math.DG/0604558

SPECIAL GRAPHS

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Dedicated to Dmitri V. Alekseevsky on the occasion of his sixty-fifth birthday

ABSTRACT. A *special* p-form is a p-form which, in some orthonormal basis $\{e_\mu\}$, has components $\varphi_{\mu_1\ldots\mu_p} = \varphi(e_{\mu_1},\ldots,e_{\mu_p})$ taking values in $\{-1,0,1\}$. We discuss graphs which characterise such forms.

1. Calibrations, special forms and graphs

A constant p-form φ in a d-dimensional Euclidean space is a *calibration* if for any p-dimensional subspace spanned by a set of orthonormalised vectors $e_1,\ldots,e_p$, the following condition holds:

$$(\varphi(e_1,\ldots,e_p))^2 \leq 1 \ , \tag{1}$$

with equality holding for at least one subspace. Let U be an oriented p–dimensional subspace of $\mathbb{R}^d$ with oriented metric volume vol_U. The set of all such subspaces is the oriented Grassmannian $\mathrm{Gr}_p\,\mathbb{R}^d$. A calibration $\varphi \in \Lambda^p\,\mathbb{R}^d$ is thus a p–form with the property that the function $\overline{\varphi} : \mathrm{Gr}_p\,\mathbb{R}^d \longrightarrow \mathbb{R}$ associated to φ and defined by $U \mapsto \overline{\varphi}(U) := \langle \varphi, \mathrm{vol}_U \rangle$ takes values in $[-1,1] \subset \mathbb{R}$ with at least one of the two extremal values ± 1 being achieved. The p-planes U for which $\overline{\varphi}(U) = \pm 1$ are said to be calibrated by φ.

Almost all examples of calibrations known are invariant under a group $G \subset \mathrm{O}(\mathbb{R}^d)$ large enough so that it is relatively simple to check the calibration condition directly. Interestingly most of these examples, in particular the calibrations characterising special holonomy manifolds, for instance the G_2–invariant Cayley 3-form in seven dimensions, defined by the structure constants of the imaginary octonions, and the Spin(7)-invariant 4-forms in eight dimensions are special forms:

This work was supported by the *Schwerpunktprogramm Stringtheorie* of the Deutsche Forschungsgemeinschaft, MPI für Mathematik, Bonn and the Belgian *Fonds National de la Recherche Scientifique*.

Definition 1. *A* special p–form φ *is a* p*–form* $\varphi \in \Lambda^p\mathbb{R}^d$ *on* d*–dimensional Euclidian space* $\mathbb{R}^d$ *in the orbit under the orthogonal group* $\mathrm{O}(d,\mathbb{R})$ *of*

$$\varphi \;=\; \sum_{1\le\mu_1<\ldots<\mu_p\le d} \varphi_{\mu_1\ldots\mu_p}\, e_{\mu_1}\wedge e_{\mu_2}\wedge\ldots\wedge e_{\mu_p} \tag{2}$$

with $\varphi_{\mu_1\ldots\mu_p} \in \{-1,0,1\}$ *and* $(e_1,\ldots,e_d)$ *an orthonormal basis.*

In other words, a p-form φ is special if there exist d orthonormal basis vectors e_μ, $\mu = 1,\ldots,d$, such that for any subset of p basis vectors $e_{\mu_1},\ldots,e_{\mu_p}$ we have

$$\varphi_{\mu_1\ldots\mu_p} \;:=\; \varphi(e_{\mu_1},\ldots,e_{\mu_p}) \;\in\; \{-1,0,1\}\ . \tag{3}$$

