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A Graded Quadrivalent Logic for Ordinal Preference Modelling: Loyola–Like Approach

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Abstract. We extend a quadrivalent logic of Belnap to graded truth values in order to handle graded relevance of positive and negative arguments provided in preferential information concerning ranking of a finite set of alternatives. This logic is used to design the preference modelling and exploitation phases of decision aiding with respect to the ranking problem. The graded arguments are presented on an ordinal scale and their aggregation leads to preference model in form of four graded outranking relations (true, false, unknown and contradictory). The exploitation procedure combines the min-scoring procedure with the leximin rule. Aggregation of positive and negative arguments as well as exploitation of the resulting outranking relations is concordant with an advice given by St. Ignatius of Loyola (1548) "how to make a good choice".

Keywords: Decision making, preference modelling, quadrivalent logic, possibility theory

1. Introduction

When ranking a finite set of alternatives from the best to the worst, one has to take into account all relevant arguments with respect to preference of one alternative over another. The arguments are, in general, conflicting and may represent two different attitudes: for a pair of alternatives (a, b), the arguments can be in favour (positive) or in disfavour (negative) of the preference $a \succcurlyeq b$. These arguments constitute a preferential information that serves to build a preference model. The preference model should then be exploited in order to recommend a ranking of alternatives from the best to the worst (Roy (1985)).

The modelling and exploitation of the preferences are two main preoccupations of scientific decision aiding. The two phases should be compatible with the kind of preferential information. Depending on decision situation and on the decision maker(s), this information can be expressed through criteria, binary relations or arguments. Moreover, it can be either cardinal or ordinal.

The aim of this article is to design the preference modelling and exploitation phases of decision aiding with respect to the ranking problem. The distinctive characteristics of the proposed approach can be recapitulated in the following points:

– preferential information has a form of positive and/or negative arguments,
– the arguments have graded relevance depending e.g. on the source of information they come from,
– in the consequence of graded relevance, the arguments are presented on an ordinal scale,

– the aggregation of positive and negative arguments leads to preference model in form of four graded outranking relations $(\mathsf{S}^T, \mathsf{S}^F, \mathsf{S}^U, \mathsf{S}^K)$ corresponding to four truth values of Belnap (Belnap (1976), Belnap (1977)): T (true), F (false), U (unknown) and K (contradictory),
– the exploitation procedure refines the min–scoring procedure by combining the score with the leximin rule; the four outranking relations are exploited sequentially, in the following order: $\mathsf{S}^T, \mathsf{S}^F, \mathsf{S}^K, \mathsf{S}^U$.

Section 2 motivates the design of this "new" method. In Section 3, we briefly sketch the quadrivalent logic (Doherty et al. (1992), Tsoukiàs (1996)) which is then adapted to cope with ordinal relevance of arguments. Section 5 considers the context of preference modelling and Section 6 is devoted to exploitation of the preference model. In Section 7, examples are given. In the concluding section it is shown that the approach proposed in this paper is concordant with an advice given by St. Ignatius of Loyola (1548) "how to make a good choice" in his Spiritual Exercises.

2. Motivations

When preferences are modelled in terms of binary relations, the key question is the existence of evidence in favour of the considered relation. For example, for S being an outranking relation ($a\mathsf{S}b$ iff a is at least as good as b), the evidence concerns the sentence $a\mathsf{S}b$ and/or $b\mathsf{S}a$, for any pair of alternatives $a, b \in A$, where A is a finite set of alternatives. Having the information on evidence of $a\mathsf{S}b$ and $b\mathsf{S}a$, one can conclude the preference of a over b ($a\mathsf{P}b$), the indifference ($a\mathsf{I}b$) and the incomparability $a\mathsf{R}b$ as follows: (Roubens and Vincke (1985))

$$a\mathsf{P}b \iff a\mathsf{S}b \quad \wedge \quad b\not\mathsf{S}a \tag{1}$$

$$a\mathsf{I}b \iff a\mathsf{S}b \quad \wedge \quad b\mathsf{S}a \tag{2}$$

$$a\mathsf{R}b \iff a\not\mathsf{S}b \quad \wedge \quad b\not\mathsf{S}a \tag{3}$$

where $\not\mathsf{S}$ denotes non–existence of evidence in favour of $a\mathsf{S}b$ and $\wedge$ means the conjunction of both facts.

In reality, the evidence is never complete, thus inducing a graded (fuzzy) relation $a\mathsf{S}b$, i.e. "a is at least as good as b, up to a certain degree of certainty". The preference relations $\mathsf{P}, \mathsf{I}, \mathsf{R}$ become fuzzy ones, but can still be determined by equations (1–3). The conjunction can be modelled by a wide range of operators, e.g. the t-norms (Dubois and Prade (1982)). The most well-known t-norm is the "*minimum*" operator, which has also been advocated in the MCDA field (Sen (1986), Pirlot (1995)).

Yet, it is reasonable to claim that both of these ways of considering evidences are not able to catch the reality of some decision problems. In fact, in these approaches, the evidence in disfavour of a sentence is semantically considered — and thus modelled — as the evidence in favour of the opposite sentence.This mental restriction may induce not only misunderstandings but, even more important, it may also imply some loss of information.

