i \phi \, \rangle \equiv \frac{1}{s_1! \ldots s_n!} \, \varphi \, \partial^i \phi = - \frac{s_i}{s_1! \ldots s_n!} \, (\partial_i \varphi) \, \phi \equiv - \langle \, \partial_i \varphi \, , \, \phi \, \rangle \, , \qquad (A.31)$$

and, if $i \neq j$,

$$\langle \, \varphi \, , \, \eta^{ij} \phi \, \rangle \equiv \frac{1}{s_1! \ldots s_n!} \, \varphi \, \eta^{ij} \phi = \frac{1}{2} \frac{s_i \, s_j}{s_1! \ldots s_n!} \, (T_{ij} \varphi) \, \phi \equiv \frac{1}{2} \langle \, T_{ij} \varphi \, , \, \phi \, \rangle \, , \qquad (A.32)$$

where the reader should notice the somewhat unusual factor $\frac{1}{2}$, originating from our choice of normalization in eq. (A.5). In a similar fashion, for γ-matrices

$$\langle \, \bar{\psi} \, , \, \gamma^i \chi \, \rangle \equiv \frac{1}{s_1! \ldots s_n!} \, \bar{\psi} \, \gamma^i \chi = \frac{s_i}{s_1! \ldots s_n!} \, (\bar{\psi} \gamma_i) \, \chi \equiv \langle \, \bar{\psi} \gamma_i \, , \, \chi \, \rangle \, . \qquad (A.33)$$

However, in the main body of this Thesis I am ignoring, for simplicity, the overall factors $\prod_{i=1}^{N} s_i!$, that should accompany the Lagrangians of multi-symmetric tensors or spinor-tensors to grant them the conventional normalization.

Finally, in this Thesis I make extensive use of a number of standard tools related to the symmetric group. These include, in particular, the Young projectors Y, that are used repeatedly to separate irreducible components in family-index space. In the

text the various components are often specified by ordered lists of the lengths of the rows for the associated diagrams enclosed between braces. For instance, the $\{3, 2\}$ component corresponds to

$$\tag{A.34}$$

In general the Young projectors Y can be built combining the contributions of different Young tableaux, that can be identified associating integer labels to the tensor indices to be projected and allowing within the given graph all their arrangements such that these integers grow from left to right and from top to bottom. In some cases, however, this simple procedure can fail to produce an orthogonal decomposition, which can still be attained by a further Graham-Schmidt orthogonalization. This difficulty is not present if, for any pair of tableaux, there is at least a couple of indices belonging to a row of the first that lie in the same column of the second, and vice versa. Let me stress that this problem is never to be faced in the constructions presented in this Thesis, as a result of the particular symmetry properties of the involved objects.

Moreover, in this Thesis I adopt the conventions of [40, 41] and Young tableaux are then defined in the *symmetric* basis. Thus, the projector corresponding to a tableau containing n boxes takes the form

$$Y_\tau = \frac{\lambda(\tau)}{n!} \, S \, A \,, \tag{A.35}$$

where S and A are the corresponding products of "row symmetrizers" and "column antisymmetrizers". As pointed out in [40, 41] the *antisymmetric* basis, where the roles of S and A are interchanged, is more convenient when dealing with multi-forms rather than with multi-symmetric fields and, as a consequence, it is never used in this Thesis. At any rate, the function $\lambda(\tau)$ appearing in eq. (A.35) gives the dimension of the associated representation of the symmetric group, that can be computed, for instance, counting the standard ways of filling the boxes of the corresponding diagram with the numbers $1, 2, \ldots, n$ in increasing order from left to right and from top to bottom. I often denote the ratio that appears in eq. (A.35) as

$$\frac{\lambda(\tau)}{n!} = \frac{1}{h(\tau)} \,, \tag{A.36}$$

where $h(\tau)$ is the "hook length" of the corresponding tableau. Similar techniques are used extensively in Section 2.4 to identify irreducible Lorentz tensors.

The hook length h also enters the formulae for the dimensions of $gl(D)$ and $o(D)$ (spinorial) representations, that might prove useful to the reader. Thus, the number of components of an irreducible $gl(D)$ *spinor-tensor* with vector indices that are $\{s_1, \ldots, s_N\}$ projected is

$$\dim_{gl(D)} [\{s_1, \ldots, s_N\}] = \frac{2^{[\frac{D}{2}]}}{h} \prod_{i=1}^{N} \frac{(D + s_i - i)!}{(D - i)!} \,, \tag{A.37}$$

while for the corresponding $o(D)$ *spinor-tensor* this number is

$$\dim_S\left[\{s_1,\ldots,s_N\}\right] = \frac{2^{\left[\frac{D}{2}\right]}}{h} \prod_{i=1}^{N} \frac{(D+s_i-N-i)!}{(D-2i)!} \prod_{j=i+1}^{N} (D+s_i+s_j-i-j+1),$$

(A.38)

to be compared with the number of components for the $o(D)$ *tensor* with the same Young projection in its vector indices:

$$\dim_T\left[\{s_1,\ldots,s_N\}\right] = \frac{1}{h} \prod_{i=1}^{N} \frac{(D+s_i-N-i-1)!}{(D-2i)!} \prod_{j=i}^{N} (D+s_i+s_j-i-j). \quad \text{(A.39)}$$

I refrain from adding further details, since these and other related standard facts are discussed extensively in the literature, and in particular in [101].

Appendix B

Useful identities

Using repeatedly the (anti)commutation rules of Appendix A yields the useful identities

$$[\, \partial_l \,,\, \eta^{i_1 j_1} \ldots \eta^{i_p j_p} \,] \;=\; \frac{1}{2} \sum_{n=1}^{p} \prod_{r \neq n}^{p} \eta^{i_r j_r} \, \delta_l{}^{(i_n} \partial^{j_n)} \,, \qquad (\mathrm{B.1})$$

$$[\, T_{i_1 j_1} \ldots T_{i_p j_p} \,,\, \partial^l \,] \;=\; \sum_{n=1}^{p} \prod_{r \neq n}^{p} T_{i_r j_r} \, \partial_{(i_n} \delta_{j_n)}{}^l \,, \qquad (\mathrm{B.2})$$

that are widely used in Section 2.2.1 and in Section 2.2.2 in the construction of the constrained Lagrangians. To built the Lagrangians for Fermi fields I also abide to the identities

$$[\, \partial_l \,,\, \gamma^{k_1 \ldots k_q} \,] \;=\; (-1)^{q+1} \, \delta_l{}^{[k_1} \gamma^{k_2 \ldots k_q]} \, \slashed{\partial} \;+\; \delta_l{}^{[k_1} \partial^{k_2} \gamma^{k_3 \ldots k_q]} \,, \qquad (\mathrm{B.3})$$

$$[\, \gamma_{k_1 \ldots k_q} \,,\, \partial^l \,] \;=\; (-1)^{q+1} \, \slashed{\partial} \, \gamma_{[k_1 \ldots k_{q-1}} \delta_{k_q]}{}^l \;+\; \gamma_{[k_1 \ldots k_{q-2}} \partial_{k_{q-1}} \delta_{k_q]}{}^l \,, \qquad (\mathrm{B.4})$$

$$[\, \gamma_{k_1 \ldots k_q} \,,\, \slashed{\partial} \,]_{(-1)^{q+1}} \;=\; 2 \, \gamma_{[k_1 \ldots k_{q-1}} \partial_{k_q]} \,, \qquad (\mathrm{B.5})$$

$$[\, \slashed{\partial} \,,\, \gamma^{k_1 \ldots k_q} \,]_{(-1)^{q+1}} \;=\; 2 \, \partial^{[k_1} \gamma^{k_2 \ldots k_q]} \,, \qquad (\mathrm{B.6})$$

where the symbols $[\, , \,]_{(-1)^{q+1}}$ can denote both commutators and anticommutators, depending on the sign of $(-1)^{q+1}$.

