ENTIRE RADIAL AND NONRADIAL SOLUTIONS FOR SYSTEMS WITH CRITICAL GROWTH

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Abstract. In this paper we establish existence of radial and nonradial solutions to the system

$$\begin{cases} -\Delta u_1 = F_1(u_1,u_2) & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = F_2(u_1,u_2) & \text{in } \mathbb{R}^N, \\ u_1 \geqslant 0,\ u_2 \geqslant 0 & \text{in } \mathbb{R}^N, \\ u_1, u_2 \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where F_1, F_2 are nonlinearities with critical behavior.

1. Introduction

The aim of this paper is to prove existence of radial and nonradial solutions to some nonlinear systems

$$\begin{cases} -\Delta u_1 = F_1(u_1,u_2) & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = F_2(u_1,u_2) & \text{in } \mathbb{R}^N, \\ u_1 \geqslant 0,\ u_2 \geqslant 0 & \text{in } \mathbb{R}^N, \\ u_1, u_2 \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where F_1, F_2 are nonlinearities with critical behavior in the Sobolev sense, $N \geqslant 3$ and $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) \text{ such that } |\nabla u| \in L^2(\mathbb{R}^N)\}$ with $2^* = \frac{2N}{N-2}$. A common feature of the systems that we will study is their invariance by *translations* and *dilations*. Papers on existence or qualitative properties of solutions to systems with critical growth in $\mathbb{R}^N$ are very few, due to the lack of compactness given by the Talenti bubbles and the difficulties arising for the lack of good variational methods. The first example of system which we consider is given by

$$\begin{cases} -\Delta u_1 = \alpha u_1^{2^*-1} + (1-\alpha) u_1^{\frac{2}{N-2}} u_2^{\frac{N}{N-2}} & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = \alpha u_2^{2^*-1} + (1-\alpha) u_2^{\frac{2}{N-2}} u_1^{\frac{N}{N-2}} & \text{in } \mathbb{R}^N, \\ u_1 \geqslant 0,\ u_2 \geqslant 0, \quad u_1, u_2 \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

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where $N \geqslant 3$ and α is a real parameter. This system, also known as Gross-Pitaevskii, arises in many physical contexts such as nonlinear optics and the Hartree-Fock theory, see [M] for its derivation, and it is very studied mainly in the cubic case, which corresponds to the critical case in $\mathbb{R}^4$ or on bounded domains where the cubic exponent is subcritical in $\mathbb{R}^3$. It is coupled when $1-\alpha \neq 0$ and cooperative when $1-\alpha > 0$. Physically, this condition means the attractive interaction of the states u_1 and u_2, while $1-\alpha < 0$ means the repulsive interaction between them. Note that System (1.1) has a gradient structure with the energy functional

$$\mathcal{E}(u_1,u_2) = \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2 - \frac{N-2}{2N}\int\limits_{\mathbb{R}^N} \alpha\big(u_1^{2^*}+u_2^{2^*}\big) + (1-\alpha)\Big(u_1^{\frac{N}{N-2}}u_2^{\frac{N}{N-2}}\Big)$$

even if it is not so easy to apply variational methods to find solutions. System (1.1) was already considered in [GLW] where the existence of infinitely many nontrivial solutions is obtained using a perturbation argument.

Another particular case of (1.1) is the following generalization of the system considered by O. Druet, E. Hebey [DH], namely

$$\begin{cases} -\Delta u_1 = \Big[\big(\alpha u_1^2 + (1-\alpha)u_2^2\big)^2\Big]^{\frac{1}{N-2}} u_1 & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = \Big[\big((1-\alpha)u_1^2 + \alpha u_2^2\big)^2\Big]^{\frac{1}{N-2}} u_2 & \text{in } \mathbb{R}^N, \\ u_1 \geqslant 0,\ u_2 \geqslant 0, \quad u_1,u_2 \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.2}$$

In [DH] the case of $\alpha = \frac{1}{2}$ was studied and the stability of solutions on manifolds was considered. Further, the radial symmetry and uniqueness of the solutions in $\mathbb{R}^N$ is proved.

We also mention the paper [CSW] where the radial symmetry of solutions is proved for a particular critical nonlinearity.

The starting point of our study is the paper [GGT] where we studied the existence of radial solutions for the $k \times k$ system of equations

$$\begin{cases} -\Delta u_i = \displaystyle\sum_{j=1}^k a_{ij} u_j^{2^*-1} & \text{in } \mathbb{R}^N, \\ u_i > 0 & \text{in } \mathbb{R}^N, \\ u_i \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.3}$$

for $i = 1,\dots,k$, where $N \geqslant 3$ and the matrix $A := (a_{ij})_{i,j=1,\dots,k}$ is symmetric and satisfies

$$\sum_{j=1}^k a_{ij} = 1 \text{ for any } i = 1,\dots,k. \tag{1.4}$$

Note that the case $k = 2$ and $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is known in the literature as nonlinearity belonging to the *critical hyperbola*.

Under the assumption (1.4) it is straightforward that system (1.3) always admits the trivial solutions

$$u_1 = \cdots = u_k = U_{\delta,y}(x) := \frac{\left[N(N-2)\delta^2\right]^{\frac{N-2}{4}}}{(\delta^2 + |x-y|^2)^{\frac{N-2}{2}}}$$

for any $\delta > 0$ and $y \in \mathbb{R}^N$. To simplify the notation, let

$$U(x) := U_{1,0}(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}. \tag{1.5}$$

A careful study of the linearized system of (1.3) at this trivial solution allows us to prove the existence of nontrivial radial solutions when the eigenvalues of the matrix A reach some specific values using bifurcation theory.

Note that System (1.3) does not have a variational structure and indeed our methods do not require it.

Even if the existence of radial solutions to some of the previous examples (1.1)–(1.3) is a new result, the main interest is the existence of nonradial ones. Nonradial solutions may be found mainly for noncooperative systems where the lack of the maximum principle can give a symmetry breaking of the solutions. Indeed, in [DH] and [CSW], the radial symmetry of the solutions is proved in a particular cooperative case.

In this paper we want to purse several goals. First, we want to introduce a new setting which allows us to consider Systems (1.1)–(1.3) jointly. Indeed all these problems admit the trivial solutions $u_1 = u_2 = U_{\delta,y}(x)$ which is the starting point to apply the bifurcation theory like in [GGT]. A general treatment of these problems is possible since we significantly improve the final part of the paper [GGT] showing that the *Lagrange multiplier* introduced to "kill" the direction of dilation invariance coming from the critical Sobolev exponent is indeed a *natural constraint* if we allow some invariance (Kelvin invariance) on the solutions. This lets us switch from a local bifurcation result in [GGT] to a global one.

This invariance is a good tool to overcome the degeneracy of critical problems in $\mathbb{R}^N$ which are invariant under dilation and can also be applied to the result in [DGG], where a Pohozaev identity gives the result only locally.

Another technical problem arises since our nonlinearities in general are not $\mathcal{C}^1$ at zero. This problem was already noticed by [GLW] and indeed their existence results are given in dimension 3 where they are able to define and to invert the linearized operator associated to their system. To overcome this problem we use a different functional setting that allows us to work only with positive values of u_1 and u_2. Observe that the functional setting of our operator is a delicate part of the proof.

Secondly we continue the study in [GGT] and we address to the existence of nonradial solutions to (1.3) using in a tricky way some *even* and *odd* symmetries. Obviously our solutions cannot be invariant with respect to odd symmetries since we are looking for positive ones. But we can introduce a suitable setting (see Eq. 2.9) in which we can make use of this invariance. This is a new aspect that has never been investigated before and that can shed light on how solutions of systems of this type are.

This use of the symmetries is the key point that allows us to distinguish between radial and nonradial solutions.

A crucial step of our method is the characterization of the kernel of the linearized operator associated to our systems. Actually, in [GGT], we find radial solutions using the classical *Crandall-Rabinowitz Theorem* which requires a one dimensional kernel. This is achieved by restricting the problem to radially symmetric functions and "killing" the direction of scale invariance.

Considering also nonradial functions the dimension of the kernel increases dramatically and it becomes very hard to control it. Moreover it is not clear whether the solution obtained considering this new kernel is *nonradial*. As said before, the use of suitable even and odd symmetries is significant and allows us to prove that in many cases the kernel contains only nonradial functions and it is *odd* dimensional. To exploit them, we need some invariance on the operator associated to our problem. This invariance naturally appears in the case of a 2×2 system while it not clear whether it applies in the general case of more equations as (1.3). For this reason we focus hereafter on the case 2×2 and we believe that a further study is needed to understand the general case. To compute the dimension of the kernel in these symmetric spaces we need a classification of symmetric spherical harmonics in $\mathbb{S}^N$ and indeed this is part of Section 4 and 5.

Finally we also give an asymptotic expansion of the solutions near the bifurcation point so as to better understand them. In this way we can distinguish different nonradial solutions by their symmetries and expansions.

2. Statement of the main results

Let us introduce our abstract setting. We consider

$$\begin{cases} -\Delta u_1 = F_1(\alpha, u_1, u_2) & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = F_2(\alpha, u_1, u_2) & \text{in } \mathbb{R}^N, \\ u_1 \geqslant 0, \ u_2 \geqslant 0, \quad u_1, u_2 \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{2.1}$$

where the F_i satisfy the following assumptions: for all $\alpha \in \mathbb{R}$ and for $i = 1, 2$,

(F1) the derivatives $\partial_\alpha F_i$, $\partial_u F_i$ and $\partial_{\alpha u} F_i$ of the map $F_i : \mathbb{R} \times (0, +\infty)^2 \to \mathbb{R} : (\alpha, u) \mapsto F_i(\alpha, u)$ exist and are continuous;

(F2) for all $\alpha \in \mathbb{R}$, there exists a neighborhood $\mathcal{A}$ of α and a constant C such that, for all $\alpha \in \mathcal{A}$ and $(u_1, u_2) \in (0, +\infty)^2$, $|\partial_u F_i(\alpha, u_1, u_2)| \leqslant C(u_1^{2^*-2} + u_2^{2^*-2})$ and $|\partial_{\alpha u} F_i(\alpha, u_1, u_2)| \leqslant C(u_1^{2^*-2} + u_2^{2^*-2})$;

(F3) $F_i(\alpha, 1, 1) = 1$;

(F4) $F_i(\alpha, \lambda u_1, \lambda u_2) = \lambda^{2^*-1} F_i(\alpha, u_1, u_2)$ for all $\lambda > 0$ and $(u_1, u_2) \in (0, +\infty)^2$;

(F5) $F_1(\alpha, u_1, u_2) = F_2(\alpha, u_2, u_1)$ for all $(u_1, u_2) \in (0, +\infty)^2$;

(F6) for all α, $\partial_\alpha \beta(\alpha) > 0$ where $\beta(\alpha) := \partial_{u_1} F_1(\alpha, 1, 1) - \partial_{u_2} F_1(\alpha, 1, 1)$.

By (F3) it is straightforward that System (2.1) admits, for any $\alpha \in \mathbb{R}$, the trivial solution $(u_1, u_2) = (U, U)$ and (F4) says that our system is scale invariant. Further, in view of Eq. (2.1), it is also translation invariant.

This generalization encompasses the following *Schrodinger* system

$$\begin{cases} -\Delta u_1 = \alpha u_1^{2^*-1} + (1-\alpha) u_1^p u_2^{2^*-1-p} & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = (1-\alpha) u_1^{2^*-1-p} u_2^p + \alpha u_2^{2^*-1} & \text{in } \mathbb{R}^N, \\ u_1 \geqslant 0,\ u_2 \geqslant 0, \quad u_1, u_2 \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{2.2}$$

with $0 \leqslant p < 2^* - 1$ and α is a real parameter. When $p = 0$ System (2.2) becomes

$$\begin{cases} -\Delta u_1 = \alpha u_1^{2^*-1} + (1-\alpha) u_2^{2^*-1} & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = (1-\alpha) u_1^{2^*-1} + \alpha u_2^{2^*-1} & \text{in } \mathbb{R}^N, \\ u_1 \geqslant 0,\ u_2 \geqslant 0, \quad u_1, u_2 \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{2.3}$$

while for $p = \frac{2}{N-2}$ we get System (1.1). Moreover System (2.1) includes System (1.2).

Our first result is the generalization of the local radial bifurcation result obtained in [GGT] for (2.3) to a global one for System (2.1). An important role in our results will be played by the Jacobi polynomials $P_j^{(\beta,\gamma)}$ that we introduce now. They are defined as

$$P_m^{(\beta,\gamma)}(\xi) = \sum_{s=0}^{m} \binom{m+\beta}{s} \binom{m+\gamma}{m-s} \left(\frac{\xi-1}{2}\right)^{m-s} \left(\frac{\xi+1}{2}\right)^{s} \tag{2.4}$$

for $m \in \mathbb{N}$, $\beta, \gamma \in \mathbb{R}^+$ and $\xi \in \mathbb{R}$.