To clarify these critical statements, let us consider an example. In a government composition, one has to choose a minister from among several candidates — let us call them *alternatives*. Each candidate has his(her) own skills, which can be considered as pieces of evidence in favour of this candidate. However, each candidate may also have drawbacks. Those are not only his(her) missing skills but also his(her) personal enemies. As far as the missing skills are concerned, they can be interpreted as advantages of some other competitors. It is quite different for enemies. As a matter of fact, the enemies of one candidate are not necessarily friends of some other ones. More precisely, if candidate a is chosen, then the enemies of a will be unsatisfied; if a has not been chosen, the personal hostility against a has no further consequence.

We wish to emphasize that negative arguments are resulting from information actively against an alternative. Arguments (positive or negative) may arise either in front of a single alternative or in front of a pair of alternatives. In section 5, we clarify this point, in a short discussion about "absolute" and "relative" arguments.

This kind of practical situations motivate the design of approaches considering both kinds of information: the arguments in favour of a preference and those in disfavour of it. They will be also called positive and negative, respectively.

A special logic — called quadrivalent logic (Tsoukiàs (1996)) — is useful to deal with both positive and negative arguments in preference modelling. Since we wish to handle graded pieces of evidence, we have to introduce a graded version of this logic.

In the next Section, we recall the basis of the quadrivalent logic and then we present our proposal for a graded quadrivalent logic. Finally, we make a brief comparison with another approach of this kind (Perny and Tsoukiàs (1998)).

3. Quadrivalent Logic – Belnap and DDT

3.1. Four Truth Values

The classical (or Boolean) logic is concerned with predicates (or sentences) that are either true or false. The truthfulness of the sentence ϕ is known and precise. Even in its graded (or fuzzy) version, the matter of this logic is "state of truth" or "reality" of ϕ. In other words, the classical logic estimates the degree of truth of the statement ϕ and the final conclusion can be either "ϕ is true" or "ϕ is false" (eventually to some degree).

If, however, one would like to investigate the sentence ϕ by searching for arguments in favour or disfavour of ϕ, then four possible states of knowledge about ϕ should be considered:

– state of ignorance, i.e. no argument in favour of ϕ nor in disfavour of ϕ is known,

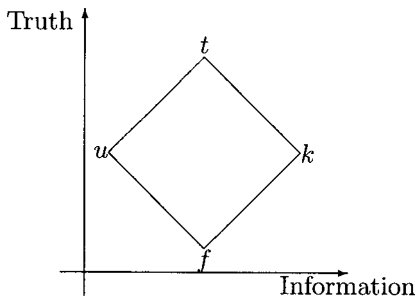

Figure 1. The bilattice of Belnap.

– state of truthfulness, i.e. only arguments in favour of ϕ are known,
– state of falsity, i.e. only arguments in disfavour of ϕ are known,
– state of contradiction, i.e. both arguments in favour and disfavour of ϕ are known.

These states of knowledge about ϕ correspond to four truth values introduced by Belnap (Belnap (1976), Belnap (1977)).

The four truth values (t, k, u, f)[1] have been ordered by Belnap on a non trivial interlaced bilattice. This structure results from the conjunction of "information" and "truth" lattices (see Figure 1).

The quadrivalent logic of Belnap has been made operational by Doherty, Driankov and Tsoukiàs (Doherty et al. (1992)) within a paraconsistent first order four valued logic called DDT logic. It has further been considered for preference modelling in the context of multicriteria decision making (Greco et al. (1998), Tsoukiàs and Vincke (1997)). In the next section we are recalling the basis of the DDT logic.

Before entering into the details, let us clarify some terminological issues. To state about the truthfulness or the falsity of a given sentence (also called "predicate" in logic), one collects pieces of information. The latter provide arguments that help to build the evidence supporting (or not) that sentence. The state of conviction evolves as knowledge is being enriched, i.e. arguments are supplied.

3.2. Quadrivalent Logic

In DDT, the unary connectives are negation-minded. There are considered three types of negations: strong ($\neg$), weak ($\not\sim$) $\mathcal{A}$and complementation ($\sim$). These negations are described by algebraic properties and summarized in Table 1. One can induce from this table the following interpretation of the negations: the strong negation swaps the values along the "truth" dimension; the weak negation reverses the existence of negative arguments and the complementation operator reverses the existence of both kinds of arguments.

The binary connectives encode conjunction ($\wedge$), disjunction ($\vee$) and implication ($\rightarrow$) and their truth tables are given in Table 2. One should notice that the semantics of the implication $\alpha \rightarrow \beta$ is "the complement of α or β": $\sim \alpha \vee \beta$.

Table 1. The truth table of $\not\sim$, $\sim$ and $\neg$.

α	$\not\sim\alpha$	$\sim\alpha$	$\neg\alpha$
t	k	f	f
k	t	u	k
u	f	k	u
f	u	t	t

Table 2. The truth tables of $\wedge$, $\vee$ and $\rightarrow$.

$\wedge$	t	k	u	f
t	t	k	u	f
k	k	k	f	f
u	u	f	u	f
f	f	f	f	f

$\vee$	t	k	u	f
t	t	t	t	t
k	t	k	t	k
u	t	t	u	u
f	t	k	u	f

$\rightarrow$	t	k	u	f
t	t	k	u	f
k	t	t	u	u
u	t	k	t	k
f	t	t	t	t

Table 3. Definition of the strong unary operators.