For the reduction of the equation of motion to the Labastida form discussed in Section 4.1, Fermi fields require the knowledge of a set of (anti)commutation relations that can be conveniently presented introducing the new operators

$$\gamma^{k_1 \ldots k_q}{}_l \;=\; \gamma^{k_1 \ldots k_q} \gamma_l \;-\; \gamma^{[k_1 \ldots k_{q-1}} S^{k_q]}{}_l \,. \qquad (\mathrm{B.7})$$

They appear in fact in the identities

$$[\, \gamma_l \,,\, \gamma^{k_1 \ldots k_q} \,]_{(-1)^{q+1}} \;=\; (\, D - q + 1 \,) \, \delta_l{}^{[k_1} \gamma^{k_2 \ldots k_q]} \;+\; 2 \, (-1)^{q+1} \gamma^{[k_1 \ldots k_{q-1}} S^{k_q]}{}_l \,, \quad (\mathrm{B.8})$$

$$[\, \gamma_{k_1 \ldots k_q} \,,\, \gamma^l \,]_{(-1)^{q+1}} \;=\; (\, D - q + 1 \,) \, \gamma_{[k_1 \ldots k_{q-1}} \delta_{k_q]}{}^l \;+\; 2 \, (-1)^{q+1} S^l{}_{[k_1} \gamma_{k_2 \ldots k_q]} \,, \quad (\mathrm{B.9})$$

$$[\,T_{lm}\,,\,\gamma^{k_1\ldots k_q}\,] \;=\; (-1)^{q+1}\,\delta_{(l|}{}^{[k_1}\gamma^{k_2\ldots k_q]}{}_{|m)}\,, \tag{B.10}$$

$$\begin{aligned}
[\,\gamma_{lm}\,,\,\gamma^{k_1\ldots k_q}\,] \;=\; &-(D-q+1)\,(D-q+2)\,\delta_l{}^{[k_1}\delta_m{}^{k_2}\gamma^{k_3\ldots k_q]}\\
&+ (-1)^q\,(D-q+2)\,\delta_{[l|}{}^{[k_1}\gamma^{k_2\ldots k_q]}{}_{|m]} + 2\,\gamma^{[k_1\ldots k_{q-1}|}{}_{[l}S^{|k_q]}{}_{m]}\\
&+ (-1)^{q+1}\,(D-q+1)\,\delta_{[l|}{}^{[k_1}\gamma^{k_2\ldots k_{q-1}}S^{k_q]}{}_{|m]}\,,
\end{aligned} \tag{B.11}$$

$$[\,S^l{}_m\,,\,\gamma^{k_1\ldots k_q}\,] \;=\; (-1)^{q+1}\,\delta_m{}^{[k_1}\gamma^{k_2\ldots k_q]\,l}\,, \tag{B.12}$$

$$[\,\gamma_{k_1\ldots k_q}\,,\,S^l{}_m\,] \;=\; (-1)^{q+1}\,\gamma_{m[k_1\ldots k_{q-1}}\delta_{k_q]}{}^l\,. \tag{B.13}$$

These are used together with the corresponding commutators involving products of metric tensors:

$$[\,\gamma_l\,,\,\eta^{i_1 j_1}\ldots\eta^{i_p j_p}\,] \;=\; \frac{1}{2}\sum_{n=1}^{p}\prod_{r\neq n}\eta^{i_r j_r}\,\delta_l{}^{(i_n}\gamma^{j_n)}\,, \tag{B.14}$$

$$\begin{aligned}
[\,T_{lm}\,,\,\eta^{i_1 j_1}\ldots\eta^{i_p j_p}\,] \;=\; &\frac{1}{2}\sum_{n=1}^{p}\Big\{\, D\prod_{r\neq n}\eta^{i_r j_r}\,\delta_l{}^{(i_n}\delta_m{}^{j_n)}\\
&+ \sum_{m<n}\prod_{r\neq m,n}\eta^{i_r j_r}\delta_{(l|}{}^{(i_m}\eta^{j_m)(i_n}\delta^{j_n)}{}_{|m)} + \prod_{r\neq n}\eta^{i_r j_r}\delta_{(l|}{}^{(i_n}S^{j_n)}{}_{|m)}\,\Big\}\,,
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
[\,\gamma_{lm}\,,\,\eta^{i_1 j_1}\ldots\eta^{i_p j_p}\,] \;=\; &\frac{1}{4}\sum_{n=1}^{p}\Big\{\, \sum_{m<n}\prod_{r\neq m,n}\eta^{i_r j_r}\delta_{[l|}{}^{(i_m}\gamma^{j_m)(i_n}\delta^{j_n)}{}_{|m]}\\
&+ 2\prod_{r\neq n}\eta^{i_r j_r}\delta_{[l|}{}^{(i_n}\gamma^{j_n)}{}_{|m]}\,\Big\}\,.
\end{aligned} \tag{B.16}$$

Finally, when dealing with Fermi fields one has to make a wide use of the following composition rules for family γ-matrices:

$$\gamma_{k_1\ldots k_q}\,\gamma_l \;=\; \gamma_{k_1\ldots k_q l} \;+\; \gamma_{[k_1\ldots k_{q-1}}T_{k_q]l}\,, \tag{B.17}$$

$$\begin{aligned}
\gamma_{k_1\ldots k_q}\,\gamma_{lm} \;=\; &\gamma_{k_1\ldots k_q lm} \;+\; \big(\gamma_{[k_1\ldots k_{q-1}|m}T_{|k_q]l} - \gamma_{[k_1\ldots k_{q-1}|l}T_{|k_q]m}\big)\\
&+ \gamma_{[k_1\ldots k_{q-2}}T_{k_{q-1}|m}T_{|k_q]l}\,.
\end{aligned} \tag{B.18}$$

Similar rules also apply when one raises indices, but the normalization of η^{ij} introduced in eq. (A.5) requires that this operation be supplemented by the substitution

$$T_{ij} \;\to\; 2\,\eta^{ij}\,. \tag{B.19}$$

In order to derive the equations of motion, or in order to impose the condition (2.133) of self-adjointness of the constrained Lagrangians, it is also convenient to keep in mind the further, standard relation

$$\gamma_0\,(\gamma_{k_1\ldots k_q})^\dagger\,\gamma_0 \;=\; (-1)^{\frac{(q-1)(q-2)}{2}}\,\gamma_{k_1\ldots k_q}\,. \tag{B.20}$$

The computational rules collected in these first two appendices suffice to compute generic (γ-)traces of the kinetic tensors $\mathcal{F}$ and $\mathcal{S}$ and of the Bianchi identities. In

particular, starting from the Fronsdal-Labastida tensor

$$\mathcal{F} = \Box\,\varphi - \partial^i\partial_i\,\varphi + \frac{1}{2}\,\partial^i\partial^j\,T_{ij}\,\varphi\,, \tag{B.21}$$

for Bose fields one can obtain

$$\prod_{r=1}^{p} T_{i_r j_r}\,\mathcal{F} = (p+1)\,\Box \prod_{r=1}^{p} T_{i_r j_r}\,\varphi - 2 \sum_{n=1}^{p} \partial_{i_n}\partial_{j_n} \prod_{r\neq n}^{p} T_{i_r j_r}\,\varphi$$

$$+ \sum_{n=1}^{p} \sum_{m<n} \left(\partial_{i_n}\partial_{(i_m}\,T_{j_m)j_n} + \partial_{j_n}\partial_{(i_m}\,T_{j_m)i_n} \right) \prod_{r\neq m,n}^{p} T_{i_r j_r}\,\varphi$$

$$- \partial^k \Bigg[\partial_k \prod_{r=1}^{p} T_{i_r j_r}\,\varphi - \sum_{n=1}^{p} \partial_{(i_n}\,T_{j_n)k} \prod_{r\neq n}^{p} T_{i_r j_r}\,\varphi \Bigg] + \frac{1}{2}\,\partial^k\partial^l\,T_{kl} \prod_{r=1}^{p} T_{i_r j_r}\,\varphi\,. \tag{B.22}$$

