A computer assisted proof of the symmetries of least energy nodal solutions on squares

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(joint work with Ariel Salort)

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Outline

The Lane-Emden problem

$$\begin{cases} -\Delta u = |u|^{p-2} u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{PDE}$$

where Ω is an open bounded set in $\mathbb{R}^N$ and $2 < p < 2^*$.
In this talk, we will especially focus on

$$\Omega =]0,1[^2$$

Solutions to (PDE) are critical points of

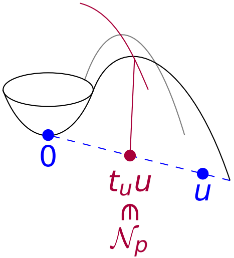
$$\mathcal{E}_p : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx - \frac{1}{p} \int_\Omega |u(x)|^p \, dx.$$

Ground state solution

$$\mathcal{E}_p(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x - \frac{1}{p}\int_\Omega |u(x)|^p \, \mathrm{d}x$$

has the property that

$$\forall u \neq 0, \ \exists! t_u > 0, \quad \mathcal{E}_p(t_u u) = \sup_{t \geqslant 0} \mathcal{E}_p(tu)$$



Nehari manifold

$$\mathcal{N}_p := \left\{ u \in H_0^1(\Omega) \setminus \{0\} \;\middle|\; \mathcal{E}_p'(u)[u] = 0 \right\}$$

Variational principle

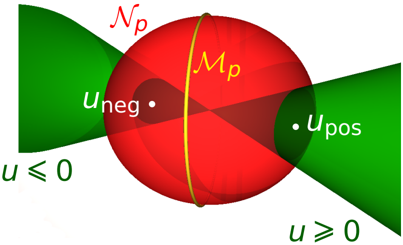
minimize $\boxed{0 \neq u \mapsto \sup_{t \geqslant 0} \mathcal{E}_p(tu)}$ i.e., $\begin{cases} \text{minimize } \mathcal{E}_p(u), \\ \text{s.t. } u \in \mathcal{N}_p. \end{cases}$

Least-energy nodal solutions (l.e.n.s.)

Nodal Nehari set

$$\mathcal{M}_p := \{u \in H^1_0(\Omega) \mid u^+ \in \mathcal{N}_p \text{ and } u^- \in \mathcal{N}_p\}$$

where $u^\pm(x) := \max\{\pm u(x), 0\}$
(so $u = u^+ - u^-$).



Variational principle

minimize $\boxed{0 \neq u \mapsto \sup_{t,s \geqslant 0} \mathcal{E}_p(tu^+ - su^-)}$ i.e., $\begin{cases} \text{minimize } \mathcal{E}_p(u), \\ \text{s.t. } u \in \mathcal{M}_p. \end{cases}$

A. Castro, J. Cossio, J.M. Neuberger. A sign-changing solution for a superlinear Dirichlet problem. Rocky Mountain J. Math. 27 (1997), no. 4, 1041–1053.

Qualitative properties of l.e.n.s.

Theorem (Bartsch, Weth, 2003)

L.e.n.s. have precisely two nodal domains and have Morse index 2.

Theorem (Aftalion, Pacella, 2004)

If Ω is a ball or an annulus in $\mathbb{R}^N$, any radial sign changing solution has Morse index $\geqslant N + 1$.

T. Bartsch, T. Weth, A note on additional properties of sign changing solutions to superlinear elliptic equations, Topol. Methods Nonlinear Anal. 22 (1) (2003) 1–14.

A. Aftalion, F. Pacella. Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains, C. R. Math. Acad. Sci. Paris 339 (5) (2004) 339–344.

Symmetry results for l.e.n.s.

Theorem (Aftalion, Pacella, 2004)

If Ω is a ball or an annulus in $\mathbb{R}^N$, l.e.n.s. are not radial. Moreover if Ω is a ball or, $N = 2$ and Ω is an annulus, the zero set $\overline{\{x \in \Omega \mid u(x) = 0\}}$ intersects the boundary.

Theorem (Bartsch, Weth, Willem, 2005)

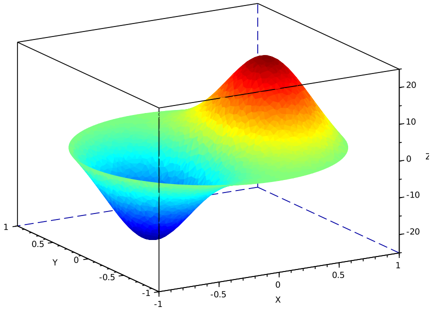
If Ω is a ball, l.e.n.s. are foliated Schwarz symmetric i.e., $u(x) = \tilde{u}(|x|, e \cdot x)$, for some $e \in \mathbb{S}^{N-1}$, and $\tilde{u}(r, \cdot)$ is nondecreasing for every $r > 0$.

A. Aftalion, F. Pacella. Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains, C. R. Math. Acad. Sci. Paris 339 (5) (2004) 339–344.

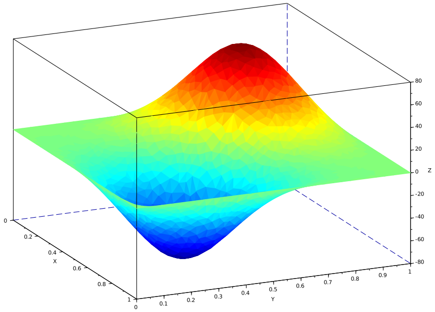
T. Bartsch, T. Weth, M. Willem. Partial symmetry of least energy nodal solutions to some variational problems. J. Anal. Math. 96 (2005), 1–18.

Numerical computation of l.e.n.s.

$$\begin{cases} -\Delta u = |u|^{p-2} u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$



$\Omega = B(0,1)$



$\Omega =]0,1[^2$

Asymptotic problem $p \to 2$ (1/2)

Let $(u_p)_{p>2}$ is a family of least-energy nodal solutions to

$$(\mathrm{PDE})_p \begin{cases} -\Delta u = |u|^{p-2} u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

For the family to be bounded and to say away from 0, we need to renormalize it:

$$\widetilde{u}_p := \lambda_2^{-1/(p-2)} u_p.$$

The family $(\widetilde{u}_p)_{p>2}$ are solutions to

$$(\mathcal{P}_p) \begin{cases} -\Delta u = \lambda_2 |u|^{p-2} u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

From now on, we will denote $u_p := \widetilde{u}_p$ since they have the same symmetries.

Asymptotic problem $p \to 2$ (2/2)

Let $(u_p)_{p>2}$ is a family of least-energy nodal solutions to

$$(\mathcal{P}_p) \begin{cases} -\Delta u = \lambda_2 |u|^{p-2} u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Then, up to a subsequence,

$$\boxed{u_p \xrightarrow[p \to 2]{} u_* \neq 0} \quad \text{in } H_0^1(\Omega),$$

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Then, up to a subsequence,

$$\boxed{u_p \xrightarrow[p \to 2]{} u_* \neq 0} \quad \text{in } H_0^1(\Omega),$$

where u_* is a solution to

$$(\mathrm{L}) \begin{cases} -\Delta u = \lambda_2 u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \qquad \text{i.e. } \boxed{u_* \in E_2}$$

i.e. u_* is a second eigenfunction of $-\Delta$ on Ω with DBC.

