



Definability of types and VC density in differential topological fields

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Abstract Given a model-complete theory of topological fields, we considered its generic differential expansions and under a certain hypothesis of largeness, we axiomatised the class of existentially closed ones. Here we show that a density result for definable types over definably closed subsets in such differential topological fields. Then we show two transfer results, one on the VC-density and the other one, on the combinatorial property NTP2.

Keywords Definable types · Topological differential fields · VC-density

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1 Introduction

As in [15], we consider *generic* differential expansions of model-complete theories of topological fields. Let \mathcal{K} be a topological \mathcal{L} -field and consider the expansion $\langle K, D \rangle$, where D is a derivation on K with a priori no interactions with the topology on K . We will always assume that we have a basis of neighbourhoods of 0 which is uniformly definable and that the topology is non-discrete.

We first consider the case where T is universally axiomatised and admits a model completion T_c . We further assume that the models of T_c satisfy an hypothesis, that

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we call Hypothesis (I), an analog of *largeness* in that context of topological fields. We show in [15] that the expansion of T to the $\mathcal{L} \cup \{D\}$ -theory T_D consisting of T together with the axioms expressing that D is a derivation, admits a model-completion $T_{c,D}^*$ that we axiomatize. Under these additional assumptions, the axiomatisation we give is quite transparent and $T_{c,D}^*$ also admits quantifier elimination. Then, more generally (weakening the assumption that T_c admits quantifier elimination to T_c model-complete) we show that $T_{c,D}$ has a model-companion that we denote $T_{c,D}^{*,\omega}$.

In this paper, we will show several transfer results from the theory T_c to the theory $T_{c,D}^*$ (respectively $T_{c,D}^{*,\omega}$).

An axiomatisation of $T_{c,D}^*$ consists in adding to the theory $T_D \cup T_c$, a scheme (DL) of axioms, which expresses that each differential polynomial has a zero close to a zero of its associated algebraic polynomial [15, Definition 5(3)]. Except for the fact that Tressl [37] considers fields endowed with several commuting derivations, it can be seen as a generalisation of his work on large differential fields. Note also that Solanki [36] extended the work of Tressl to a topological setting similar to ours.

A former well-studied case taking for T_c the theory RCF of real-closed fields, is the theory $CODF$ of closed ordered differential fields, who has been axiomatized by Singer [35]. Singer [35] also noted that even though isolated types are dense in the Stone space of types of the theory RCF , this no longer holds for $CODF$. Along the same lines as the proof of Singer for $CODF$, we first show that in general isolated types are not dense in Stone spaces of the theory $T_{c,D}^*$.

Brouette [8] described definable types over definably closed subsets of models of $CODF$ and in particular he showed that they are dense. He used the description of definable types over real-closed fields (and more generally over definably closed subsets of models of a complete o-minimal theory) due to van den Dries, Marker, Steinhorn [23] and Pillay [29].

Here, using the description of certain definable types over models of T_c , we show that definable types over definably closed subsets of models of $T_{c,D}^*$ are dense in the corresponding Stone spaces.

The density property of definable types has been a key ingredient in recent proofs of elimination of imaginaries (see for instance [17]). In a recent paper with Brouette and Cubidès [6], we strengthen the result of Brouette [31] cited above and give a new proof of the elimination of imaginaries for $CODF$.

Note that whenever T_c has the NIP property, it transfers to $T_{c,D}^*$ [15, 26] since the theory $T_{c,D}^*$ admits quantifier elimination.

Then, assuming that T_c is NIP , we relate the dual VC-density of \mathcal{L}_D -definable subsets of models of $T_{c,D}^*$ to their \mathcal{L} -counterparts, under the further hypothesis that T_c has finite Skolem functions and the local continuity property. In order to prove that result we need a precise description of the \mathcal{L}_D -definable sets and these two last assumptions allow us to use a cell-decomposition theorem for models of T_c due to Mathews [24, Theorem 7.1].

In the last section, we consider the transfer of the $NT P_2$ property; that combinatorial property was introduced by Shelah as a dividing line in the hierarchy of unstable theories. More recently, $NT P_2$ -theories have been shown to be a natural framework for developing the properties of forking [10]. In that last section, we work simply under the hypothesis that T_c is model-complete and we show that if T_c is $NT P_2$, then the

theory $T_{c,D}^{*,\omega}$ is also $NT P_2$. Then we apply our transfer result when T_c is the theory of bounded pseudo-real closed fields PRC_e with $e \geq 2$ distinct orderings. In her thesis [27], Montenegro [28] showed that bounded pseudo-real closed fields are exactly those pseudo-real closed fields which are $NT P_2$. Moreover pseudo-real-closed fields which are not real-closed are not NIP [28].

Finally, let us mention the following question. In [15, Sections 9, 10], we axiomatized the model-companion of the generic differential expansions of certain theories of ordered fields and valued fields endowed with finitely many distinct orderings and valuations. For instance, we showed that the theory of maximal pseudo-real closed fields \overline{OF}_e endowed with a derivation, has a model-companion. A natural question is what happens in the case of pseudo p -adically closed fields. Using a similar strategy as in the ordered case, Montenegro [28] proved that bounded pseudo p -adically closed fields are $NT P_2$ and those valued fields are endowed with finitely many p -adic valuations.

2 Preliminaries

Let \mathcal{L} be the language $\mathcal{L}_{\text{rings}} \cup \{R_i; i \in I\} \cup \{c_j; j \in J\}$ where $\mathcal{L}_{\text{rings}} := \{+, -, \cdot, 0, 1\}$, the c_j 's are constants and the R_i are n_i -ary predicates, $n_i > 0$. Let \mathcal{K} be an $\mathcal{L} \cup \{-1\}$ -structure such that its $\mathcal{L}_{\text{rings}}$ -reduct is a field of characteristic zero. Let \mathcal{V} of a basis of neighbourhoods of 0 in K and assume that $\langle K, \mathcal{V} \rangle$ is a topological \mathcal{L} -field as introduced in [15, Section 2]. We will assume here that the topology is non-discrete. Recall that every relation R_i (respectively its complement $\neg R_i$), with $i \in I$, is interpreted in \mathcal{K} , as the union of an open set O_{R_i} (respectively $O_{\neg R_i}$) and an algebraic subset $\{\bar{x} \in K^{n_i} : \bigwedge_k r_{i,k}(\bar{x}) = 0\}$ of K^{n_i} (respectively $\{\bar{x} \in K^{n_i} : \bigwedge_l s_{i,l}(\bar{x}) = 0\}$ of K^{n_i}), where $r_{i,k}, s_{i,l} \in K[X_1, \dots, X_{n_i}]$.

Examples of topological \mathcal{L} -fields are: ordered fields, ordered valued fields, valued fields, p -valued fields, fields endowed with several distinct valuations or several distinct orders [15, Section 2].

Let $\langle L, \mathcal{W} \rangle$ be a topological \mathcal{L} -extension of $\langle K, \mathcal{V} \rangle$ [15, Definition 2.3]. For a subset $\tilde{\mathcal{W}}$ of \mathcal{W} satisfying natural compatibility conditions with respect to the language \mathcal{L} that we called $Comp(K)$ (see [15, Definition 2.4] and [15, Notation 2.6]), we define for $a, b \in L$, the relation $a \sim_{\tilde{\mathcal{W}}} b$ by $(a - b \in V \text{ for all } V \in \tilde{\mathcal{W}})$.

Let T be an \mathcal{L} -theory (respectively a universal $\mathcal{L} \cup \{-1\}$ -theory) which admits a model-companion (respectively a model-completion) T_c . We will in addition assume that there is an \mathcal{L} -formula $\varphi(x, \bar{y})$ such that for $\mathcal{K} \models T$, the set of subsets of the form $\varphi(K, \bar{a}) := \{x \in K : \mathcal{K} \models \varphi(x, \bar{a})\}$ with $\bar{a} \subset K$ can be chosen as a basis \mathcal{V} of neighbourhoods of 0.

Let $L \models T_c$ extending \mathcal{K} , and endow L with the following basis of neighbourhoods of 0: $\tilde{\mathcal{V}} := \{\varphi(L, \bar{b}) : \bar{b} \in L\}$. Denote by $\tilde{\mathcal{V}}(K) := \{\varphi(L, \bar{a}) \text{ where } \bar{a} \text{ varies in } K\}$; this subset of neighbourhoods of 0 (in L) satisfies $Comp(K)$ (this follows from the fact that L is an elementary extension of \mathcal{K}).

Further, we will work under the extra-assumption that models of T_c satisfies *Hypothesis (I)*; it generalizes in our topological setting of the notion of *large fields* introduced by Pop [33], see [15, Definition 2.21] and [15, Section 2.3].

Recall that a field K is large if and only if it is existentially closed in the field of Laurent series $K((t))$ [33, Proposition 1.1] and equivalently in any iterated Laurent series field extension $K((t_1))((t_2)) \cdots ((t_n))$, for some natural number $n \geq 1$ (also denoted by $K((\mathbb{Z}^n))$) [37, Proposition 5.3]. (This second equivalence is straightforward using Frayne's embedding theorem: if a structure \mathcal{A} is existentially closed in \mathcal{B} ($\mathcal{A} \subseteq_{ec} \mathcal{B}$), then there is an embedding of \mathcal{B} in an ultrapower of \mathcal{A} , which is the identity on \mathcal{A} .)