This result is at the root of the derivation, in eq. (2.92), of the constrained Lagrangians via the identification of a constrained self-adjoint Einstein-like operator. As a consequence, it is also crucial in the derivation of the equations of motion that follow from the various bosonic Lagrangians presented in this Thesis. Furthermore, starting from the Bianchi identities

$$\mathscr{B}_i\ :\ \partial_i\,\mathcal{F} - \frac{1}{2}\,\partial^j\,T_{ij}\,\mathcal{F} = -\frac{1}{12}\,\partial^j\partial^k\partial^l\,T_{(ij}\,T_{kl)}\,\varphi\,, \tag{B.23}$$

in the constrained theory one can obtain the result

$$T_{i_1 j_1}\dots T_{i_p j_p}\,\mathscr{B}_k\ :\ \partial_k\,T_{i_1 j_1}\dots T_{i_p j_p}\,\mathcal{F} - \frac{1}{2}\sum_{n=1}^{p}\partial_{(i_n}\,T_{j_n)k} \prod_{r\neq n}^{p} T_{i_r j_r}\,\mathcal{F}$$

$$- \frac{1}{2}\,\partial^l\,T_{i_1 j_1}\dots T_{i_p j_p}\,T_{kl}\,\mathcal{F} = 0\,, \tag{B.24}$$

that was already displayed in eq. (2.84), and that plays a crucial role in the derivation of the constrained Lagrangians via the request of gauge invariance.

In a similar fashion, starting from the Fang-Fronsdal-Labastida tensor

$$\mathcal{S} = i\left\{ \slashed{\partial}\,\psi - \partial^i\,\gamma_i\,\psi \right\}\,, \tag{B.25}$$

for Fermi fields one can obtain

$$\prod_{r=1}^{p} T_{i_r j_r}\,\gamma_{k_1\dots k_q}\,\mathcal{S} = (-1)^q\,i\,(q+1)\,\slashed{\partial}\,\gamma_{k_1\dots k_q} \prod_{r=1}^{p} T_{i_r j_r}\,\psi$$

$$+ i\,(q+1)\,\gamma_{[k_1\dots k_{q-1}}\partial_{k_q]} \prod_{r=1}^{p} T_{i_r j_r}\,\psi - i\sum_{n=1}^{p} \gamma_{[k_1\dots k_{q-1}}T_{k_q](i_n}\partial_{j_n)} \prod_{r\neq n}^{p} T_{i_r j_r}\,\psi$$

$$- i\sum_{n=1}^{p} \gamma_{k_1\dots k_q(i_n}\partial_{j_n)} \prod_{r\neq n}^{p} T_{i_r j_r}\,\psi - i\,\partial^l\,\gamma_{[k_1\dots k_{q-1}}T_{k_q]l} \prod_{r=1}^{p} T_{i_r j_r}\,\psi$$

$$- i\,\partial^l\,\gamma_{k_1\dots k_q l} \prod_{r=1}^{p} T_{i_r j_r}\,\psi \tag{B.26}$$

In analogy with the bosonic case, this result is at the root of the derivation, in eq. (2.133), of the constrained Lagrangians via the identification of a constrained self-adjoint Rarita-Schwinger-like operator. As a consequence, it is also crucial in the derivation of the equations of motion that follow from the various fermionic Lagrangians presented in this Thesis. The γ-traces of the Bianchi identities

$$\mathscr{B}_i \,:\, \partial_i \mathcal{S} - \frac{1}{2}\,\slashed{\partial}\,\gamma_i \mathcal{S} - \frac{1}{2}\,\partial^j T_{ij}\,\mathcal{S} - \frac{1}{6}\,\partial^j\,\gamma_{ij}\,\mathcal{S} \,=\, \frac{i}{6}\,\partial^j\partial^k\,T_{(ij}\,\gamma_{k)}\,\psi\,, \qquad \text{(B.27)}$$

can be also easily computed applying the rules displayed in this appendix. However, even at the constrained level the result takes the far more involved form

$$(-1)^q \prod_{r=1}^{p} T_{i_r j_r}\,\gamma_{k_1\ldots k_q}\,\mathscr{B}_l \,:\, \frac{1}{6}\sum_{n=1}^{p}\,\partial_{(i_n}\gamma_{j_n)\,k_1\ldots k_q\,l}\,\prod_{r\neq n}^{p} T_{i_r j_r}\,\mathcal{S}$$

$$+\,\partial_{[k_1}\gamma_{k_2\ldots k_q\,l\,]}\prod_{r=1}^{p} T_{i_r j_r}\,\mathcal{S} \,+\, \frac{q-1}{6}\,\partial_{[k_1}\gamma_{k_2\ldots k_q]l}\prod_{r=1}^{p} T_{i_r j_r}\,\mathcal{S}$$

$$-\,\frac{(-1)^q}{2}\sum_{n=1}^{p}\gamma_{k_1\ldots k_q}\,\partial_{(i_n}T_{j_n)l}\prod_{r\neq n}^{p} T_{i_r j_r}\,\mathcal{S} \,+\, \frac{1}{6}\sum_{n=1}^{p}\partial_{(i_n}\gamma_{j_n)[k_1\ldots k_{q-1}}T_{k_q]l}\prod_{r\neq n}^{p} T_{i_r j_r}\,\mathcal{S}$$

$$-\,\frac{1}{6}\sum_{n=1}^{p}\gamma_{l\,[k_1\ldots k_{q-1}}T_{k_q](i_n}\partial_{j_n)}\prod_{r\neq n}^{p} T_{i_r j_r}\,\mathcal{S} \,+\, \frac{q+1}{6}\,\partial_{[k_1}\gamma_{k_2\ldots k_{q-1}}T_{k_q]l}\prod_{r=1}^{p} T_{i_r j_r}\,\mathcal{S}$$

$$+\,\frac{(-1)^q}{6}\sum_{n=1}^{p}\gamma_{[k_1\ldots k_{q-2}}T_{k_{q-1}|l}T_{|k_q](i_n}\partial_{j_n)}\prod_{r\neq n}^{p} T_{i_r j_r}\,\mathcal{S}$$

$$-\,\slashed{\partial}\left\{\frac{q+3}{6}\,\gamma_{k_1\ldots k_q\,l} + \frac{q-1}{6}\,\gamma_{[k_1\ldots k_{q-1}}T_{k_q]l}\right\}\prod_{r=1}^{p} T_{i_r j_r}\,\mathcal{S}$$

$$-\,\frac{(-1)^q}{6}\,\partial^m\gamma_{k_1\ldots k_q\,lm}\prod_{r=1}^{p} T_{i_r j_r}\,\mathcal{S} \,-\, \frac{(-1)^q}{2}\,\partial^m\gamma_{k_1\ldots k_q}\,T_{lm}\prod_{r=1}^{p} T_{i_r j_r}\,\mathcal{S}$$

$$-\,\frac{1}{6}\,\partial^m\big(\gamma_{l\,[k_1\ldots k_{q-1}}T_{k_q]m} - \gamma_{m\,[k_1\ldots k_{q-1}}T_{k_q]l}\big)\prod_{r=1}^{p} T_{i_r j_r}\,\mathcal{S}$$

$$+\,\frac{(-1)^q}{6}\,\partial^m\gamma_{[k_1\ldots k_{q-2}}T_{k_{q-1}|l}T_{|k_q]m}\prod_{r=1}^{p} T_{i_r j_r}\,\mathcal{S} \,=\, 0\,. \qquad \text{(B.28)}$$

The projections of this expression that allow to build the Lagrangians via the request of gauge invariance will be computed in the next appendix.