Given a basis $\{e_\mu\}$, there are clearly only a finite number (obviously less than $3^{\frac{d!}{p!(d-p)!}}$) of orbits of special p–forms under $\mathrm{O}(d,\mathbb{R})$ parametrised by the components $\varphi_{\mu_1\ldots\mu_p} \in \{-1,0,1\}$. Apparently different special p–forms may nevertheless be in the same orbit under $\mathrm{O}(d,\mathbb{R})$, because the subgroup $\mathrm{O}(d,\mathbb{Z}) \subset \mathrm{O}(d,\mathbb{R})$ of orthogonal matrices with integer coefficients maps the special form φ in equation (2) again into a special form with possibly different components. The group $\mathrm{O}(d,\mathbb{Z})$ is isomorphic to the semidirect product of the permutation group acting naturally on d copies of $\mathbb{Z}_2$. The action of $(\sigma,\eta_1,\ldots,\eta_d) \in S_d \ltimes \mathbb{Z}_2^d \cong \mathrm{O}(d,\mathbb{Z})$ on the antisymmetric tensor indices of φ is given by $\varphi_{i_1\,\ldots\,i_p} \mapsto \eta_{i_1}\cdots\eta_{i_p}\,\varphi_{\sigma(i_1)\,\ldots\,\sigma(i_p)}$, where $\sigma\in S_d$ and $\eta_i^2 = 1$, $i = 1,\ldots,d$.

Let us now give an alternative description of special forms. An oriented p-subset of $\{1,2,\ldots,d\}$ is given by the p elements $s = \{\mu_1,\ldots,\mu_p\}$ such that $1\le\mu_1<\mu_2<\cdots<\mu_p\le d$. The space of all such p-subsets is the *vertex space* $\mathcal{P}^p(\{1,\ldots,d\})$. These oriented subsets of $\{1,2,\ldots,d\}$ are in bijective correspondence to oriented coordinate subspaces $\mathbb{R}^p \subset \mathbb{R}^d$ via $s = \{\mu_1,\ldots,\mu_p\} \longmapsto e_{\mu_1}\wedge\ldots\wedge e_{\mu_p}$. A special p–form can be thought of as a function from $\mathcal{P}^p(\{1,\ldots,d\})$ to $\varphi_{\mu_1\ldots\mu_p} \in \{-1,0,1\}$. Consequently a special p–form is specified completely by the two sets $\mathcal{I}^+$ and $\mathcal{I}^-$ of oriented subsets $\{\mu_1,\ldots,\mu_p\} \subset \{1,\ldots,d\}$ which have respectively $\varphi_{\mu_1\ldots\mu_p} = +1$ and -1. We denote by $\mu^{(a)}, a = 1,\ldots,|\varphi|$, the elements in $\mathcal{I} := \mathcal{I}^+\cup\mathcal{I}^-$, the support of φ and

$$\varphi = \sum_{a=1}^{|\varphi|} \varphi_{\mu_1^{(a)}\ldots\mu_p^{(a)}}\ e_{\mu_1^{(a)}}\wedge\cdots\wedge e_{\mu_p^{(a)}}\ , \tag{4}$$

where $|\varphi| := |\mathcal{I}|$ is the weight of φ. For every permutation $\sigma\in S_d$ of the basis vectors $e_1,\ldots,e_d$, there exists a corresponding permutation of the oriented p-subsets.

Interestingly, we can define a metric on $\mathcal{P}^p(\{1,\ldots,d\})$ by setting the distance between two oriented p-subsets, s and $\tilde{s}$, to be $(s,\tilde{s}) = p - \#(s\cap\tilde{s})$, where $\#(s\cap\tilde{s})$ is the number of elements in the intersection of the sets s and $\tilde{s}$. We can visualise the restriction of this metric to the set $\mathcal{I}$ by drawing a graph with labeled edges, the vertices corresponding to the elements of $\mathcal{I}$ and the edges running between vertices labeled by a *distance* strictly less than p. Unfortunately the graph of a special p–form φ does not specify the components $\varphi_{\mu_1 \dots \mu_p} \in \{-1, 0, 1\}$ completely up to the action of $\mathrm{O}(d, \mathbb{Z})$; we still need to specify some relative sign. Nevertheless the graph gives a very condensed way of encoding the characteristics of a special p–form.