Now let us recall below *Hypothesis (I)* (in a slightly less general form which will suffice in the present setting). First, we fix some notations. Let \mathcal{K} be a model of T_c and consider the iterated Laurent series field extension $K((\mathbb{Z}^n))$ endowed with the valuation map v taking its values in the lexicographic product \mathbb{Z}^n of n copies of $\langle \mathbb{Z}, +, -, <, 0, 1 \rangle$. We endow $K((\mathbb{Z}^n))$ with the following fundamental system of neighbourhoods \mathcal{W} of zero:

$$W_{V,0} := \{a \in K((t_1)) \cdots ((t_n)) : \alpha_0 \in V \text{ and } v(a) \geq 0\} \text{ with } V \in \mathcal{V},$$

$$W_\gamma := \{a \in K((t_1)) \cdots ((t_n)) : v(a) \geq \gamma\} \text{ with } \gamma \in (\mathbb{Z}^n)^{\geq 0}.$$

We will denote the corresponding topological structure by $\langle K((\mathbb{Z}^n)), \mathcal{W} \rangle$ and let $\mathcal{W}_{K,0} := \{W_{V,0} : V \in \mathcal{V}\}$. It is easy to see that $\langle K((\mathbb{Z}^n)), \mathcal{W} \rangle$ is a topological \mathcal{L}_{rings} -extension of $\langle K, \mathcal{V} \rangle$. In [15, Lemma 2.10], we show that the relation on $K((\mathbb{Z}^n))$ defined by $a \sim_{\mathcal{W}_{K,0}} b$ satisfies *Comp(K)*.

Definition 2.1 [15, Definition 2.21] *The class \mathcal{C} of models of T_c satisfies Hypothesis (I) if for every element $\langle K, \mathcal{V} \rangle$ of \mathcal{C} the following holds: given the topological \mathcal{L}_{rings} -extension $\langle K((\mathbb{Z}^n)), \mathcal{W} \rangle$ of \mathcal{K} and a polynomial $f(X) \in K((\mathbb{Z}^n))[X]$ with coefficients in $W_{K,0}$ if we have $f(a) \sim_{\mathcal{W}_{K,0}} 0$ and $f^2(a) \sim_{\mathcal{W}_{K,0}} 0$ for some element $a \in W_{K,0}$, then there exists a topological \mathcal{L}_{rings} -extension $\langle \tilde{L}, \tilde{\mathcal{V}} \rangle$ of $\langle K((\mathbb{Z}^n)), \mathcal{W} \rangle$ such that*

- (1) $\langle \tilde{L}, \tilde{\mathcal{V}} \rangle$ is a topological \mathcal{L} -extension of $\langle K, \mathcal{V} \rangle$ belonging to \mathcal{C} ;
- (2) there exists a subset $\tilde{\mathcal{V}}_K$ of $\tilde{\mathcal{V}}$ satisfying *Comp(K)*;
- (3) $t_i \sim_{\tilde{\mathcal{V}}_K} 0$, $i = 1, \dots, n$;
- (4) there exists an element b of \tilde{L} with $f(b) = 0$ and $a \sim_{\tilde{\mathcal{V}}_K} b$.

Remark 2.2 Note that if the language \mathcal{L} does not contain any relation symbols and if the models of T_c are in addition large fields, then the class of models of T_c satisfies *Hypothesis (I)*. Indeed, when K is a large field, we have $K \subseteq_{ec} K((\mathbb{Z}^n))$. Therefore by Frayne's theorem, there is a non-principal ultrapower K^* of K such that $K \subseteq K((\mathbb{Z}^n)) \subseteq K^*$. In order to check that if $K \models T_c$, then it satisfies *Hypothesis (I)*, we set $\tilde{L} = K^*$ and we endow K^* with the basis of neighbourhoods $\tilde{\mathcal{V}} := \{\varphi(K^*, \vec{d}) : \vec{d} \in K^*\}$ and we take $\tilde{\mathcal{V}}_K := \tilde{\mathcal{V}}(K) = \{\varphi(K^*, \vec{d}) : \vec{d} \in K\}$. We use the fact that $K((\mathbb{Z}^n))$ is henselian and that if $a, b \in K((\mathbb{Z}^n))$ and $a \sim_{\mathcal{W}_{K,0}} b$, then $a \sim_{\tilde{\mathcal{V}}(K)} b$. Indeed if $a \sim_{\mathcal{W}_{K,0}} b$, then $K((\mathbb{Z}^n)) \models \varphi(a - b, \vec{d})$ for any $\vec{d} \in K$. Since the formula φ can be taken existential, it still holds in K^* and so $a \sim_{\tilde{\mathcal{V}}(K)} b$.

We consider expansions of topological \mathcal{L} -fields to $\mathcal{L} \cup \{-1, D\}$ -structures, where D is a new unary function symbol which will satisfy the axioms of a derivation:

$$\forall a \forall b \ D(a + b) = D(a) + D(b), \forall a \forall b \ D(a.b) = a.D(b) + D(a).b.$$

We will call such generic expansions, *differential topological \mathcal{L} -fields*. Let $\mathcal{L}_D := \mathcal{L} \cup \{-1, D\}$, $\mathcal{L}_{rings,D} := \mathcal{L}_{rings} \cup \{D\}$.

Notation 2.3 Let $K\{X_1, \dots, X_n\}$ be the ring of differential polynomials over K in n differential indeterminates X_1, \dots, X_n over K , namely it is the ordinary polynomial ring in indeterminates $X_i^{(j)}$, $1 \leq i \leq n$, $j \in \omega$, with by convention $X_i^{(0)} := X_i$. One can extend the derivation D of K to this ring by setting $D(X_i^{(j)}) := X_i^{(j+1)}$ and using the additivity and the Leibnitz rule.

Set $\mathbf{X} := X_1, \dots, X_n$. Let $f(\mathbf{X}) \in K\{\mathbf{X}\} \setminus K$ and suppose that f is of order m , then we write $f(\mathbf{X}) = f^*(X_1, \dots, X_1^{(m)}, \dots, X_n, \dots, X_n^{(m)})$ for some ordinary polynomial $f^*(X_1, \dots, X_{n \cdot (m+1)})$ in $K[X_1, \dots, X_{n \cdot (m+1)}]$. We will make the following abuse of notation: if $\bar{b} \in K^n$, then $f^*(\bar{b})$ means that we evaluate the polynomial f^* at the tuple $\bar{b}^{\nabla m} := (b_1, \dots, b_1^{(m)}, \dots, b_n, \dots, b_n^{(m)})$, where $b^{(i)} := D(b^{(i-1)})$, $b^{(0)} := b$, $i \geq 1$ and $\bar{b} := (b_1, \dots, b_n)$. (Sometimes, we simply denote the tuple $\bar{b}^{\nabla m}$ by \bar{b}^∇ .) If $n = 1$, recall that the separant s_f of f is defined as $s_f := \frac{\partial f}{\partial X_1^{(m)}}$. Finally for $f(\mathbf{X}) \in K\{\mathbf{X}\}$, we denote by $f^D(\mathbf{X})$ the differential polynomial obtained from $f(\mathbf{X})$ by applying the derivative D to its coefficients.

Let $\phi(x_1, \dots, x_n)$ be a quantifier-free \mathcal{L}_D -formula, for each x_i , $1 \leq i \leq n$, let m_i be the maximal natural number m such that $x_i^{(m)}$ occurs in an atomic subformula.

Then, we denote by $\phi^*((x_{i,j})_{i=1, j=0}^{n, m_i})$ the formula we obtain from ϕ by replacing each $x_i^{(j)}$ by $x_{i,j}$.

Let $R := K[\bar{X}]$. Recall that $R[X_k]$ satisfies the generalized Euclidean algorithm. Namely, given $f(X_k), g(X_k) \neq 0$ two polynomials in $R[X_k]$ and letting $b \in R$ be the leading coefficient of $g(X_k)$ (viewed as a polynomial in X_k), then there exists $d \in \mathbb{N}$ and polynomials $q(X_k), r(X_k) \in R[X_k]$ such that $b^d \cdot f(X_k) = q(X_k) \cdot g(X_k) + r(X_k)$ with $\deg(r(X_k)) < \deg(g(X_k))$ [18, Theorem 2.14].

Let T be a theory of topological \mathcal{L} -fields admitting a model companion T_c . Let T_D (respectively $T_{c,D}$) be the \mathcal{L} -theory T (respectively T_c) together with the axioms stating that D is a derivation. Under the assumption that the models of T_c satisfy Hypothesis (I), we show that T_D admits a model companion $T_{c,D}^{*,\omega}$ [15, Theorem 9.3]. In the special case of $\mathcal{L} = \mathcal{L}_{rings}$ and T_c a model complete theory of large fields, the existence of the model companion of $T_{c,D}$ is due to Tressl [37, Theorem 7.2]. This was recently revisited in [5] where one can find, in particular, a geometric axiomatisation of the model-companion and a proof if $\mathcal{K} \models T_{c,D}^{*,\omega}$, then the subfield of constants C_K is always a model of T_c .

In the case where T is universally axiomatisable in $\mathcal{L} \cup \{-1\}$ and T_c is its model completion, we first show that any model of T_D embeds into one satisfying the scheme (DL) [15, Proposition 3.9], namely for each differential polynomial $f(X) = f^*(X, X^{(1)}, \dots, X^{(n)}) \in K\{X\}$, for every $W \in \mathcal{V}$,

$$(\exists \alpha_0, \dots, \alpha_n (f^*(\alpha_0, \dots, \alpha_n) = 0 \wedge s_f^*(\alpha_0, \dots, \alpha_n) \neq 0)) \Rightarrow \\ \left(\exists z (f(z) = 0 \wedge s_f(z) \neq 0 \wedge \bigwedge_{i=0}^n (z^{(i)} - \alpha_i \in W)) \right).$$

Note that in [15], the scheme (DL) is not quite given as above (the coefficients of the polynomials f vary over a smaller subset), but in an equivalent form [15, Proposition 3.14].

Since we assumed that the topology is first-order definable, the scheme of axioms (DL) can be expressed in a first-order way. Let $T_{c,D}^*$ be the \mathcal{L}_D -theory consisting of $T_{c,D}$ together with the scheme (DL) , then $T_{c,D}^*$ is the model completion of T_D (in particular $T_{c,D}^*$ admits quantifier elimination). A consequence of that axiomatisation is that in a model \mathfrak{U} of $T_{c,D}^*$, the subfield of constants C_U is dense in \mathfrak{U} [15, Corollary 3.13].