Theorem 2.1. *Assume* (F1)–(F6). *The point* (α^*, U, U) *is a* radial *bifurcation point from the curve of trivial solutions* (α, U, U) *to System* (2.1) *if* α^* *satisfies*

$$\beta(\alpha^*) = \frac{(2n+N)(2n+N-2)}{N(N-2)} \tag{2.5}$$

for some $n \in \mathbb{N}$, *where* β *is defined in* (F6). *More precisely there exists a continuously differentiable curve defined for* ε *small enough*

$$(-\varepsilon_0, \varepsilon_0) \to \mathbb{R} \times \left(D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)\right)^2 : \varepsilon \mapsto \big(\alpha(\varepsilon), u_1(\varepsilon), u_2(\varepsilon)\big)$$

passing through (α^*, U, U), *i.e.,* $\big(\alpha(0), u_1(0), u_2(0)\big) = (\alpha^*, U, U)$, *such that, for all* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $(u_1(\varepsilon), u_2(\varepsilon))$ *is a radial solution to* (2.1) *with* $\alpha = \alpha(\varepsilon)$. *Moreover,*

$$\begin{cases} u_1(\varepsilon) = U + \varepsilon W_n(|x|) + \varepsilon \phi_{1,\varepsilon}(|x|), \\ u_2(\varepsilon) = U - \varepsilon W_n(|x|) + \varepsilon \phi_{2,\varepsilon}(|x|), \end{cases} \tag{2.6}$$

with W_n *being the function*

$$W_n(|x|) := \frac{1}{(1+|x|^2)^{\frac{N-2}{2}}} P_n^{\left(\frac{N-2}{2}, \frac{N-2}{2}\right)} \left(\frac{1-|x|^2}{1+|x|^2}\right) \tag{2.7}$$

where $\phi_{1,\varepsilon}, \phi_{2,\varepsilon}$ *are functions uniformly bounded in* $D^{1,2}(\mathbb{R}^N)$ *with respect to* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, *and such that* $\phi_{i,0} = 0$ *for* $i = 1, 2$. *Finally the bifurcation is global and the Rabinowitz alternative holds.*

The values α^* in (2.5) are all of those for which the linearized system at the trivial solution (U, U) is non-invertible showing that condition (2.5) is also necessary.

Corollary 2.2. *For any $n \in \mathbb{N}$, let*

$$\alpha_n^* = \begin{cases} \frac{(2n+N-2)(2n+N)}{2N(N+2-p(N-2))} + \frac{N+2}{2(N+2-p(N-2))} - \frac{p(N-2)}{N+2-p(N-2)} & \text{in (2.2)}, \\ \frac{(2n+N-2)(2n+N)}{2N^2} + \frac{N-2}{2N} & \text{in (1.1)}, \\ \frac{(2n+N-2)(2n+N)+N(6-N)}{8N} & \text{in (1.2)}. \end{cases} \tag{2.8}$$

Then (α^*, U, U) is a radial bifurcation point of Systems (2.2), (1.1) and (1.2) from its curve of trivial solutions (α, U, U) if $\alpha^* = \alpha_n^*$ for some $n \in \mathbb{N}$. Moreover, the expansion around the bifurcation point given by Theorem 2.1 holds and the curve is global.

Remark 2.3. An interesting fact is that in (2.2) the exponent p does not enter in a relevant way in the proof of the previous results and indeed the solutions we find have, near a bifurcation point, the same expansion for every value of p. In this way we have a path of solutions connecting (2.3) with (1.1) showing that these solutions are not due the variational structure of (2.3).

The next step is to find nonradial solutions. In [GL] was proved that in the cooperative case (i.e., when $1-\alpha > 0$), System (2.2) admits only radial solutions. Note that, for all $n \geqslant 1$, $1-\alpha_n^* < 1-\alpha_1^* = 0$ where α_n^* is defined by (2.8). Then α_n^* are good "candidates" to find nonradial solutions. Moreover, at each value α_n^* the linearized system possesses many nonradial solutions and the kernel becomes richer and richer as $n \to \infty$ (see Proposition 3.1). However, one technical problem in looking for nonradial solutions is that the kernel of the linearized problem at a degeneracy point always contains the radial function W_n defined by (2.7). So our aim becomes to choose a suitable subspace of the kernel in which W_n does not lies. This will be done by using in a tricky way some odd-symmetries. It is possible indeed to apply such symmetries to a linear combination of the components u_1, u_2 even if the solutions we are interested in are positive.

Here is our basic idea: if one writes

$$\begin{cases} u_1 = U + \frac{z_1+z_2}{2} \\ u_2 = U + \frac{z_1-z_2}{2} \end{cases} \tag{2.9}$$

then the system satisfied by z_1, z_2 admits solutions obtained by imposing the following symmetries on (z_1, z_2):

$$\forall (x', x_N) \in \mathbb{R}^N, \qquad z_1(x', x_N) = z_1(|x'|, -x_N) \text{ and } z_2(x', x_N) = -z_2(|x'|, -x_N), \tag{2.10}$$

(more general symmetries will be imposed later; see Section 4.2 for more details). The *crucial* remark is that the new system in (z_1, z_2) obtained by (2.9) is *invariant* for the symmetries in (2.10) (see (3.1)–(3.5)). This use of odd symmetries is unclear if we considered directly System (2.1).

In order to state our first nonradial bifurcation result, we use in $\mathbb{R}^N$ the spherical coordinates $(r, \varphi, \theta_1, \dots, \theta_{N-2}) \in [0, +\infty) \times [0, 2\pi) \times [0, \pi)^{N-2}$. We have

Theorem 2.4. *Assume (F1)–(F6) and let α_n^* be the unique solution to (2.5) for some $n \in \mathbb{N}$. The point (α_n^*, U, U) is a* nonradial *bifurcation point for the curve of trivial solutions (α, U, U) to System (2.1) when $n \bmod 4 \in \{1, 2\}$. More precisely, there exist a*

continuum $\mathcal{C}$ *of nonradial solutions* (u_1, u_2) *to System* (2.1)*, bifurcating from* (α_n^*, U, U)*; the bifurcation is global and the Rabinowitz alternative holds. Finally for any sequence of solutions* $(\alpha_k, u_{1,k}, u_{2,k}) \to (\alpha_n^*, U, U)$*, we have that (up to a subsequence)*

$$\begin{cases} u_{1,k} = U + \varepsilon_k Z_n(x) + o(\varepsilon_k), \\ u_{2,k} = U - \varepsilon_k Z_n(x) + o(\varepsilon_k), \end{cases} \tag{2.11}$$

as $k \to \infty$ *where* $\varepsilon_k = \|z_{2,k}\|_X \to 0$ *(see* (2.9) *and* (3.13)*) and* $Z_n \not\equiv 0$ *is the function*

$$Z_n(x) = \sum_{h=1,\ h \text{ odd}}^{n} a_h \frac{r^h}{(1+r^2)^{h+\frac{N-2}{2}}} P_{n-h}^{\left(h+\frac{N-2}{2}, h+\frac{N-2}{2}\right)}\left(\frac{1-r^2}{1+r^2}\right) P_h^{\left(\frac{N-3}{2}, \frac{N-3}{2}\right)}(\cos\theta_{N-2}) \tag{2.12}$$

for some coefficients $a_h \in \mathbb{R}$*.*

Observe that the functions $P_h^{\left(\frac{N-3}{2}, \frac{N-3}{2}\right)}(\cos\theta_{N-2})$ are the spherical harmonics that are $O(N-1)$-invariant.

Corollary 2.5. *Let* $n \in \mathbb{N}$ *and* α_n^* *as defined in Corollary 2.2. Then the same claims of Theorem 2.4 hold for Systems* (2.2) *and* (1.2)*.*

It is possible to prove a similar result using more symmetries. Here we ask the following ones: $\forall x = (x', x_{N-m+1}, \dots, x_N) \in \mathbb{R}^{N-m} \times \mathbb{R}^m$,

$$\begin{aligned} &z_1(x) = z_1(|x'|, \pm x_{N-m+1}, \dots, \pm x_N), \text{ and} \\ &z_2(x', x_{N-m+1}, x_N) = -z_2(|x'|, -x_{N-m+1}, \dots, x_N), \\ &\dots \\ &z_2(x', x_{N-m+1}, x_N) = -z_2(|x'|, x_{N-m+1}, \dots, -x_N)\}. \end{aligned}$$

Imposing these symmetries on the functions z_1, z_2 defined in (2.9), we get the following result:

Theorem 2.6. *Let* $1 \leqslant m \leqslant N$ *and let* α_n^* *be the unique solution to* (2.5) *for some* $n \geqslant m$*. Suppose that*

$$\binom{m + \left\lfloor \frac{n-m}{2} \right\rfloor}{m} \text{ is an odd integer.} \tag{2.13}$$

Then for any m *there exists a* continuum $\mathcal{C}_m$ *of nonradial solutions that satisfies System* (2.1)*, bifurcating from* (α_n^*, U, U) *and the bifurcation is global and the Rabinowitz alternative holds. Moreover the continua* $\mathcal{C}_m$ *are distinct and we have that, up to a subsequence,* (u_1, u_2) *has the same expansion as in* (2.11) *where*

$$Z_n(x) = \sum_{h=1}^{n} a_h \frac{r^h}{(1+r^2)^{h+\frac{N-2}{2}}} P_{n-h}^{\left(h+\frac{N-2}{2}, h+\frac{N-2}{2}\right)}\left(\frac{1-r^2}{1+r^2}\right) Y_h(\theta) \tag{2.14}$$

and the spherical harmonics $Y_h(\theta)$ *are* $O(N-m)$ *invariant and odd in the last* m *variables.*

Corollary 2.7. *Let* $n \in \mathbb{N}$ *and* α_n^* *as defined in Corollary 2.2. Then the same claims of Theorem 2.6 hold for System* (2.2) *and* (1.2)*.*

For the reader's convenience, we state the previous theorem when $m = 2$.

Corollary 2.8. *Let $m = 2$ in Theorem 2.6. Then if*

$$n \bmod 8 \in \{2, 3, 4, 5\} \tag{2.15}$$

the claim of Theorem 2.6 holds and Z_n in this case is given by

$$Z_n(x) = \sum_{h=1}^{n} a_h \frac{r^h}{(1+r^2)^{h+\frac{N-2}{2}}} P_{n-h}^{\left(h+\frac{N-2}{2}, h+\frac{N-2}{2}\right)}\left(\frac{1-r^2}{1+r^2}\right) Y_h(\theta) \tag{2.16}$$

for some coefficients $a_h \in \mathbb{R}$, where $Y_h(\theta)$ are spherical harmonics which are $O(N-2)$ invariant and are odd with respect to x_N and to x_{N-1}.

We conclude by giving one more existence result which produces a nonradial solutions for every value of n. These solutions are found imposing an odd symmetry with respect to an angle in spherical coordinates and also a periodicity assumption. They are different from the previous ones since they have a different expansion.

Theorem 2.9. *Assume (F1)–(F6) and α_n^* be the unique solution to (2.5) for some $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, $n \geqslant 2$, there exists a* continuum *$\mathcal{D}_n$ of nonradial solutions to System (2.1), bifurcating from (α_n^*, U, U). When ε is small enough this continuum is a continuously differentiable curve*

$$(-\varepsilon_0, \varepsilon_0) \to \mathbb{R} \times \left(D^{1,2}_{\mathrm{rad}}(\mathbb{R}^N)\right)^2 : \varepsilon \mapsto \big(\alpha(\varepsilon), u_1(\varepsilon), u_2(\varepsilon)\big)$$

passing through (α_n^*, U, U), i.e., $\big(\alpha(0), u_1(0), u_2(0)\big) = (\alpha_n^*, U, U)$, such that, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $(u_1(\varepsilon), u_2(\varepsilon))$ is a nonradial solution to (2.1) with $\alpha = \alpha(\varepsilon)$. Moreover,

$$\begin{cases} u_1(\varepsilon) = U + \varepsilon Z_n(x) + \varepsilon \phi_{1,\varepsilon}(x), \\ u_2(\varepsilon) = U - \varepsilon Z_n(x) + \varepsilon \phi_{2,\varepsilon}(x), \end{cases}$$

with

$$Z_n(r, \varphi, \Theta) = a \frac{r^n}{(1+r^2)^{n+\frac{N-2}{2}}} \sin(n\varphi)(\sin\theta_1)^n \cdots (\sin\theta_{N-2})^n, \qquad a \in \mathbb{R}, \tag{2.17}$$

(here we use the spherical coordinates $(r, \varphi, \Theta) = (r, \varphi, \theta_1, \dots, \theta_{N-2})$ in $\mathbb{R}^N$). Moreover the bifurcation is global and the Rabinowitz alternative holds.

Remark 2.10. Note that the function $Y_n(\varphi, \Theta) = \sin(n\varphi)(\sin\theta_1)^n \cdots (\sin\theta_{N-2})^n$ is the unique spherical harmonic of order n which is odd and periodic of period $\frac{2\pi}{n}$ with respect to the angle φ. Moreover, in Cartesian coordinates we have that $Y_n(x) = \Im\mathrm{m}(x_1 + ix_2)^n$.

Corollary 2.11. *Let $n \in \mathbb{N}$ and α_n^* as defined in Corollary 2.2. Then the same claims of Theorem 2.9 hold for System (2.2) and (1.2).*

Remark 2.12. It is difficult to give a formula with the exact number of solutions which takes in account all the previous theorems. Here we describe a particular case: choose $n = 4$ in (2.5) and $N \geqslant 4$ then we have the existence of at least *five* solutions bifurcating by (U, U) as follows:

i) one radial solution (Theorem 2.1),
ii) one nonradial solution with z_1 even in all the coordinates and z_2 odd with respect to x_{N-1} and x_N and even in other coordinates (Corollary 2.8),

iii) one nonradial solution with z_2 odd with respect to $x_{N-3}, \dots, x_N$ and even in other coordinates (Theorem 2.6 with $m = 3$),
iv) one nonradial solution in $\mathbb{R}^N$ with $N \geqslant 4$ with z_2 odd with respect to $x_{N-4}, \dots, x_N$ and even in other coordinates (Theorem 2.6 with $m = 4$),
v) one nonradial solution where z_1 and z_2 are periodic of period $\frac{2\pi}{4}$ with respect to the angle φ and z_2 is odd in φ (Theorem 2.9).