Abstract symmetry ($p \approx 2$)

Theorem (D. Bonheure, V. Bouchez, C. Grumiau, J. Van Schaftingen, '08)

For every $\rho > 0$, there exists $\bar{p} > 2$ such that, for any solution u_p to

$$\begin{cases} -\Delta u = \lambda_2 |u|^{p-2} u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

with $p < \bar{p}$, such that $\alpha := P_{E_2} u_p$ satisfies

- $\alpha \in E_2 \setminus B(0, \rho)$;
- $T\alpha = \alpha$;
- $T(E_2) = E_2$;
- $T(E_2^{\perp}) = E_2^{\perp}$;
- $\forall u \in H_0^1(\Omega),\ \mathcal{E}_p(Tu) = \mathcal{E}_p(u)$;

for an isomorphism $T : H_0^1(\Omega) \to H_0^1(\Omega)$, then $Tu_p = u_p$.

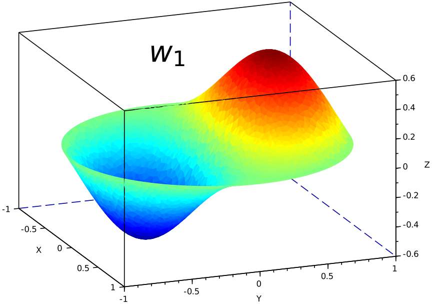
D. Bonheure, V. Bouchez, C. Grumiau, J. Van Schaftingen, Asymptotics and symmetries of least energy nodal solutions of Lane-Emden problems with slow growth, Communications in Contemporary Mathematics 10 (2008), no. 04, 609–631.

The second eigenspace E_2 on the ball

$$E_2 := \left\{ u : \Omega \to \mathbb{R} \;\middle|\; -\Delta u = \lambda_2 u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \right\}$$

When $\Omega = B(0,1) \subseteq \mathbb{R}^N$,
$E_2 = \operatorname{span}\{w_1, \dots, w_N\}$ where,
in spherical coordinates
$(r, \theta) := (|x|, x/|x|)$,

$$w_i(r\theta) = r^{-\frac{N-2}{2}} J_{N/2}(\sqrt{\lambda_2} r)\, S_i(\theta),$$



where J_ν are the Bessel functions of the first kind
and S_i is the map $x \mapsto x_i$ restricted to the sphere.

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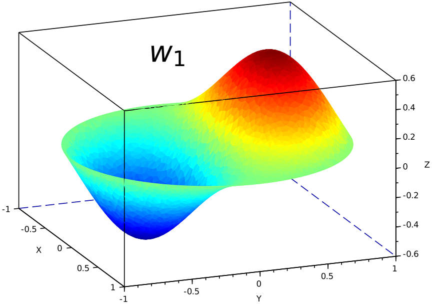


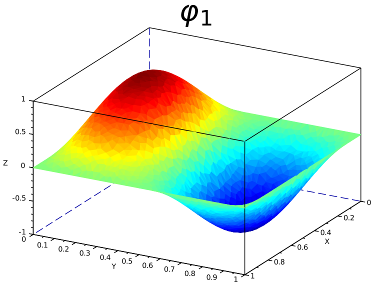
where J_ν are the Bessel functions of the first kind
and S_i is the map $x \mapsto x_i$ restricted to the sphere.

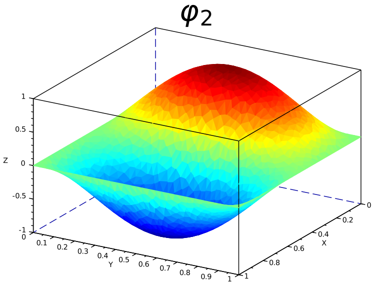
Theorem: For $p \approx 2$, u_p is anti-symmetric w.r.t. a diameter and symmetric in the $N-1$ orthogonal directions.

The second eigenspace E_2 on the square (1/2)

When $\Omega =]0,1[^2$, $E_2 = \operatorname{span}\{\varphi_1, \varphi_2\}$ where

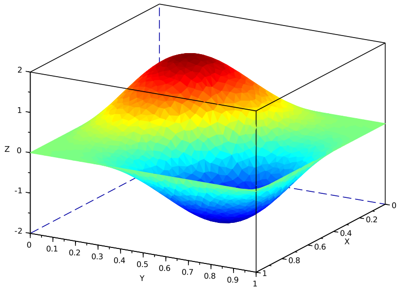
$$\varphi_1(x,y) = \sin(\pi x)\sin(2\pi y) \quad \text{and} \quad \varphi_2(x,y) = \sin(2\pi x)\sin(\pi y).$$



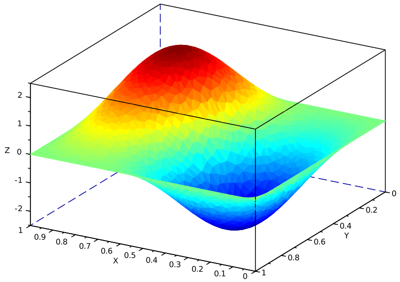


The second eigenspace E_2 on the square (2/2)

$\varphi_1 - \varphi_2$



$\varphi_1 - 2\varphi_2$



Questions

- What function is u_* in E_2?
- How are the symmetries of u_* and u_p related?

Back to the variational formulation

Let us recall that u_p minimize

$$\mathcal{E}_p : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x - \frac{\lambda_2}{p}\int_\Omega |u(x)|^p \, \mathrm{d}x$$

sur $\mathcal{M}_p$. Let us remark that $\mathcal{E}_2 = 0$ on E_2. Let us perform an expansion w.r.t. p for $u \in E_2$:

$$\mathcal{E}_p(u) = \underbrace{\mathcal{E}_2(u)}_{=0} + \partial_p \mathcal{E}_p(u)\big|_{p=2}(p-2) + o(p-2)$$

Using this idea and the fact that u_p is characterized by $\min_{u\in\mathcal{M}_p} \mathcal{E}_p(u)$, one gets that u_* is a solution to the minimization problem

$$\min_{u\in\mathcal{N}_*} \mathcal{E}_*(u) \qquad \text{where } \mathcal{E}_*(u) := \partial_p \mathcal{E}_p(u)\big|_{p=2}.$$

D. Bonheure, V. Bouchez, C. Grumiau, J. Van Schaftingen, Asymptotics and symmetries of least energy nodal solutions of Lane-Emden problems with slow growth, Communications in Contemporary Mathematics 10 (2008), no. 04, 609–631.

The reduced functional

$$\mathcal{E}_p : H^1_0(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x - \frac{\lambda_2}{p}\int_\Omega |u(x)|^p \,\mathrm{d}x$$

$$\Downarrow$$

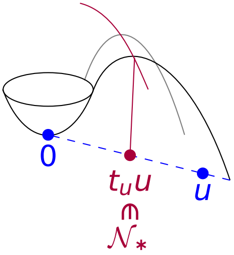
$$\partial_p \mathcal{E}_p(u) = \lambda_2 \Bigl(\frac{1}{p^2}\int_\Omega |u(x)|^p \,\mathrm{d}x - \frac{1}{2p}\int_\Omega |u(x)|^p \log|u|^2 \,\mathrm{d}x\Bigr)$$

$$\Downarrow$$

$$\mathcal{E}_*(u) = \partial_p \mathcal{E}_p(u)\big|_{p=2} = \frac{\lambda_2}{4}\int_\Omega u^2 - u^2 \log u^2 \,\mathrm{d}x.$$

We drop a factor $\frac{\lambda_2}{2}$ which does not change the minimization problem.