Lemma 2.4 *Under the above hypothesis, let $\mathcal{M} \models T_{c,D}^*$ and let C_M be its subfield of constants. Then the isolated points are not dense in the Stone space of 1-types $S_1^{T_{c,D}^*}(C_M)$.*

Proof The proof is analogous to the proof when $T_{c,D}^* = CODF$ [35]. Consider the clopen subset $[x^{(1)} = 1]$ in $S_1^{T_{c,D}^*}(C_M)$; any realisation of that formula in a model of $T_{c,D}^*$ containing M is an (algebraically) transcendental element over C_M . Assume that we have a formula $\chi(x)$ (with parameters in C_M) isolating a point in that clopen subset. Then the formula $\chi(x)$ is equivalent to $(x^{(1)} = 1 \wedge \theta(x))$, where θ is an open formula (with parameters in C_M) with the property that $\theta^*(M)$ is a non-empty open subset of some cartesian product of M . Moreover since we may assume that only the variable x (and not any term of the form $x^{(n)}$) occurs in θ , we may assume that $\theta^*(M)$ is actually a subset of M and equal to $\theta(M)$. Since the topology of M is definable, non-discrete and C_M dense in M , we can find two non-empty disjoint open definable subsets $\theta_1(M)$, $\theta_2(M)$ included in $\theta(M)$, with θ_1 , θ_2 two open formulas with parameters in C_M . By the scheme (DL) , both formulas $(x^{(1)} = 1 \wedge \theta_1(x))$, $(x^{(1)} = 1 \wedge \theta_2(x))$ are realized in M . Indeed, we can find two algebraic solutions namely $(a, 1)$ and $(b, 1)$ in M^2 with $\theta_1(a)$ and $\theta_2(b)$. Therefore there exist two differential solutions u_1, u_2 in M with $u_1^{(1)} = 1$, $u_2^{(1)} = 1$, close to respectively $(a, 1)$ and $(b, 1)$, which implies that $\theta_1(u_1)$ and $\theta_2(u_2)$ also hold, a contradiction. \square

Notation 2.5 Let $K \subset L$ be a pair of differential fields and $\bar{a} \in L$, then we denote $\mathcal{I}_K^D(\bar{a})$ the differential ideal: $\{p(\bar{X}) \in K\{\bar{X}\} : p(\bar{a}) = 0\}$. Let $\langle f \rangle$ denote the differential ideal generated by f . The ideal $\mathcal{I}_K^D(\bar{a})$ is a prime differential ideal of the form, for some $f \in K\{X\}$, $\mathcal{I}(f) := \{g \in K\{X\} : g.s_f^k \in \langle f \rangle, \text{ for some } k \in \mathbb{N}\}$ [22, Lemma 1.4].

We will denote by $\mathfrak{U} \upharpoonright \mathcal{L}$ the \mathcal{L} -reduct of \mathfrak{U} . Given a subset B of \mathfrak{U} , we will denote by $\mathcal{L}(B)$ (respectively $\mathcal{L}_D(B)$) the expansion of \mathcal{L} (respectively \mathcal{L}_D) by constants for each element of B . In the following lemma, we relate the algebraic closure in models of $T_{c,D}^*$ to the algebraic closure in their \mathcal{L} -reducts.

Fact 2.6 [7, Lemma 5.5] *Let $\mathcal{A} \models T_D$ and let \mathfrak{U} be a model of $T_{c,D}^*$ extending \mathcal{A} . Then the algebraic closure $\text{acl}^{\mathfrak{U}}(A)$ is equal to $\text{acl}^{\mathfrak{U} \upharpoonright \mathcal{L}}(A)$.*

Proof For convenience of the reader we reproduce the proof below. Let $a \in \text{acl}^{\mathfrak{U}}(A)$ and let $\phi(x)$ be an $\mathcal{L}_D(A)$ -formula such that $\phi(a)$ holds in \mathfrak{U} and which has only finitely

many realizations. Since $T_{c,D}^*$ admits quantifier elimination, $\phi(x)$ is equivalent to a finite disjunction of formulas of the form:

$$\bigwedge_{i \in I} p_i(x) = 0 \wedge \theta(x),$$

where $\theta^*(\mathfrak{U})$ is an open subset of some cartesian product of \mathfrak{U} and $p_i(X) \in A\{X\}$ [15, Theorem 4.1]. If $I = \emptyset$, then we obtain a contradiction since $\theta(\mathfrak{U})$ is infinite (it is a direct consequence of the scheme (DL) that near every tuple, one can find a tuple of the form d^∇ [15, Lemma 3.12]). Therefore we may assume that $I \neq \emptyset$; consider $\mathcal{I}_A^D(a)$. This is a prime ideal of the form $\mathcal{I}(f)$, for some $f \in A\{X\}$ [22, Lemma 1.4]. Note that $f^*(a^\nabla) = 0 \wedge s_f^*(a^\nabla) \neq 0$. If the set of solutions in \mathfrak{U} of the formula $f^*(\bar{y}) = 0 \wedge \theta^*(\bar{y})$ is finite, then $a \in \text{acl}_{\mathcal{L}}(A)$. If not, by the scheme (DL), there exist elements $b \in \mathfrak{U}$ satisfying $f(b) = 0$ (and $\theta(b)$) in a neighbourhood of each of these points included in $\theta^*(\mathfrak{U})$. So we get a contradiction with the finiteness of the number of solutions of $\phi(x)$. \square

The following is folklore (see for instance [19, Lemma 6.2.9] and also [8] in the case of RCF). It reduces the description of definable n -types to definable 1-types.

Let $\mathcal{M} \models T_c$ and assume that for any subset B of M we have a prime model extension, that we will denote by $\langle B \rangle$, to a model of T_c . Let \bar{m} be a tuple of elements of M , we denote by $tp_B^{\mathcal{M}}(\bar{m})$ the type of \bar{m} over B in \mathcal{M} and by $B\bar{m}$ the subset $B \cup \{\bar{m}\}$.

Lemma 2.7 *Let $\mathcal{M} \models T_c$. Let $A \subset M$, assume that \mathcal{M} is $|A|^+$ -saturated. Let $\bar{a}_1, \bar{a}_2 \in M$ and suppose that $tp_{\langle A \rangle}^{\mathcal{M}}(\bar{a}_1)$ is definable over $\langle A \rangle$ and that $tp_{\langle A\bar{a}_1 \rangle}^{\mathcal{M}}(\bar{a}_2)$ is definable over $\langle A\bar{a}_1 \rangle$. Then $tp_{\langle A \rangle}^{\mathcal{M}}(\bar{a}_1, \bar{a}_2)$ is definable over A .*

Proof Let $\phi(\bar{v}_1, \bar{v}_2, \bar{w})$ be an \mathcal{L} -formula. We will show that we can find an $\mathcal{L}(A)$ -formula $d\phi(\bar{w})$ such that for any $\bar{a} \in \langle A \rangle$, $\phi(\bar{v}_1, \bar{v}_2, \bar{a}) \in tp_{\langle A \rangle}^{\mathcal{M}}(\bar{a}_1, \bar{a}_2)$ iff $\mathcal{M} \models d\phi(\bar{a})$.

By hypothesis, $tp_{\langle A\bar{a}_1 \rangle}^{\mathcal{M}}(\bar{a}_2)$ is definable over $\langle A\bar{a}_1 \rangle$. So we have an $\mathcal{L}(\langle A\bar{a}_1 \rangle)$ -formula $d^1\phi(\bar{w})$ such that $d^1\phi(\bar{a}_1, \bar{a})$ holds iff $\phi(\bar{a}_1, \bar{v}_2, \bar{a}) \in tp_{\langle A\bar{a}_1 \rangle}^{\mathcal{M}}(\bar{a}_2)$. Let $\bar{d} \in \langle A\bar{a}_1 \rangle$ be the parameters occurring in $d^1\phi$ and rewrite that formula as an \mathcal{L} -formula $\tilde{d}^1\phi(\bar{w}, \bar{d})$. Since \bar{d} belongs to $\langle A\bar{a}_1 \rangle$, its type is isolated over $A\bar{a}_1$ by an $\mathcal{L}(A\bar{a}_1)$ -formula $\psi(\bar{x})$. Rewrite that formula as an $\mathcal{L}(A)$ -formula $\tilde{\psi}(\bar{x}, \bar{a}_1)$. Consider the formula $\chi(\bar{a}_1, \bar{a}) := \forall \bar{x} (\tilde{\psi}(\bar{x}, \bar{a}_1) \rightarrow \tilde{d}^1\phi(\bar{a}_1, \bar{a}, \bar{x}))$. Since $tp_{\langle A \rangle}^{\mathcal{M}}(\bar{a}_1)$ is definable over $\langle A \rangle$, there is an $\mathcal{L}(\langle A \rangle)$ -formula $d^2\chi(\bar{z})$ such that $d^2\chi(\bar{a})$ holds if and only if $\chi(\bar{a}_1, \bar{a}) \in tp_{\langle A \rangle}^{\mathcal{M}}(\bar{a}_1)$. Since $\langle A \rangle$ is a prime extension of A , we may assume that the only parameters occurring in $d^2\chi$ belong to A ; the corresponding $\mathcal{L}(A)$ -formula is the formula we looked for. \square

3 Axiomatization

As recalled in the preceding section, examples of theories T and of the corresponding theories T_c [15, Section 2] to which we may apply our set-up, are in each case, the \mathcal{L} -theory T of :

- (1) ordered fields with $\mathcal{L} := \mathcal{L}_< := \mathcal{L}_{rings} \cup \{<\}$, with T_c the theory RCF of real-closed fields,
- (2) ordered valued fields with $\mathcal{L} := \mathcal{L}_< \cup \{div\}$, where div is a binary predicate defined as $x \text{ div } y$ iff $v(x) \leq v(y)$, where v is a valuation, with T_c the theory $RCVF$ of real-closed valued fields,
- (3) valued fields with $\mathcal{L} := \mathcal{L}_{rings} \cup \{div\}$, with T_c the theory $ACVF_0$ of algebraically closed fields of characteristic $(0, 0)$,
- (4) p -valued fields with $\mathcal{L} := \mathcal{L}_{rings} \cup \{div\} \cup \{P_n : n \geq 2, n \in \mathbb{N}\}$ with T_c the theory pCF_d of p -adically closed fields of characteristic $(0, p)$ and p -rank d .