In the following table, which does not pretend to be exhaustive, we show the number of solutions bifurcating from (U, U) arising from Theorems 2.1–2.9.

	$N = 3$	$N = 4$	$N = 5$
$n = 2$	4	4	4
$n = 3$	4	4	4
$n = 4$	4	5	5
$n = 5$	4	5	6
$n = 6$	3	4	5
$n = 7$	2	3	3

Remark 2.13. Note that the our results for System (1.1) hold for any dimension $N \geqslant 3$, extending some recent results of [GLW]. Finally, as observed in [GLW], when the dimension $N \geqslant 4$, System (1.1) becomes linear or sublinear in some of its components and this fact produces problem in defining and estimating the linearization. In some sense, we can say that the bifurcation theory suits well this problem. We remark moreover that the solutions founded in [GLW] are always different from ours since their expansion is of the following type $u_1 = U + \varepsilon\phi_1$ and $u_2 = \sum_k U_{\delta_k, y_k} + \varepsilon\phi_2$.

The paper is organized as follows: in Section 3 we recall some preliminaries and introduce the functional setting to find the nonradial solution. In Section 4 we define the symmetric spaces and prove Theorems 2.1, 2.4 and 2.6. In Section 5 we prove Theorem 2.9.

3. Preliminary results and the functional setting

To study System (2.1), we perform the following change of variables

$$\begin{cases} z_1 = u_1 + u_2 - 2U, \\ z_2 = u_1 - u_2, \end{cases} \tag{3.1}$$

that turns (2.1) into the system

$$\begin{cases} -\Delta z_1 = f_1(|x|, z_1, z_2) & \text{in } \mathbb{R}^N, \\ -\Delta z_2 = f_2(|x|, z_1, z_2) & \text{in } \mathbb{R}^N, \\ z_1, z_2 \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{3.2}$$

where

$$f_1(|x|,z_1,z_2) := F_1\Big(\alpha, U+\frac{z_1+z_2}{2},\ U+\frac{z_1-z_2}{2}\Big) + F_2\Big(\alpha, U+\frac{z_1+z_2}{2},\ U+\frac{z_1-z_2}{2}\Big) - 2U^{2^*-1}, \tag{3.3}$$

$$f_2(|x|,z_1,z_2) := F_1\Big(\alpha, U+\frac{z_1+z_2}{2},\ U+\frac{z_1-z_2}{2}\Big) - F_2\Big(\alpha, U+\frac{z_1+z_2}{2},\ U+\frac{z_1-z_2}{2}\Big). \tag{3.4}$$

One important feature in looking for nonradial solutions is that, using (F5), this change of variables gives the following invariance:

$$\begin{aligned} f_1(|x|,z_1,-z_2) &= f_1(|x|,z_1,z_2),\\ f_2(|x|,z_1,-z_2) &= -f_2(|x|,z_1,z_2). \end{aligned} \tag{3.5}$$

Solutions to (2.1) are zeros of the operator

$$T(\alpha,z_1,z_2) := \begin{pmatrix} z_1-(-\Delta)^{-1}\big(f_1(|x|,z_1,z_2)\big)\\ z_2-(-\Delta)^{-1}\big(f_2(|x|,z_1,z_2)\big)\end{pmatrix}.$$

Clearly, $T(\alpha,0,0)=(0,0)$ for all $\alpha\in\mathbb{R}$ (thanks to (F3) and (F4)). A necessary condition for the bifurcation is that the linearized operator $\partial_z T(\alpha,0,0)$ is not invertible. This corresponds to study the system:

$$\begin{cases} -\Delta w_1 = \frac{\partial f_1}{\partial z_1}(|x|,0,0)\,w_1 + \frac{\partial f_1}{\partial z_2}(|x|,0,0)\,w_2 & \text{in } \mathbb{R}^N,\\ -\Delta w_2 = \frac{\partial f_2}{\partial z_1}(|x|,0,0)\,w_1 + \frac{\partial f_2}{\partial z_2}(|x|,0,0)\,w_2 & \text{in } \mathbb{R}^N,\\ w_1,w_2\in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{3.6}$$

A simple computation shows

$$\frac{\partial f_1}{\partial z_1}(\alpha,0,0) = \frac{1}{2}\left[\frac{\partial F_1}{\partial u_1}(\alpha,U,U)+\frac{\partial F_1}{\partial u_2}(\alpha,U,U)+\frac{\partial F_2}{\partial u_1}(\alpha,U,U)+\frac{\partial F_2}{\partial u_2}(\alpha,U,U)\right],$$

$$\frac{\partial f_1}{\partial z_2}(\alpha,0,0) = \frac{1}{2}\left[\frac{\partial F_1}{\partial u_1}(\alpha,U,U)-\frac{\partial F_1}{\partial u_2}(\alpha,U,U)+\frac{\partial F_2}{\partial u_1}(\alpha,U,U)-\frac{\partial F_2}{\partial u_2}(\alpha,U,U)\right],$$

and a very similar expression holds for $\frac{\partial f_2}{\partial z_i}(\alpha,0,0)$ for $i=1,2$. First observe that from (F5) we get

$$\frac{\partial F_1}{\partial u_1}(\alpha,U,U) = \frac{\partial F_2}{\partial u_2}(\alpha,U,U) \quad \text{and} \quad \frac{\partial F_1}{\partial u_2}(\alpha,U,U) = \frac{\partial F_2}{\partial u_1}(\alpha,U,U).$$

Then, differentiating (F4) with respect to λ we get

$$(\partial_{u_1}F_1+\partial_{u_2}F_1)(\alpha,U,U) = (2^*-1)U^{2^*-2}F_1(\alpha,1,1) = \frac{N+2}{N-2}U^{\frac{4}{N-2}}.$$

Moreover, using again (F4):

$$\partial_{u_j}F_i(\alpha,\lambda u_1,\lambda u_2) = \lambda^{2^*-2}\partial_{u_j}F_i(\alpha,u_1,u_2) \quad \text{for } i=1,2 \text{ and } j=1,2,$$

and in particular

$$\partial_{u_j} F_i(\alpha, U, U) = U^{2^*-2}\, \partial_{u_j} F_i(\alpha, 1, 1).$$

Putting together all these remarks, it is straightforward that system (3.6) becomes

$$\begin{cases} -\Delta w_1 = \frac{N+2}{N-2}\, U^{\frac{4}{N-2}} w_1 & \text{in } \mathbb{R}^N, \\ -\Delta w_2 = \beta(\alpha)\, U^{\frac{4}{N-2}} w_2 & \text{in } \mathbb{R}^N, \\ w_1, w_2 \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{3.7}$$

with $\beta(\alpha)$ defined in (F6).

System (3.7) is degenerate for any α, since the problem is invariant by translations and dilations. Indeed, it is well known that the first equation admits the solutions $W(x) := \frac{1-|x|^2}{(1+|x|^2)^{N/2}}$ and $W_i(x) = \frac{\partial U}{\partial x_i}$ for $i = 1, \dots, N$. The second equation instead has solutions if and only if $\beta(\alpha)$ is an eigenvalue of the linearized equation of the classical critical problem at the standard bubble U. Using the classification of the eigenvalues and eigenfunctions in [GGT, Theorem 1.1], one gets that the second equation admits nontrivial solutions if and only if $\beta(\alpha) = \lambda_n \frac{N+2}{N-2}$ with $\lambda_n := \frac{(2n+N-2)(2n+N)}{N(N+2)}$ for some $n \in \mathbb{N}$. So we have the following classification result for (3.7).

Proposition 3.1. *Let β_n be given by*

$$\beta_n := \frac{(2n+N)(2n+N-2)}{N(N-2)}. \tag{3.8}$$

i) *When $\beta(\alpha) \neq \beta_n$ for all $n \in \mathbb{N}$, all solutions to (3.7) are given by*

$$(w_1, w_2) = \left(\sum_{i=1}^N a_i \frac{\partial U}{\partial x_i} + bW,\ 0 \right) \tag{3.9}$$

for some real constants $a_1, \dots, a_N, b$, where W is the radial function defined by

$$W(x) := \frac{1}{d} \left(x \cdot \nabla U + \frac{N-2}{2} U \right) = \frac{1-|x|^2}{(1+|x|^2)^{N/2}} \tag{3.10}$$

with $d := \frac{1}{2} N^{(N-2)/4} (N-2)^{(N+2)/4}$.

ii) *When $\beta(\alpha) = \beta_n$ for some $n \in \mathbb{N}$, all solutions to (3.7) are given by*

$$(w_1, w_2) = \left(\sum_{i=1}^N a_i \frac{\partial U}{\partial x_i} + bW,\ \sum_{k=0}^n A_k W_{n,k}(r) Y_k(\theta) \right) \tag{3.11}$$

for some real constants $a_1, \dots, a_N, b, A_0, \dots, A_n$, where $W_{n,k}$ are

$$W_{n,k}(r) := \frac{r^k}{(1+r^2)^{k+\frac{N-2}{2}}} P_{n-k}^{(k+\frac{N-2}{2},\, k+\frac{N-2}{2})} \left(\frac{1-r^2}{1+r^2} \right) \tag{3.12}$$

for $k = 0, \dots, n$. Here, as usual, $Y_k(\theta)$ denotes a spherical harmonic related to the eigenvalue $k(k+N-2)$ and $P_j^{(a,b)}$ are the Jacobi polynomials.

In [GGT] we restricted to the radial functions and since the kernel of the second equation in (3.7) at the values β_n is one dimensional, Crandall-Rabinowitz' Theorem allowed us to prove the bifurcation result. In the nonradial setting, the kernel of the second equation in (3.7) is very rich. We prove a bifurcation result using the Leray Schauder degree, when this kernel has an odd dimension.

Of course, in this case, we need some compactness of the operator T. Since we seek positive solutions to System (2.1) and the maximum principle does not apply, the standard space $D^{1,2}(\mathbb{R}^N)$ does not seem to be the best one. For this reason we use a suitable weighted functional space. Set

$$D := \Big\{ u \in L^\infty(\mathbb{R}^N) \;\Big|\; \sup_{x\in\mathbb{R}^N} \frac{|u(x)|}{U(x)} < +\infty \Big\}$$

endowed with the norm $\|u\|_D := \sup_{x\in\mathbb{R}^N} \frac{|u(x)|}{U(x)}$ and define

$$X := D^{1,2}(\mathbb{R}^N) \cap D. \tag{3.13}$$

Then X is a Banach space when equipped with the norm $\|u\|_X := \max\{\|u\|_{1,2}, \|u\|_D\}$ where $\|u\|_{1,2} = (\int_{\mathbb{R}^N} |\nabla u|^2)^{1/2}$ is the classical norm in $D^{1,2}(\mathbb{R}^N)$.

Definition 3.2. Let us denote by $\mathcal{X}$ the space

$$\mathcal{X} := \big\{ (z_1, z_2) \in X^2 \;\big|\; \exists \delta > 0,\ |z_2| \leqslant (2-\delta)U + z_1 \big\}$$

and define the operator

$$T : \mathbb{R} \times \mathcal{X} \to X \times X$$

as

$$T(\alpha, z_1, z_2) := \begin{pmatrix} z_1 - (-\Delta)^{-1}\big(f_1(|x|, z_1, z_2)\big) \\ z_2 - (-\Delta)^{-1}\big(f_2(|x|, z_1, z_2)\big) \end{pmatrix}. \tag{3.14}$$

Note that if $(z_1, z_2) \in \mathcal{X}$, both quantities $U + \frac{z_1+z_2}{2}$ and $U + \frac{z_1-z_2}{2}$ are positive so that $F_i(\alpha, U + \frac{z_1+z_2}{2}, U + \frac{z_1-z_2}{2})$ are well defined on $\mathbb{R}^N$ and C^1. Moreover, $\mathcal{X}$ is an open subset of X^2.

The zeros of the operator T correspond to the solutions to System (2.1). As said before, Problem (2.1) is degenerate for any α. To overcome this degeneracy we will use some symmetry and invariance properties. The solutions we will find will inherit the symmetry and the invariance. To overcome the degeneracy of the first equation in (3.7), which is due to the scale invariance of the problem, we use the *Kelvin transform* $k(z)$ of z, namely

$$k(z)(x) := \frac{1}{|x|^{N-2}}\, z\left(\frac{x}{|x|^2}\right) \tag{3.15}$$

and we denote by $X_k^\pm \subseteq X$ the subset of functions in X which are invariant (up to the sign) by a Kelvin transform, i.e.

$$X_k^+ := \{ z \in X \mid k(z) = z \} \quad \text{and} \quad X_k^- := \{ z \in X \mid k(z) = -z \}. \tag{3.16}$$

Observe that $U \in X_k^+$, $W \in X_k^-$ and, using the fact that the Jacobi polynomials $P_j^{(a,b)}$ are *even* if j is even and *odd* if j is odd, an easy computation shows that $W_{n,k} \in X_k^+$ if $n-k$ is *even* while $W_{n,k} \in X_k^-$ if $n-k$ is *odd*.

First we prove some properties of the operator T.