Appendix C

Proofs of some results of Chapter 3

In this last appendix I would like to provide the proofs of some results that I used in chapter 3 to fix the structure of the unconstrained Lagrangians. In particular I will deal with

- the structure of the compensator terms of the bosonic Lagrangians, that is fixed by eq. (3.40);

- the structure of the compensator terms of the fermionic Lagrangians, that is fixed by eq. (3.79);

- the two-column projections of the multiple γ-traces of the Bianchi identities for Fermi fields, that lead to eq. (2.128) and to eq. (3.68).

Proof of eq. (3.40)

For Bose fields the structure of the compensator terms entering the unconstrained Lagrangians is fixed by the identity

$$
\begin{aligned}
& \langle\, Y_{\{3,2^{\,p-1}\}}\, T_{i_1 j_1} \ldots T_{i_p j_p}\, \Lambda_k\ ,\ \partial_k\, \mathcal{A}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p}\, \rangle \\
&= \frac{p}{p+2}\, \langle\, T_{i_2 j_2} \ldots T_{i_p j_p}\, T_{(i_1 j_1}\, \Lambda_{k)}\ ,\ \partial_k\, \mathcal{A}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p}\, \rangle\,,
\end{aligned} \tag{C.1}
$$

that allows to rewrite the terms in the gauge variation (3.39) in a way that manifestly contains symmetrized traces of the gauge parameters. The scalar product plays a crucial role here, since makes it possible to extract the $\{3, 2^{p-1}\}$ component enforcing the projection associated to a *single* Young tableau. In fact, while the full $\{3, 2^{p-1}\}$ projection results from a sum of different tableaux, one can choose them in such a way that only one of them contributes to the scalar product. In particular, one needs only the tableau

i_1	j_1	k
$\vdots$	$\vdots$	
i_p	j_p	

$$\tag{C.2}$$

which corresponds to the choice of standard labeling $i_n = 2\,n-1$, $j_n = 2\,n$ for $1 \leq n \leq p$ and $k = 2\,p+1$. In fact, any symmetrization involving three indices of the set $(\,i_m, j_n\,)$ in the right entry of the scalar product would induce a symmetrization beyond a given line of the tableau

$$
\begin{array}{|c|c|}
\hline
i_1 & j_1 \\
\hline
\vdots & \vdots \\
\hline
i_p & j_p \\
\hline
\end{array}
\tag{C.3}
$$

that identifies the structure of $\mathcal{A}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p}$. In order to proceed, it is then convenient to apply the projection associated to the tableau (C.2) directly on the right entry of the scalar product. In fact, the operations needed to build it differ from those implied by the tableau (C.3) only in the symmetrization of the three indices $(\,i_1, j_1, k\,)$. More precisely, denoting the tableau (C.2) by τ_1 and the tableau (C.3) by τ_2 and using the notation of eq. (A.35) one can recognize that acting on products of traces

$$
Y_{\tau_2} = 2\,\frac{\lambda(\tau_2)}{(\,2\,p\,)!}\,\widetilde{S}_{\tau_2}\,A_{\tau_2}\,.
\tag{C.4}
$$

Notice that the product $\widetilde{S}_{\tau_2}$ of "row symmetrizers" does not include the operator $S_{(i_1, j_1)}$ because this symmetrization is already induced by the others. When applied to $\partial_k\,T_{i_1 j_1} \ldots T_{i_p j_p}$, the product of the two Young projectors Y_{τ_1} and Y_{τ_2} then gives

$$
Y_{\tau_1}\,Y_{\tau_2} = \frac{\lambda(\tau_1)}{(\,2\,p+1\,)!}\,S_{(i_1, j_1, k)}\,\widetilde{S}_{\tau_2}\,A_{\tau_2}\,Y_{\tau_2} = \frac{\lambda(\tau_1)}{2\,(\,2\,p+1\,)\,\lambda(\tau_2)}\,S_{(i_1, j_1, k)}\,(\,Y_{\tau_2}\,)^2.
\tag{C.5}
$$

The ratio that appears in eq. (C.5) is

$$
\frac{\lambda(\tau_1)}{(\,2\,p+1\,)\,\lambda(\tau_2)} = \frac{p}{p+2}\,,
\tag{C.6}
$$

so that

$$
Y_{\tau_1}\,\partial_k\,\mathcal{A}^{[p]}{}_{i_1 j_1, \ldots, i_p j_p} = \frac{p}{p+2}\,\partial_{(\,k}\,\mathcal{A}^{[p]}{}_{i_1 j_1\,), i_2 j_2, \ldots, i_p j_p}\,.
\tag{C.7}
$$

Proof of eq. (3.79)

For Fermi fields the structure of the compensator terms entering the unconstrained Lagrangians is fixed by the identity

$$
\langle\,Y_{\{3, 2^{p-1}, 1^q\}}\,T_{i_1 j_1} \ldots T_{i_p j_p}\,\bar{\epsilon}_l\,\gamma_{k_1 \ldots k_q}\,,\ \partial_l\,(\,\gamma^{[q]}\,\mathcal{W}^{[p]}\,)_{i_1 j_1, \ldots, i_p j_p\,; k_1 \ldots k_q}\,\rangle
$$

$$
= \langle\,(\,p+1\,)\,T_{i_2 j_2} \ldots T_{i_p j_p}\,T_{(i_1 j_1}\,\bar{\epsilon}_{l)}\,\gamma_{k_1 \ldots k_q} - T_{i_2 j_2} \ldots T_{i_p j_p}\,T_{[\,k_1\,|\,(\,l}\,\bar{\epsilon}_{i_1}\,\gamma_{j_1}\,)\,\gamma_{|\,k_2 \ldots k_q\,]}\,,
$$

$$
\frac{p}{3\,(\,p+1\,)(\,p+q+2\,)}\,\partial_{(\,l\,|}\,(\,\gamma^{[q]}\,\mathcal{W}^{[p]}\,)_{|\,i_1 j_1\,), \ldots, i_p j_p\,; k_1 \ldots k_q}\,\rangle\,.
\tag{C.8}
$$

that appears in eq. (3.79). As in eq. (C.1), the presence of the scalar product makes it possible to extract the $\{3, 2^{p-1}\}$ component acting on the left entry with the projector

associated to the *single* Young tableau

$$
\begin{array}{|c|c|c|}
\hline
i_1 & j_1 & l \\
\hline
\vdots & \vdots & \multicolumn{1}{c}{} \\
\cline{1-2}
i_p & j_p & \multicolumn{1}{c}{} \\
\cline{1-2}
k_1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-1}
k_q & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}
\tag{C.9}
$$

In fact, one can decompose $Y_{\{3,2^{p-1}\}}$ into a sum of Young tableaux where only the projector associated to (C.9) is not annihilated when contracted against the right entry of the scalar product, where $(\gamma^{[q]}\mathcal{W}^{[p]})$ carries the projection

$$
\begin{array}{|c|c|}
\hline
i_1 & j_1 \\
\hline
\vdots & \vdots \\
\hline
i_p & j_p \\
\hline
k_1 & \multicolumn{1}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{1}{c}{} \\
\cline{1-1}
k_q & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}
\tag{C.10}
$$

since all other tableaux would result in symmetrizations beyond a given a line.