One can note a few straightforward consequences of the axiomatisation of $T_{c,D}^*$, in some of the above cases.

Remark 3.1

- (1) If $\mathcal{K} \models RCVF_D^*$, then $\mathcal{K} \models CODF$.

Proof Let $\mathcal{K} \models RCVF_D^*$, in particular \mathcal{K} is an ordered differential field, so \mathcal{K} embeds into a model $\tilde{\mathcal{K}}$ of $CODF$. Let us show that $\mathcal{K} \models CODF$. Let us take the axiomatisation of Singer [35]. So consider a system of the form $f(X) = 0 \wedge \bigwedge_{i=1}^m g_i(X) > 0$ with $ord(g_i) \leq ord(f)$ and $f(X), g_i(X) \in K\{X\}$. Suppose this system has an algebraic solution \tilde{a} in \tilde{K} such that $s_f(\tilde{a}) \neq 0$. Then since both K and \tilde{K} are models of RCF , there is an algebraic solution in K and so by the scheme (DL) , there is a differential solution in K . \square

- (2) If $K \models ACVF_D^*$, then $K \models DCF_0$.

Proof Consider a system of the form $f(X) = 0 \wedge g(X) \neq 0$ with $ord(g) < ord(f)$, $f, g \in K\{X\}$. This system has a solution a in the differential closure \hat{K} of K . Let $h(X) \in K\{X\}$ be a differential polynomial such that $\mathcal{I}(h) = \mathcal{I}_K^D(a)$ and consider the system $h(X) = 0 \wedge s_h(X) \neq 0$. So \hat{K} has an algebraic solution of the corresponding algebraic system. Since K is an existentially closed subfield of \hat{K} , K has an algebraic solution and so a differential one by the scheme (DL) . \square

- (3) If $\mathcal{K} \models RCVF_D^*$, then by (1), $\mathcal{K} \models CODF$ and we have both that $\mathcal{K}(i) \models DCF_0$ [35] and $\mathcal{K}(i) \models ACVF_0$ [25, Corollary 1.3]. However, is it the case that $\mathcal{K}(i) \models ACVF_D^*$?

Note that in each of the cases RCF , $RCVF$ $ACVF_0$ and pCF_d , there exist prime models extensions over subfields A [39, 40]. In particular, the type of a tuple of elements of $\langle A \rangle$ over A will be isolated over A . In all of these cases $\langle A \rangle$ is equal to the relative field algebraic closure of A , which implies that if $\mathfrak{U} \models T_{c,D}^*$, then $C_{\mathfrak{U}} \models T_c$. Note also that in all cases but $ACVF_0$, we have $\langle A \rangle = dcl(A)$, where dcl denotes the model-theoretic definable closure.

4 Definable types

In this section, we will show that in the case certain types are definable in models of T_c , definable types are dense in the Stone spaces of n -types $S_n^{T_c, D}(A)$, where A is a definably closed subset of a model of $T_{c, D}^*$. We will first prove the statement for 1-types and then Lemma 2.7 will entail it holds for n -types.

In the following, types will always be complete types unless we use the term *partial types*.

Definition 4.1 Let $\mathcal{M} \models T_c$. Recall that the set of subsets of the form $\varphi(M, \bar{a}) := \{x \in M : \mathcal{M} \models \varphi(x, \bar{a})\}$ with $\bar{a} \subset M$ can be chosen as a basis \mathcal{V} of neighbourhoods of 0. Let A be a substructure of \mathcal{M} . We will say that a partial 1-type over A is a 0^+ -type over A if it contains the set of all $\mathcal{L}(A)$ -formulas $\varphi(x, \bar{a})$, with $\bar{a} \subset A$ together with $x \neq 0$.

Example 1 Let \mathcal{M} be an expansion of an ordered field. Then a 0^+ -type (over M) is any partial type containing the set of formulas: $\{0 < |x| < m : m \in M_{>0}\}$.

Definition 4.2 [23, Preliminaries] Let \mathcal{M} be an ordered field. A partial type $p(x)$ over M is a cut if it contains the set of formulas: $\{c_1 < x < c_2 : c_1 \in C_1, c_2 \in C_2\}$ where $C_1 < C_2$ are two nonempty disjoint subsets of M with $C_1 \cup C_2 = M$ and C_1 (respectively C_2) has no maximum (respectively no minimum) in M . Note that this is sometimes called an irrational cut. Recall that \mathcal{M} is Dedekind complete in $\mathcal{M}(\bar{a})$ if no cut of \mathcal{M} is realised in $\mathcal{M}(\bar{a})$.

Extending previous results of van den Dries in the context of real-closed fields, Marker and Steinhorn have described definable types in o-minimal theories. Let T be an o-minimal theory and let $p(\bar{v}) \in S_n^T(M)$, where $\mathcal{M} \models T$. Then $p(\bar{v})$ is definable iff \mathcal{M} is Dedekind complete in $\langle \mathcal{M}\bar{a} \rangle$ where \bar{a} any realisation of $p(\bar{v})$ [23, Theorem 2.1]. Moreover if $A \subset M$ and $A = dcl(A)$, then $p(\bar{v})$ is definable over A iff A is Dedekind complete in $dcl(A, \bar{a})$ for any \bar{a} realising p ([23, Theorem 4.1], [29]).

If $\mathcal{M} := (M, <, \dots)$ is a weakly o-minimal structure, then Mellor described definable 1-types as follows. Let C be a convex subset of M . Then, a set A is initial (respectively final) on C if $(\forall x \in C) (\exists y \in A \cap C) (x > y)$ (respectively $(x < y)$).

Let $B \subset M$ and $p(x) \in S_1(B)$, a left (respectively right) generic type of C over B is the set of formulas with parameters in B that define sets which are initial (respectively final) on C . Mellor showed that $p(x)$ is B -definable if for some convex B -definable subset C of M , $p(x)$ is the right or left generic type of C (over B). Moreover, if $Th(\mathcal{M})$ is a model-complete weakly o-minimal theory and B an elementary substructure of \mathcal{M} , then these are exactly the B -definable 1-types [25, Proposition 2.13].

Let \mathcal{M} be an expansion of a non-trivially valued field. Then there are three kinds of non-realized 1-types over M : the valuational, residual and limit types. The valuational types are realised by elements increasing the value group, a realisation of a residual type increases the residue field and a limit type is realised in an immediate extension. One can show that a realisation of a limit type over M is a limit of a pseudo-Cauchy sequence of M [21, Theorem 1].

Example 2 Let \mathcal{M} be a non-trivially valued field, then a 0^+ -type over M is a special kind of valational type, namely a type containing the set of formulas: $\{v(x) > v(m) : m \in M \setminus \{0\}\}$.

In $ACVF_0$, the definable 1-types have been described over models; they are either residual or valational. Moreover, valational types are not definable if and only if they determine a cut in the value group [12, Corollary 2.7].

If \mathcal{M} be a model of ${}_pCF_1$. Then again the definable types have been described in an expansion of the valued field language with coefficients maps [4]. Since we are working in the language with the predicates P_n (the Macintyre language), a more adequate reference is [3, Proposition 4.6 and its proof].

Example 3 Let \mathcal{M} be a model of ${}_pCF_1$. Then the complete 0^+ -types are of the form: $\{v(x) > v(m) : m \in M \setminus \{0\}\} \cup \{P_n(x.e_n) : e_n \in \mathbb{N} \setminus \{0\}, n \in \mathbb{N} \setminus \{0, 1\}\}$.

Proposition 4.3 *Let T be a universal theory of topological \mathcal{L} -fields, satisfying Hypothesis (I) and admitting a model completion T_c . Assume in addition that T_c has prime model extensions and that in any model \mathcal{K} of T_c , there exists a complete 0^+ -type $q_K(x)$ over K , which is $\mathcal{L}(K)$ -definable in T_c . Let $\mathcal{M} \models T_{c,D}^*$ and let $A \subset M$ be a model of $T_{c,D}$. Then the definable types over A are dense in $S_1^{T_{c,D}^*}(A)$.*

Proof Let $\mathcal{M} \models T_{c,D}^*$ and let $A \subset M$ be a model of $T_{c,D}$. Let $\phi(x, \bar{y})$ be an \mathcal{L}_D -formula and let $\bar{a} \in A$. Let $[\phi(x, \bar{a})]$ a clopen subset of $S_1^{T_{c,D}^*}(A)$. We may assume that $\phi(x, \bar{y})$ is quantifier-free and of the form: $\bigwedge_{i \in I} p_i(x) = 0 \wedge \theta(x)$, where $p_i[X] \in A\{X\} \setminus \{0\}$ and $\theta^*(M)$ is an open subset in some cartesian product of M .

Let u be a realisation of $\phi(x, \bar{a})$ in \mathcal{M} and first note that we may further assume that $I \neq \emptyset$.

Indeed, suppose otherwise that $I = \emptyset$. Let n be the maximum order of the variable x occurring in the formula θ . Then consider the clopen subset of $[\phi(x, \bar{a})]$ defined by $[\theta(x) \wedge x^{(n+1)} = 0]$. We will show that there is a definable type (over A) in that clopen subset.

So we reduce ourselves to the case where the differential ideal $\mathcal{I}_A^D(u)$ of $A\{X\}$ is non-trivial. Let $f(X)$ be an irreducible polynomial such that $\mathcal{I}(f) = \mathcal{I}_A^D(u)$ (see Notation 2.5). Assume that the order of f is equal to $n+1$, $n \in \mathbb{N}$. Note that if $f(u) = 0$, then $u^{(n+1+k)} \in A(u, \dots, u^{(n+1)})$, $k \geq 1$, where $A(u, \dots, u^{(n+1)})$ denotes the field generated by A and $u, \dots, u^{(n+1)}$. So without loss of generality we will assume that in $\theta(x)$ the variable x occurs with order at most $n+1$.