Lemma 3.3. *The operator T given by (3.14) is well defined and continuous from $\mathbb{R}\times\mathcal{X}$ to X^2. Moreover, $\partial_\alpha T$, $\partial_z T$ and $\partial_{\alpha z}T$ exist and are continuous. Finally, T maps $\mathbb{R}\times\big(\mathcal{X}\cap(X_k^+\times X_k^\pm)\big)$ to $X_k^+\times X_k^\pm$.*

Proof. First notice that, (F4) implies $\lim_{\lambda\to 0}F_i(\alpha,\lambda u_1,\lambda u_2)=0$. Thus, using (F2), one gets

$$\begin{aligned}|F_i(\alpha,u_1,u_2)| &= \Big|\int_0^1 \partial_\lambda\big(F_i(\alpha,\lambda u_1,\lambda u_2)\big)\,\mathrm{d}\lambda\Big|\\ &\leqslant \int_0^1 \big|\partial_{u_1}F_i(\alpha,\lambda u_1,\lambda u_2)[u_1]+\partial_{u_2}F_i(\alpha,\lambda u_1,\lambda u_2)[u_2]\big|\,\mathrm{d}\lambda\\ &\leqslant C(u_1^{2^*-2}+u_2^{2^*-2})(u_1+u_2)\\ &\leqslant C(u_1^{2^*-1}+u_2^{2^*-1}).\end{aligned}\tag{3.17}$$

(Different occurrences of C may denote different constants.) Given that z_1,z_2 and U belong to X, (3.17) implies that $\big|F_i\big(\alpha,U+\frac{z_1+z_2}{2},U+\frac{z_1-z_2}{2}\big)\big|\leqslant CU^{2^*-1}$ and thus, using (3.3) and (3.4),

$$|f_i(|x|,z_1,z_2)|\leqslant CU^{2^*-1}\quad\text{ for } i=1,2.$$

Then $f_i(|x|,z_1,z_2)$ belong to $L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ and there exists a unique $g_i\in D^{1,2}(\mathbb{R}^N)$ for $i=1,2$ such that g_i is a weak solution to

$$-\Delta g_i=f_i(|x|,z_1,z_2)\quad\text{in }\mathbb{R}^N.\tag{3.18}$$

The solution g_i enjoys the following representation:

$$g_i(x)=\frac{1}{\omega_N(N-2)}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-2}}f_i(|y|,z_1,z_2)\,\mathrm{d}y$$

where ω_N is the area of the unit sphere in $\mathbb{R}^N$. This implies

$$|g_i(x)|\leqslant C\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-2}}U^{2^*-1}(y)\,\mathrm{d}y=CU(x)$$

and $g_i\in X$ showing that T is well defined from $\mathcal{X}$ to $X\times X$.

Next we have to show that the operator T maps Kelvin invariant (up to a sign) functions into functions that are Kelvin invariant (with the same sign). It is enough to show that $\big((-\Delta)^{-1}(f_1(|x|,z_1,z_2)),(-\Delta)^{-1}(f_2(|x|,z_1,z_2))\big)$ maps $\mathcal{X}\cap(X_k^+\times X_k^\pm)$ into $X_k^+\times X_k^\pm$. Assume $(z_1,z_2)\in\mathcal{X}\cap(X_k^+\times X_k^\pm)$ and let, as before, $g_i=(-\Delta)^{-1}(f_i(|x|,z_1,z_2))$. Then $g_i\in X$ is a weak solution to (3.18) and letting $\widetilde{g}_i:=k(g_i)$, the Kelvin transform of g_i we have that $\widetilde{g}_i$ weakly solves

$$-\Delta\widetilde{g}_i=-\frac{1}{|x|^{N+2}}\Delta g_i\Big(\frac{x}{|x|^2}\Big)=\frac{1}{|x|^{N+2}}\,f_i\left(\frac{x}{|x|^2},z_1\Big(\frac{x}{|x|^2}\Big),z_2\Big(\frac{x}{|x|^2}\Big)\right)$$

An easy consequence of (F4) is that

$$\frac{1}{|x|^{N+2}}\, F_i\left(\alpha, \Big(U+\frac{z_1+z_2}{2}\Big)\Big(\frac{x}{|x|^2}\Big), \Big(U+\frac{z_1-z_2}{2}\Big)\Big(\frac{x}{|x|^2}\Big)\right)$$
$$= F_i\Big(\alpha, k(U)+\frac{k(z_1)+k(z_2)}{2}, k(U)+\frac{k(z_1)-k(z_2)}{2}\Big).$$

This, together with the fact that U and z_1 are Kelvin invariant while z_2 is Kelvin invariant up to a sign (depending which space $X_k^{\pm}$ we are dealing with) shows that

$$\frac{1}{|x|^{N+2}}\, f_i\left(\frac{x}{|x|^2}, z_1\Big(\frac{x}{|x|^2}\Big), z_2\Big(\frac{x}{|x|^2}\Big)\right) = f_i\big(|x|, z_1(x), \pm z_2(x)\big)$$

where $\pm$ depends on the space $X_k^{\pm}$ we consider. Then, using (3.5), it follows that

$$\frac{1}{|x|^{N+2}}\, f_1\left(\frac{1}{|x|}, z_1\Big(\frac{x}{|x|^2}\Big), z_2\Big(\frac{x}{|x|^2}\Big)\right) = f_1\big(|x|, z_1(x), z_2(x)\big)$$

while

$$\frac{1}{|x|^{N+2}}\, f_2\left(\frac{x}{|x|^2}, z_1\Big(\frac{x}{|x|^2}\Big), z_2\Big(\frac{x}{|x|^2}\Big)\right) = \pm f_2\big(|x|, z_1(x), z_2(x)\big).$$

This implies that $\widetilde{g}_1$ weakly solves $-\Delta\widetilde{g}_1 = f_1(|x|, z_1(x), z_2(x))$ and $\widetilde{g}_2$ solves $-\Delta\widetilde{g}_2 = \pm f_2(|x|, z_1(x), z_2(x))$. The uniqueness of solutions in $D^{1,2}(\mathbb{R}^N)$ then implies $\widetilde{g}_1 = g_1$ and $\widetilde{g}_2 = \pm g_2$ which shows that $g_1 \in X_k^+$ and $g_2 \in X_k^{\pm}$. This concludes the first part of the proof.

Let us now prove the continuity of T on $\mathbb{R}\times\mathcal{X}$. Let $\alpha_n \to \alpha$ in $\mathbb{R}$ and $(z_{1,n}, z_{2,n}) \to (z_1, z_2)$ in $\mathcal{X}$ as $n\to\infty$, and set

$$g_{i,n} := (-\Delta)^{-1} f_{i,n} \quad \text{where } f_{i,n}(x) := f_i(|x|, z_{1,n}, z_{2,n}) \text{ with } \alpha = \alpha_n.$$

Since $z_{i,n} \to z_i$ in $D^{1,2}(\mathbb{R}^N)$, the convergence also holds in $L^{2^*}(\mathbb{R}^N)$. Using (3.17) and Lebesgue's dominated convergence theorem and its converse, one deduces that $f_{i,n} \to f_i$ in $L^{\frac{2N}{N+2}}$. Therefore $g_{i,n} \to g_i$ in $D^{1,2}$ and $T(\alpha_n, z_n) \to T(\alpha, z)$ in $D^{1,2}$. Now let us show the convergence in D. We have that

$$\begin{aligned}\frac{|g_{i,n}(x)-g_i(x)|}{U(x)} &\leqslant \frac{1}{\omega_N(N-2)U(x)} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \frac{|f_{i,n}(y)-f_i(y)|}{U(y)^{2^*-1}}\, U(y)^{2^*-1}\,\mathrm{d}y \\ &\leqslant C \sup_{y\in\mathbb{R}^N} \frac{|f_{i,n}(y)-f_i(y)|}{U(y)^{2^*-1}}.\end{aligned} \tag{3.19}$$

Moreover, using (F4), one gets

$$\begin{aligned}\frac{|f_{i,n}(y)-f_i(y)|}{U(y)^{2^*-1}} \leqslant \sum_{i=1}^{2} \Big| &F_i\Big(\alpha_n, 1+\frac{z_{1,n}+z_{2,n}}{2U},\ 1+\frac{z_{1,n}-z_{2,n}}{2U}\Big) \\ &- F_i\Big(\alpha, 1+\frac{z_1+z_2}{2U},\ 1+\frac{z_1-z_2}{2U}\Big)\Big|\end{aligned}$$

Thanks to the convergence in D, $z_{j,n}/U \to z_j/U$ uniformly for $j = 1, 2$. Thus $1 + \frac{z_{1,n} \pm z_{2,n}}{2U} \to 1 + \frac{z_1 \pm z_2}{2U}$ uniformly on $\mathbb{R}^N$. The continuity of the maps F_i then imply that both terms of the sum converge uniformly to 0.

The existence and continuity of the derivatives is proved in a similar way. □

Next we show a compactness result for the operator $(z_1, z_2) \mapsto \big((-\Delta)^{-1} f_1, (-\Delta)^{-1} f_2\big)$. Here we need some decay estimates on solutions of a semilinear elliptic equation.

Lemma 3.4 ([ST]). *If $0 < p < N$ and h is a non negative, radial function belonging to $L^1(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} \frac{h(y)}{|x-y|^p}\,\mathrm{d}y = O\left(\frac{1}{|x|^p}\right) \quad \text{as } |x| \to +\infty.$$

Now we can prove our compactness result:

Lemma 3.5. *For all α, the operator*

$$M(z_1, z_2) := \big((-\Delta)^{-1} f_1(|x|, z_1, z_2), (-\Delta)^{-1} f_2(|x|, z_1, z_2)\big) \tag{3.20}$$

is compact from $\mathcal{X}$ to X^2.

Proof. 1. From Lemma 3.3, we have that $M : \mathcal{X} \to X^2$ is continuous. Now let $(z_n) = (z_{1,n}, z_{2,n})$ be a bounded sequence in $\mathcal{X}$ and let us prove that, up to a subsequence, $g_n := M(z_n)$ converges strongly to some $g \in X \times X$. On one hand, since (z_n) is bounded in $D^{1,2} \times D^{1,2}$, going if necessary to a subsequence, one can assume that (z_n) converges weakly to some $z = (z_1, z_2)$ in $D^{1,2} \times D^{1,2}$ and $z_n \to z$ almost everywhere. On the other hand, $(\|z_n\|_{D\times D})$ is also bounded which means that $|z_{i,n}| \leqslant CU$ where C is independent of i and n and so, using (3.17), $|f_i(|x|, z_n)| \leqslant CU^{2^*-1}$. Lebesgue's dominated convergence theorem then implies that $f_i(|x|, z_n)$ converges strongly to $f_i(|x|, z)$ in $L^{\frac{2N}{N+2}}$ for $i = 1, 2$. From the continuity of $(-\Delta)^{-1} : L^{\frac{2N}{N+2}} \to D^{1,2}$, one concludes that $g_n \to g$ in $D^{1,2} \times D^{1,2}$. The inequality $|z_{i,n}| \leqslant CU$ also implies

$$|g_{i,n}(x)| \leqslant C \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}}\,|f_i(z_n(y))|\,\mathrm{d}y \leqslant C \int_{\mathbb{R}^N} \frac{U^{2^*-1}(y)}{|x-y|^{N-2}}\,\mathrm{d}y = C\,U(x),$$

and passing to the limit yields $g_i \in D$.

2. It remains to show that $\|g_n - g\|_{D\times D} \to 0$. First, Hölder's inequality allows to get the estimate:

$$\begin{aligned}
&|g_{i,n}(x) - g_i(x)| \\
&\quad \leqslant C \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \left| f_i(|y|, z_n(y)) - f_i(|y|, z(y)) \right| \mathrm{d}y \\
&\quad = C \int_{\mathbb{R}^N} \frac{U^{2^*-1-\varepsilon}(y)}{|x-y|^{N-2}} \frac{\left| f_i(|y|, z_n(y)) - f_i(|y|, z(y)) \right|}{U^{2^*-1-\varepsilon}(y)} \mathrm{d}y \\
&\quad \leqslant C \left(\int_{\mathbb{R}^N} \left| \frac{U^{\frac{N+2}{N-2}-\varepsilon}(y)}{|x-y|^{N-2}} \right|^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^N} \left| \frac{\left| f_i(|y|, z_n(y)) - f_i(|y|, z(y)) \right|}{U^{2^*-1}(y)} U^{\varepsilon}(y) \right|^q \right)^{\frac{1}{q}}
\end{aligned}$$

where $\varepsilon > 0$ will be chosen small and $q > 1$ large such that $\varepsilon q = 2^*$. Note that (3.17) implies $|f_i(|y|, z(y))| \leqslant C(U + |z_1| + |z_2|)^{2^*-1} + CU^{2^*-1}$ and so the ratio in the right integral is bounded on $\mathbb{R}^N$. Thus the integrand of the right integral is bounded by $C^q\, U^{\varepsilon q}(y) \leqslant CU^{2^*}(y) \in L^1(\mathbb{R}^N)$ where C is independent of n. Lebesgue's dominated convergence theorem then implies that this integral converges to 0 as $n \to \infty$.

The proof will be complete if we show:

$$\int_{\mathbb{R}^N} \left| \frac{U^{\frac{N+2}{N-2}-\varepsilon}(y)}{|x-y|^{N-2}} \right|^{\frac{q}{q-1}} \mathrm{d}y \leqslant \frac{C}{(1+|x|)^{(N-2)\frac{q}{q-1}}} = CU^{\frac{q}{q-1}}(x). \tag{3.21}$$

This inequality follows from Lemma 3.4 because $h := U^{\left(\frac{N+2}{N-2}-\varepsilon\right)\frac{q}{q-1}} \in L^1(\mathbb{R}^N)$ i.e., $(N-2)\left(\frac{N+2}{N-2} - \varepsilon\right)\frac{q}{q-1} > N$, and $(N-2)\frac{q}{q-1} < N$ are possible if ε is small enough and q is large enough. □

4. The role of symmetries

The operator T is a compact perturbation of the identity and, as proved in Lemma 3.3, maps $\mathbb{R} \times \left(\mathcal{X} \cap (X_k^+ \times X_k^\pm)\right)$ into $X_k^+ \times X_k^\pm$.