$T\alpha$	$\equiv$	α	$\wedge$	$\sim\neg\alpha$
$K\alpha$	$\equiv$	$\not\sim\alpha$	$\wedge$	$\not\sim\neg\alpha$
$U\alpha$	$\equiv$	$\sim\not\sim\alpha$	$\wedge$	$\sim\not\sim\neg\alpha$
$F\alpha$	$\equiv$	$\sim\alpha$	$\wedge$	$\neg\alpha$

Table 4. Truth table of the strong unary operators.

α	$T\alpha$	$K\alpha$	$U\alpha$	$F\alpha$
t	t	f	f	f
k	f	t	f	f
u	f	f	t	f
f	f	f	f	t

3.3. Two-Valued Fragment

The most interesting tool to produce sound results in this logic is the two-valued fragment, created by means of strong unary operators encoding truthfulness, contradiction, ignorance and falsity of sentence α. For example, $T\alpha$ encodes the truthfulness of α defined as $\alpha\wedge\sim\neg\alpha$. Other operators are defined in Table 3 and Table 4 provides their truth table.

4. Graded Quadrivalent Logic – Loyola-Like Approach

In the previous context, the sources of information are assumed to be equally relevant. Therefore all the evidences enjoy the same strength. However, first, it is more realistic to rank the information sources according to their respective relevance, and second, the

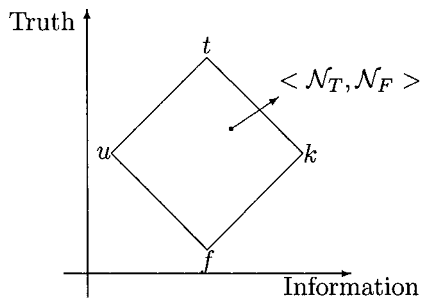

Figure 2. The graded interlaced bilattice.

arguments issued by the same source can have different convincing power. This motivates the consideration of graded evidences taking values from the interval [0, 1]; while 1 corresponds to full evidence, 0 signifies no evidence at all. We admit that the degree (e.g. of relevance) has an ordinal meaning only.

4.1. Four Graded Truth Values

In order to handle graded evidences on an ordinal scale, one can use the degree of necessity proposed within possibility theory (Dubois and Prade (1988)). Each point of the interlaced bilattice representing the four truth values (see Figure 2) is then characterized by a pair $< \mathcal{N}_T, \mathcal{N}_F >$ of necessity degrees of truthfulness and falsity, respectively. Each necessity degree accumulates the ordinal strength of the related evidences, in the way described below.

When a new argument of strength σ in favour of the truthfulness of a sentence ϕ is raised, the resulting necessity of the truthfulness of ϕ is set equal to max $\{\mathcal{N}_T(\phi), \sigma\}$, where $\mathcal{N}_T(\phi)$ is the necessity of the truthfulness of ϕ before the new argument has been raised. On the other hand, argument of strength ρ against ϕ leads to an update of the same type:

$$\mathcal{N}_T \leftarrow \max\{\mathcal{N}_T, \sigma\} \tag{4}$$

$$\mathcal{N}_F \leftarrow \max\{\mathcal{N}_F, \rho\} \tag{5}$$

Let us stress that, in our context, an argument (in favour or in disfavour of a sentence ϕ) results from a piece of information. A given piece of information may provide different arguments. Redundant or repetitive pieces of information can bring stronger or new arguments, due to this repetition. In fact, the latter is also a piece of information, eventually providing argument.

However, one should take care that the same piece of information is not reused to artificially reinforce its related arguments. Because of its idempotency (Fodor and Roubens (1994)), the max-operator in equations (4–5) prevents such a phenomenon.

4.2. Graded Four-Valued Logic

Consistently with definitions of the negation and connection operators introduced in paragraph 3.2, and in order to generalize in an ordinal way the truth table shown in Tables 1 and 2, we propose the following extensions.

For any sentences α and β, characterized by the pairs of necessity degrees $< \mathcal{N}_T(\alpha), \mathcal{N}_F(\alpha) >$ and $< \mathcal{N}_T(\beta), \mathcal{N}_F(\beta) >$, respectively, we have:

$$\neg\alpha \ : < \mathcal{N}_F(\alpha), \mathcal{N}_T(\alpha) > \tag{6}$$

$$\not\sim \alpha \ : < \mathcal{N}_T(\alpha), 1 - \mathcal{N}_F(\alpha) > \tag{7}$$

$$\sim \alpha \ : < 1 - \mathcal{N}_T(\alpha), 1 - \mathcal{N}_F(\alpha) > \tag{8}$$

and

$$\alpha \wedge \beta \ : < \min\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}, \max\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\} > \tag{9}$$

$$\alpha \vee \beta \ : < \max\{\mathcal{N}_T(\alpha), \mathcal{N}_T(\beta)\}, \min\{\mathcal{N}_F(\alpha), \mathcal{N}_F(\beta)\} > \tag{10}$$

The necessity of truthfulness of the conjunction of α and β is computed consistently with the possibility theory (Dubois and Prade (1988)): for any pair of events A (e.g. sentence α is true) and B (sentence β is true),

$$\mathcal{N}(A \cap B) = \min\{\mathcal{N}(A), \mathcal{N}(B)\}.$$

Semantically, it encodes that arguments in favour of $\alpha \wedge \beta$ have to be in favour of α and in favour of β. As far as disjunction is concerned, one has

$$\mathcal{N}(A \cup B) \geq \max\{\mathcal{N}(A), \mathcal{N}(B)\}.$$

The least specific — i.e. the least imperative — necessity degree satisfying this relation defines the necessity of truthfulness of the disjunction of α and β. Since the lowest necessity degree is the least specific one[2], we have

$$\mathcal{N}(A \cup B) = \inf\{\mathrm{z} \mid \mathrm{z} \geq \max\{\mathcal{N}(A), \mathcal{N}(B)\}\} = \max\{\mathcal{N}(A), \mathcal{N}(B)\}.$$

The principle of *minimum specificity* (Dubois and Prade (1991)) can be reasoned in our context, since it implies a non–informative operation. All the arguments about each sentence α, β are known and no new argument will emerge from joint knowledge of both sentences.