Since $A \models T_c$, there is a $n+2$ -tuple \bar{d} in A such that $f^*(\bar{d}) = 0 \wedge s_f^*(\bar{d}) \neq 0 \wedge \theta^*(\bar{d})$.

Let \mathcal{N} be a $|A|^+$ -saturated extension of A and let $\epsilon_0 \in N$ realising $q_A(x)$, then ϵ_1 realising $q_{A(\epsilon_0)}$, and iterating n times ϵ_n realising $q_{A(\epsilon_0, \dots, \epsilon_{n-1})}(x)$. Note that $\epsilon_0, \dots, \epsilon_n$ are algebraically independent over A . Using the fact that the models of T_c satisfy Hypothesis (I), we show in [15, Lemma 3.7], that there exists a model $\tilde{\mathcal{N}}$ of $T_{c,D}$ extending $\langle A(\epsilon_0, \dots, \epsilon_n) \rangle$ and an element $b \in \tilde{\mathcal{N}}$ such that $f(b) = 0 \wedge s_f(b) \neq 0$ with $b^{(i)} = d_i + \epsilon_i$, $0 \leq i \leq n$, $(b^{(n+1)})$ close to d_{n+1} with respect to A . This implies that we have in addition $\theta(b)$. Note that $b^{(n+1)}$ is algebraic over $A(b^{(0)}, \dots, b^{(n)})$ and so its \mathcal{L} -type is isolated (and so definable) over $A(b^{(0)}, \dots, b^{(n)})$.

Claim 4.4 The \mathcal{L} -type of $(b, b^{(1)}, \dots, b^{(n+1)})$ over A is $\mathcal{L}(A)$ -definable.

Proof of Claim The field $A(b, b^{(1)}, \dots, b^{(n)})$ is equal to $A(b - d_0, b^{(1)} - d_1, \dots, b^{(n)} - d_n)$. By assumption, each $b^{(i)} - d_i$ realizes a definable 0^+ -type over $A(b - d_0, b^{(1)} - d_1, \dots, b^{(i-1)} - d_{i-1})$, $n \geq i \geq 1$, and $b - d_0$ realizes a definable 0^+ -type over A . So by Lemma 2.7, the type of $(b, b^{(1)}, \dots, b^{(n)})$ over A is $\mathcal{L}(A)$ -definable. Finally $b^{(n+1)}$ is algebraic over $A(b, b^{(1)}, \dots, b^{(n)})$ and applying again Lemma 2.7, we get that the type of $(b, b^{(1)}, \dots, b^{(n+1)})$ over A is $\mathcal{L}(A)$ -definable. \square

Claim 4.5 The \mathcal{L}_D -type of b over A is $\mathcal{L}_D(A)$ -definable.

Proof of Claim The \mathcal{L}_D -type of b over A in $T_{c,D}^*$ is determined by the $(n+2)$ - \mathcal{L} -type of $(b, b^{(1)}, \dots, b^{(n+1)})$ in T_c over A . (Note that A is a differential subfield of M .) \square

Let $\psi(x, \bar{a})$ be an open $\mathcal{L}(A)$ -formula belonging to $tp(b/A)$. Then since $T_{c,D}^*$ admits quantifier elimination, we may assume that $\psi(x, \bar{a})$ is of the form: $\bigvee_{\ell \in L} (\bigwedge_{i \in I_\ell} p_i(x) = 0 \wedge \chi_\ell(x))$, with for all $\ell \in L$, $i \in I_\ell$, $p_i(X) \in A\{X\} \setminus \{0\}$ and for all $j \in J_\ell$, χ_ℓ^* defines an open subset. Since $f(b) = 0$ and $s_f(b) \neq 0$, $b^{(n+k)} \in A(b, \dots, b^{(n+1)})$, $k \geq 2$, so without loss of generality, we may assume that the orders of $p_i(X)$ are at most $n+1$ and that the order of the variable x in χ_ℓ is at most $n+1$. Therefore we may rewrite $p_i(x) = p_i^*(x, \dots, x^{(n+1)})$ and $\chi_\ell(x) = \chi_\ell^*(x, \dots, x^{(n+1)})$, we can transform the \mathcal{L}_D -formula $\psi(x, \bar{a})$ into an \mathcal{L} -formula $\psi^*(x, \dots, x^{(n+1)}, \bar{a}, \dots, \bar{a}^{(m)})$ with the property that $\psi(x, \bar{a}) \in tp_A^{T_{c,D}^*}(b)$ iff $\psi^*(x, \dots, x^{(n+1)}, \bar{a}, \dots, \bar{a}^{(m)}) \in tp_A^{T_c}(b, \dots, b^{(n+1)})$. By the previous claim, this last type is definable over A . So there exists an $\mathcal{L}(A)$ -formula $d\psi^*$ such that $\psi^*(x, \dots, x^{(n+1)}, \bar{a}, \dots, \bar{a}^{(m)}) \in tp_A^{T_c}(b, \dots, b^{(n+1)})$ iff $\mathcal{M} \models d\psi^*(\bar{a}, \dots, \bar{a}^{(m)})$. Finally rewrite $d\psi^*$ as an $\mathcal{L}_D(A)$ -formula $d\psi$ and we get $\psi(x, \bar{a}) \in tp^{T_{c,D}^*}(b)$ iff $\mathcal{M} \models d\psi(\bar{a})$. \square

Corollary 4.6 Under the same hypothesis on T_c , \mathcal{M} and A , the definable types over A are dense in $S_n^{T_{c,D}^*}(A)$.

Proof It follows from the above proposition and Lemma 2.7. Indeed by induction assume that the definable types are dense in $S_{n-1}(\bar{A})$, for $n \geq 2$ and $\bar{A} \models T_{c,D}$. Let $\phi(\bar{x})$ be an $\mathcal{L}(A)$ -formula with $\bar{x} := (x_1, x_2, \dots, x_n)$. Consider the formula $\exists x_2 \dots \exists x_n \phi(x_1, x_2, \dots, x_n)$. By the above proposition, there is a definable type, say $p(x_1)$, over A in the corresponding clopen subset. Let \mathcal{N} be a $|M|^+$ -saturated elementary extension of \mathcal{M} and let $a_1 \in N$ be a realisation of $p(x_1)$. Then consider $B := \langle A(a_1) \rangle$. By assumption, there is $q(x_2, \dots, x_n)$ a definable type over B containing the formula $\phi(a_1, x_2, \dots, x_n)$. Let (a_2, \dots, a_n) be a realization of $q(x_2, \dots, x_n)$ in N . By Lemma 2.7, $tp_A^{\mathcal{M}}(a_1, \dots, a_n)$ is definable over A and contains $\phi(\bar{x})$. \square

Corollary 4.7 For $T_c = RCF, RCVF, ACVF, {}_pCF_1$ and $\mathcal{M} \models T_{c,D}^*$, the definable types over A are dense in $S_n^{T_{c,D}^*}(A)$, with $A \subset M$ a model of $T_{c,D}$, $n \geq 1$. \square

In the case $T_{c,D}^* = CODF$, Brouette obtained the following characterisation of definable types [8, Chapter 4]. The main ingredients were the analog of Proposition 4.3,

the description of definable types for RCF recalled at the beginning of the section and the fact that models of $CODF$ are definably complete [31] (i.e. any bounded definable non-empty subset in the domain of a model has an upper bound).

Fact 4.8 [9] *Let $p(x) \in S_1^{CODF}(A)$, where A is a differential real-closed ordered field. Then $p(x)$ is definable over A if and only if A is Dedekind complete in $A(v, \dots, v^{(n)}, \dots)$, for any element v realizing $p(x)$.* \square

5 VC-density

Aschenbrenner et al. [1] calculate bounds for Vapnik-Chervonenkis (VC) densities for o-minimal, weakly o-minimal, P-minimal theories. The tool they use is the link between VC-density and the uniform definability property of types over finite sets (UDTFS property) introduced by Guingona [14, Definition 2.1], which they generalised to finite sets of formulas [1, Definition 5.1].

In view of the transfer of the NIP property that we recall below, a natural question is to relate VC-density bounds in T_c to ones in $T_{c,D}^*$. In the next section, we will work with the theory $T_{c,D}^{*,\omega}$ and so we state our result in a slightly more general context.

Fact 5.1 [15] *Let T_c be a model-complete theory of topological \mathcal{L} -fields satisfying Hypothesis (I). Assume that any quantifier-free formula in $T_{c,D}$ is NIP , then any quantifier-free formula in $T_{c,D}^{*,\omega}$ is NIP .*

Proof This follows from [15, Lemma 4.2, Theorem 9.3]. \square

In the further case when T_c admits quantifier elimination and is NIP , it implies that $T_{c,D}^*$ is NIP (since it admits quantifier elimination too). Therefore, we can apply the above result to the theories: $T_c = RCF$, ${}_pCF_d$, $RCVF$, $ACVF_0$.

Let us fix notation and recall some basic definitions. In the following T will denote a complete \mathcal{L} -theory with only infinite models and let \mathcal{M} be a model of T . Given a (partitioned) \mathcal{L} -formula $\phi(\bar{x}; \bar{y})$, let $\phi^{dual}(\bar{x}; \bar{y})$ be the dual formula, namely $\phi(\bar{y}; \bar{x})$.

Let $B \subset M^{|\bar{y}|}$ and let $S_\phi(B)$ be the set of ϕ -types over B , namely the set of consistent sets of formulas of the form $\{\phi(\bar{x}; \bar{b}) : b \in B'\} \cup \{\neg\phi(\bar{x}; \bar{b}) : \bar{b} \in B \setminus B'\}$. If Δ is a finite set of partitioned formulas with \bar{x}, \bar{y} of some fixed length, one defines Δ -types similarly.

Set $\pi_{\phi^{dual}(\bar{x}; \bar{y})}(t) := \max\{|S_\phi(B)| : B \subset M^{|\bar{y}|}, |B| = t\}$.