Reduced variational formulation (1/3)



Reduced functional

$$\mathcal{E}_* : E_2 \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega u^2 - u^2 \log u^2$$

Reduced Nehari manifold

$$\mathcal{N}_* := \left\{u \in E_2 \setminus \{0\} \;\middle|\; \mathcal{E}_*'(u)[u] = 0\right\}$$

Criteria: u_* is a solution to

$$\text{minimize } \boxed{0 \neq u \mapsto \sup_{t \geqslant 0} \mathcal{E}_*(tu)} \qquad \text{i.e.,} \qquad \begin{cases} \text{minimize } \mathcal{E}_*(u) \\ \text{s.t. } u \in \mathcal{N}_* \end{cases}$$

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Reduced variational formulation (2/3)

$$\mathcal{E}_*(u) = \frac{1}{2}\int_\Omega u^2 - u^2\log u^2 \qquad \mathcal{E}'_*(u)[u] = -\int_\Omega u^2 \log u^2 = 0.$$

For any $u \in E_2 \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_*$. Since

$$\mathcal{E}'_*(tu)[tu] = -2t^2 \int_\Omega u^2(\log t + \log|u|)$$

one gets

$$t_u = \exp\left(-\frac{\int_\Omega u^2 \log|u|}{\int_\Omega u^2}\right).$$

and

$$\mathcal{E}_*(t_u u) = \frac{1}{2}\int_\Omega (t_u u)^2 = \frac{1}{2}\exp\left(-2\frac{\int_\Omega u^2 \log|u|}{\int_\Omega u^2}\right)\int_\Omega u^2$$

Reduced variational formulation (3/3)

If $\int_\Omega u^2 = 1$ (i.e., u is on the unit L^2-sphere),

$$S_*(u) := \frac{1}{2}\log\Bigl(2\sup_{t\geqslant 0}\mathcal{E}_*(tu)\Bigr) = -\int_\Omega u^2 \log|u|\,\mathrm{d}x$$

We want to minimize S_* on the L^2-unit sphere of E_2.
Since

$$S_*(ru) = r^2 S_*(u) - r^2\log r,$$

one may as well minimize on the sphere of radius r.

$u_* = t_u u$ where u is a minimizer, hence has the same symmetries.

Reduced variational formulation (3/3)

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✔ Localization
? u_p has the same symmetries as u_*.

Lyapunov–Schmidt reduction (1/3)

Theorem (A. Salort, C.T., 2019)

Assume the functional $\mathcal{E}_*$ defined previously is $C^2(E_2;\mathbb{R})$. For any non-degenerate critical point $u_* \in E_2$ of $\mathcal{E}_*$, there exists a neighborhood V_* of u_* in $H^1_0(\Omega)$ and a continuous curve $\gamma : [2, 2+\varepsilon[\to H^1_0(\Omega)$, $\varepsilon > 0$, such that $\gamma(2) = u_*$ and

$$\forall p \in]2, 2+\varepsilon[,\ \forall u \in V_*, \qquad \boxed{u \text{ solves } (\mathcal{P}_p) \iff u = \gamma(p)}$$

where

$$(\mathcal{P}_p) \begin{cases} -\Delta u = \lambda_2 |u|^{p-2} u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

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$$\forall p \in]2, 2+\varepsilon[,\ \forall u \in V_*, \qquad u \text{ solves } (\mathcal{P}_p) \iff u = \gamma(p)$$

where

$$(\mathcal{P}_p) \begin{cases} -\Delta u = \lambda_2 |u|^{p-2} u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Corollary: Let T be a symmetry of u_* ($Tu_* = u_*$). For $p \in]2, 2+\varepsilon[$, u_p has the same symmetry ($Tu_p = u_p$).

Lyapunov–Schmidt reduction (2/3)

$$\mathcal{E}_p'(u)[\eta] = \int_\Omega \nabla u \nabla \eta \,\mathrm{d}x - \lambda_2 \int_\Omega |u|^{p-2} u \eta \,\mathrm{d}x$$

Sketch of the proof.

- Decompose $u = v + w$ with $v \in E_2$ and $w \in E_2^\perp$.
- Split the equation:

$$\mathcal{E}_p'(v+w) = 0 \quad \Leftrightarrow \quad \begin{cases} \mathcal{G}(p, v, w) := \mathcal{E}_p'(v+w)\big|_{E_2^\perp} = 0, \\ \qquad\qquad\qquad\quad \mathcal{E}_p'(v+w)\big|_{E_2} = 0. \end{cases}$$

- Use the implicit function theorem to prove that, when $p \approx 2$ and $w \approx 0$,

$$\mathcal{G}(p, v, w) = 0 \quad \Leftrightarrow \quad w = \omega(p, v).$$

Lyapunov–Schmidt reduction (3/3)

- It remains to solve

$$\mathcal{H}(p,v) := \mathcal{E}_p'\big(v + \omega(p,v)\big)\big|_{E_2} = 0.$$

  One can show $\mathcal{H}(2,v) = 0$ for all $v \in E_2$.

- For $p > 2$, this is equivalent to find the roots of

$$\mathcal{K}(p,v) := \begin{cases} \dfrac{\mathcal{H}(p,v)}{p-2} & \text{if } p > 2, \\ \partial_p \mathcal{H}(2,v) & \text{if } p = 2. \end{cases}$$

- One has $\partial_p \mathcal{H}(2,v) = \mathcal{E}_*'(v)$ so non-degenerate critical points of $\mathcal{E}_*(v)$ give rise to local curves of solutions. □

Numerical simulation (1/2)

➨ Symmetries and non-degeneracy of u_*?

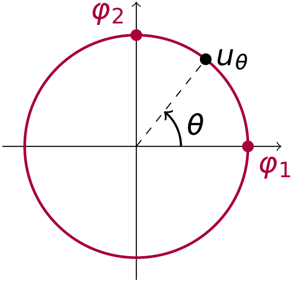
$$\min_{|u|_{L^2}=r} S_*(u) \qquad \text{where } S_*(u) = -\int_\Omega u^2 \log|u| \,\mathrm{d}x$$

➨ For $\Omega =]0,1[^2$:

Because $|\varphi_1|_{L^2} = r$, $|\varphi_2|_{L^2} = r$ (with $r = \frac{1}{2}$) and $\varphi_1 \perp \varphi_2$ in L^2,

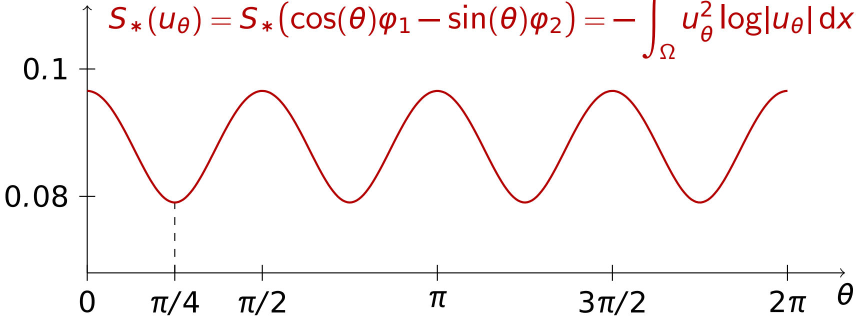
$$u_\theta := \cos\theta\,\varphi_1 - \sin\theta\,\varphi_2$$

parameterizes the L^2-sphere of E_2 of radius r.