4.3. Graded Two-Valued Fragment

The graded four-valued logic will also be made operational by means of a two-valued fragment (see Section 3.3). One should notice that the "two values" of this fragment are graded in an ordinal way, i.e. "true or false, up to a certain degree".

We define the two-valued fragment operators consistently with the definitions of Table 3 as well as Equations (6–10). For example, we have

$$T\alpha : \alpha \wedge \sim \neg\alpha$$
$$: < \min\{\mathcal{N}_T(\alpha), 1 - \mathcal{N}_F(\alpha)\}, \max\{1 - \mathcal{N}_T(\alpha), \mathcal{N}_F(\alpha)\} >$$

Since the two necessity degrees of Tα are complementary (i.e. they always sum up to 1), we can drop the necessity degree of falsity. Similarly for all the strong unary operators,

$$T\alpha \ : \ \min\{\mathcal{N}_T(\alpha), 1 - \mathcal{N}_F(\alpha)\} \tag{11}$$

$$K\alpha \ : \ \min\{\mathcal{N}_T(\alpha), \mathcal{N}_F(\alpha)\} \tag{12}$$

$$U\alpha \ : \ \min\{1 - \mathcal{N}_T(\alpha), 1 - \mathcal{N}_F(\alpha)\} \tag{13}$$

$$F\alpha \ : \ \min\{1 - \mathcal{N}_T(\alpha), \mathcal{N}_F(\alpha)\} \tag{14}$$

Therefore, we really come to a two-valued fragment, since each sentence $T\alpha$, $K\alpha$, $U\alpha$ or $F\alpha$ is characterized by one degree. They can be modelled by a unique lattice.

4.4. Another Approach

Perny and Tsoukiàs (1998) have also presented a continuous extension of the four-valued logic for the same context of preference modelling. However, both approaches are strongly different.

First of all, in the framework of (Perny and Tsoukiàs (1998)), the arguments in favour (resp. in disfavour) of a predicate S are contained in the set S^+ (resp. S^-) of the models of S (resp. of $\neg S$, the negation of S). Let us mention that a model of S is, in our terms, an argument in favour of S. The measures of the sets S^+ and S^- correspond to our necessity degrees $< \mathcal{N}_T(\alpha), \mathcal{N}_F(\alpha) >$.

Perny and Tsoukiàs consider a collection of set-theoretic properties that the Belnap's logic enjoys and they try to extend these properties to fuzzy membership degrees. For example, they require that

$$S^t \cup S^k = S^+$$

which gives in our notation

$$T\alpha \vee K\alpha : \mathcal{N}_T(\alpha)$$

This means that all the arguments in favour of a sentence α have to be exploited by the strong operators T and K. No erosion can occur in the information in favour of alpha, even if there are present arguments against α.

One could think first to transform this relation using a De Morgan triple (t, s, n). The conjunction, the disjunction and the negation are then modelled by means of a t-norm, a t-conorm and a negation (Fodor and Roubens (1994)). This leads to the following functional equation

$$x = s(t(x, n(y)), t(x, y)) \tag{15}$$

which holds in the crisp case. However, it has no solution in the fuzzy case as shown in (Alsina (1985)). In other words, there doesn't exist a De Morgan triple (t, s, n) such that (15) holds for any pair of values $(x, y) \in [0, 1]^2$.

According to Fodor and Roubens (1994) [p.73], one can relax (15) to get a feasible functional equation, by using the minimum t-norm for conjunction and a Łukasiewicz triple (LT,LS,LN) for the rest of the relation. Therefore, Perny and Tsoukiás reformulate their framework to rely on two different conjunction operators. In other words, the conjunction action will be differently performed according to the context. For example, conjunction in the definition of $T\alpha$ and $K\alpha$ will not be carried out similarly.

Moreover, the solution they proposed is unique *modulo ψ an automorphism of* [0, 1]. It seems that no obvious reason (apart from the sake of simplicity, which implies $\psi(x) = x$) can be pointed out to choose one specific automorphism. In any case, the information from the arguments is not considered as purely ordinal. It is assumed to be cardinal, since

$$LT(x, y) = \max(x + y - 1, 0).$$

In our approach, we focus on the meaning of the necessity degrees and on the interpretation of the approach in terms of graded arguments in favour or in disfavour of a sentence ϕ. Instead of an analytical perspective, our semantic point of view results in an ordinal framework where the min and max operators command attention. By the way, the *minimum specificity principle* brings on a kind of erosion, since we have now

$$\begin{aligned} T\alpha \vee K\alpha &= \max(\min(\mathcal{N}_T(\alpha), 1 - \mathcal{N}_F(\alpha)), \min(\mathcal{N}_T(\alpha), \mathcal{N}_F(\alpha))) \\ &= \min(\mathcal{N}_T(\alpha), \max(\mathcal{N}_F(\alpha), 1 - \mathcal{N}_F(\alpha))) \\ &\leq \mathcal{N}_T(\alpha). \end{aligned}$$