The dual VC-density of ϕ is equal to the infimum of all real numbers r such that $\frac{\pi_{\phi^{dual}(\bar{x}; \bar{y})}(t)}{t^r}$ is bounded (as a function of t) and denoted by $vc(\phi^{dual})$.

Let $vc(m) := \sup\{vc(\phi^{dual}(\bar{x}; \bar{y})) : \phi(\bar{x}; \bar{y}) \text{ is an } \mathcal{L}\text{-formula, } |\bar{x}| = m\}$, $m \in \mathbb{N} \setminus \{0\}$.

An \mathcal{L} -structure \mathcal{M} has the VCd property if any finite set $\Delta(\bar{x}; \bar{y})$ of \mathcal{L} -formulas with $|\bar{x}| = 1$ has a uniform definition of $\Delta(\bar{x}; B)$ -types over finite sets with d parameters [2, 5.2]. If T has the VCd property, then $m \leq vc(m) \leq d.m$ for every $m \in \mathbb{N} \setminus \{0\}$ [2, Corollary 5.8].

The following result on the dual VC-density for weakly o-minimal theories T (for instance $T = RCF$ or $T = RCVF$) was proven by Aschenbrenner et al., using the fact that such theories have the $VC1$ property.

Fact 5.2 [2, Theorem 6.1] *Let T be a weakly o-minimal theory. Then, the dual VC-density of a formula $\phi(\bar{x}; \bar{y})$ is bounded by $|\bar{x}|$.*

Interpreting $ACVF_0$ in $RCVF$, they deduce that in $ACVF_0$, the dual VC-density of a formula $\phi(\bar{x}; \bar{y})$ is bounded by $2|\bar{x}| - 1$ [2, Corollary 6.3].

Then using that P -minimal theories have the VC 2 property, they show the following bound on the dual VC-density.

Fact 5.3 [2, Theorem 7.3] *Let $T = {}_pCF_1$. Then, the dual VC-density of a formula $\phi(\bar{x}; \bar{y})$ is bounded by $2 \cdot |\bar{x}| - 1$.*

In order to relate the dual VC-density in models of $T_{c,D}^*$ to the bounds obtained in models of T_c , we need to recall how to associate with an \mathcal{L}_D -definable set in a model of $T_{c,D}^*$, a (closely related) \mathcal{L} -definable set.

Notation 5.4 Let \mathcal{M} be a topological \mathcal{L} -field, we denote the projection maps as follows: $\pi_{(i_1, \dots, i_k)} : M^n \rightarrow M^k : (x_1, \dots, x_n) \rightarrow (x_{i_1}, \dots, x_{i_k})$, $1 \leq i_1 < \dots < i_k \leq n$,

$$\pi_k : M^n \rightarrow M^k : (x_1, \dots, x_n) \rightarrow (x_1, \dots, x_k), \quad 1 \leq k \leq n.$$

For ease of notation, we will also denote $\pi_{(i_1, \dots, i_k)}$ by $\pi_{\bar{c}}$, where \bar{c} is a tuple of 0's and 1's with the 1's at the i_1, \dots, i_k positions and 0's elsewhere.

Recall that a cell C is a definable subset of M^n (with parameters) either consisting of single point in M^n (in which case it has dimension 0), or such that $\pi_{(i_1, \dots, i_k)}(C)$ is an open subset of M^k , for some $1 \leq k \leq n$ and $1 \leq i_1 < \dots < i_k \leq n$ and the projection map $\pi_{(i_1, \dots, i_k)}$ a definable homeomorphism. In the latter case, we call k the \mathcal{L} -dimension of C , which we denote by $\mathcal{L}\text{-dim}(C)$ [24, Definition 6.2].

The topological \mathcal{L} -field \mathcal{M} has the cell decomposition property (CDP) if any A -definable subset $X \subset M^n$, $A \subset M$, can be partitioned into finitely many cells and if given any A -definable function f from X to \mathcal{M} there exists a partition of X into finitely many A -definable cells X_i such that $f|_{X_i}$ is continuous [24, Definition 6.3].

In the remainder of the section, we will assume the following hypotheses on T_c , a model-complete theory of topological \mathcal{L} -fields satisfying Hypothesis (I): T_c admits quantifier elimination, it has finite Skolem functions and the local continuity property of zeroes of polynomials. These additional hypotheses are needed in order to apply a result of Mathews [24, Theorem 7.1] who showed that any model of T_c has the cell decomposition property. He also proved that if the model-theoretic closure $\text{acl}^{\mathcal{L}}$ has the exchange property, then there is a well-defined notion of dimension ($\mathcal{L}\text{-dim}$) which coincides with the topological dimension [24, Theorem 8.8].

This can be applied to $T_c = RCF$, or $RCVF$ or ${}_pCF_d$. Indeed in these three cases one has (definable) Skolem functions. Van den Dries proved it for ${}_pCF_d$ [39, Theorem 3.1]. He applied a general result for theories T which admits quantifier elimination, proving [39, Theorem 2.1] the equivalence between the following two properties:

- (i) T has definable Skolem functions and
- (ii) any substructure A of a model of T , has a prime model extension to a model of T which is rigid over it.

The following Proposition is proven in [31, Lemma 2.1, proof of Theorem 2.4] for $T_{c,D}^* = CODF$ and in [30] in the general case. Recall that a differential tuple in a differential field M is a tuple of the form a^∇ with $a \in M$.

Proposition 5.5 [31, Proposition 3.13] *Let T_c be as above, let $\mathcal{M} \models T_{c,D}^*$ and let K be a differential subfield of M . Given an $\mathcal{L}_D(K)$ -definable set $S \subset M^k$, $k \geq 1$, there exists an $\mathcal{L}(K)$ -definable subset $S^* \subset M^{n_1+\dots+n_k}$ such that S is included and dense $\pi_{(i_1,\dots,i_k)}(S^*)$, with $i_1 = 1 < i_2 = n_1 + 1 < \dots < i_k = n_1 + \dots + n_{k-1} + 1$. Moreover, S^* is a finite union of $\mathcal{L}(K)$ -cells C in which the differential tuples are dense. Each cell C is included in $dcl_{\mathcal{L}(K)}(\pi_{\bar{c}_{i1}\bar{0}\dots\bar{0}\bar{c}_{k1}}(C))$ with $\pi_{\bar{c}_{i1}\bar{0}\dots\bar{0}\bar{c}_{k1}}(C)$ an open set, \bar{c}_{i1} , a tuple of 1's of length $n_{i1} \leq n_i$ and $\bar{c}_{i1}\bar{0}$ of length n_i , $1 \leq i \leq k$. In particular, $\mathcal{L}\text{-dim}(C) = \sum_{i=1}^k n_{i1}$.*

This proposition is proven by induction on k . Given an \mathcal{L}_D -formula ϕ such that $S = \phi(M)$, we modify the \mathcal{L} -formula ϕ^* given in Notation 2.3 to a more intrinsic \mathcal{L} -formula denoted by ϕ_{mod}^* such that $S^* := \phi_{mod}^*(M)$ has the required properties. We use the cell decomposition property of the models of T_c . Given a cell occurring in the decomposition of $\phi^*(M)$ we modify it to a cell C where the differential tuples in $S^\nabla \cap C$ are dense in C . Below, in the special case when $k = 1$ and $n_1 = n$, we describe each cell C as follows:

either $\mathcal{L}\text{-dim}(C) = n$ and so C is an open subset of M^n (this is the case when the cell occurring in the decomposition of $\phi^*(M)$ is already an open cell and so we simply take it since by the scheme (DL) , the differential tuples are dense),

or $\mathcal{L}\text{-dim}(C) = 0$ and C is a singleton of the form

$$\bar{v} := (u, g_1(u), \dots, g_{n-1}(u)),$$

where $g_s(x)$ are rational functions of x , $u^{(s)} = g_s(u)$, $1 \leq s \leq n-1$, and $u, \bar{v} \in acl(K)$,

or $0 < \mathcal{L}\text{-dim}(C) = \mathcal{L}\text{-dim}(\pi_m(C)) = m < n$ (depending on C) and C consists of tuples

$$\bar{v} := (\bar{u}, f(\bar{u}), g_1(\bar{u}, f(\bar{u})), \dots, g_{n-m-1}(\bar{u}, f(\bar{u}))),$$

where \bar{u} belongs to $\pi_m(C)$, $f(\bar{y})$ is a definable Skolem function, $g_s(\bar{y}, z)$ rational functions, with $\ell(\bar{y}) = \ell(\bar{u})$, $1 \leq s \leq n-m-1$. Moreover $f(\bar{y})$ is continuous and $g_s(\bar{y}, f(\bar{y}))$ are well-defined on $\pi_m(C)$. For a tuple $\bar{u} \in \pi_m(C)$ of the form u^{∇_m} , $g_s(u^{\nabla_{m+1}}) = u^{(m+s+1)}$, $1 \leq s \leq n-m-1$ and in any neighbourhood of \bar{u} , there is a tuple of the form a^{∇_m} such that $a^{(m+1)} = f(a^{\nabla_m})$.

The \mathcal{L} -dimension of the cells appearing in the decomposition of S^* appearing in Proposition 5.5, has the following interpretation. Let $\bar{u} \in S$ and consider the subfields $K^{[t]} := K(\bar{u}, \dots, \bar{u}^{\nabla t})$ of M generated by the differential subfield K and $\bar{u}^{\nabla t}$, $0 \leq \ell \leq t$. The transcendence degree of $K^{[t]}$ over K stabilises for t sufficiently big and is asymptotically given by a linear polynomial of the form $\alpha \cdot t + \beta$, called the Kolchin polynomial of \bar{u} over K [32, Proposition 2.4] (see also [20]). The coefficient α is the differential transcendence degree of $L := K(\bar{u}^{\nabla \ell}; \ell \in \omega)$ [32, Proposition 2.4]. The

tuple \bar{u}^∇ belongs to some cell C appearing in the decomposition of S^* and assume it is a \mathcal{L} -generic point of C . In the description of C above, the interpretation of α is also given by $\alpha = |I_1|$ where $I_1 := \{1 \leq i \leq k : n_{i1} = n_i\}$ [16, Proposition 3.12] and we get an interpretation of the constant term β as $\beta = \sum_{i \in I_0} n_{i1}$, where $I_0 := \{1 \leq i \leq k : n_{i1} < n_i\}$.