In analogy with the previous discussion, the operations needed to build the projection associated to the tableau (C.9) differ from those implied by the tableau (C.10) only in the symmetrization of the three indices (i_1, j_1, l). More precisely, denoting the tableau (C.9) by τ_1 and the tableau (C.10) by τ_2 and using the notation of eq. (A.35), when acting on the left entry of eq. (C.8)

$$
Y_{\tau_2} \;=\; \frac{2}{h(\tau_2)}\, \widetilde{S}_{\tau_2}\, A_{\tau_2}\,.
\tag{C.11}
$$

The product $\widetilde{S}_{\tau_2}$ of "row symmetrizers" does not include the operator $S_{(i_1,j_1)}$ simply because this symmetrization is already induced by the others. Acting with Y_{τ_1} gives instead

$$
Y_{\tau_1}\, Y_{\tau_2} \;=\; \frac{1}{h(\tau_1)}\, S_{(i_1,j_1,l)}\, \widetilde{S}_{\tau_2}\, A_{\tau_2}\, Y_{\tau_2} \;=\; \frac{h(\tau_2)}{2\,h(\tau_1)}\, S_{(i_1,j_1,l)}\, \big(Y_{\tau_2}\big)^2 \,,
\tag{C.12}
$$

so that, taking into account the ratio between the hook lengths, one ends up with

$$
\big\langle\, Y_{\{3,2^{p-1},1^q\}}\, T_{i_1 j_1}\ldots T_{i_p j_p}\, \bar{\epsilon}_l\, \gamma_{k_1\ldots k_q}\,,\ \partial_l\,(\,\gamma^{[q]}\mathcal{W}^{[p]}\,)_{i_1 j_1,\,\ldots,\,i_p j_p\,;\,k_1\ldots k_q}\,\big\rangle
\tag{C.13}
$$

$$
=\; \frac{p\,(p+q+1)}{(p+1)(p+q+2)}\, \big\langle\, Y_{\tau_2}\, T_{i_1 j_1}\ldots T_{i_p j_p}\, \bar{\epsilon}_l\, \gamma_{k_1\ldots k_q}\,,\ \partial_{(l|}\,(\,\gamma^{[q]}\mathcal{W}^{[p]}\,)_{|i_1 j_1\,),\,\ldots,\,i_p j_p\,;\,k_1\ldots k_q}\,\big\rangle\,.
$$

In the bosonic case these steps sufficed to recover the Labastida constraints on the gauge parameters, due to the presence of only identical T tensors. However, here one needs to refine the arguments, because the family indices (i_1, j_1) that are symmetrized

with l can be carried both by a trace T and by an antisymmetrized γ-trace, as can be easily seen absorbing the symmetrizations induced by τ_2 in the right entry of the scalar product, so that

$$
\langle \, Y_{\{3,2^{p-1},1^q\}} \, T_{i_1 j_1} \ldots T_{i_p j_p} \, \bar{\epsilon}_l \, \gamma_{k_1 \ldots k_q} \, , \, \partial_l \, (\, \gamma^{[q]} \, \mathcal{W}^{[p]} \,)_{i_1 j_1, \ldots, i_p j_p \, ; \, k_1 \ldots k_q} \, \rangle
$$
$$
= \, \langle \, \frac{2^p \, (\, q + 1 \,) \, !}{(\, p + q + 1 \,) \, !} \, T_{[\, i_1 \, | \, j_1} \ldots T_{\, | \, i_p \, | \, j_p} \, \bar{\epsilon}_l \, \gamma_{\, | \, k_1 \ldots k_q \,]} \, ,
$$
$$
\frac{p \, (\, p + q + 1 \,)}{(\, p + 1 \,)(\, p + q + 2 \,)} \, \partial_{(\, l \, |} \, (\, \gamma^{[q]} \, \mathcal{W}^{[p]} \,)_{\, | \, i_1 j_1 \,), \ldots, \, i_p j_p \, ; \, k_1 \ldots k_q} \, \rangle \, . \tag{C.14}
$$

In order to proceed, it is convenient to separate the terms in the antisymmetrization that differ in the position of $(\, i_1, j_1 \,)$,

$$
T_{[\, i_1 \, | \, j_1} \ldots T_{\, | \, i_p \, | \, j_p} \, \bar{\epsilon}_l \, \gamma_{\, | \, k_1 \ldots k_q \,]} \, = \, T_{i_1 j_1} \, T_{[\, i_2 \, | \, j_2} \ldots T_{\, | \, i_p \, | \, j_p} \, \bar{\epsilon}_l \, \gamma_{\, | \, k_1 \ldots k_q \,]}
$$
$$
- \, \sum_{n \, = \, 2}^{p} (-1)^{\, n \, p} \, T_{i_1 j_n} \, T_{j_1 \, [\, i_2} \, T_{i_3 \, | \, j_{n+1}} \ldots T_{\, | \, i_p \, | \, j_{n-1}} \, \bar{\epsilon}_l \, \gamma_{\, | \, k_1 \ldots k_q \,]}
$$
$$
- \, T_{[\, i_2 \, | \, j_2} \ldots T_{\, | \, i_p \, | \, j_p} \, T_{\, | \, k_1 \, | \, j_1} \, \bar{\epsilon}_l \, \gamma_{i_1 \, | \, k_1 \ldots k_q \,]} \, . \tag{C.15}
$$

Forcing again the symmetrization in $(\, i_1, j_1, l \,)$, in the scalar product the second group of terms in eq. (C.15) becomes proportional to the first term, because

$$
\sum_{n \, = \, 2}^{p} (-1)^{\, n \, p} \, T_{[\, i_2 \, | \, j_n} \ldots T_{\, | \, i_{p-1} \, | \, j_{n-2}} \ldots T_{\, | \, i_p \, | \, (\, i_1} \, T_{j_1 \, | \, j_{n-1}} \bar{\epsilon}_{\, | \, l \,)} \, \gamma_{\, | \, k_1 \ldots k_q \,]}
$$
$$
= \, \sum_{n \, = \, 2}^{p} (-1)^{\, n \, p} \, T_{[\, i_2 \, | \, j_n} \ldots T_{\, | \, i_{p-1} \, | \, j_{n-2}} \ldots T_{\, | \, i_p \, | \, (\, i_1} \, T_{j_1 l} \, \bar{\epsilon}_{\, j_{n-1} \,)} \, \gamma_{\, | \, k_1 \ldots k_q \,]}
$$
$$
- \, \sum_{n \, = \, 2}^{p} (-1)^{\, n \, p} \, T_{(\, i_1 j_1} \, T_{l \,) \, [\, i_2} \, T_{i_3 \, | \, j_{n+1}} \ldots T_{\, | \, i_p \, | \, j_{n-1}} \, \bar{\epsilon}_{j_n} \, \gamma_{\, | \, k_1 \ldots k_q \,]}
$$
$$
- \, (\, p - 1 \,) \, T_{[\, i_2 \, | \, j_n} \ldots T_{\, | \, i_p \, | \, j_{n-1}} \, T_{(\, i_1 j_1} \, \bar{\epsilon}_{l \,)} \, \gamma_{\, | \, k_1 \ldots k_q \,]} \, , \tag{C.16}
$$

and all antisymmetrizations over four indices (including those induced on a couple of traces by a symmetrization over three indices) are annihilated by the right entry of the scalar product (C.14), so that only the last term survives.