Consider a graph Γ composed of a set of vertices $V = \{v_i\ ;\ i = 1, \dots r\}$ connected by the maximum possible number of edges, $r(r-1)/2$, each labeled by a positive number $d(v_i, v_j)$, the *distance* between the vertices v_i and v_j at its ends. A graph is *admissible* if any triangle with edges labeled by distances d_i, d_j, d_k satisfies the triangle inequalities

$$1 \leq d_i \leq d_j + d_k \quad \text{and cyclic permutations.} \tag{5}$$

Definition 2. *A* realisation *of a graph* Γ *is a map*

$$\begin{aligned} \rho : \Gamma &\to \mathcal{P}^p(\{1, \dots, d\}) \\ v &\mapsto s_v \end{aligned} \tag{6}$$

which assigns to any vertex v an oriented p-subset s_v such that the distance between any two vertices $d(v, w)$ is equal to the distance between the corresponding oriented p-subsets (s_v, s_w) and that

$$\#\left(\bigcup_{v\in\Gamma} s_v\right) = d \quad , \quad \#\left(\bigcap_{v\in\Gamma} s_v\right) = 0\ . \tag{7}$$

Two realisations are equivalent if there exist a permutation $\sigma \in S_d$ of the oriented p-subsets which maps one onto the other.

Consider the power set $\mathcal{P}(V)$, the set of all subsets $S \subset V$ of vertices of the graph Γ,

$$\mathcal{P}(V) = \{\{\emptyset\}, \{v\}, \{w\}, \dots, \{v, w\}, \{x, y\}, \dots, \{v, w, x\}, \dots, \{V\}\}\ . \tag{8}$$

Clearly, $\#\{\mathcal{P}(V)\} = 2^r$. For every realisation, a graph function f associates a nonnegative integer to every $S \in \mathcal{P}(V)$ as follows

$$\begin{aligned} F(S) &:= \left(\left(\bigcap_{v\in S} s_v\right) \bigcap \left(\bigcup_{v\notin S} s_v\right)^{\mathrm{C}}\right) \geq 0 \\ f(S) &:= \#(F(S))\ , \end{aligned} \tag{9}$$

where C denotes the complementary subset. Clearly, this function measures the number of indices which occur in every s_v for $v \in S$ but do not occur in any s_v for $v \notin S$. Trivially, we have

$$\begin{aligned} f(\emptyset) &= 0 \\ f(V) &= 0 \\ \sum_{S \in \mathcal{P}(V)} f(S) &= d \ , \end{aligned} \tag{10}$$

because $\bigcap_{v \in S} s_v$ is empty for the two first cases and $\sum_{S \in \mathcal{P}(V)} f(S)$ contains all the indices $\{1 \ldots d\}$ and each index contributes to f for one and only one subset S.

Theorem 1. *To every graph function f, with non negative integer values, which satisfies*

$$d(v, \tilde{v}) = p \quad - \sum_{\{S \in \mathcal{P}(V) | v, \tilde{v} \in S\}} f(S) \ , \tag{11}$$

there corresponds a class of equivalent realisations of the graph.

In particular, if $v = \tilde{v}$

$$d(v, v) = p \quad - \sum_{\{S \in \mathcal{P}(V) | v \in S\}} f(S) = 0 \ . \tag{12}$$

As a consequence, all realisations of a graph are simply obtained by finding all the non negative solutions to (11). Every realisation yields a simple and direct construction of a special form, up to choices of signs. Examples can easily be generated [DNW].

2. Democratic Graphs

Consider a graph Γ with vertices $v_i, i = 1, \ldots, r$, and the set of nonzero distances $\{d(v_i, v_j) \ ; \ i < j\}$. The $r \times r$ *distance matrix*

$$M^{[r]}_{ij} = d(v_i, v_j) \tag{13}$$

is clearly symmetric, with diagonal elements equal to 0.

Definition 3. *A* symmetry σ *of a graph* Γ *is a permutation of the vertices* $v_i \mapsto \tilde{v}_i$ *which leaves the distance matrix invariant, i.e.*

$$d(\tilde{v}_i, \tilde{v}_j) = d(v_i, v_j) \ . \tag{14}$$

If there exists a realisation of a graph with symmetry σ, such that $f(\sigma S) = f(S)$ for any $S \in \mathcal{P}(V)$, then the realisation has a permutation of the indices which induces σ on the monomials.