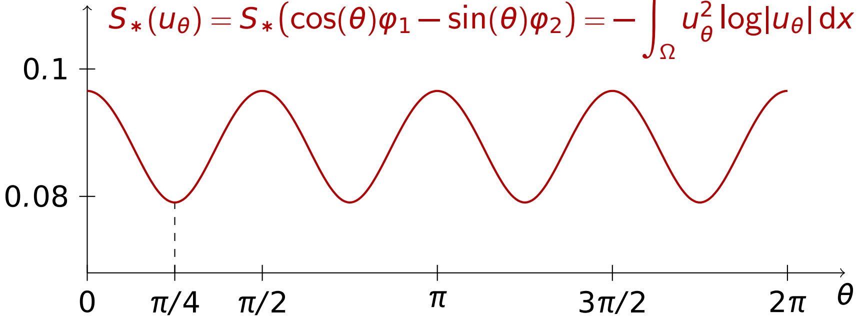
Numerical simulation (2/2)

$$S_*(u_\theta) = S_*\big(\cos(\theta)\varphi_1 - \sin(\theta)\varphi_2\big) = -\int_\Omega u_\theta^2 \log|u_\theta| \,\mathrm{d}x$$



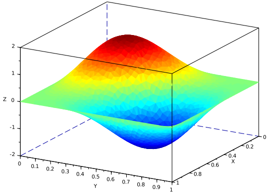
Numerical simulation (2/2)

$$S_*(u_\theta) = S_*\big(\cos(\theta)\varphi_1 - \sin(\theta)\varphi_2\big) = -\int_\Omega u_\theta^2 \log|u_\theta| \,dx$$

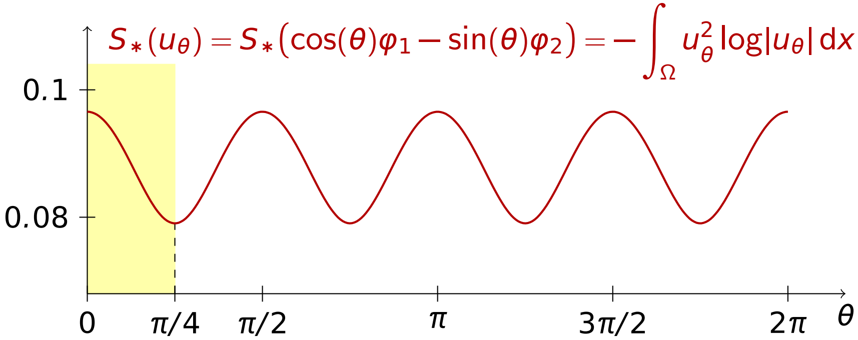


Recall that $u_{\pi/4} = \frac{\sqrt{2}}{2}(\varphi_1 - \varphi_2)$ is anti-symmetric w.r.t. a diagonal.

➠ Conjecture



Symmetries of $S_*(u_\theta)$



Because the problem is invariant by rotations of $\pi/2$ and axial symmetries and S_* is even, one has:

- S_* is $\pi/2$-periodic;
- $S_*(\frac{\pi}{4} - \theta) = S_*(\frac{\pi}{4} + \theta)$.

Crash course in interval arithmetic (1/3)

Observation: floating point computations may be inaccurate due to rounding error.

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function

$$f(x,y) = 333.75\,y^6 + x^2(11x^2y^2 - y^6 - 121\,y^4 - 2) + 5.5\,y^8$$

In double precision, evaluating $f(77617, 33096)$ yields $-1.180592 \cdot 10^{21}$. The correct value is -2.

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Basic idea: Compute an interval $[\underline{z}, \overline{z}]$ containing the true value:

$$f(x, y) \in [\underline{z}, \overline{z}],$$

the rounding of each endpoint taking care of rounding errors.
➠ guaranteed bounds

Crash course on interval arithmetic (2/3)

Extend operations to intervals:

$$[\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \subseteq [\underline{x} +_{\downarrow} \underline{y}, \overline{x} +^{\uparrow} \overline{y}]$$

$$[\underline{x}, \overline{x}] \cdot [\underline{y}, \overline{y}] = \left[\min\{\underline{x}\,\underline{y}, \underline{x}\,\overline{y}, \overline{x}\,\underline{y}, \overline{x}\,\overline{y}\}, \max\{\underline{x}\,\underline{y}, \underline{x}\,\overline{y}, \overline{x}\,\underline{y}, \overline{x}\,\overline{y}\}\right]$$

sin, cos, …

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sin, cos, ...

Fundamental property: Let $x \mapsto f(x)$ be a function and $I \mapsto \mathbf{f}(I)$ an interval extension of f. That means:

$$\forall I \text{ interval}, \qquad \forall x \in I,\ f(x) \in \mathbf{f}(I)$$

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Dependency problem:

- $[\underline{x},\overline{x}]-[\underline{x},\overline{x}]=[\underline{x}-\overline{x},\overline{x}-\underline{x}]\supseteq[0,0]$ but $\neq$ (unless $\underline{x}=\overline{x}$).
- $\big([\underline{x},\overline{x}]\big)^2\subseteq[\underline{x},\overline{x}]\cdot[\underline{x},\overline{x}]$ but in general $\neq$.
- etc.

Crash course on interval arithmetic (3/3)

For our original example:

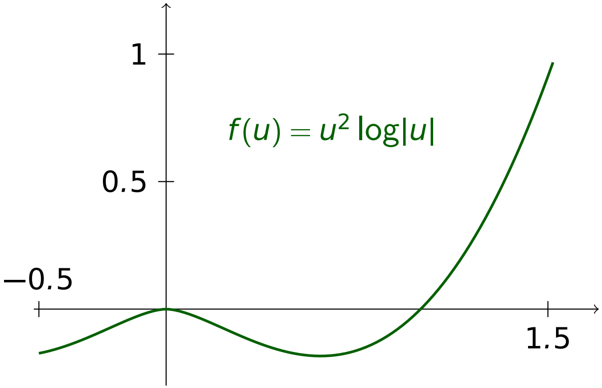
$$f(x,y) = 333.75\,y^6 + x^2(11x^2y^2 - y^6 - 121\,y^4 - 2) + 5.5\,y^8$$

In double precision interval arithmetic,
$\mathbf{f}([77617],[33096]) = [-5.902957 \cdot 10^{21}, 4.722367 \cdot 10^{21}]$.

➠ Need to adapt standard algorithms.

Evaluation of basic functions

Recall that $S_*(u) = -\int_\Omega f(u)\,dx$ where $f(u) := u^2 \log|u|$.