In conclusion, the left entry of eq. (C.14) gives rise to two terms when one forces the symmetrization in $(\, i_1, j_1, l \,)$,

$$
T_{[\, i_1 \, | \, j_1} \ldots T_{\, | \, i_p \, | \, j_p} \, \bar{\epsilon}_l \, \gamma_{\, | \, k_1 \ldots k_q \,]} \, \longrightarrow \, \frac{1}{6} \, \Big\{ \, (\, p + 1 \,) \, T_{[\, i_2 \, | \, j_n} \ldots T_{\, | \, i_p \, | \, j_{n-1}} \, T_{(\, i_1 j_1} \, \bar{\epsilon}_{l \,)} \, \gamma_{\, | \, k_1 \ldots k_q \,]}
$$
$$
- \, T_{[\, i_2 \, | \, j_n} \ldots T_{\, | \, i_p \, | \, j_{n-1}} \, T_{\, | \, k_1 \, | \, (\, i_1} \bar{\epsilon}_{j_1} \, \gamma_{l \,) \, | \, k_1 \ldots k_q \,]} \, \Big\} \, , \tag{C.17}
$$

and imposing the remaining symmetrizations one can recover in each of them the

projection associated to the Young tableau

$$
\begin{array}{|c|c|}
\hline i_2 & j_2 \\
\hline \vdots & \vdots \\
\hline i_p & j_p \\
\hline k_1 \\
\cline{1-1}
\vdots \\
\cline{1-1}
k_q \\
\cline{1-1}
\end{array}
\tag{C.18}
$$

However, this projection is automatically enforced by the right entry of the scalar product, so that

$$
\begin{aligned}
& \left\langle\, Y_{\{3,2^{p-1},1^q\}}\, T_{i_1 j_1} \ldots T_{i_p j_p}\, \bar{\epsilon}_l\, \gamma_{k_1 \ldots k_q}\, ,\, \partial_l\, (\gamma^{[q]}\, \mathcal{W}^{[p]})_{i_1 j_1,\,\ldots,\, i_p j_p\,;\, k_1 \ldots k_q}\, \right\rangle \\
&= \frac{p\,(p+q+1)}{(p+1)(p+q+2)}\, \Big\langle\, \frac{1}{3\,(p+q+1)}\, Y_{\tau_3}\, \Big\{\, (p+1)\, T_{[i_1 \,|\, j_1} \ldots T_{|\, i_p \,|\, j_p}\, \bar{\epsilon}_l\, \gamma_{|\, k_1 \ldots k_q\,]} \\
&\quad -\, T_{[i_2 \,|\, j_n} \ldots T_{|\, i_p \,|\, j_{n-1}}\, T_{|\, k_1 \,|\, (i_1 \bar{\epsilon}_{j_1}\, \gamma_l)\,|\, k_1 \ldots k_q\,]}\, \Big\}\, ,\, \partial_{(l\,|}\, (\gamma^{[q]}\, \mathcal{W}^{[p]})_{|\, i_1 j_1\,),\,\ldots,\, i_p j_p\,;\, k_1 \ldots k_q}\, \Big\rangle \\
&= \Big\langle\, (p+1)\, T_{i_2 j_2} \ldots T_{i_p j_p}\, T_{(i_1 j_1}\, \bar{\epsilon}_{l)}\, \gamma_{k_1 \ldots k_q}\, -\, T_{i_2 j_2} \ldots T_{i_p j_p}\, T_{[k_1 \,|\, (i_1\, \bar{\epsilon}_{j_1}\, \gamma_{l)\,|\, k_2 \ldots k_q\,]}}\, , \\
&\quad \frac{p}{3\,(p+1)(p+q+2)}\, \partial_{(l\,|}\, (\gamma^{[q]}\, \mathcal{W}^{[p]})_{|\, i_1 j_1\,),\,\ldots,\, i_p j_p\,;\, k_1 \ldots k_q}\, \Big\rangle,
\end{aligned}
\tag{C.19}
$$

where τ_3 denotes the tableau of eq. (C.18) and the last two lines lead to the result in eqs. (3.79) or (C.8), using eq. (B.17).

Two-column projected consequences of the Bianchi identities for Fermi fields

In Sections 2.2.2 and 3.1.2 I presented some two-column projected consequences of the Bianchi identities (B.27), and now I would like to show how they can be extracted from their multiple γ-traces (B.28). As anticipated in Section 3.1.2, in the fermionic case eqs. (B.28) actually admit the couple of two-column projections that can be obtained from the action of an already two-column projected combination of γ-traces. More precisely, acting with $T_{i_1 j_1} \ldots T_{i_p j_p}\, \gamma_{k_1 \ldots k_q}$ on the $\mathcal{B}_l$ of eq. (B.27) one can obtain both two-column projections appearing in

$$
\{2^p, 1^q\} \otimes \{1\} = \{2^p, 1^{q+1}\} \oplus \{2^{p+1}, 1^{q-1}\} \oplus \{3, 2^{p-1}, 1^q\}.
\tag{C.20}
$$

On the other hand, it is convenient to deal with Young projections containing the maximum number of antisymmetrizations compatible with the tensorial structure of the expressions under scrutiny. In this case one is led to consider the $\{2^p, 1^{q+1}\}$ projection of (B.28), that can be computed acting with the projector associated to the

single Young tableau

$$
\begin{array}{|c|c|}
\hline
i_1 & j_1 \\
\hline
\vdots & \vdots \\
\hline
i_p & j_p \\
\hline
k_1 & \multicolumn{1}{c}{} \\
\cline{1-1}
\vdots & \multicolumn{1}{c}{} \\
\cline{1-1}
k_q & \multicolumn{1}{c}{} \\
\cline{1-1}
l & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}
\tag{C.21}
$$

on account of the symmetry properties of the combination $T_{i_1 j_1} \dots T_{i_p j_p} \gamma_{k_1 \dots k_q} \mathscr{B}_l$. One can thus combine the terms of (B.28) resorting to techniques similar to those adopted to deal with the traces of the bosonic Bianchi identities in Section 3.1.1. First of all, the structure of the tableau (C.21) implies that all symmetrizations in the generic set of three indices (i_m, j_m, k_n) vanish. One can use this fact to immediately relate various terms as in eq. (2.86). However, for Fermi fields this kind of manipulations does not suffice to make all contributions with the same number of traces and an antisymmetrized γ-trace with a given number of indices proportional. Still, one can also consider that, in order to build the projection associated to (C.21), an antisymmetrization over l and all the k_n indices is required. The $\{2^p, 1^{q+1}\}$ projections of expressions that become proportional after this operation remain proportional with the same overall factor even after the full projection is enforced. For instance, denoting the column antisymmetrizer with $A_{\{2^p, 1^q\}}$ as in eq. (A.35), one can easily recognize that

$$
A_{\{2^p, 1^q\}} \, \partial_l \, T_{i_1 j_1} \dots T_{i_p j_p} \, \gamma_{k_1 \dots k_q} \, \mathcal{S} \; = \; p! \, (q-1)! \, \partial_{[l} \, T_{i_1 | j_1} \dots T_{|i_p | j_p} \, \gamma_{|k_1 \dots k_q]} \, \mathcal{S} \, ,
$$

$$
A_{\{2^p, 1^q\}} \, \partial_{[k_1} \gamma_{k_2 \dots k_q] l} \prod_{r=1}^{p} T_{i_r j_r} \, \mathcal{S} \; = \; (-1)^q \, p! \, q! \, \partial_{[l} \, T_{i_1 | j_1} \dots T_{|i_p | j_p} \, \gamma_{|k_1 \dots k_q]} \, \mathcal{S} \, , \tag{C.22}
$$

and this implies

$$
Y_{\{2^p, 1^q\}} \, \partial_{[k_1} \gamma_{k_2 \dots k_q] l} \prod_{r=1}^{p} T_{i_r j_r} \, \mathcal{S} \; = \; (-1)^q \, q \, Y_{\{2^p, 1^q\}} \, \partial_l \, T_{i_1 j_1} \dots T_{i_p j_p} \, \gamma_{k_1 \dots k_q} \, \mathcal{S} \, . \tag{C.23}
$$