We want to find solutions to our problem as zeroes of T and we will use the bifurcation theory. As explained in the introduction, we want to find both radial and nonradial solutions. In particular, to obtain the nonradial ones, we use some symmetry properties of the operator T that can be obtained by (3.5).

We state the definition in a general way and we will then apply to some specific cases so to obtain different solutions. Let us introduce some notations. Let $\mathcal{S}$ be a subgroup of $O(N)$, where $O(N)$ is the orthogonal group of $\mathbb{R}^N$, and let

$$X_{\mathcal{S}} := \left\{ v \in X_k^+ \;\middle|\; \forall s \in \mathcal{S},\ \forall x \in \mathbb{R}^N,\ v(s^{-1}(x)) = v(x) \right\} \tag{4.1}$$

be the set of functions invariant by the action of $\mathcal{S}$. Let $\sigma : \mathcal{S} \to \{-1, 1\}$ be a group morphism and define a second action of $\mathcal{S}$ on X by

$$(s \diamond v)(x) := \sigma(s)\, v\big(s^{-1}(x)\big). \tag{4.2}$$

The invariant subspace of $X_k^+ \times X_k^\pm$ of interest is

$$\mathcal{Z} := \big\{ z = (z_1, z_2) \in X_k^+ \times X_k^\pm \;\big|\; \forall s \in \mathcal{S},\, z_1(s^{-1}(x)) = z_1(x) \text{ and } \sigma(s)\, z_2(s^{-1}(x)) = z_2(x) \big\}. \tag{4.3}$$

Then we can prove the following result:

Lemma 4.1. *The operator T defined in (3.14) maps $\mathbb{R} \times (\mathcal{X} \cap \mathcal{Z})$ into $\mathcal{Z}$.*

Proof. We will show that $T = (T_1, T_2)$ is equivariant under the action of $\mathcal{S}$, namely

$$\begin{aligned} & T_1\big(\alpha, z_1(s^{-1}(x)), \sigma(s) z_2(s^{-1}(x))\big) = T_1\big(\alpha, z_1(x), z_2(x)\big), \\ \text{and} \quad & T_2\big(\alpha, z_1(s^{-1}(x)), \sigma(s) z_2(s^{-1}(x))\big) = \sigma(s)\, T_2\big(\alpha, z_1(x), z_2(x)\big). \end{aligned}$$

Let $z = (z_1, z_2) \in \mathcal{X}$. First, notice that, thanks to (3.5), the functions f_1 and f_2 defined in (3.3)–(3.4) satisfy

$$\begin{aligned} & f_1\big(|x|, z_1(s^{-1}(x)), \sigma(s) z_2(s^{-1}(x))\big) = f_1(|x|, z(x)), \\ \text{and} \quad & f_2\big(|x|, z_1(s^{-1}(x)), \sigma(s) z_2(s^{-1}(x))\big) = \sigma(s) f_2(|x|, z(x)). \end{aligned}$$

Second, because the Laplacian is equivariant under the action of the group $O(N)$, it readily follows that $(-\Delta)^{-1}\big(\sigma(s) f(s(x))\big) = \sigma(s)\big((-\Delta)^{-1} f(s(x))\big)$ for any σ, $s \in \mathcal{S}$ and $f \in L^{2N/(N+2)}$.

Putting these observations together concludes the proof. □

Lemma 4.2. *Assume $\beta(\alpha) \neq \beta_n$ for all $n \in \mathbb{N}$, with β_n be as defined in (3.8), and that the subspace of solutions in $X_{\mathcal{S}}$ to the first equation of (3.7) has only the trivial solution. Still denote T the operator defined in (3.14) restricted to $\mathcal{X} \cap \mathcal{Z}$. Then the linear map $\partial_z T(\alpha, 0, 0) : \mathcal{Z} \to \mathcal{Z}$ is invertible, where $\partial_z T(\alpha, 0, 0)$ is the Fréchet derivative of T with respect to z at $(\alpha, 0, 0)$.*

Proof. For any $(w_1, w_2) \in X^2$, one has, see (3.7),

$$\partial_z T(\alpha, 0, 0) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 - (-\Delta)^{-1}\Big(\frac{N+2}{N-2} U^{\frac{4}{N-2}}\, w_1\Big) \\ w_2 - (-\Delta)^{-1}\Big(\beta(\alpha)\, U^{\frac{4}{N-2}}\, w_2\Big) \end{pmatrix} \tag{4.4}$$

with $\beta(\alpha)$ as defined in (F6). Since $\partial_z T(\alpha, 0, 0)$ is a compact perturbation of the identity (see Lemma 3.5 in [GGT]), it suffices to prove that $\ker\big(\partial_z T(\alpha, 0, 0)\big) = \{(0, 0)\}$ in $\mathcal{Z}$ whenever $\beta(\alpha) \neq \beta_n$. Let $(w_1, w_2) \in \mathcal{Z} \subseteq X_k^+ \times X_k^\pm$. Notice that $\partial_z T(\alpha, 0, 0) \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ if and only if (w_1, w_2) is a solution to (3.7). By assumption we have that $w_1 \equiv 0$ and Proposition 3.1 says that the only solutions to the second equation are given by (3.9) as we assumed $\beta(\alpha) \neq \beta_n$. This gives the claim. □

Remark 4.3. From Lemma 4.2 we have that, when $\beta(\alpha) \neq \beta_n$ for all n,

$$\deg\big(T(\alpha, \cdot), \widetilde{B}, 0\big) = \deg\big(\partial_z T(\alpha, 0, 0), \widetilde{B}, 0\big) = (-1)^{m(\alpha)} \tag{4.5}$$

where $\widetilde{B}$ is a suitable ball in $\mathcal{Z}$ centered at the origin and $m(\alpha)$ the sum of the algebraic multiplicities of all eigenvalues λ belonging to $(0,1)$ of the problem

$$\begin{cases} -\Delta w_1 = \lambda \frac{N+2}{N-2} U^{\frac{4}{N-2}} w_1 & \text{in } \mathbb{R}^N, \\ -\Delta w_2 = \lambda \, \beta(\alpha) \, U^{\frac{4}{N-2}} w_2 & \text{in } \mathbb{R}^N, \\ (w_1, w_2) \in \mathcal{Z}. \end{cases} \tag{4.6}$$

Proposition 4.4. *Assume the same hypotheses as in Lemma 4.2. Let $n \in \mathbb{N}$ and α_n^* be such that $\beta(\alpha_n^*) = \beta_n$ (recall that β_n is defined in (3.8)). For $\varepsilon > 0$ small enough, the following holds*

$$m(\alpha_n^* + \varepsilon) = m(\alpha_n^* - \varepsilon) + \gamma(n) \tag{4.7}$$

where $\gamma(n)$ is the algebraic multiplicity of the solutions to $-\Delta w = \beta_n U^{\frac{4}{N-2}} w$ such that $(0, w) \in \mathcal{Z}$.

Proof. As the first equation of (4.6) does not depend on α, its contribution is the same to the values $m(\alpha_n^* \pm \varepsilon)$. Concerning the second one, since $\beta(\alpha)$ is a continuous increasing function we have get that $\beta(\alpha_n^* + \varepsilon) \searrow \beta(\alpha_n^*)$ and then the contribution of the second equation to $m(\alpha_n^* + \varepsilon)$ is given by the algebraic multiplicity of the eigenvalues $\lambda = \left\{ \frac{1}{\beta(\alpha_n^* + \varepsilon)}, \dots, \frac{\beta_n}{\beta(\alpha_n^* + \varepsilon)} \right\}$. In the same way, for ε small enough we have that $m(\alpha_n^* - \varepsilon)$ is given by the algebraic multiplicity of the eigenvalues $\lambda = \left\{ \frac{1}{\beta(\alpha_n^* + \varepsilon)}, \dots, \frac{\beta_{n-1}}{\beta(\alpha_n^* + \varepsilon)} \right\}$. This gives the claim. □

Proposition 4.5. *Assume the same hypotheses as in Lemma 4.2 and let us suppose that $\gamma(n)$ is an odd integer. Then the point (α_n^*, U, U) is a bifurcation point from the curve of trivial solutions (α, U, U) to System (2.1). Moreover the bifurcation is global, the Rabinowitz alternative holds, and for any sequence $(\alpha_k, u_{1,k}, u_{2,k})$ of solutions converging to (α_n^*, U, U), we have that*

$$\begin{cases} u_{1,k} = U + \frac{z_{1,k} + z_{2,k}}{2} \\ u_{2,k} = U + \frac{z_{1,k} - z_{2,k}}{2} \end{cases}$$

and, up to a subsequence,

$$\begin{cases} u_{1,k} = U + \varepsilon_k Z_n + o(\varepsilon_k), \\ u_{2,k} = U - \varepsilon_k Z_n + o(\varepsilon_k), \end{cases} \tag{4.8}$$

as $k \to \infty$ where Z_n is a solution to the second equation in (3.7) such that $(0, Z_n) \in \mathcal{Z}$, $\|Z_n\|_X = 1$ and $\varepsilon_k = \|z_{2,k}\|_X \to 0$.

Proof. From (4.5) and (4.7), it is standard to see that the curve of trivial solutions for the operator $T : \mathbb{R} \times (\mathcal{X} \cap \mathcal{Z}) \to \mathcal{Z}$ bifurcates at the values α_n^* with $\beta(\alpha_n^*) = \beta_n$ for any n such that $\gamma(n)$ is odd, see [K, Theorem II.3.2] and the bifurcation is global. The Rabinowitz alternative finally follows from [K, Theorem II.3.3].

Next let us show the expansion (2.11). Let $(z_{1,k}, z_{2,k})$ be solutions obtained by the bifurcation result to (2.1) as $\alpha_k \to \alpha_n^*$ (recall that $(z_{1,k}, z_{2,k}) \to (0,0)$ in the space X).

First we show that

$$\frac{\|z_{1,k}\|}{\|z_{2,k}\|} \leqslant C \tag{4.9}$$

where C is a constant independent of k and $\|\cdot\| = \|\cdot\|_X$. First, notice that $z_{2,k} \not\equiv 0$ because, if it was, $z_{1,k} \in X_{\mathcal{S}}$ would satisfy $-\Delta z_1 = f_1(|x|, z_1, 0)$ but the assumption that the first equation of (3.7) has only the trivial solution in $X_{\mathcal{S}}$ implies that this equation only has trivial solutions for $\alpha \approx \alpha_n^*$. This contradicts the fact that $(z_{1,k}, z_{2,k})$ lies on the branch of nontrivial solutions.

To show (4.9), let us argue by contradiction: let us suppose that, up to subsequence, $\frac{\|z_{1,k}\|}{\|z_{2,k}\|} \to +\infty$. Set $w_{1,k} = \frac{z_{1,k}}{\|z_{1,k}\|}$, $w_{2,k} = \frac{z_{2,k}}{\|z_{2,k}\|}$. The system satisfied by $w_{1,k}$ and $w_{2,k}$ is

$$\begin{aligned}
-\Delta w_{1,k} = \frac{1}{\|z_{1,k}\|}\Bigg[& F_1\Big(\alpha_k, U + \|z_{1,k}\|\frac{w_{1,k} + \frac{\|z_{2,k}\|}{\|z_{1,k}\|}w_{2,k}}{2},\ U + \|z_{1,k}\|\frac{w_{1,k} - \frac{\|z_{2,k}\|}{\|z_{1,k}\|}w_{2,k}}{2}\Big) \\
& + F_2\Big(\alpha_k, U + \|z_{1,k}\|\frac{w_{1,k} + \frac{\|z_{2,k}\|}{\|z_{1,k}\|}w_{2,k}}{2},\ U + \|z_{1,k}\|\frac{w_{1,k} - \frac{\|z_{2,k}\|}{\|z_{1,k}\|}w_{2,k}}{2}\Big) \\
& - 2U^{2^*-1}\Big)\Bigg]
\end{aligned} \tag{4.10a}$$

$$\begin{aligned}
-\Delta w_{2,k} = \frac{1}{\|z_{2,k}\|}\Bigg[& F_1\Big(\alpha_k, U + \|z_{2,k}\|\frac{\frac{\|z_{1,k}\|}{\|z_{2,k}\|}w_{1,k} + w_{2,k}}{2},\ U + \|z_{2,k}\|\frac{\frac{\|z_{1,k}\|}{\|z_{2,k}\|}w_{1,k} - w_{2,k}}{2}\Big) \\
& - F_2\Big(\alpha_k, U + \|z_{2,k}\|\frac{\frac{\|z_{1,k}\|}{\|z_{2,k}\|}w_{1,k} + w_{2,k}}{2},\ U + \|z_{2,k}\|\frac{\frac{\|z_{1,k}\|}{\|z_{2,k}\|}w_{1,k} - w_{2,k}}{2}\Big)\Bigg]
\end{aligned} \tag{4.10b}$$

$$\|w_{1,k}\| = \|w_{2,k}\| = 1 \tag{4.10c}$$

Going if necessary to a subsequence, we can assume $w_{1,k} \rightharpoonup w_1$ and $w_{2,k} \rightharpoonup w_2$ in $D^{1,2}$ for some $(w_1, w_2) \in \mathcal{Z}$. Arguing as in the first part of the proof of Lemma 3.5, we deduce that $w_{1,k} \to w_1$ and $w_{2,k} \to w_2$ in $L^{2^*}(\mathbb{R}^N)$ and in $D^{1,2}$. Using that $F_i(\alpha_k, U, U) = U^{2^*-1}$ for $i = 1, 2$, we can pass to the limit on Eq. (4.10a) and show that $w_1 \in X_{\mathcal{S}}$ satisfies

$$-\Delta w_1 = \left[\frac{\partial F_1}{\partial u_1}(\alpha_n^*, U, U) + \frac{\partial F_1}{\partial u_2}(\alpha_n^*, U, U) + \frac{\partial F_2}{\partial u_1}(\alpha_n^*, U, U) + \frac{\partial F_2}{\partial u_2}(\alpha_n^*, U, U)\right]\frac{w_1}{2}$$

Moreover, arguing as in the second part of the proof of Lemma 3.5 on (4.10a), we can show that $\|w_{1,k} - w_1\|_D \to 0$. Thus $w_{1,k} \to w_1$ in X and $\|w_1\| = 1$. As in Section 3, using the properties of F we have that $w_1 \in X_{\mathcal{S}}$ satisfies

$$-\Delta w_1 = \frac{N+2}{N-2}\, U^{\frac{4}{N-2}}\, w_1 \quad \text{in } \mathbb{R}^N,$$

This is a contradiction since in $X_{\mathcal{S}}$ the previous equation admits only the trivial solution. So (4.9) holds.