Definition 4. *A graph is* democratic *if, for every pair of vertices v_i, v_j, there exists some symmetry σ which maps $v_i \mapsto \tilde{v}_i = v_j$.*

Let $\{d_a\}$ denote the set of unequal distances.

Proposition 1. *A necessary condition for a graph with r vertices to be democratic is that $n_a^{(i)}$, the number of vertices at distance d_a from vertex v_i, is independent of the choice of v_i. A graph satisfying this condition, i.e. $n_a^{(i)} = n_a$, will be called* predemocratic.

Note that $\sum_a n_a^{(i)} = r-1$ and that a predemocratic graph is not necessarily democratic.

First, consider graphs with an even number of vertices, $r = 2n$. Then, from every vertex there are $r-1$ edges labeled by $r-1$ distances. To have democracy, every vertex should have the same set of distances to its neighbouring vertices. Let $d_i := d(v_1, v_{i+1})$, $i = 1, \ldots, r-1$, be the distances between v_1 and v_{i+1}. An example of a predemocratic graph with an even number of vertices $r = 2n$ has distance matrix, up to relabeling, of the form

$$\begin{aligned} d(v_i, v_j) &= (1-\delta_{ij})\; d_{\,j+i-2 \pmod{r-1}} \\ d(v_i, v_r) &= (1-\delta_{ir})\; d_{\,2i-2 \pmod{r-1}} \end{aligned} \tag{15}$$

with $d_0 \equiv d_{r-1}$. For $r = 4$, the corresponding graph is the unique predemocratic graph and it is also democratic. For higher r's, these graphs are in general not democratic.

Now, consider graphs with an odd number of vertices, $r = 2n+1$.

Lemma 1. *If the number of vertices r is odd, a predemocratic graph has all n_a's even.*

Proof: For a predemocratic graph with r vertices, the total number of edges of distance d_a is clearly $n_a r/2$. □

For r odd, if we set $n_a = 2$, for all a, there are $(r-1)/2$ unequal distances d_a. Distance matrices with $n_a = 4, 6, \ldots$, with all distances d_a different, can always be obtained from the distance matrices with $n_a = 2$ by setting some d_a's to be equal.

We now classify all distance matrices with $n_a = 2$ for all a and all distances d_a different. Under these assumptions, the edges of length d_a for given a form a closed (possibly disconnected) curve $\mathcal{C}_a$ containing every vertex once. Let us call the number of vertices in a connected piece of curve $\mathcal{C}_a$ its pathlength, which is obviously between 3 and r.

Lemma 2. *A necessary condition for a predemocratic graph with an odd number of vertices r and $n_a = 2$ to be democratic is that the curve $\mathcal{C}_a$, for every a, has connected pieces of equal pathlength $3 \leq L_a \leq r$. In other words, L_a is a divisor of r and $\mathcal{C}_a$ consists of r/L_a disconnected pieces.*

Proof: The proof follows from the definition of a democratic graph: the curve $\mathcal{C}_a$ as seen from any vertex has the same form, independent of the choice of the vertex.

□

To give an example of a democratic graph with $r = 2n + 1$ vertices and $n_a = 2$ for all a we choose n distinct positive integers $d_1, \dots, d_n$ and define the distance matrix $M^{[r]}$ by

$$M^{[r]}_{ii} = 0 \qquad , \qquad M^{[r]}_{ij} = d_{\min\{|i-j|, 2n+1-|i-j|\}} \tag{16}$$

for $0 \leq i, j \leq 2n$. Evidently the matrix $M^{[r]}$ has a cyclic isometry group $\mathbb{Z}_r$ shifting the vertices $v_i \longmapsto v_{i+1 (\mathrm{mod}\, r)}$. Assuming that the distances $d_1, \dots, d_n$ can be chosen in such a way that there exists a graph function f for the $M^{[r]}$ invariant under $\mathbb{Z}_r$ we get a realisation for $M^{[r]}$, which is a democratic graph with symmetry group containing $\mathbb{Z}_r$.