Using these techniques one can finally obtain

$$
Y_{\{2^p, 1^{q+1}\}} \, T_{i_1 j_1} \dots T_{i_p j_p} \, \gamma_{k_1 \dots k_q} \, \mathscr{B}_l \; :
$$

$$
\left[(q+1) + \frac{q(q-1)}{6} + \frac{p}{2} + \frac{pq}{6} \right] Y_{\{2^p, 1^{q+1}\}} \, \partial_l \, T_{i_1 j_1} \dots T_{i_p j_p} \, \gamma_{k_1 \dots k_q} \, \mathcal{S}
$$

$$
+ \, \frac{1}{6} \, Y_{\{2^p, 1^{q+1}\}} \sum_{n=1}^{p} \partial_{(i_n |} \prod_{r \neq n}^{p} T_{i_r j_r} \, \gamma_{|j_n) l k_1 \dots k_q} \, \mathcal{S}
$$

$$
- \, \frac{q+3}{6} \, Y_{\{2^p, 1^{q+1}\}} \left\{ \partial\!\!\!/ \, T_{i_1 j_1} \dots T_{i_p j_p} \, \gamma_{l k_1 \dots k_q} \, \mathcal{S} + \partial^m \, T_{i_1 j_1} \dots T_{i_p j_p} \, T_{lm} \, \gamma_{k_1 \dots k_q} \, \mathcal{S} \right\}
$$

$$
- \, \frac{1}{6} \, Y_{\{2^p, 1^{q+1}\}} \, \partial^m \, T_{i_1 j_1} \dots T_{i_p j_p} \, \gamma_{k_1 \dots k_q \, lm} \, \mathcal{S} \; = \; 0 \, . \tag{C.24}
$$

As in eq. (3.25), in the two divergence terms only the two-column projected combination of traces and antisymmetrized γ-traces can contribute to the full two-column projection involving also the divergence. In a similar fashion, a two-column projection in all indices is enforced in the first gradient term. In fact, the manifest symmetries for its lower family indices are

$$\{2^p, 1^{q+1}\} \otimes \{1\} = \{2^p, 1^{q+2}\} \oplus \{2^{p+1}, 1^q\} \oplus \{3, 2^{p-1}, 1^{q+1}\}, \qquad \text{(C.25)}$$

but on the other hand this term originates from the product of $(p+1)$ traces and a q-fold γ-trace. As a result, the irreducible components with less than four columns that it admits come from the decomposition

$$\{2^{p+1}\} \otimes \{1^q\} = \{2^{p+1}, 1^q\} \oplus \{3, 2^p, 1^{q-1}\} \oplus \{3^2, 2^{p-1}, 1^{q-2}\} \oplus \dots, \qquad \text{(C.26)}$$

so that the corresponding term is actually $\{2^{p+1}, 1^q\}$ projected. This is no longer true for the last term of eq. (C.24), where the components displayed in eq. (C.25) and those carried by the product of p traces and a $(q+2)$-fold γ-trace,

$$\{2^p\} \otimes \{1^{q+2}\} = \{2^p, 1^{q+2}\} \oplus \{3, 2^{p-1}, 1^{q+1}\} \oplus \{3^2, 2^{p-2}, 1^q\} \oplus \dots, \qquad \text{(C.27)}$$

share two admissible components. However, in the constrained setting one can elude this problem and simply select the $\{2^p, 1^{q+2}\}$ component, because it is the only non-vanishing one. Therefore eq. (C.24) directly leads to eq. (2.128), that suffices to annihilate the gauge variation of the constrained Lagrangians.

On the contrary, in the unconstrained setting the situation is more subtle, since one is left with

$$\frac{(q+3)(p+q+2)}{6} Y_{\{2^p, 1^{q+1}\}} \partial_l (\gamma^{[q]} \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p; k_1 \dots k_q}$$

$$+ \frac{1}{6} Y_{\{2^p, 1^{q+1}\}} \sum_{n=1}^{p} \partial_{(i_n |} (\gamma^{[q+2]} \mathcal{S}^{[p-1]})_{\dots, i_r \neq n j_r \neq n, \dots; | j_n) l k_1 \dots k_q}$$

$$- \frac{q+3}{6} \left\{ \slashed{\partial} (\gamma^{[q+1]} \mathcal{S}^{[p]})_{i_1 j_1, \dots, i_p j_p; l k_1 \dots k_q} - \partial^m (\gamma^{[q]} \mathcal{S}^{[p+1]})_{i_1 j_1, \dots, i_p j_p, lm; k_1 \dots k_q} \right\}$$

$$- \frac{1}{6} Y_{\{2^p, 1^{q+1}\}} \partial^m T_{i_1 j_1} \dots T_{i_p j_p} \gamma_{k_1 \dots k_q lm} \mathcal{S}$$

$$= \frac{i}{6} Y_{\{2^p, 1^{q+1}\}} T_{i_1 j_1} \dots T_{i_p j_p} \gamma_{k_1 \dots k_q} \partial^m \partial^n T_{(lm} \gamma_{n)} \psi. \qquad \text{(C.28)}$$

In order to reproduce the expression that appears in the gauge variation (3.66), one should extract the $\{3, 2^{p-1}, 1^{q+1}\}$ component from the last gradient term. Then one should express it as a combination of γ-traces of $T_{(ij} \gamma_{k)} \psi$ in order to fix the structure of the gauge transformations of the Lagrange multipliers λ_{ijk}. Rather than following this path, it is more convenient to exploit the possibility of producing different two-column projected consequences of the Bianchi identities. In fact, one can consider

$$Y_{\{2^p, 1^{q+1}\}} \sum_{n=1}^{p} \prod_{r \neq n}^{p} T_{i_r j_r} \gamma_{k_1 \dots k_q l (i_n} \mathscr{B}_{j_n)}, \qquad \text{(C.29)}$$

that displays the same manifest symmetries as $Y_{\{2^p,1^{q+1}\}} T_{i_1 j_1} \ldots T_{i_p j_p} \gamma_{k_1 \ldots k_q} \mathscr{B}_l$. Using the techniques described at the beginning of this appendix it can be cast in the form

$$
p\,(\,q+1\,)\Bigg\{ \frac{(\,q+3\,)\,(\,p+q+2\,)}{6}\, Y_{\{\,2^p,\,1^{q+1}\}}\, \partial_l\,(\,\gamma^{[q]}\,\mathcal{S}^{[p]}\,)_{i_1 j_1,\,\ldots,\,i_p j_p\,;\,k_1\ldots k_q}
$$

$$
+\,\frac{1}{6}\, Y_{\{\,2^p,\,1^{q+1}\}}\, \sum_{n=1}^{p}\, \partial_{(\,i_n\,|}\,(\,\gamma^{[q+2]}\,\mathcal{S}^{[p-1]}\,)_{\ldots,\,i_r\neq n j_r \neq n,\,\ldots;\,|\,j_n\,)\,l\,k_1\ldots k_q}
$$