Hence, up to a subsequence, we have that $\frac{\|z_{1,k}\|}{\|z_{2,k}\|} \to \delta \geqslant 0$. Passing to the limit in (4.10b), we get that

$$-\Delta w_2 = \frac{\partial F_1}{\partial u_1}(\alpha_n^*, U, U)\frac{\delta w_1 + w_2}{2} + \frac{\partial F_1}{\partial u_2}(\alpha_n^*, U, U)\frac{\delta w_1 - w_2}{2} - \frac{\partial F_2}{\partial u_1}(\alpha_n^*, U, U)\frac{\delta w_1 + w_2}{2} - \frac{\partial F_2}{\partial u_2}(\alpha_n^*, U, U)\frac{\delta w_1 - w_2}{2}.$$

and, arguing again as in the second part of the proof of Lemma 3.5, $w_{2,k} \to w_2$ in X with $\|w_2\| = 1$. As before, using the properties of F, we have that w_2 solves

$$-\Delta w_2 = \beta(\alpha)\, U^{\frac{4}{N-2}} w_2 \quad \text{in } \mathbb{R}^N,$$

and hence $w_2 = Z_n$ where Z_n is a solution to the second equation in (3.7) such that $(0, Z_n) \in \mathcal{Z}$ and $\|Z_n\| = 1$. Then $z_{2,k} = \|z_{2,k}\|(Z_n + o(1))$. Next we show that

$$z_{1,k} = o(1)\|z_{2,k}\|. \tag{4.11}$$

This is clear if $\lim\limits_{k\to+\infty} \frac{\|z_{1,k}\|}{\|z_{2,k}\|} = 0$ since in this case

$$\|z_{1,k}\| = \frac{\|z_{1,k}\|}{\|z_{2,k}\|}\|z_{2,k}\| = o(1)\|z_{2,k}\|. \tag{4.12}$$

On the other hand, it is not possible that $\frac{\|z_{1,k}\|}{\|z_{2,k}\|} \geqslant D > 0$ because in this case we can pass to the limit in (4.10a) and as before we get a contradiction. This shows (4.11). Coming back to the definition of $(u_{1,k}, u_{2,k})$ we have that (4.8) holds with $\varepsilon_k = \|z_{2,k}\|$. □

Now we specify some subgroups $\mathcal{S}$ that satisfy the assumptions of Lemma 4.2. Observe that when $\beta(\alpha) \neq \beta_n$ the second equation in (3.7) does not possess solutions. The first equation instead admits in X_k^+ the solutions $\sum_{i=1}^N a_i \frac{x_i}{(1+|x|^2)^{N/2}}$. Then, the assumptions of Lemma 4.2 are satisfied if the functions $\frac{x_i}{(1+|x|^2)^{N/2}}$ do not belong to X_S. The first example is the radial case which allows to prove Theorem 2.1. The other examples, which are provided for every $N \geqslant 3$, prove the existence of different nonradial solutions.

4.1. **The radial case.** Following the previous notation we let $\mathcal{S} = O(N)$ and $\sigma : \mathcal{S} \to \{-1, 1\}$ be the group morphism such that $\sigma(s) := 1$ for all $s \in O(N)$. Thus

$$X_{\mathcal{S}} = \big\{v \in X \;\big|\; \forall x \in \mathbb{R}^N,\ v(x) = v(|x|)\big\},$$
$$\mathcal{Z} \equiv \mathcal{Z}^{\pm}_{\text{rad}} = \big\{z \in X_k^+ \times X_k^{\pm} \;\big|\; \forall x \in \mathbb{R}^N,\ z(x) = z(|x|)\big\}.$$

Proof of Theorem 2.1. To prove the bifurcation result we define the operator T in (3.14) in the space $\mathcal{Z}^+_{\text{rad}} \subseteq X_k^+ \times X_k^+$ when n is even and in the space $\mathcal{Z}^-_{\text{rad}} \subseteq X_k^+ \times X_k^-$ when n is odd. Recalling the discussion at the beginning of Section 3, we have that the linearized operator $\partial_z T(\alpha, 0, 0)$ is invertible if and only if system (3.7) does not admit solutions in $\mathcal{Z}^+_{\text{rad}}$ when n is even ($\mathcal{Z}^-_{\text{rad}}$ in case of n odd). From Proposition 3.1 we know that the first equation in (3.7) does not depend on α and admits the unique radial solution $W(|x|)$ which does not belong to X_k^+. The second equation in (3.7) instead admits solutions if and only if $\beta(\alpha) = \beta_n$ and the corresponding radial solution is $W_n(|x|) := W_{n,0}(r)$. Hence the assumption of Lemma 4.2 are satisfied. Moreover from (3.12) and the definition of

the Jacobi polynomials we have that $W_n \in X_k^+$ if n is even and $W_n \in X_k^-$ if n is odd showing that $\gamma(n) = 1$ for any n. Further, using the monotonicity of $\beta(\alpha)$, the global bifurcation result and the Rabinowitz alternative follows from Theorem II.3.2 and Theorem II.3.3 of [K]. Finally the fact that the curve is continuously differentiable near the bifurcation point follows from the bifurcation result of Crandall-Rabinowitz for one-dimensional kernel since the operator T is differentiable and the transversality condition holds in $\mathcal{Z}$ because

$$\partial_{\alpha z} T(\alpha, 0, 0) \begin{pmatrix} 0 \\ W_n \end{pmatrix} = -\partial_\alpha \beta(\alpha) \begin{pmatrix} 0 \\ (-\Delta)^{-1}\Big(U^{\frac{4}{N-2}} W_n\Big) \end{pmatrix},$$

and so

$$\left(\begin{pmatrix} 0 \\ W_n \end{pmatrix} \,\Big|\, \partial_{\alpha z} T(\alpha_n^*, 0, 0) \begin{pmatrix} 0 \\ W_n \end{pmatrix} \right)_{(D^{1,2})^2} = -\partial_\alpha \beta(\alpha_n^*) \int_{\mathbb{R}^N} U^{\frac{4}{N-2}} W_n^2 \, \mathrm{d}x \neq 0. \qquad \square$$

Proof of Corollary 2.2 . It is easy to check that (F1)–(F5) are satisfied. One readily computes that

$$\beta(\alpha) = \begin{cases} 2(2^* - 1 - p)\alpha - (2^* - 1 - 2p) & \text{in (2.2)}, \\ 2^*\alpha - 1 & \text{in (1.1)}, \\ \frac{8}{N-2}\alpha - \frac{6-N}{N-2} & \text{in (1.2)}. \end{cases} \tag{4.13}$$

and so (F6) is also satisfied. Moreover (2.5) holds if and only if $\alpha^* = \alpha_n^*$ where α_n^* is defined by (2.8). Corollary 2.2 immediately follows. $\square$

4.2. **The first nonradial case.** Let h be the reflection through the hyperplane $x_N = 0$, $\mathcal{S}_1 := \langle O(N-1), h\rangle$ be the subgroup generated by $O(N-1)$ and h, and $\sigma_1 : \mathcal{S}_1 \to \{-1, 1\}$ be the group morphism such that $\sigma_1(s) := 1$ if $s \in O(N-1)$ and $\sigma_1(h) := -1$ (σ_1 is easily seen to be well defined because h commutes with any element of $O(N-1)$). Thus

$$\begin{aligned} X_{\mathcal{S}_1} &= \big\{ v \in X_k^+ \,\big|\, \forall x = (x', x_N) \in \mathbb{R}^N,\ v(x', x_N) = v(|x'|, -x_N) \big\}, \\ \mathcal{Z} \equiv \mathcal{Z}_1^\pm &= \big\{ z \in X_k^+ \times X_k^\pm \,\big|\, \forall x = (x', x_N) \in \mathbb{R}^N,\ z_1(x', x_N) = z_1(|x'|, -x_N) \text{ and} \\ &\qquad\qquad z_2(x', x_N) = -z_2(|x'|, -x_N) \big\}. \end{aligned}$$

Observe that the odd symmetry helps to kill the radial solution in the kernel of the linearized system while the even symmetries help to avoid the solutions given by the translation invariance of the problem. Indeed since functions in $X_{\mathcal{S}_1}$ are even with respect to each x_i, $i = 1, \dots, N$ and belong to X_k^+ from Proposition 3.1, it is easily deduced that the solutions in $X_{\mathcal{S}_1}$ of the first equation of (3.7) (see (3.9)) are the trivial ones. Thus Lemma 4.2 applies and by Proposition 4.5 the bifurcation result can be proved when $\gamma(n)$ is odd.

Proposition 4.6. *With this choice of $\mathcal{S} = \mathcal{S}_1$ and $\sigma = \sigma_1$, we have that $\gamma(n)$ is odd if and only if $n = 4\ell + 1$ or $n = 4\ell + 2$ for $\ell = 0, 1, \dots$*

Proof. In $\mathbb{R}^N$, we consider the spherical coordinates $(r, \varphi, \theta_1, \dots, \theta_{N-2})$ with $r = |x| \in [0, +\infty)$, $\varphi \in [0, 2\pi]$, and $\theta_i \in [0, \pi]$ as $i = 1, 2, \dots, N-2$ with

$$\begin{cases} x_1 = r\cos\varphi \sin\theta_1 \cdots \sin\theta_{N-2} \\ x_2 = r\sin\varphi \sin\theta_1 \cdots \sin\theta_{N-2} \\ \quad\vdots \\ x_{N-1} = r\sin\theta_{N-2}\cos\theta_{N-3} \\ x_N = r\cos\theta_{N-2}. \end{cases} \tag{4.14}$$

Proposition 3.1 says that the solutions to $-\Delta w = \beta_n U^{\frac{4}{N-2}} w$ are, in radial coordinates, linear combinations of the $n+1$ functions

$$[0, +\infty) \times \mathbb{S}^{N-1} \to \mathbb{R} : (r, \varphi, \theta_1, \dots, \theta_{N-2}) \mapsto W_{n,k}(r) Y_k(\varphi, \theta_1, \dots, \theta_{N-2}) \tag{4.15}$$

for $k = 0, \dots, n$, where $Y_k(\varphi, \theta_1, \dots, \theta_{N-2})$ are spherical harmonics with eigenvalue $k(k+N-2)$. For any k, there is only a single (up to a scalar multiple) spherical harmonic which is $O(N-1)$-invariant and it is given by the function:

$$Y_k(\varphi, \theta_1, \dots, \theta_{N-2}) = Y_k(\theta_{N-2}) = P_k^{(\frac{N-3}{2}, \frac{N-3}{2})}(\cos\theta_{N-2}) \\ \text{where } r\cos\theta_{N-2} = x_N \text{ with } \theta_{N-2} \in [0, \pi], \tag{4.16}$$

and $P_k^{(\frac{N-3}{2}, \frac{N-3}{2})}$ are the Jacobi Polynomials, see [G] for example. Then, the algebraic multiplicity of the solutions to $-\Delta w = \beta_n U^{\frac{4}{N-2}} w$ that are $O(N-1)$-invariant is $n+1$. By definition of the space $\mathcal{Z}_1$, the solution $\big(0, W_{n,k}(r) Y_k(\theta_{N-2})\big)$ belongs to $\mathcal{Z}_1$ if and only if Y_k is odd with respect to x_N, that is iff $Y_h(\theta_{N-2}) = -Y_h(\pi - \theta_{N-2})$. Since the Jacobi Polynomials are *even* if k is even and *odd* if k is odd, $Y_k(\theta_{N-2})$ is odd with respect to x_N if and only if k is odd. This implies that to compute $\gamma(n)$ we only have to consider the odd indices k.

The radial part corresponding to the index k is given by

$$W_{n,k}(r) = \frac{r^k}{(1+r^2)^{k+\frac{N-2}{2}}} P_{n-k}^{(k+\frac{N-2}{2}, k+\frac{N-2}{2})}\left(\frac{1-r^2}{1+r^2}\right).$$

If $n = 2j$, we consider the operator T defined in $X_k^+ \times X_k^-$. In this way, $\frac{1}{|x|^{N-2}} \cdot W_{n,k}\big(\frac{x}{|x|^2}\big) = -W_{n,k}(x)$ since $n-k$ is odd for any k odd. Then $\gamma(n) = \sum_{k=0,\ k \text{ odd}}^{n} 1 = j$ and it is odd if and only if $j = 2\ell + 1$, or equivalently $n = 4\ell + 2$.