Our disjunction operator "max" is not able to rebuild the total value of $\mathcal{N}_T(\alpha)$. But its choice has been advocated by the minimum specificity principle. In fact, the discrepancy between $T\alpha$ or $K\alpha$ and $\mathcal{N}_T(\alpha)$ can be used to measure the consistency of the information as well as the robustness of the arguments. To our mind, $T\alpha \vee K\alpha$ quantifies the presence of truthfulness in α, which, because of the negative arguments, is less than the total amount of positive arguments.

To complete this discussion, it is worth noting that other kinds of "truth" measures could be used (e.g., probabilities and belief functions). The choice of necessity degrees is advocated by the information kind we want to cope with. The arguments in favour or in disfavour of a sentence ϕ are ordinal, i.e. the only assumption we make is that the arguments can be ordered with respect to their relevance.

In our opinion, stronger requirements can mislead the undergone negotiations. People would focus then on the exact value of argument strength, instead of looking efficiently for arguments and pieces of evidence.

5. Preference Modelling

The pairwise comparison of alternatives a and b regards two particular sentences "a is at least as good as b" $(a\mathsf{S}b)$ and "b is at least as good as a" $(b\mathsf{S}a)$. The truthfulness of both sentences can be endorsed by the evaluation of a and b on criteria, by explicit preference structures or, more generally, by arguments. Each sentence can be either corroborated or invalidated by arguments in favour or in disfavour of the sentence, respectively.

In other words, when comparing a pair of alternatives a and b, one may answer to four different and generally independent queries: are there arguments favourable or unfavourable to $a\mathsf{S}b$ or to $b\mathsf{S}a$. More often than not, one assumes that arguments in disfavour of a sentence (e.g. $a\mathsf{S}b$) are identical to arguments in favour of the opposite sentence (e.g. $b\mathsf{S}a$). However, the complete set of four questions cannot be reduced without any loss of generality.

In some contexts, this interrogative principle could be thought as similar to comparison of each alternative to a neutral point $\mathcal{Z}$. Here, the arguments in favour or in disfavour of $a\mathsf{S}b$ are called "absolute" and state to what extent a is more acceptable or more unacceptable to $\mathcal{Z}$, than b (and vice versa for $b\mathsf{S}a$). This assumption of a neutral point may be meaningful. It conveys the ability of the DM to have no inclination for one alternative over others. It doesn't mean that the point $\mathcal{Z}$ is somewhere in the centre of the alternative set A. The position of $\mathcal{Z}$ is actually unknown but it is assumed to exist.

In other contexts, where the neutral point assumption is useless, the arguments do involve only the considered alternatives. In this case, arguments can be called "relative", since they regard a pair of alternatives.

Let us assume now that arguments both in favour and in disfavour of both sentences $a\mathsf{S}b$ and $b\mathsf{S}a$ have been brought forward. Then, each sentence can be characterized by its degrees of truthfulness and falsity:

$$a\mathsf{S}b \ : < \mathcal{N}_T(a\mathsf{S}b), \mathcal{N}_F(a\mathsf{S}b) > \tag{16}$$

$$b\mathsf{S}a \ : < \mathcal{N}_T(b\mathsf{S}a), \mathcal{N}_F(b\mathsf{S}a) > \tag{17}$$

Table 5. Ten preference situations: I means indifference, K – ambiguity, U – ignorance, R – incomparability, P – strict preference; H – weak preference, J – semi preference, L – semi-weak preference; Q – weak incomparability and V – semi incomparability.

	$b\mathsf{S}^T a$	$b\mathsf{S}^K a$	$b\mathsf{S}^U a$	$b\mathsf{S}^F a$
$a\mathsf{S}^T b$	$a\mathrm{I}b$	$a\mathrm{H}b$	$a\mathrm{J}b$	$a\mathrm{P}b$
$a\mathsf{S}^K b$	$b\mathrm{H}a$	$a\mathrm{K}b$	$a\mathrm{L}b$	$a\mathrm{Q}b$
$a\mathsf{S}^U b$	$b\mathrm{J}a$	$b\mathrm{L}a$	$a\mathrm{U}b$	$a\mathrm{V}b$
$a\mathsf{S}^F b$	$b\mathrm{P}a$	$b\mathrm{Q}a$	$b\mathrm{V}a$	$a\mathrm{R}b$

From the degrees, we may build *four outranking relations*. For each pair of alternatives $(a, b) \in A^2$, we have

$$a\mathsf{S}^T b = T(a\mathsf{S}b) \ : \ \min(\mathcal{N}_T(a\mathsf{S}b), 1 - \mathcal{N}_F(a\mathsf{S}b)) \tag{18}$$

$$a\mathsf{S}^K b = K(a\mathsf{S}b) \ : \ \min(\mathcal{N}_T(a\mathsf{S}b), \mathcal{N}_F(a\mathsf{S}b)) \tag{19}$$

$$a\mathsf{S}^U b = U(a\mathsf{S}b) \ : \ \min(1 - \mathcal{N}_T(a\mathsf{S}b), 1 - \mathcal{N}_F(a\mathsf{S}b)) \tag{20}$$

$$a\mathsf{S}^F b = T(a\mathsf{S}b) \ : \ \min(1 - \mathcal{N}_T(a\mathsf{S}b), \mathcal{N}_F(a\mathsf{S}b)) \tag{21}$$

Obviously, we have similar relations for the reverse pair (b, a).