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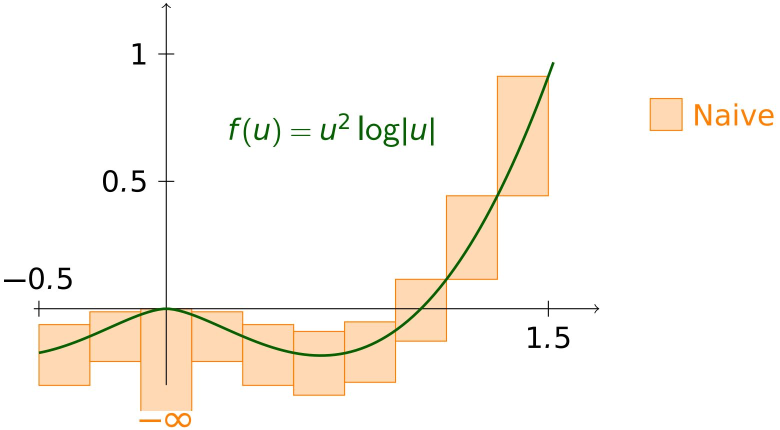

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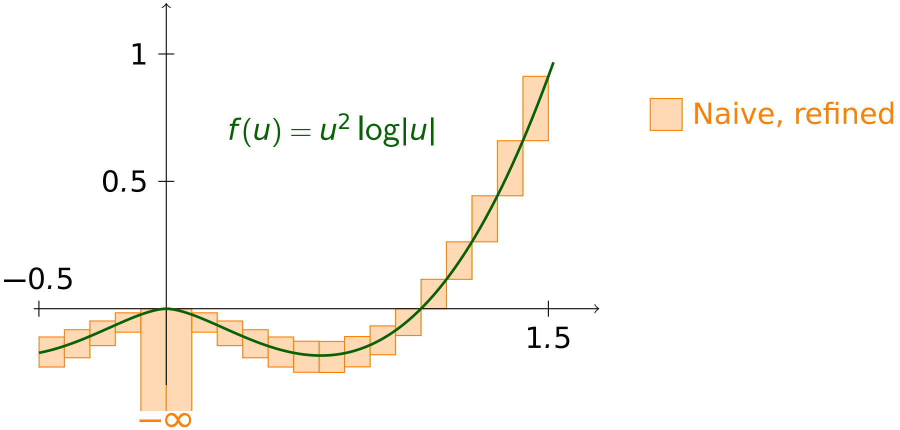

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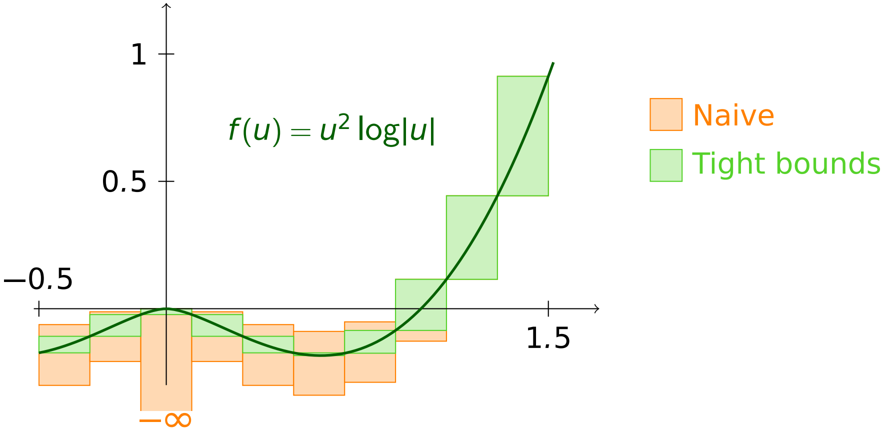

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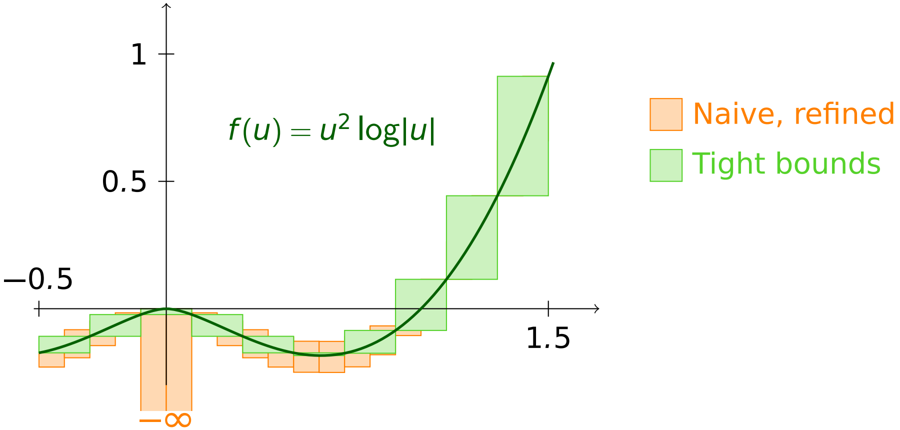

Adaptive integration (1/2)

Compute $S_*(u) = -\int_\Omega u^2 \log|u| \,\mathrm{d}x$ where $u = \cos\theta\, \varphi_1 - \sin\theta\, \varphi_2$.

Basic scheme: partition Ω in a union of "small" P and estimate each integral with

$$\frac{1}{|P|}\int_P g(x)\,\mathrm{d}x \in g(P) \qquad \big(P = I_1 \times I_2 \Rightarrow g(P) \subseteq \mathbf{g}(I_1, I_2)\big).$$

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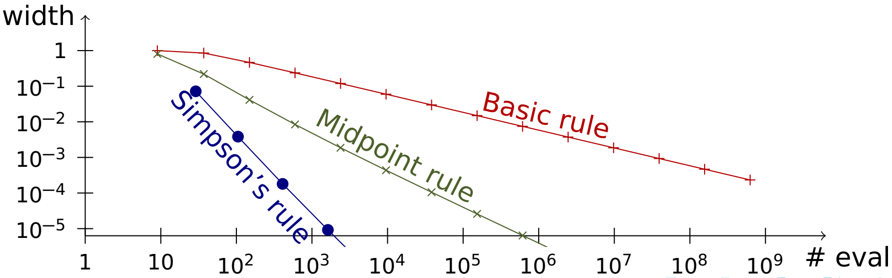
Higher order schemes: require some regularity (e.g., $g \in \mathcal{C}^2$).