If, instead, n is odd, then $n-k$ is even for any k odd and so we consider the operator T defined in $X_k^+ \times X_k^+$. Indeed, in this case, $W_{n,k}(r) \in X_k^+$ for every k odd and so $\gamma(n) = j+1$ and it is odd if and only if $j = 2\ell$, equivalently $n = 4\ell + 1$ and this concludes the proof. □

Proof of Theorem 2.4. As explained before we are in position to apply Proposition 4.5 using Proposition 4.6. The expansion in (2.11) follows again from Proposition 4.5. Finally let us show that our continuum of solutions contains *nonradial* functions. If by contradiction we have that u_1 and u_2 are both radial we get that $z_2 = u_1 - u_2$ is also radial. But z_2 is odd in the last variable and so we get that $z_2 \equiv 0$. Then $u_1 = u_2$ and

by (F3)–(F4) we deduce that $F_i(\alpha, u_1, u_1) = u_1^{2^*-1}$. This implies that $u_1 = u_2 = U$, a contradiction. □

4.3. **The general case: proof of Theorem 2.6.** Since the general case involves hard notations, for reader's convenience we consider first the case $m = 2$ and prove Corollary 2.8. The general case does not involve additional difficulties and we just will sketch it.

Let h_1 (resp. h_2) be the reflection through the hyperplane $x_N = 0$ (resp. $x_{N-1} = 0$), $\mathcal{S}_2 = \langle O(N-2), h_1, h_2\rangle$ and $\sigma_2 : \mathcal{S}_2 \to \{-1, 1\}$ be the group morphism that satisfies $\sigma_2(s) = 1$ whenever $s \in O(N-2)$ and $\sigma_2(h_1) = \sigma_2(h_2) = -1$. Thus

$$\begin{aligned} X_{\mathcal{S}_2} = \big\{ v \in X \;\big|\; \forall x = (x', x_{N-1}, x_N) \in \mathbb{R}^N,\ & v(x', x_{N-1}, x_N) = v(|x'|, -x_{N-1}, x_N), \\ & v(x', x_{N-1}, x_N) = v(|x'|, x_{N-1}, -x_N)\big\}, \end{aligned}$$

$$\begin{aligned} \mathcal{Z} \equiv \mathcal{Z}_2 = \big\{ z \in X_k^+ \times X_k^\pm \;\big|\; & \forall x = (x', x_{N-1}, x_N) \in \mathbb{R}^N, \\ & z_1(x', x_{N-1}, x_N) = z_1(|x'|, -x_{N-1}, x_N), \\ & z_1(x', x_{N-1}, x_N) = z_1(|x'|, x_{N-1}, -x_N), \\ & z_2(x', x_{N-1}, x_N) = -z_2(|x'|, -x_{N-1}, x_N), \text{ and} \\ & z_2(x', x_{N-1}, x_N) = -z_2(|x'|, x_{N-1}, -x_N)\big\}, \end{aligned}$$

With this choice, arguing as in the previous case we have that the only solution in $X_{\mathcal{S}_2}$ to the first equation of (3.7) is the trivial one. As a consequence, Proposition 4.5 applies and a bifurcation occurs when $\gamma(n)$ is odd.

It remains to compute $\gamma(n)$. To do this we will compute the dimension of $\mathcal{Y}_k^{\mathcal{S}_2}(\mathbb{R}^N)$, the space of spherical harmonics on $\mathbb{R}^N$ related to the eigenvalue $k(k+N-2)$ which are invariant by the action of $\mathcal{S}$ induced by σ (thus, for $\mathcal{S} = \mathcal{S}_2$, we select the spherical harmonics which are invariant under the action of $O(N-2)$ and odd with respect to x_N and x_{N-1}).

First, let use prove the following decomposition lemma:

Lemma 4.7. *Let $\mathcal{P}^{\mathcal{S}_2}(\mathbb{R}^N)$ be the space of the polynomials in N variables which are invariant by the action of $O(N-2)$ and such that $\forall x \in \mathbb{R}^N$, $v(h_i(x)) = -v(x)$, for $i = 1, 2$. Then*

$$\mathcal{P}^{\mathcal{S}_2}(\mathbb{R}^N) = x_N x_{N-1} \mathbb{R}[r^2, x_{N-1}^2, x_N^2] \quad \text{where } r^2 = x_1^2 + \cdots + x_{N-2}^2 \tag{4.17}$$

and $\mathbb{R}[a_1, \ldots, a_k]$ denotes the space of polynomials in the variables $a_1, \ldots, a_k$.

Proof. The proof is similar as in Lemma 6.4 in [SW]. If $p(x)$ is a polynomial in $x_N x_{N-1}\mathbb{R}[r^2, x_{N-1}^2, x_N^2]$ then it has an odd degree in x_N and x_{N-1} and so it satisfies $p(h_i(x)) = -p(x)$ for $i = 1, 2$. Moreover it depends on even powers of $x_1^2 + \cdots + x_{N-2}^2$ and so it is invariant with respect to any $s \in O(N-2)$. Thus $x_N x_{N-1}\mathbb{R}[r^2, x_{N-1}^2, x_N^2] \subseteq \mathcal{P}^{\mathcal{S}_2}(\mathbb{R}^N)$.

Conversely, let $p \in \mathcal{P}^{\mathcal{S}_2}(\mathbb{R}^N)$. Since $p(h_i(x)) = -p(x)$ for $i = 1, 2$ then each term in p has to contain an odd power of x_{N-1} and x_N. We can then define the polynomial $q(x) := \frac{p(x)}{x_{N-1}x_N}$ which is even in x_{N-1} and x_N. Now let $s \in O(N-2)$ such that $s(x_1, \ldots, x_{N-2}) = (r, 0, \ldots, 0)$ with $r^2 = x_1^2 + \cdots + x_{N-2}^2$. Then q is invariant so that $q(x_1, \ldots, x_N) = q\big(s(x_1, \ldots, x_{N-2}), x_{N-1}, x_N\big) = q(r, 0 \ldots, 0, x_{N-1}, x_N) =$

$q(-r, 0 \dots, 0, x_{N-1}, x_N)$ where the last equality comes from the fact that the map $(x_1, x_2, \dots, x_{N-2}) \mapsto (-x_1, x_2, \dots, x_{N-2})$ belongs to $O(N-2)$. Then q has to be even in r and this implies that $q \in \mathbb{R}[r^2, x_{N-1}^2, x_N^2]$. □

Proposition 4.8. *With this choice of $\mathcal{S} = \mathcal{S}_2$ and $\sigma = \sigma_2$, $\gamma(n)$ is odd if and only if $n = 8\ell + 2$, $n = 8\ell + 3$, $n = 8\ell + 4$ or $n = 8\ell + 5$ for $\ell = 0, 1, \dots$*

Proof. Recall that $\mathcal{Y}_k(\mathbb{R}^N)$, the space of spherical harmonics of eigenvalue $k(k+N-2)$ for $-\Delta_{\mathbb{S}^{N-1}}$ consists of harmonic homogeneous polynomials of degree k. As stated in Proposition 5.5 of [ABR], the space $\mathcal{P}_k$ of homogeneous polynomials of degree k can be decomposed as a direct sum of $\mathcal{Y}_k(\mathbb{R}^N)$ with a subspace isomorphic to $\mathcal{P}_{k-2}$. This decomposition still holds when restricted to polynomials that are $O(N-2)$-invariant and odd with respect to x_N and x_{N-1}. This follows easily using the formula (5.6) of [ABR]. As a consequence,

$$\dim \mathcal{Y}_k^{\mathcal{S}_2}(\mathbb{R}^N) = \dim \mathcal{P}_k^{\mathcal{S}_2}(\mathbb{R}^N) - \dim \mathcal{P}_{k-2}^{\mathcal{S}_2}(\mathbb{R}^N) \tag{4.18}$$

where $\mathcal{P}_k^{\mathcal{S}_2}(\mathbb{R}^N)$ is the space of homogeneous polynomials on $\mathbb{R}^N$ of degree k which are $O(N-2)$-invariant and odd with respect to x_N and x_{N-1}.

In view of (4.18), we have to compute the dimension of $\mathcal{P}_k^{\mathcal{S}_2}(\mathbb{R}^N)$ using the decomposition in Lemma 4.7.

It is not difficult to show that for any $h \in \mathbb{N}$ we have $\mathcal{P}_{2h+1}^{\mathcal{S}_2}(\mathbb{R}^N) = \{0\}$ since any polynomial in it must contain $x_{N-1}x_N$ and powers of $x_1^2 + \dots + x_{N-2}^2$ and this is not possible if the degree of the polynomial is odd. So we have proved that $\dim \mathcal{Y}_{2h+1}^{\mathcal{S}_2}(\mathbb{R}^N) = 0$ for any h and N.

Then let us compute $\dim \mathcal{Y}_{2h}^{\mathcal{S}_2}(\mathbb{R}^N)$. Again from Lemma 4.7, we have that $\mathcal{P}_k^{\mathcal{S}_2}(\mathbb{R}^N) = \operatorname{span}\big\{x_N^{2h+1}\, x_{N-1}^{k-2\ell-2h-1}\, r^{2\ell} \;\big|\; h = 0, \dots, \frac{k-2}{2} \text{ and } \ell = 0, \dots, \frac{k-2h-2}{2}\big\}$ so that

$$\dim \mathcal{P}_k^{\mathcal{S}_2}(\mathbb{R}^N) = \sum_{h=0}^{\frac{k-2}{2}} \sum_{\ell=0}^{\frac{k-2h-2}{2}} 1 = \frac{k}{4}\Big(\frac{k}{2} + 1\Big)$$

and using (4.18) we get for k even

$$\dim \mathcal{Y}_k^{\mathcal{S}_2}(\mathbb{R}^N) = \frac{k}{4}\Big(\frac{k}{2} + 1\Big) - \frac{k-2}{4}\Big(\frac{k-2}{2} + 1\Big) = \frac{k}{2}. \tag{4.19}$$

In this case the unique spherical harmonics which contribute to the computation of $\gamma(n)$ are those of index k even. The corresponding radial part is $W_{n,k}(r)$ which belongs to X^+ if n is even and to X^- if n is odd. Then, when n is even we define the operator T in the space $X^+ \times X^+$ and we have that

$$\gamma(n) = \sum_{k=0}^{n} \dim \mathcal{Y}_k^{\mathcal{S}_2}(\mathbb{R}^N) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \dim \mathcal{Y}_{2j}^{\mathcal{S}_2}(\mathbb{R}^N) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} j = \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)$$

Then $\gamma(n)$ is odd when $n = 8j+2$ and $n = 8j+4$. When n is odd instead, we define the operator T in the space $X^+ \times X^-$ and we have again that

$$\gamma(n) = \sum_{k=0}^{n} \dim \mathcal{Y}_k^{\mathcal{S}_2}(\mathbb{R}^N) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \dim \mathcal{Y}_{2j}^{\mathcal{S}_2}(\mathbb{R}^N) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} j = \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right)$$

Then $\gamma(n)$ is odd when $n = 8j+3$ and $n = 8j+5$ concluding the proof. □

Proof of Corollary 2.8. It is the same as the one of Theorem 2.4 (using Proposition 4.8). □

Now we sketch the general case of Theorem 2.6. Let $N \geqslant 3$ and $2 \leqslant m \leqslant N-1$. For $i = 1, \dots, m$ let h_i be the reflection through the hyperplane $x_{N+1-i} = 0$, $\mathcal{S}_m = \langle O(N-m), h_1, \dots, h_m \rangle$, and $\sigma_m : \mathcal{S}_m \to \{-1, 1\}$ be the group morphism defined by $\sigma_m(s) = 1$ for $s \in O(N-m)$ and $\sigma_m(h_i) = -1$. Thus

$$\begin{aligned} X_{\mathcal{S}_m} = \big\{ v \in X_k^+ \;\big|\; & \forall x = (x', x_{N-m+1}, \dots, x_N) \in \mathbb{R}^{N-m} \times \mathbb{R}^m,\ \forall i_1, \dots, i_m \in \mathbb{N}, \\ & v(x) = v(|x'|, (-1)^{i_1} x_{N-m+1}, \dots, (-1)^{i_m} x_N) \big\}, \\ \mathcal{Z}_m = \big\{ z \in X_k^+ \times X_k^\pm \;\big|\; & \forall x = (x', x_{N-m+1}, \dots, x_N) \in \mathbb{R}^{N-m} \times \mathbb{R}^m,\ \forall i_1, \dots, i_m \in \mathbb{N}, \\ & z_1(x) = z_1(|x'|, (-1)^{i_1} x_{N-m+1}, \dots, (-1)^{i_m} x_N), \text{ and} \\ & z_2(x) = (-1)^{i_1 + \dots + i_m} z_2(|x'|, (-1)^{i_1} x_{N-m+1}, \dots, (-1)^{i_m} x_N) \big\}. \end{aligned}$$

As before we have that there is no nontrivial solution in $X_{\mathcal{S}_m}$ to the first equation of (3.7). Hence by Proposition 4.5 we only have to compute $\gamma(n)$. Analogously to the case $m = 2$ we use the following decomposition lemma:

Lemma 4.9. *Let $\mathcal{P}^{S_m}(\mathbb{R}^N)$ be the space of the polynomials in N variables which are invariant under the action of $O(N-m)$ and such that $\forall x \in \mathbb{R}^N$, $v(h_i(x)) = -v(x)$ for all $i = 1, \dots, m$. Then*

$$\mathcal{P}^{\mathcal{S}_m}(\mathbb{R}^N) = x_{N-m+1} \cdots x_N \, \mathbb{R}[r^2, x_{N-m+1}^2, \dots, x_N^2] \quad \text{where } r^2 = x_1^2 + \dots + x_{N-m}^2 \tag{4.20}$$

and $\mathbb{R}[a_1, \dots, a_k]$ denotes the space of polynomials in the variables $a_1, \dots, a_k$.