The semantic of these relations is easy to understand. For example, relation S^T encodes how much we can be confident in the truthfulness of $a\mathsf{S}b$; meanwhile, relation S^K carries the contradictions about the same preference.

6. Exploitation of the Preference Model

In the crisp case, Tsoukiàs and Vincke (1995) proposed to identify 10 preference situations, ranging from indifference and incomparability to strict preference. These preference situations (see Table 5) refine the interpretation and the analysis of the DM's preferences. The same 10 situations can be considered in the graded case.

6.1. Exploitation

This very general approach to exploitation of the preference model is, however, quite uneasy to apply in practical context. It is not obvious, indeed, how to workout a recommendation for the DM on the basis of the ten graded preference relations. In order to make the exploitation procedure more operational, we propose a simpler approach based on a scoring procedure.

First of all, the preference of a over b is well-grounded if there is truthfulness in the outranking of a over b (i.e. $a\mathsf{S}^T b$) and falsity in the outranking of b over a (i.e. $b\mathsf{S}^F a$). For each alternative a, we compute its score according to the outranked alternatives w.r.t. S^T and to the outranking alternatives w.r.t. S^F. In other words, the score of a is estimated from the vector

$$\bar{a} = (u_b, u_c, \dots, v_b, v_c, \dots)$$

where u_b (resp. v_b) is the degree of $a\mathsf{S}^T b$ (resp. $b\mathsf{S}^F a$). One denotes

$$\bar{a} \equiv (a_1, \dots, a_n)$$

where $n/2$ is equal to the number of alternatives different from a, i.e. $n = 2(|A| - 1)$.

The score is used to propose a ranking ($\succcurlyeq$) from the high-scored alternative to the low-scored ones. If one needs to break ties in the ranking based on S^T and S^F, one can use the two remaining outranking relations. However, since we prefer clear situations, we will prefer alternatives with the lowest score according to S^K or to S^U. To work with S^K or with S^U is a choice that depends on the DM's spirit. Does he(she) prefer to have arguments, even contradictory, or to be ignorant?

6.2. Scoring

The leftover problem is the scoring procedure. Since we have worked in an ordinal context, we would like to use an ordinal scoring procedure. Typically, the min-scoring is a good candidate (Sen (1986), Pirlot (1995), Bouyssou and Pirlot (1997)). However, it lacks discriminating power. This phenomenon, also called "drowning effect", has led to some refinements of this approach among which the leximin procedure is the most appealing (Behringer (1981), Sen (1986), Behringer (1990), Dubois et al. (1996), Dubois and Fortemps (1999))[3].

In the comparison of two alternatives a and b, the problem boils down to a meaningful comparison of their respective evaluation vectors $\bar{a}$ and $\bar{b}$. The classical min-scoring procedure declares that a is preferred to b, if the smallest element of $\bar{a}$ is greater than the smallest element of $\bar{b}$:

$$a \succcurlyeq b \text{ if and only if } \min_{i=1,\dots,n} a_i \geq \min_{i=1,\dots,n} b_i$$

In other words, the elements of $\bar{a}$ (resp. of $\bar{b}$) are ranked in an increasing way to build the vector $\vec{a}$ (resp. $\vec{b}$). The elements of $\vec{a}$ are denoted $(a_{[1]}, \dots, a_{[n]})$. Therefore, one has

$$\min_{i=1,\dots,n} a_i = a_{[1]} \leq \dots \leq a_{[n]} = \max_{i=1,\dots,n} a_i$$

and, for the min-scoring procedure,

$$a \succcurlyeq b \text{ if and only if } a_{[1]} \geq b_{[1]}.$$

The leximin-scoring refines the previous approach. It gives the same strict preferences than the min-scoring but may help to break some ties. The idea is to discard the first common elements of $\vec{a}$ and $\vec{b}$ and make the decision on the base of the first different elements of $\vec{a}$ and $\vec{b}$:

$$a \succcurlyeq b \text{ if and only if } \exists j \leq n : \forall i < j : a_{[i]} = b_{[i]} \text{ and } a_{[j]} \geq b_{[j]}.$$

7. Illustrative Examples

First of all, one of the two objectives of our proposed method is to ease the DM's interview. As already illustrated in the government composition example, the comparison of two alternatives does not always boil down to the arguments in favour of *a* versus *b* and in favour of *b* versus *a*. It is sometimes very meaningful to ask also about the drawbacks of each alternative in front of the other. The following examples will try to point out this new issue: project negotiation in a company, voting in a competition and soccer championship.

7.1. Project Negotiation

When comparing two candidate projects, the different members of the company may have various opinions. It makes sense to specify both the advantages — what should it bring to us, to our customers ? — and the drawbacks — what are the bad impacts on the environment ? what are the costs ? — of projects. Each piece of evidence is valued by managers from different levels of hierarchy in the company. This gives a graded-argument decision framework.

7.2. Government Composition

The following scheme may occur in a wide bunch of contexts, ranging from political voting to TV-program selection by a family. In each case, the alternatives have their absolute or relative advantages and drawbacks, pros and cons, . . .