Let $\mathcal{M} \models T_{c,D}^*$, assume that \mathcal{M} sufficiently saturated and let $\phi(x; \bar{y})$ be an \mathcal{L}_D -formula. Let $K \subset M$ and $S = \phi(M, \bar{k})$, with $\bar{k} \subset K$. Let $\phi_{mod}^*(x^\nabla; \bar{y}^\nabla)$ be the formula constructed above, with $n_{11} = |x^\nabla|$.

Proposition 5.6 *Let T_c be as above. Then, in models of $T_{c,D}^*$, the dual VC-density of the \mathcal{L}_D -formula $\phi(x; \bar{y})$ is equal to the dual VC-density of $\phi_{mod}^*(x^{\nabla m-1}; \bar{y}^\nabla)$, where m is equal to $\mathcal{L}\text{-dim}(\phi_{mod}^*)$ (viewed as a formula in \bar{x}).*

Proof Let $\phi(x; \bar{y})$ be an \mathcal{L}_D -formula, let $\mathcal{M} \models T_{c,D}^*$ and let $B \subset M^{|\bar{y}|}$. Since $T_{c,D}^*$ admits quantifier elimination, one may assume that $\phi(x; \bar{y})$ is of the form $\bigvee_{j \in J} (\bigwedge_{i \in I_j} p_{ij}(x, \bar{y}) = 0 \wedge \theta_j(x, \bar{y}))$, with for all $i \in I_j$, $p_{ij}(X; \bar{Y}) \in \mathbb{Q}\{X; \bar{Y}\} \setminus \{0\}$ and for all $j \in J$, θ_j defines an open subset in some cartesian product of M .

We associate with $\phi(x; \bar{y})$ an \mathcal{L} -formula $\phi^*(\bar{x}; \bar{z})$ as in Notation 2.3, where \bar{x} , \bar{z} are tuples of variables of the same length as x^∇ , respectively \bar{y}^∇ . Set $B^\nabla := \{(\bar{b}^\nabla) : \bar{b} \in B\}$ and note that $|B| = |B^\nabla|$. We apply Proposition 5.5 to both formulas ϕ and $\neg\phi$, in order to obtain \mathcal{L} -formula $\phi_{mod}^*(\bar{x}, \bar{z})$, $(\neg\phi)_{mod}^*$ with the following properties. The definable set $\phi_{mod}^*(M)$ (respectively $(\neg\phi)_{mod}^*(M)$) is a finite union of cells \tilde{C} and $\mathcal{L}\text{-dim}(\tilde{C}) = \sum_{i=1}^{\ell+1} n_{i1}$, where ℓ is the length of \bar{y} . The number of cells depends only on the formulas ϕ_{mod}^* , $(\neg\phi)_{mod}^*$ in particular the number of cells of dimension 0.

Fix a finite set $B := \{b_1, \dots, b_\ell\}$, consider a fiber of the form $\phi_{mod}^*(\bar{a}; M)$ and let $I_+ := \{i : \phi_{mod}^*(\bar{a}; b_i^\nabla), b_i \in B\}$. For each $i \in I_+$, let \tilde{C}_i be one of the cells such that $(\bar{a}, b_i^\nabla) \in \tilde{C}_i$, with \tilde{C}_i occurring in the decomposition of $\phi_{mod}^*(M)$. First, if for some i , $\pi_{n_1}(\tilde{C}_i)$ is of dimension 0, then since the differential points are dense in \tilde{C}_i , \bar{a} is a differential point. Otherwise we let $n_{1i} > 0$ be such that $\pi_{n_{1i}}(\tilde{C}_i)$ is an open set. Let $m > 0$ be the minimum of all such n_{1i} . So $\bigcap_{i \in I_+} \phi_{mod}^*(M, b_i^\nabla)$ is infinite and contains \bar{a} . Using the scheme (DL) we can construct a differential point a^∇ close to \bar{a} and satisfying the formula $\bigwedge_{i \in I_+} \phi_{mod}^*(\bar{x}, b_i^\nabla)$. It may happen that such a^∇ does satisfy $\phi_{mod}^*(M, b_j^\nabla)$ for $j \in I_-$. This corresponds to the case when $\bar{a} \in \pi_{n_1}(D)$, for D a cell occurring in the cell decomposition of $(\neg\phi)_{mod}^*$ of dimension 0. Since the number of such cells is bounded independently of $|B|$, the difference between $|S_{\phi_{mod}^*}(B^\nabla)|$ and $|S_\phi(B)|$ is constant. \square

Example 4 Let $T_{c,D}^*$ be the theory CODF of closed ordered differential fields. Below we give some examples of, given an \mathcal{L}_D -formula, how to construct an \mathcal{L} -formula ϕ_{mod}^* .

Let $q_0(Y), q_1(Y), q_2(Y) \in \mathbb{Q}\{Y\} \setminus \{0\}$ and assume that the order of these differential polynomials is respectively m_0, m_1, m_2 . Let $m = \max\{m_0, m_1, m_2\}$ and $\bar{y} = (y_0, \dots, y_m)$. Fix a differential tuple $b^\nabla := (b_0, b_0^{(1)}, \dots, b_0^{(m)})$.

Let $\varphi(x; y)$ be the formula $\varphi_1(x; y) \vee \varphi_2(x; y)$, where $\varphi_1(x; y) := (D(x) = q_0(y) \wedge q_1(y) < x < q_2(y))$ and $\varphi_2(x; y) := (p(x) = q_0(y) \wedge q_1(y) < D(x) < q_2(y))$. Set $\varphi^* := \varphi_1^* \vee \varphi_2^*$, where

$\varphi_1^*(x_0, x_1; b^\nabla) := ((q_1^*(b^\nabla) < x_0 < q_2^*(b^\nabla) \wedge x_1 = q_0^*(b^\nabla)) \vee (x_0 = q_1^*(b^\nabla) \wedge D(q_1(b)) = q_0(b)))$. In this case we take $\varphi_1^* = \varphi_{1_{mod}}^*$ and we have that $\mathcal{L} - \dim(\varphi_{mod}^*) = 1$. We use the axiomatisation of *CODF* to show that we can find close to any element in the open interval $]q_1(b) \ q_2(b)[$ an element whose derivative is equal to $q_0(b)$.

Let $\varphi_2^*(x_0, x_1; b^\nabla) := (p(x_0) = q_0^*(b^\nabla) \wedge q_1^*(b^\nabla) < x_1 < q_2^*(b^\nabla))$. In this case we consider the following disjunction $(p(x_0) = q_0^*(b^\nabla) \wedge s_p(x_0) \neq 0 \wedge x_1 = \frac{D(q_0(b))}{s_p(x_0)} \wedge q_1^*(b^\nabla) < x_1 < q_2^*(b^\nabla))$ or $(p(x_0) = q_0^*(b^\nabla) \wedge s_p(x_0) = 0 \wedge q_1^*(b^\nabla) < x_1 < q_2^*(b^\nabla))$. In order to obtain φ_{mod}^* we have to put the second disjunct into a form similar to the first disjunct in order to be able to calculate $D(x_0)$. In any case, $\mathcal{L} - \dim(\varphi_{mod}^*)$ is either 0 or $-\infty$, depending on whether a condition of the form $q_1^*(b^\nabla) < \frac{(\partial_x q_0)^*(b^\nabla)}{s_p(a)} < q_2^*(b^\nabla)$ is satisfied for one of the non singular roots a of $p(x) = 0$ and some tuple b^∇ .

Corollary 5.7 *For T_c a weakly o-minimal theory, the dual VC-density of a formula $\phi(x; \bar{y})$ in $T_{c,D}^*$ is bounded by $\mathcal{L} - \dim(\phi_{mod}^*)$.*

Proof By [2, Theorem 6.1] and proposition above, the dual VC-density of such formula is bounded by $|\bar{x}|$ and this bound is optimal as noted in [2, Section 1.4]. \square

Corollary 5.8 *For $T_c = {}_pCF_1$, the dual VC-density of a formula $\phi(x; \bar{y})$ in $T_{c,D}^*$ is bounded by $2 \cdot \mathcal{L} - \dim(\phi_{mod}^*) - 1$*

Proof We apply [2, Theorem 7.3] and the above proposition. \square

6 Transfer of $NT P_2$

Definition 6.1 *Let T be a complete theory, then T is TP_2 , if it has a formula $\phi(\bar{x}; \bar{y})$ with TP_2 . A formula $\phi(\bar{x}; \bar{y})$ has TP_2 if there exist an array $(\bar{a}_{\ell j})_{\ell, j < \omega}$ in a model \mathcal{M} of T and $|\bar{y}| = |\bar{a}_{\ell j}|$, and $k \in \omega \setminus \{0, 1\}$ such that:*

- *for all $\ell \in \omega$, the set of formulas $\phi(\bar{x}; \bar{a}_{\ell j})_{j \in \omega}$ is k -inconsistent,*
- *for every $f : \omega \rightarrow \omega$, the set $\{\phi(\bar{x}; \bar{a}_{\ell f(\ell)})\}$ is consistent.*

If a theory T is TP_2 , then there is a formula $\phi(x; \bar{y})$ with TP_2 with $|x| = 1$ [10, Theorem 2.9 and Lemma 3.2].

From now on, we let T_c be a model-complete theory of topological \mathcal{L} -fields satisfying Hypothesis (I) (as defined in Sect. 2), where the topology is non-discrete and uniformly definable. (Note that here we are no longer assuming that T_c admits quantifier elimination). Let $T_{c,D}$ be the corresponding theory of differential topological \mathcal{L} -fields and $T_{c,D}^{*,\omega}$ its model-companion as described in [15, Theorem 9.3]. We show that the $NT P_2$ property transfers from T_c to $T_{c,D}^{*,\omega}$. Then we apply our result to the case of certain theories of bounded pseudo-real closed fields, in particular when T_c is the theory of maximal pseudo-real closed fields.