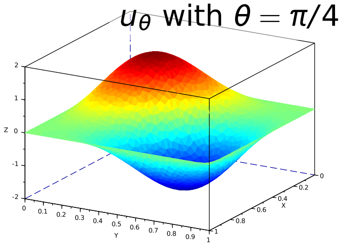


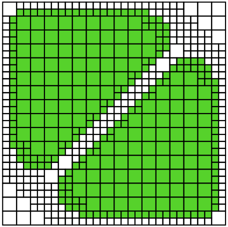
Adaptive integration (2/2)

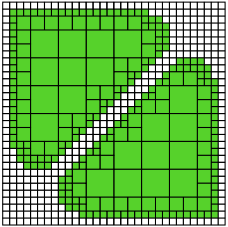
1D Simpson's rule:

$$\frac{1}{h}\int_a^{a+h} f(x)\,\mathrm{d}x - \frac{1}{6}\big(f(a) + 4f(a+\tfrac{1}{2}h) + f(a+h)\big) = -\frac{1}{2880}h^4 f^{(4)}(\xi)$$

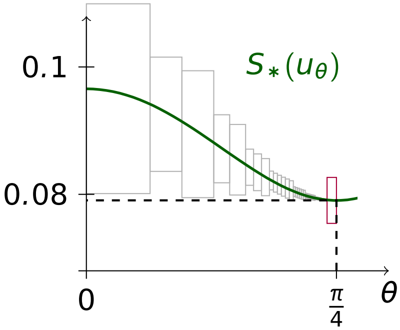
For $S_*(u) = -\int_\Omega f(u(x))\,\mathrm{d}x$ where $f(u) := u^2 \log|u|$, the function $x \mapsto f(u(x))$ is not C^2 whenever $u(x) = 0$.







Asymptotic problem on $\Omega =]0,1[^2$



Determine a small interval I such that $\pi/4 \in I$ and

$$\forall \theta \in [0, \pi/4] \setminus I, \quad \mathcal{E}_*(\theta) > \mathcal{E}_*(\pi/4)$$

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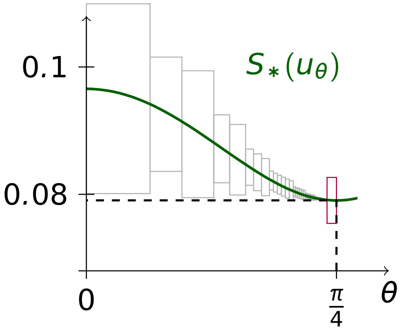

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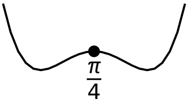

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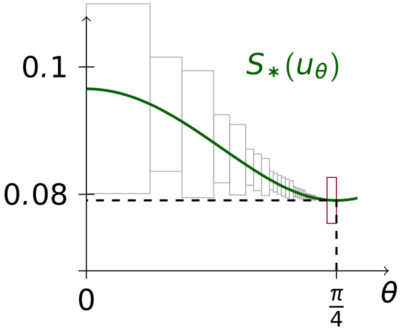

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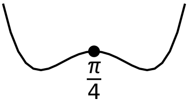


Solution: Show that

$$\forall \theta \in I, \quad \partial_\theta^2 \big(S_*(u_\theta)\big) > 0.$$

Computing the second derivative

Recall that:

$$S_*(u) = -\int_\Omega u^2 \log|u| \, dx$$

Let $u_\theta = \cos\theta\, \varphi_1 - \sin\theta\, \varphi_2$ and $u'_\theta := \partial_\theta u_\theta$. Taking into account that $\int u_\theta^2 = r^2$ and $\int (u'_\theta)^2 = r^2$, one computes

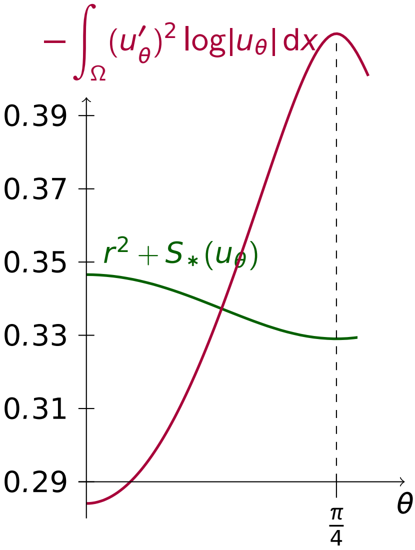
$$\boxed{\partial_\theta^2 \big(S_*(u_\theta)\big)} = 2\Big(-r^2 - S_*(u_\theta) - \int_\Omega (u'_\theta)^2 \log|u_\theta| \, dx\Big).$$

Thus

$$\partial_\theta^2 \big(S_*(u_\theta)\big) > 0 \quad \Leftrightarrow \quad -\int_\Omega (u'_\theta)^2 \log|u_\theta| \, dx > r^2 + S_*(u_\theta).$$

Note that the second derivative is singular.

Positiveness test for the second derivative



$$-\int_\Omega (u_\theta')^2 \log|u_\theta| \,dx > r^2 + S_*(u_\theta).$$

If on a subdivision P used to compute the integral, one has $\log|u_\theta(P)| = \,]{-\infty}, \overline{\alpha}]$, then

$$-\int_\Omega (u_\theta')^2 \log|u_\theta| \,dx = [\underline{\beta}, +\infty[.$$

This is fine since we care about the lower bound!

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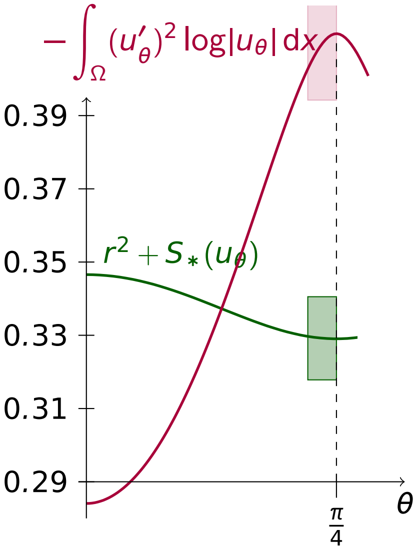

$$-\int_\Omega (u'_\theta)^2 \log|u_\theta| \,\mathrm{d}x$$
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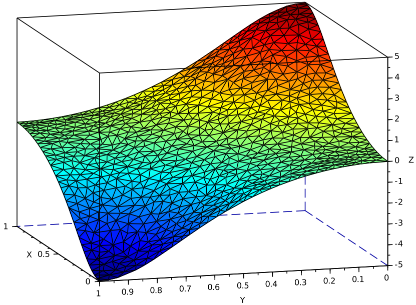
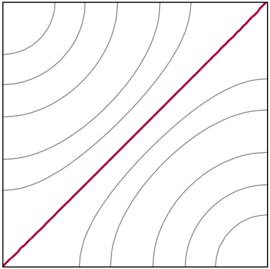
Neumann boundary conditions

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega. \end{cases}$$





Symmetry breaking (1/3)

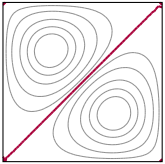
$$\begin{cases} -\Delta\tilde{u} = \lambda_2(\varepsilon)\,|\tilde{u}|^{p-2}\tilde{u} & \text{in } R_\varepsilon, \\ \tilde{u} = 0 & \text{on } \partial R_\varepsilon, \end{cases}$$

where $R_\varepsilon =]0,1[\times]0,1+\varepsilon[$ and $\lambda_2(\varepsilon)$ is the second eigenvalue of $-\Delta$ on R_ε with DBC.
Change of variables $u(x,y) = \tilde{u}\big(x,(1+\varepsilon)y\big)$:

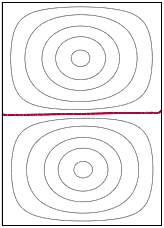
$$\begin{cases} -u_{xx} - \frac{1}{(1+\varepsilon)^2}u_{yy} = \lambda_2(\varepsilon)\,|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega =]0,1[^2$.