Proposition 4.10. *With this choice of $\mathcal{S} = \mathcal{S}_m$ and $\sigma = \sigma_m$, $\gamma(n)$ is odd if and only if*

$$\binom{m + \lfloor \frac{n-m}{2} \rfloor}{m} \text{ is an odd integer.}$$

Proof. As in the proof of Proposition 4.8 we have that

$$\dim \mathcal{Y}_k^{\mathcal{S}_m}(\mathbb{R}^N) = \dim \mathcal{P}_k^{\mathcal{S}_m}(\mathbb{R}^N) - \dim \mathcal{P}_{k-2}^{\mathcal{S}_m}(\mathbb{R}^N) \tag{4.21}$$

where $\mathcal{P}_k^{\mathcal{S}_m}(\mathbb{R}^N)$ is the space of homogeneous polynomials on $\mathbb{R}^N$ of degree k which are $O(N-m)$-invariant and odd with respect to $x_{N-m+1}, \dots, x_N$. Because of Lemma 4.9, all non-zero polynomials invariant under the action induced by σ on $\mathcal{S}_m$ must have degree at least m and so $\mathcal{P}_k^{\mathcal{S}_m}(\mathbb{R}^N) = \{0\}$ for $k = 0, \dots, m-1$, and $\dim \mathcal{P}_m^{\mathcal{S}_m}(\mathbb{R}^N) = 1$. Moreover, as in the case $m = 2$, $\mathcal{P}_{m+2h+1}^{\mathcal{S}_m}(\mathbb{R}^N) = \{0\}$ for any $h \in \mathbb{N}$. For $\mathcal{P}_{m+2h}^{\mathcal{S}_m}(\mathbb{R}^N)$, the decomposition in Lemma 4.9 implies that is is isomorphic to $\mathcal{P}_h(a_1, \dots, a_{m+1})$, the space

of homogeneous polynomials of degree h in $m+1$ variables. Thus $\dim \mathcal{P}^{\mathcal{S}_m}_{m+2h}(\mathbb{R}^N) = \dim \mathcal{P}_h(a_1,\dots,a_{m+1}) = \binom{h+m}{m}$. Then, using (4.21), we get

$$\dim \mathcal{Y}^{\mathcal{S}_m}_{m+2h}(\mathbb{R}^N) = \binom{h+m}{m} - \binom{h-1+m}{m} = \binom{h+m-1}{m-1}, \qquad h \in \mathbb{N}.$$

This implies that $\gamma(n) = 0$ for $n \leqslant m-1$. As when $m=2$ we get

$$\gamma(n) = \sum_{h=0}^{\lfloor \frac{n-m}{2} \rfloor} \dim \mathcal{Y}^{\mathcal{S}_m}_{m+2h}(\mathbb{R}^N) = \sum_{h=0}^{\lfloor \frac{n-m}{2} \rfloor} \binom{h+m-1}{m-1}.$$

Now we use the so called *hockey-stick* identity

$$\sum_{i=r}^{\ell} \binom{i}{r} = \binom{\ell+1}{r+1}$$

which implies

$$\gamma(n) = \sum_{h=0}^{\lfloor \frac{n-m}{2} \rfloor} \binom{h+m-1}{m-1} = \binom{m+\lfloor \frac{n-m}{2} \rfloor}{m}.$$

Finally the proof of Theorem 2.6 follows as in Theorem 2.4. □

Proof of Theorem 2.6. From Proposition 4.10 we have that $\gamma(n)$ is odd when $\binom{m+\lfloor \frac{n-m}{2} \rfloor}{m}$ is odd. Then the proof follows from Proposition 4.5. □

5. Other solutions

The use of other symmetry subgroups of $O(N)$ makes it possible to find different solutions. As an example we give another choice that generates nonradial solutions non equivalent to the previous ones.

For $m \geqslant 1$, let R_m be the rotation of angle $\frac{2\pi}{m}$ in φ, h_i the reflection with respect to $x_i = 0$, $i = 2,\dots,N$. Set $\mathcal{S}_m = \langle R_m, h_2, h_3, \dots, h_N \rangle$, and $\sigma_m : \mathcal{S}_m \to \{-1,1\}$ be the group morphism defined by $\sigma_m(R_m) = 1$, $\sigma_m(h_2) = -1$, and $\sigma_m(h_i) = 1$ for $i = 3,\dots,N$. (One easily checks that σ_m is well defined using $R_m h_2 R_m = h_2$.) Thus, using spherical coordinates, see (4.14),

$$\begin{aligned} X_{\mathcal{S}_m} = \Big\{ v \in X_k^+ \;\Big|\; \forall x = (r,\varphi,\theta_1,\dots,\theta_{N-2}) \in \mathbb{R}^N, \\ v(r,\varphi,\theta_1,\dots,\theta_{N-2}) = v\left(r, 2\pi-\varphi, \pi-\theta_1,\dots,\pi-\theta_{N-2}\right), \\ v(r,\varphi,\theta_1,\dots,\theta_{N-2}) = v\Big(r, \varphi+\frac{2\pi}{m}, \pi-\theta_1,\dots,\pi-\theta_{N-2}\Big) \Big\} \end{aligned}$$

$$\begin{aligned}\mathcal{Z} \equiv \mathcal{Z}_m = \Big\{ z \in X_k^+ \times X_k^+ \;\Big|\; \forall x = (r, \varphi, \theta_1, \dots, \theta_{N-2}) \in \mathbb{R}^N, \\ z_1(x) = z_1\left(r, 2\pi - \varphi, \pi - \theta_1, \dots, \pi - \theta_{N-2}\right), \\ z_1(x) = z_1\Big(r, \varphi + \frac{2\pi}{m}, \pi - \theta_1, \dots, \pi - \theta_{N-2}\Big), \\ z_2(x) = -z_2\left(r, 2\pi - \varphi, \pi - \theta_1, \dots, \pi - \theta_{N-2}\right), \\ z_2(x) = z_2\Big(r, \varphi + \frac{2\pi}{m}, \pi - \theta_1, \dots, \pi - \theta_{N-2}\Big)\Big\}.\end{aligned}$$

Let us show that, for any $m \geqslant 2$, the first equation in (3.7) admits only the trivial solution. By Proposition 3.1, we have that $w = \sum_{i=1}^N a_i \frac{x_i}{(1+|x|^2)^{N/2}} + bW$. By (4.14) and the definition of $X_{\mathcal{S}_m}$, we get that $a_1 = a_2 = 0$ (using the invariance with respect to R_m) and $a_3 = \cdots = a_{N-2} = 0$ (using that $\cos\theta_i \neq \cos(\pi - \theta_i)$, for any $i = 1, \dots, \theta_{N-2}$). Finally $b = 0$ since $W \notin X_k^+$. Thus the assumptions of Proposition 4.2 are satisfied. To apply Proposition 4.4, we also need:

Proposition 5.1. *Let $m \geqslant 2$, $n = m$, $\mathcal{S} := \mathcal{S}_n$ and $\sigma := \sigma_n$. Then*

$$\gamma(n) = 1. \tag{5.1}$$

Proof. By Proposition 3.1 all solutions to the second equation of (3.7) corresponding to α_m^* are given by $\sum_{k=0}^m A_k W_{m,k}(r) Y_k(\varphi, \theta_1, \dots, \theta_{N-2})$. We know from [W] (see also [AG] for another use of this expansion in bifurcation theory) that

$$\begin{aligned}&Y_k(\varphi, \theta_1, \dots, \theta_{N-2}) \\ &\quad = \sum_{\substack{j=0,\dots,k \\ i_0 \leqslant i_1 \cdots \leqslant i_{N-2} \\ i_0 = j,\ i_{N-2} = k}} \prod_{\ell=1}^{N-2} G_{i_\ell}^{i_{\ell-1}}(\cos\theta_\ell, \ell - 1) \left(B_j^{i_1 \dots i_{N-3}} \cos j\varphi + C_j^{i_1 \dots i_{N-3}} \sin j\varphi \right),\end{aligned} \tag{5.2}$$

where $G_i^0(\cdot, \ell)$ are the Gegenbauer polynomials namely,

$$\sum_{i=0}^\infty G_i^0(\omega, \ell) x^i = (1 - 2x\omega + x^2)^{-(1+\ell)/2},$$

while

$$G_i^k(\omega, \ell) = (1 - \omega^2)^{k/2} \frac{\mathrm{d}^k}{\mathrm{d}\omega^k} G_i^0(\omega, \ell).$$

By definition of the space $\mathcal{Z}_m$, the solution $\left(0, \sum_{k=0}^m A_j W_{m,k}(r) Y_k(\varphi, \theta_1, \dots, \theta_{N-2})\right)$ belongs to $\mathcal{Z}_m$ if and only if $Y_k(\varphi, \theta_1, \dots, \theta_{N-2})$ is $2\pi/m$ periodic in φ, changes sign under the transformation $\varphi \mapsto 2\pi - \varphi$, and is invariant under the transformations $\theta_i \mapsto \pi - \theta_i$. The first two imply that $Y_k(\varphi, \theta_1, \dots, \theta_{N-2})$ must not be constant in φ and $k \geqslant j \geqslant m$. Thus solutions to the second equation in (3.7) with $\mathcal{Z}_m$-invariance are multiple of $W_{m,m}(r) Y_m(\theta)$. Moreover, the unique nonzero coefficient in (5.2) is $C_m^{m \dots m}$.

Because $G_i^0(\cdot, \ell)$ is a polynomial of degree ℓ, $G_m^m(\omega, \ell)$ is a constant multiple of $(1 - \omega^2)^{m/2}$. A straightforward computation shows that

$$Y_m(\varphi, \theta_1, \dots, \theta_{N-2}) = (\sin\theta_{N-2})^m \cdots (\sin\theta_1)^m \sin(m\varphi) = \Im\mathrm{m}(x_1 + \mathbf{i}\, x_2)^m.$$

Observe that $Y_m(\varphi, \theta_1, \dots, \theta_{N-2})$ is invariant with respect to the reflection h_i for $i = 3, \dots, N$, with respect to the rotation R_m and it is odd in φ so that $(0, W_{m,m}(r)Y_m(\theta))$ belongs to $\mathcal{Z}_m$. Recalling that $W_{m,m}(r) = \frac{r^m}{(1+r^2)^{m+\frac{N-2}{2}}} \in X_k^+$, we have that $\gamma(m) = 1$. □

Proof of Theorem 2.9. From the previous discussion we have that the assumption of Lemma 4.2 are satisfied. Then the proof follows as in the case of Theorem 2.1 since we have a one dimensional kernel. □

References

[AG] A. Amadori, F. Gladiali, Nonradial sign changing solutions to Lane Emden equation, preprint

[ABR] S. Axler, P. Bourdon, W. Ramey, Harmonic Function Theory. Second edition. Graduate Texts in Mathematics, 137. Springer-Verlag, New York, 2001

[CSW] M. Chipot, I. Shafrir, G. Wolansky, On the solutions of Liouville systems, J. Diff. Eqns 140 (1997), 59-105.

[D] E.N. Dancer, Global breaking of symmetry of positive solutions on two-dimensional annuli, Differential Integral Equations, 5, (1992), no. 4, 903-913.

[DGG] E.N. Dancer, F. Gladiali, M. Grossi, On the Hardy Sobolev equation, Proc. R. Soc. Edinburgh, Section A:Mathematics

[DH] O. Druet, E. Hebey, Stability for strongly coupled critical elliptic systems in a fully inhomogeneous medium. Anal. PDE 2 (2009), 305-359.

[G] F. Gladiali, Separation of branches of $O(N-1)$-invariant solutions for a semilinear elliptic equation, preprint

[GGT] F. Gladiali, M. Grossi, C. Troesler, A non-variational system involving the critical Sobolev exponent. The radial case, to appear in Jour. Anal. Math.

[GL] Y. Guo, J. Liu , Liouville type theorem for positive solutions of elliptic systems in $\mathbb{R}^N$, Comm. Partial Diff. Eqns 33 (2008), 263–284.

[GLW] Y. Guo, B. Li, J. Wei, Entire nonradial solutions for non-cooperative coupled elliptic system with critical exponents in $\mathbb{R}^3$. J. Differential Equations 256 (2014), no. 10, 3463-3495.

[M] C.R. Menynk, Nonlinear pulse propagation in birefringent optical fibers, IEEEEJ. Quant. Electr. 23 (1987), 174-176

[K] H. Kielhöfer, Bifurcation theory. An introduction with applications to partial differential equations. Second edition. Applied Mathematical Sciences, 156. Springer, New York, 2012.

[ST] D. Siegel, E. Talvila, Pointwise growth estimates of the Riesz potential. Dynam. Contin. Discrete Impuls. Systems 5 (1999), 185-194.

[SW] J. Smoller, A. Wasserman, Bifurcation and symmetry-breaking, Invent. Math. 100, (1990), 63-95.

[W] N. Wheeler, Algebraic theory of spherical harmonics Reed College Physics Department (1996).

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