In the case of a generic predicate P , A. Chernikov showed that if T is geometric (namely *acl* satisfies the exchange property and T eliminates the quantifier \exists^∞) and $NT P_2$, then T_P is $NT P_2$ [10, Theorem 7.3].

Proposition 6.2 *Let T_c be a model-complete theory of topological \mathcal{L} -fields satisfying Hypothesis (I). Assume that T_c has NP_2 , then $T_{c,D}^{*,\omega}$ has NP_2 .*

Proof By the way of contradiction, we suppose we have an \mathcal{L}_D -formula $\phi(x; \bar{y})$ which is TP_2 with $|x| = 1$. Let \mathcal{M} be a sufficiently saturated model of $T_{c,D}^{*,\omega}$, $k \in \omega \setminus \{0, 1\}$ and an array $(\bar{a}_{\ell j})_{j \in \omega}$ such that

- (i) for all $\ell < \omega$, the set of formulas $(\phi(x; \bar{a}_{\ell j}))_{j \in \omega}$ is k -inconsistent,
- (ii) for all $f : \omega \rightarrow \omega$, $\{\phi(x; \bar{a}_{\ell f(\ell)}) : \ell \in \omega\}$ is consistent.

Since $T_{c,D}^{*,\omega}$ is model-complete, the \mathcal{L}_D -formula $\phi(x; \bar{y})$ is equivalent to an existential formula. Since the disjunction of two NP_2 formulas is NP_2 [10, Lemma 7.1], we may assume $\phi(x; \bar{y})$ is of the form $\exists \bar{z} (\chi(x, \bar{z}; \bar{y}) \wedge \theta(x, \bar{z}; \bar{y}))$, where χ is a conjunction of differential polynomial equations, $\theta^*(M)$ is an open subset and $\bar{z} := (z_1, \dots, z_m)$.

Let $\chi^*(x^{\nabla n_1}, \bar{z}^{\nabla n_3}; \bar{y}^{\nabla n_2})$ be the corresponding \mathcal{L}_{rings} -formula (which expresses that a certain differential tuple belongs to an algebraic set); we may assume that $n_1 = n_2 = n_3 = n$. Let $\phi^*(\bar{x}; \bar{y})$ be the formula $\exists \bar{z} (\chi^*(\bar{x}, \bar{z}; \bar{y}) \wedge \theta^*(\bar{x}, \bar{z}; \bar{y}))$, where \bar{x} has length $n + 1$, \bar{y} (respectively \bar{z}) has length $(n + 1) \cdot |\bar{y}|$ (respectively $(n + 1) \cdot |\bar{z}|$).

Since the theory T_c is NP_2 , there exists ℓ such that $\{\phi^*(\bar{x}; \bar{a}_{\ell j}) : j \in \omega\}$ is consistent. Let us show that this implies that $\{\phi(x; \bar{a}_{\ell j}) : j \in \omega\}$ is finitely consistent and so consistent which is a contradiction with assumption (i).

Let J be a finite subset of \mathbb{N} and consider the formula $\bigwedge_{j \in J} \phi^*(\bar{x}; \bar{a}_{\ell j}^{\nabla n})$. Let (\bar{d}, \tilde{b}) an element of M realising the formula $\chi^*(\bar{d}, \tilde{b}; \bar{a}_{\ell j}^{\nabla n}) \wedge \theta^*(\bar{d}, \tilde{b}; \bar{a}_{\ell j}^{\nabla n})$. Consider the algebraic set W given by: $\bigwedge_{j \in J} \chi^*(\bar{x}, \tilde{b}; \bar{a}_{\ell j}^{\nabla n})$ (defined over M). By applying the generalised Euclidean algorithm, we may assume that W is given by a finite union of subsets W_i of the form $p_i(x, \dots, x_k, \tilde{b}; \bar{a}_{\ell j}^{\nabla n}) = 0 \ \& \ \partial_{x_k} p_i(x, \dots, x_k, \tilde{b}; \bar{a}_{\ell j}^{\nabla n}) \neq 0 \ \& \ \bigwedge_t q_t(\tilde{b}; \bar{a}_{\ell j}^{\nabla n}) = 0 \ \& \ q(\tilde{b}; \bar{a}_{\ell j}^{\nabla n}) \neq 0$. The tuple \bar{d} belongs to one of the W_i . By the axiomatisation of $T_D^{*,\omega}$ [15, Lemma 9.2], which reduces in the one variable case to the scheme (DL) (see Sect. 2), we can find a differential solution $u^{\nabla n} \in W_i$ that we can choose close enough to \bar{d} in order to be in $\theta^*(M, \tilde{b}; \bar{a}_{\ell j}^{\nabla n})$. So we get that $\chi^*(u^{\nabla n}, \tilde{b}; \bar{a}_{\ell j}^{\nabla n}) \wedge \theta^*(u^{\nabla n}, \tilde{b}; \bar{a}_{\ell j}^{\nabla n})$ holds. Then we fix the differential tuple $u^{\nabla n}$ and we re-iterate the above procedure with the first element b_1 of the tuple \tilde{b} , namely we express the tuple \tilde{b} as (\bar{b}_1, \tilde{b}') with $|\bar{b}_1| = n + 1$, $|\tilde{b}'| = (n + 1) \cdot (|\bar{b}| - 1)$. We replace \bar{b}_1 with a *close enough* differential tuple $v_1^{\nabla n}$ such that $\chi^*(u^{\nabla n}, v_1^{\nabla n}, \tilde{b}'; \bar{a}_{\ell j}^{\nabla n}) \wedge \theta^*(u^{\nabla n}, v_1^{\nabla n}, \tilde{b}'; \bar{a}_{\ell j}^{\nabla n})$ holds.

Continuing in this way with the successive elements of the tuple \tilde{b} , we get a differential tuple $\bar{v}^{\nabla n}$ close to \tilde{b} and such that $\bigwedge_{j \in J} \chi^*(u^{\nabla n}, \bar{v}^{\nabla n}; \bar{a}_{\ell j}^{\nabla n}) \wedge \theta^*(u^{\nabla n}, \bar{v}^{\nabla n}; \bar{a}_{\ell j}^{\nabla n})$ holds. So $\bigwedge_{j \in J} \phi(u, \bar{a}_{\ell j})$ holds in M , which leads to a contradiction. \square

Now we apply the above result taking for T_c a model-complete theory of pseudo-real closed fields. Recall that the class of pseudo-real-closed fields has been axiomatised by Prestel [34] and then independently by S. Basarab. A pseudo-real-closed field (PRC -field) is bounded if it has only finitely many algebraic extensions of each degree and it is maximal if it has no proper algebraic extensions. (Note that an algebraic extension

of a PRC -field is again PRC .) It is easy to show that if a PRC -field is bounded, then it can be only endowed with finitely many distinct orderings. Denote by PRC_e the theory of pseudo-real-closed fields endowed with exactly e distinct orderings.

In his thesis [38], van den Dries introduced the theory OF_e of formally real fields endowed with e distinct orderings and showed that this theory has a model-companion: \overline{OF}_e , which coincides with the theory of maximal PRC_e fields, as observed later by Jarden [18].

Montenegro [27] noted that PRC -fields which are not algebraically closed nor real-closed have the independence property. (They can be interpreted in PAC fields and then one applies a former result of Duret [13]). Furthermore, she proved that bounded PRC -fields are exactly the PRC fields with $NT P_2$ [27], answering a question of Chernikov et al. [11].

In [15], we considered differential generic expansions of fields endowed with several distinct orderings. Let $\mathcal{L} := \mathcal{L}_{rings} \cup \{<_1, \dots, <_e\}$, $e \geq 2$. The generic expansion of the theory OF_e with a derivation D : $\overline{OF}_{e,D}$ has a model-companion $\overline{OF}_{e,D}^{*,\omega}$ [15, Theorem 9.3] (the notation used there was $\overline{OF}_{e,D}^\omega$).

Adding to the language \mathcal{L} , a countable set C of new constants, we can generalize that result as follows. Let K be a bounded PRC -field endowed with exactly e orderings and let K_0 be a countable elementary substructure of K . Then the $\mathcal{L}_{rings}(C)$ -theory T_C of K is model-complete, where C is interpreted by the elements of K_0 [28, Corollary 3.6]. (One first uses a former result of Jarden that the $\mathcal{L}(C)$ -theory of K is model-complete and then one shows that the orderings $<_i$, $1 \leq i \leq e$, are existentially definable in $\mathcal{L}_{rings}(C)$ [28, Lemma 3.5]). In order to show that $T_{C,D}$ has a model-companion, one either uses the result of Tressl [37, Theorem 7.2] or one checks that T_C satisfies Hypothesis (I) and this follows from Remark 2.2 since PRC -fields are large fields.

Corollary 6.3 *Let K be a bounded PRC -field and let K_0 be a countable elementary substructure of K . Let T_C the $\mathcal{L}_{rings}(C)$ -theory of K , where C is interpreted by the elements of K_0 . Then the theory $T_{C,D}^*$ is $NT P_2$.*

Let \overline{OF}_e be the theory of maximal PRC_e fields, $e \geq 1$. Then \mathcal{L}_D -theory $\overline{OF}_{e,D}^{,\omega}$ is $NT P_2$.*

Proof This follows from the preceding proposition and the fact that the theories T_C , respectively \overline{OF}_e are $NT P_2$ [28, Theorem 4.22]. \square

Remark 6.4 Let T_C be as in the preceding corollary. Then any quantifier-free formula in $T_{C,D}^{*,\omega}$, respectively in $\overline{OF}_{e,D}^{*,\omega}$ is NIP .

Proof We apply Fact 5.1 and the fact that the corresponding property holds for T_C , respectively \overline{OF}_e by [28, Corollary 4.12]. \square

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