Let us come back to our political example outlined in the introduction. A three-party coalition is negotiating the new government composition. The discussion is currently focusing on appointment of the minister of education. Five possible candidates are first ranked according to their competencies by the three presidents of the coalition parties — $P1$, $P2$ and $P3$. Besides the competence issues, the presidents have also to consider the "political tension" inside the coalition that can result from the appointment of one candidate. This means that the candidates are also ranked according to the estimated number of other appointed ministers who do not share the political opinions of the

Table 6. Candidates $\{a, b, c, d, e\}$ ranked by presidents $\{P1, P2, P3\}$.

	Competencies				Political tension		
	$P1$	$P2$	$P3$		$P1$	$P2$	$P3$
a	1	4	4	a	5	2	3
b	2	2	2	b	4	4	4
c	3	1	2	c	3	5	4
d	4	3	1	d	2	2	5
e	5	5	5	e	1	1	1

candidate and who could thus inconvenience the governmental cooperation. In both kinds of ranking, ex-aequos are allowed. Table 6 provides the two different rankings of the candidates by each of three presidents.

Clearly the competency-based ranking gives argument in favour of $a\mathsf{S}b$; the ratio of presidents that agree to say that a is at least as good as b is equal to the grade of this positive argument. The argument in disfavour of $a\mathsf{S}b$ follows from the ranking with respect to induced political tension; its grade is equal to the ratio of presidents finding that a could induce a greater tension than b (see Table 7).

The four outranking relations $\mathsf{S}^T, \mathsf{S}^K, \mathsf{S}^U, \mathsf{S}^F$ are build from these arguments. They are written in Table 8.

From these outranking relations, we can "leximin-score" the alternatives according to S^T and S^F. For example, alternative a will be characterized with the elements of the first row of S^T — these are the truth degrees of $a\mathsf{S}.$ — as well as of the first column of S^F — those are the false degrees of $.\mathsf{S}a$. In fact, each alternative is scored by the increasingly ordered set of these elements (see Table 9).

From this table, one can derive the following weak ranking — denoted $\succcurlyeq$ — among the alternatives:

$$c \succcurlyeq b \succcurlyeq \begin{matrix} a \\ \\ d \end{matrix} \succcurlyeq e$$

If one needs a strict ranking, the contradictory (S^K) or the unknown (S^U) outranking relation may be used to break up the ties, according to the risk-attitude of the DM. For this

Table 7. The arguments in favour and in disfavour of $a\mathsf{S}b$.

	$\mathcal{N}_T(.\mathsf{S}.)$						$\mathcal{N}_F(.\mathsf{S}.)$				
	a	b	c	d	e		a	b	c	d	e
a	3/3	1/3	1/3	1/3	3/3	a	0	2/3	2/3	1/3	0
b	2/3	3/3	2/3	2/3	2/3	b	2/3	0	2/3	1/3	0
c	2/3	2/3	3/3	2/3	3/3	c	1/3	1/3	0	1/3	0
d	2/3	1/3	1/3	3/3	3/3	d	2/3	2/3	2/3	0	0
e	0	0	0	0	3/3	e	3/3	3/3	3/3	3/3	0

Table 8. Outranking relations.

S^T	a	b	c	d	e
a	3/3	1/3	1/3	1/3	3/3
b	1/3	3/3	1/3	2/3	3/3
c	2/3	2/3	3/3	2/3	3/3
d	1/3	1/3	1/3	3/3	3/3
e	0	0	0	0	3/3

S^K	a	b	c	d	e
a	0	1/3	1/3	1/3	0
b	2/3	0	2/3	1/3	0
c	1/3	1/3	0	1/3	0
d	2/3	1/3	1/3	0	0
e	0	0	0	1/3	0

S^U	a	b	c	d	e
a	0	1/3	1/3	2/3	0
b	1/3	0	1/3	1/3	0
c	1/3	1/3	0	1/3	0
d	1/3	1/3	1/3	0	0
e	0	0	0	0	0

S^F	a	b	c	d	e
a	0	2/3	2/3	1/3	0
b	1/3	0	1/3	1/3	0
c	1/3	1/3	0	1/3	0
d	1/3	2/3	2/3	0	0
e	3/3	3/3	3/3	3/3	0

example, we choose to use the S^K relation, modelling a preference for the least contradictory situation. For example, alternative a is evaluated by the row and the first column of S^K, describing how much the situation of a is contradictory. The vector of each alternative (a or d) is then ordered in a decreasing way. This gives

$$d \succcurlyeq a$$

according to a *leximax* comparison procedure (this procedure is obviously the dual of the leximin one); see Table 10.

7.3. Soccer Championship

Our last example tries to advocate a new proposal for the ranking of soccer teams in a national (or international) championship. Classically, the information available is the results of each match A *vs* B, where A and B are any distinct team in the championship. Therefore, we have for each pair of teams (A, B), the results of both matches A *vs* B and B *vs* A. Denote by x_{AB} (resp. y_{AB}) the score of the host team A (resp. the visiting team B) in the match A *vs* B.

This example is characterized by "relative" arguments in favour or in disfavour of a team in front of another, since the information we have is linked to a pair of alternatives.

Table 9. Candidates scored according to S^T and S^F.