We write $\lambda_2 = \lambda_2(0) = 5\pi^2$, the second eigenvalue on the square.

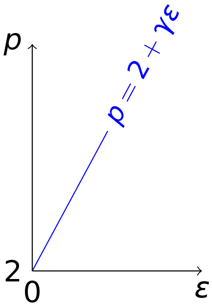


$\varepsilon = 0,\ p \to 2$



$\varepsilon > 0,\ p \to 2$

Symmetry breaking (2/3)

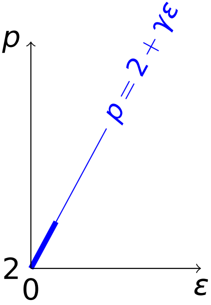


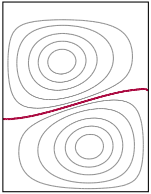
$$\mathcal{E}_{*,\gamma}(u) = -\int_\Omega (\partial_y u)^2 \,\mathrm{d}x - \frac{\lambda'}{2}\int_\Omega u^2 \,\mathrm{d}x + \gamma\lambda_2 \underbrace{\frac{1}{4}\int_\Omega u^2(1 - \log u^2)\,\mathrm{d}x}_{=\frac{1}{2}\mathcal{E}_*(u)}$$

where $\lambda' = \partial_\varepsilon \lambda_2(0) = -8\pi^2$. Thus

$$\mathcal{S}_{*,\gamma}(u_\theta) = \pi^2\Big(\tfrac{3}{2}\sin^2\theta - 2 + \gamma\tfrac{5}{2}\mathcal{E}_*(u_\theta)\Big)$$

Symmetry breaking (2/3)





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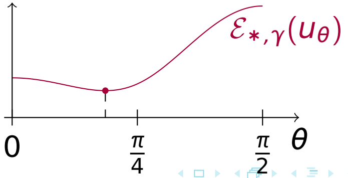


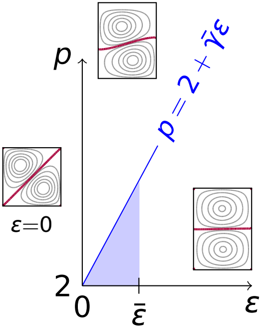
Symmetry breaking (3/3)

Theorem (A. Salort, C.T., 2019)

There exists $\bar\gamma > 0$ and $\bar\varepsilon > 0$ such that, for any (ε, p) in the triangle defined by $\varepsilon \in \,]0, \bar\varepsilon]$ and $2 < p \leqslant 2 + \bar\gamma\varepsilon$, every l.e.n.s. to

$$\begin{cases} -u_{xx} - \frac{1}{(1+\varepsilon)^2} u_{yy} = \lambda_2(\varepsilon)\,|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is symmetric with respect to the longest median and antisymmetric with respect to the shortest one.



A "staged" Lyapunov–Schmidt reduction (1/5)

$$\mathcal{E}_{\varepsilon,\gamma}(u) = \frac{1}{2}\int_\Omega A_\varepsilon \nabla u \cdot \nabla u \,\mathrm{d}x - \frac{\lambda_2(\varepsilon)}{p}\int_\Omega |u|^p \,\mathrm{d}x \qquad \text{where } p = 2 + \gamma\varepsilon$$

As before, when $\varepsilon \approx 0$, $\gamma \in [0, \bar{\gamma}]$ and $v \in \bar{B}_R \subseteq E_2$,

$$\mathcal{E}'_{\varepsilon,\gamma}(v + w)\big|_{E_2^\perp} = 0 \quad \Leftrightarrow \quad w = \omega(\varepsilon, \gamma, v).$$

It remains to solve

$$\mathcal{H}(\varepsilon, \gamma, v) := \mathcal{E}'_{\varepsilon,\gamma}\big(v + \omega(\varepsilon, \gamma, v)\big)\big|_{E_2} = 0 \qquad \text{where } v \in E_2.$$

As $\mathcal{H}(0, \gamma, v) = 0$, one can define

$$\mathcal{K}(\varepsilon, \gamma, v) := \begin{cases} \mathcal{H}(\varepsilon, \gamma, v)/\varepsilon & \text{if } \varepsilon > 0,\\ \partial_\varepsilon \mathcal{H}(0, \gamma, v) = \boxed{\mathcal{E}'_{*,\gamma}(v)} & \text{if } \varepsilon = 0. \end{cases}$$

A "staged" Lyapunov–Schmidt reduction (2/5)

$$\mathcal{K}(\varepsilon, \gamma, v) := \begin{cases} \mathcal{H}(\varepsilon, \gamma, v)/\varepsilon & \text{if } \varepsilon > 0, \\ \mathcal{E}'_{*,\gamma}(v) & \text{if } \varepsilon = 0. \end{cases}$$

where

$$\mathcal{E}_{*,\gamma}(v) = \frac{1}{2}\int_\Omega A'\nabla v\cdot\nabla v\,\mathrm{d}x - \frac{\lambda'}{2}\int_\Omega v^2\,\mathrm{d}x + \gamma\lambda_2\frac{1}{4}\int_\Omega v^2(1-\log v^2)\,\mathrm{d}x$$

➨ When $\gamma > 0$, $\mathcal{E}_{*,\gamma}$ has a mountain pass structure. If u_{γ_*} is a non-degenerate critical point of $\mathcal{E}_{*,\gamma_*}$, one has

$$\mathcal{K}(\varepsilon, \gamma, v) = 0 \quad\Leftrightarrow\quad v = \sigma(\varepsilon, \gamma).$$

By compactness, this is valid for

$$\varepsilon \in [0, \bar\varepsilon],\ \gamma \in [\gamma_0, \bar\gamma] \text{ and } \|v - u_\gamma\| \leqslant \rho$$

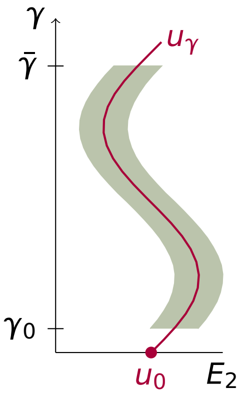
where $\gamma_0 > 0$ is as small as we want.

A “staged” Lyapunov–Schmidt reduction (3/5)

➨ When $\gamma = 0$, $\mathcal{E}_{*,\gamma}$ is quadratic.

$$\mathcal{E}_{*,0}(v) = \frac{1}{2}\int_\Omega A'\nabla v \cdot \nabla v \,\mathrm{d}x - \frac{\lambda'}{2}\int_\Omega v^2 \,\mathrm{d}x$$

$\mathcal{E}'_{*,0}$ vanishes on $E_0 \subset E_2$ (so all its critical points are degenerate).