$$
-\,\frac{q+3}{6}\,\Big[\,\partial\!\!\!/\,(\,\gamma^{[q+1]}\,\mathcal{S}^{[p]}\,)_{i_1 j_1,\,\ldots,\,i_p j_p\,;\,l k_1\ldots k_q} +\, \partial^m(\,\gamma^{[q]}\,\mathcal{S}^{[p+1]}\,)_{i_1 j_1,\,\ldots,\,i_p j_p,\,l m\,;\,k_1\ldots k_q}\,\Big]\Bigg\}
$$

$$
-\,\frac{p\,(\,q+3\,)}{6}\, Y_{\{2^p,1^{q+1}\}}\, \partial^m\, T_{i_1 j_1}\ldots T_{i_p j_p}\, \gamma_{k_1\ldots k_q\, l m}\,\mathcal{S}
$$

$$
+\,\frac{(-1)^q}{3}\, Y_{\{2^p,1^{q+1}\}}\, \partial^m \sum_{n=1}^{p}\prod_{r\neq n}^{p} T_{i_r j_r}\, T_{m\,(\,i_n}\,\gamma_{j_n\,)\,k_1\ldots k_q\, l}\,\mathcal{S}
$$

$$
=\,\frac{i}{6}\, Y_{\{2^p,1^{q+1}\}}\, \sum_{n=1}^{p}\prod_{r\neq n}^{p} T_{i_r j_r}\, \gamma_{k_1\ldots k_q\, l\,(\,i_n\,|}\, \partial^m\partial^n\,\big(\,T_{|\,j_n\,)\,(\,m}\,\gamma_{n\,)} +\, T_{m n}\,\gamma_{|\,j_n\,)}\,\big)\,. \quad \text{(C.30)}
$$

The terms that already came in the form $(\,\gamma^{[m]}\,\mathcal{S}^{[n]}\,)$ in eq. (C.28) enter exactly the same combination also here, but one can rearrange these two expressions in order to project the gradient terms that do not appear in the desired form. Indeed, in the combination

$$
Y_{\{2^p,1^{q+1}\}}\Bigg\{ (\,p+2\,)\,(\,q+3\,)\, T_{i_1 j_1}\ldots T_{i_p j_p}\,\gamma_{k_1\ldots k_q}\,\mathscr{B}_l -\, \sum_{n=1}^{p}\prod_{r\neq n}^{p} T_{i_r j_r}\,\gamma_{k_1\ldots k_q\, l\,(\,i_n}\,\mathscr{B}_{j_n\,)}\Bigg\}
$$

$$
\text{(C.31)}
$$

the terms with p traces and a $(q+2)$-fold γ-trace,

$$
-\,\frac{1}{3}\, Y_{\{2^p,1^{q+1}\}}\, \partial^m\Bigg\{ (\,q+3\,)\, T_{i_1 j_1}\ldots T_{i_p j_p}\,\gamma_{k_1\ldots k_q\, l m}\,\mathcal{S}
$$

$$
+\,(-1)^{q+1} \sum_{n=1}^{p}\prod_{r\neq n}^{p} T_{i_r j_r}\, T_{m\,(\,i_n}\,\gamma_{j_n\,)\,k_1\ldots k_q\, l}\,\mathcal{S}\Bigg\}\,,
$$

$$
\text{(C.32)}
$$

are actually $\{2^p,1^{q+2}\}$ projected. In order to verify this statement, notice that one can impose the $\{2^p,1^{q+1}\}$ projection again via the single Young tableau (C.21). Then, using the notation of eq. (A.35) and applying the column antisymmetrizer associated to (C.21) to the expression within parentheses gives

$$
A_{\{2^p,1^{q+1}\}}\, \partial^m\Bigg\{ (\,q+3\,)\, T_{i_1 j_1}\ldots T_{i_p j_p}\,\gamma_{k_1\ldots k_q\, l m}\,\mathcal{S} +\,\ldots\Bigg\}
$$

$$
=\, p!\,(\,q+1\,)!\,\partial^m\Bigg\{ (\,q+3\,)\, T_{[\,i_1\,|\,j_1}\ldots T_{|\,i_p\,|\,j_p}\,\gamma_{|\,k_1\ldots k_q\, l\,]\,m}\,\mathcal{S}
$$

$$
+\,(-1)^{q+1}\,(\,q+2\,) \sum_{n=1}^{p} T_{[\,i_1\,|\,j_n}\ldots T_{|\,i_{p-1}\,|\,j_{n-2}}\, T_{m\, j_{n-1}}\,\gamma_{|\,i_p\, k_1\ldots k_q\, l\,]}\,\mathcal{S}
$$

$$
+\,(-1)^{q+1} \sum_{n=1}^{p} T_{[\,i_1\,|\,j_n}\ldots T_{|\,i_{p-1}\,|\,j_{n-2}}\, T_{m\,|\,i_p\,|}\,\gamma_{j_{n-1}\,|\,k_1\ldots k_q\, l\,]}\,\mathcal{S}\Bigg\}\,. \quad \text{(C.33)}
$$

When acting with the row symmetrizer S, the last two terms become proportional, since they only differ because of an antisymmetrization over all the i_n indices and a single j_n index, that is annihilated by S. In conclusion, the right-hand side of eq. (C.33) is equivalent to

$$(q + 3) \, p \, ! \, (q + 1) \, ! \, \partial^m T_{[i_1 | j_1} \dots T_{| i_p | j_p} \, \gamma_{| k_1 \dots k_q \, l m]} \, \mathcal{S} \, , \tag{C.34}$$

and the remaining symmetrizations induced by S build an expression proportional to that associated to a Young tableau with the index m attached at the end of the longer column of (C.21). Due to the symmetry properties of the underlying combination, this suffices to build the Young projector $Y_{\{2^p, 1^{q+2}\}}$. In order to fix the proportionality factor and to reach the result displayed in eq. (3.68), one can then compare eq. (C.34) with an expression obtained acting with the column antisymmetrizer of the $\{2^p, 1^{q+2}\}$ projector,

$$A_{\{2^p, 1^{q+2}\}} \, T_{i_1 j_1} \dots T_{i_p j_p} \, \gamma_{k_1 \dots k_q \, l m} \, \mathcal{S} = p \, ! \, (q + 2) \, ! \, T_{[i_1 | j_1} \dots T_{| i_p | j_p} \, \gamma_{| k_1 \dots k_q \, l m]} \, \mathcal{S} \, . \tag{C.35}$$

Taking into account the hook lengths of the corresponding expressions,

$$h \, (\, \{2^p, 1^{q+1}\} \,) = \frac{p \, ! \, (p + q + 2) \, !}{q + 2} \, , \qquad h \, (\, \{2^p, 1^{q+2}\} \,) = \frac{p \, ! \, (p + q + 3) \, !}{q + 3} \, , \tag{C.36}$$

finally gives

$$Y_{\{2^p, 1^{q+1}\}} \, \partial^m \bigg\{ (q + 3) \, T_{i_1 j_1} \dots T_{i_p j_p} \, \gamma_{k_1 \dots k_q \, l m} \, \mathcal{S} - \sum_{n=1}^{p} \prod_{r \neq n}^{p} T_{i_r j_r} \, T_{m \, (i_n} \, \gamma_{j_n) \, l \, k_1 \dots k_q} \, \mathcal{S} \bigg\}$$

$$= (p + q + 3) \, \partial^m Y_{\{2^p, 1^{q+2}\}} \, T_{i_1 j_1} \dots T_{i_p j_p} \, \gamma_{k_1 \dots k_q \, l m} \, \mathcal{S} \, , \tag{C.37}$$

that when added to the other contributions contained in eqs. (C.28) and (C.30) proves the result in eq. (3.68).

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