$\vec{a}$:	1/3	1/3	1/3	1/3	1/3	1/3	3/3	3/3
$\vec{b}$:	1/3	1/3	1/3	2/3	2/3	2/3	3/3	3/3
$\vec{c}$:	1/3	2/3	2/3	2/3	2/3	2/3	3/3	3/3
$\vec{d}$:	1/3	1/3	1/3	1/3	1/3	1/3	3/3	3/3
$\vec{e}$:	0	0	0	0	0	0	0	0

Table 10. Candidates a and d scored according to S^K.

$\overleftarrow{a}$	2/3	2/3	1/3	1/3	1/3	1/3	0	0
$\overleftarrow{d}$	2/3	1/3	1/3	1/3	1/3	1/3	1/3	0

One could decide for instance to evaluate the arguments in favour of $A\mathsf{S}B$ as related to Δ_{AB}, the difference between the sum of goals of A and the sum of goals of B:

$$\Delta_{AB} = (x_{AB} + y_{BA}) - (x_{BA} + y_{AB})$$

The relation between $\mathcal{N}_T(A\mathsf{S}B)$ and Δ_{AB} can be defined as follows:

$$\mathcal{N}_T(A\mathsf{S}B) = \begin{cases} 0 & \Delta_{AB} < 0 \\ 1/4 & \Delta_{AB} = 0 \\ 1/2 \quad \text{if} & \Delta_{AB} = 1 \\ 3/4 & \Delta_{AB} = 2 \\ 1 & \Delta_{AB} > 2 \end{cases}$$

Similar relations are used to define Δ_{BA} and $\mathcal{N}_T(B\mathsf{S}A)$.

Arguments in disfavour are linked to the number of shots that the goalkeeper of A couldn't stop from the players of B, while A was home, i.e. to y_{AB}, in the following way:

$$\mathcal{N}_F(A\mathsf{S}B) = \begin{cases} 0 & y_{AB} = 0 \\ 1/4 & y_{AB} = 1 \\ 1/2 \quad \text{if} & y_{AB} = 2 \\ 3/4 & y_{AB} = 3 \\ 1 & y_{AB} > 3 \end{cases}$$

8. Conclusions and Some Comparative Remarks

In this paper, we presented a "new" methodology to cope with decision problems, where information about the alternatives are (or become) available in terms of arguments in favour (positive) or in disfavour (negative) of pairwise preferences with ordinal degree of relevance. This preferential information is modelled in the context of a graded extension of Belnap's four-valued logic and gives birth to four outranking relations. They are used sequentially, from the most certain to the most uncertain ones, by means of a refined scoring procedure.

Examples have illustrated the different steps of the proposed methodology and provided *arguments in favour* of its relevance in some decision context. In particular, we pointed out that the pairwise comparison of two alternatives has to be assessed by four (generally) independent questions.

But up to now, this is a quite classical concluding section. It is the authors' pleasure to acknowledge some links between their works and an old manuscript by St. Ignatius of Loyola (1548). His "*first way to make a sound and good election*" (items 178–183) contains 6 points. We would like now to summarize the different points of his advice, emphasize the fourth and fifth ones and describe the strong concordance with our approach.

With respect to a pair of alternatives[4] (a, b), Ignatius proceeds as follows:

First point Verify if a and b are both feasible and non dominated.
Second point Free your mind from "a priori" feelings and find yourself as in the middle of a balance.
Third point Be positively and honestly prepared to discussing faithfully various arguments with your intellect.
Fourth point Consider how many advantages and utilities follow for you from holding the proposed alternative a, and, consider likewise, on the contrary, the disadvantages and dangers which there are in having the alternative a.
Doing the same in the second part, that is, looking at the advantages and utilities that are in having b, and likewise, on the contrary, the disadvantages and dangers in having b.
Fifth point After you have thus discussed and reckoned up on all sides about the alternatives proposed, look where reason more inclines: and so, according to the greater inclination of reason, deliberation should be made on the proposed alternatives.
Sixth point Conclude your decision.

The fourth point corresponds to our pairwise comparison of alternatives a and b, obtaining the graded arguments both in favour (advantages) and in disfavour (disadvantages) of $a\mathsf{S}b$ as well as of $b\mathsf{S}a$. The greater inclination of reason in the fifth point advocates for an ordinal aggregation of evidences (to build $\mathsf{S}^T, \mathsf{S}^K, \mathsf{S}^U, \mathsf{S}^T$) and for an exploitation of the aggregates aiming at choosing according to the strongest evidence. In other words, the min-scoring procedure (and its leximin refinement) is consistent with Ignatius' advice.

While the use of positive and negative reasons is concordant with common sense in different cultures (Aristotle (350), Ignatius of Loyola (1548), Franklin (1887), Raju (1954)), the exploitation of the resulting preference relation by a procedure like min-scoring is a distinctive feature of the Ignatius's advice.

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Notes

1. t stands for true, k for contradictory, u for unknown and f for false.
2. As a matter of fact, higher a necessity degree, more certain is the related event or more true is the sentence. In other words, lower a necessity degree, less specific and less enforcing it is.
3. Another refinement called "discrimin" has been proposed in (Behringer, 1977; Behringer, 1990).
4. In the very context of Ignatius, alternative a is to hold a proposed office or benefice, while alternative b is not to hold it. Thus, even if they concern the same office or benefice, the alternatives a and b are exclusive, as alternatives concerning any two different choices.

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