A “staged” Lyapunov–Schmidt reduction (3/5)

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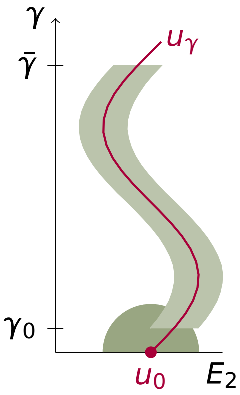
$\mathcal{E}'_{*,0}$ vanishes on $E_0 \subset E_2$ (so all its critical points are degenerate).
A *second* Lyapunov–Schmidt reduction is performed on E_2: the solutions for $\gamma \approx 0$ are driven by $\mathcal{E}'_{*}|_{E_0}$. We require u_0 to be a non-degenerate critical point of $\mathcal{E}_{*}|_{E_0}$. Then



$$\mathcal{K}(\varepsilon, \gamma, v) = 0 \quad \Leftrightarrow \quad v = \sigma(\varepsilon, \gamma)$$

when $\varepsilon \in [0, \bar{\varepsilon}]$, $\gamma \in\,]0, \gamma_0]$, $\|v - u_0\| \leqslant \rho$.

A “staged” Lyapunov–Schmidt reduction (4/5)

Theorem (A. Salort, C.T., 2019)

Assume that $\mathbb{R} \to C(\Omega, \mathrm{Sym}_N) : \varepsilon \mapsto A_\varepsilon$ *and* $\varepsilon \mapsto \lambda_2(\varepsilon)$ *are differentiable in a neighborhood of* 0 *and that there is* $\bar\gamma > 0$ *and a continuous map*

$$[0, \bar\gamma] \to E_2 \setminus \{0\} : \gamma \mapsto u_\gamma$$

such that, for all $\gamma \in]0, \bar\gamma]$, u_γ *is a non-degenerate critical point of* $\mathcal{E}_{*,\gamma} : E_2 \to \mathbb{R}$ *with* $A' = (\partial_\varepsilon A_\varepsilon)|_{\varepsilon=0}$ *and* $\lambda' = \partial_\varepsilon \lambda_2(0)$. *Let* $E_0 := \{u \in E_2 \mid \mathcal{E}'_{*,0}(u) = 0\}$ *and assume further that*

$$\text{for all } \varepsilon > 0 \text{ small,} \quad E_0 \subseteq \ker\bigl(u \mapsto -\mathrm{div}(A_\varepsilon \nabla u) - \lambda_2(\varepsilon) u\bigr) \tag{1}$$

and $u_0 \in E_0$ *is a non-degenerate critical point of* $\mathcal{E}_*|_{E_0}$.

A "staged" Lyapunov–Schmidt reduction (5/5)

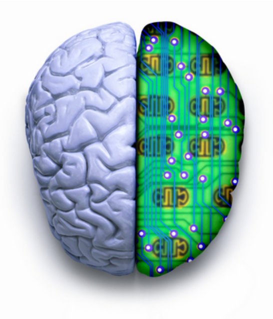
Theorem (cont'd)

Then there exists $\bar\varepsilon > 0$, $\rho > 0$, *and a continuous function* $\sigma : [0,\bar\varepsilon] \times [0,\bar\gamma] \to H^1_0(\Omega) : (\varepsilon,\gamma) \mapsto \sigma(\varepsilon,\gamma)$ *such that*

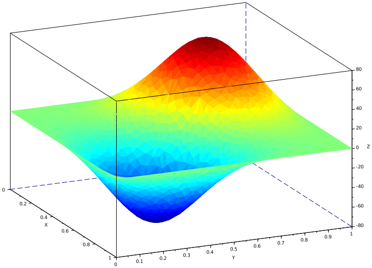
1. *for all* $\gamma \in [0,\bar\gamma]$, $\sigma(0,\gamma) = u_\gamma$,
2. *for all* $\varepsilon \in [0,\bar\varepsilon]$, $\sigma(\varepsilon,0) = u_0$,
3. *for all* $\varepsilon \in \,]0,\bar\varepsilon]$, $\gamma \in \,]0,\bar\gamma]$ *and* $u \in H^1_0(\Omega)$ *such that* $\|u - u_\gamma\| \leqslant \rho$, *one has*

$$u \text{ is a critical point of } \mathcal{E}_{\varepsilon,\gamma} \quad \Leftrightarrow \quad u = \sigma(\varepsilon,\gamma).$$

Thank you for your attention!



$\vdash$



Thanks to the conference organizers !

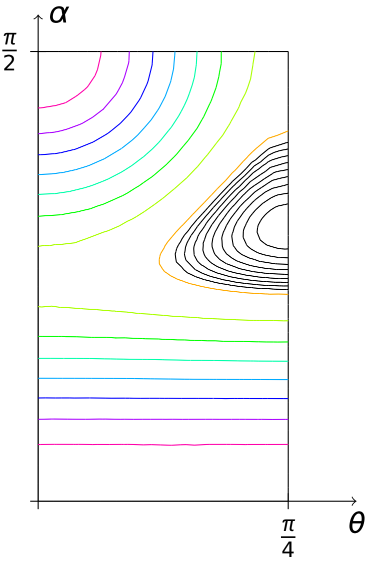
The 3D case

On $\Omega =]{-1}, 1[^3$, $E_2 = \operatorname{span}\{\varphi_1, \varphi_2, \varphi_3\}$ where

$$\varphi_1(x, y, z) := \sin(\pi x)\cos\Big(\frac{\pi}{2} y\Big)\cos\Big(\frac{\pi}{2} z\Big)$$
$$\varphi_2(x, y, z) := \cos\Big(\frac{\pi}{2} x\Big)\sin(\pi y)\cos\Big(\frac{\pi}{2} z\Big)$$
$$\varphi_3(x, y, z) := \cos\Big(\frac{\pi}{2} x\Big)\cos\Big(\frac{\pi}{2} y\Big)\sin(\pi z)$$

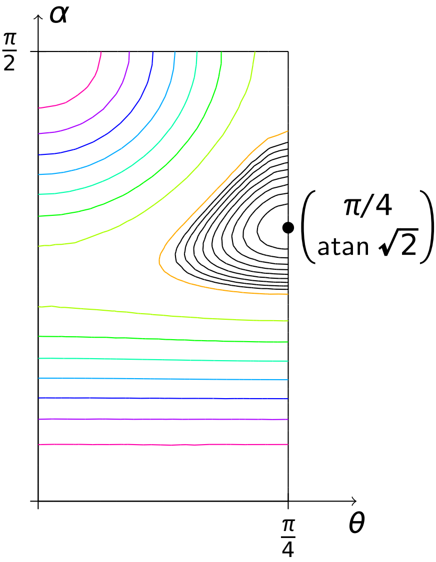
Let $u_{\theta,\alpha} := (\cos\theta\varphi_1 + \sin\theta\varphi_2)\sin\alpha + \cos\alpha\varphi_3$.

The 3D case: minimizers



$$u_{\theta,\alpha} := (\cos\theta\,\varphi_1 + \sin\theta\,\varphi_2)\sin\alpha + \cos\alpha\,\varphi_3$$

The 3D case: minimizers



$$u_{\theta,\alpha} := (\cos\theta\,\varphi_1 + \sin\theta\,\varphi_2)\sin\alpha + \cos\alpha\,\varphi_3$$

The minimum seems to be achieved for

$$(\theta,\alpha) = \left(\frac{\pi}{4}, \operatorname{atan}\sqrt{2}\right)$$

i.e., for

$$\varphi_1 + \varphi_2 + \varphi_3.$$

The 3D case: minimizers

The zero set of $\varphi_1 + \varphi_2 + \varphi_3$ is pictured below.