Theorem 2. *For an odd prime number $r = 2n + 1$ every democratic graph with r vertices satisfying $n_a = 2$ for all a (and n distinct distances d_a) has a distance matrix of the form $M^{[r]}_{ij}$ with a suitable choice of the positive integers $d_1, \dots, d_n$.*

Proof: Essentially we only need to show that the symmetry group of a democratic graph with a prime number $r = 2n + 1$ of vertices and $n_a = 2$ for all a must contain a cyclic subgroup of order r acting transitively on the vertices. Clearly all curves $\mathcal{C}_a$ must be connected circles of length r, because r being prime has no proper divisors. We fix two vertices v_0 and v_1 and the curve $\mathcal{C}_a$ containing the edge between them. By democracy there exists a symmetry σ sending v_0 to $\sigma(v_0) = v_1$ and mapping the curve $\mathcal{C}_a$ to itself, because it is a symmetry and all d_a are distinct. Thus σ can only be the cyclic shift by one step along the curve $\mathcal{C}_a$, which clearly generates a cyclic group of symmetries of order r acting transitively on the vertices. Setting $v_i := \sigma^i(v_0)$ for $0 \le i \le 2n$ we conclude that $M^{[r]}_{ij} := d(v_i, v_j) = d(v_0, v_{j-i}) = d(v_{2n+1-j+i}, v_0)$ for all $0 \le i \le j \le 2n$. □

Slightly more generally we can consider graphs with a group of symmetries isomorphic to $\mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2}$ acting transitively on the vertices. Of course if r_1 and r_2 are relatively prime, then the group of symmetries considered is isomorphic to $\mathbb{Z}_{r_1 r_2}$. Nevertheless we expect new features compared to the classification above, because $r_1 r_2$ is no longer prime. With a group $\mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2}$ acting transitively on the vertices it is convenient to label the vertices by tuples $(i_1, i_2) \in \{1, \ldots, r_1\} \times \{1, \ldots, r_2\}$. Straightening this out by replacing (i_1, i_2) with $i := i_2 + r_2(i_1 - 1)$ we get a distance matrix of the form

$$M^{[r_1][r_2]} = \begin{pmatrix} M^{[r_2]} & Q_1 & Q_2 & Q_3 & \ldots & Q_2^t & Q_1^t \\ Q_1^t & M^{[r_2]} & Q_1 & Q_2 & \ddots & Q_3^t & Q_2^t \\ Q_2^t & Q_1^t & M^{[r_2]} & Q_1 & \ddots & Q_4^t & Q_3^t \\ Q_3^t & Q_2^t & Q_1^t & M^{[r_2]} & \ddots & Q_5^t & Q_4^t \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ Q_2 & Q_3 & Q_4 & Q_5 & \ddots & M^{[r_2]} & Q_1 \\ Q_1 & Q_2 & Q_3 & Q_4 & \ddots & Q_1^t & M^{[r_2]} \end{pmatrix} , \qquad (17)$$

where $M^{[r_2]}$ is the $r_2 \times r_2$ distance matrix defined in (16) which depends on $(r_2-1)/2$ arbitrary distances. The $r_2 \times r_2$ matrices Q_i, $i = 1, \ldots, (r_1-1)/2$, depend on r_2 arbitrary parameters. The first row of every Q_i is arbitrary and the r_2-1 following rows are obtained by cyclically permuting the elements of the first row.

Clearly a graph function f for a matrix of the form (17), with the property that $f(\sigma S) = f(S)$ for all sets of vertices S and all $\sigma \in \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2}$, defines an equivalence class of democratic graphs. Conversely, for a democratic graph with $r = r_1 r_2$ vertices, r_1, r_2 prime and $n_a = 2$ for all a, the distance matrix must be of the form $M^{[r]}$ in (16) or $M^{[r_1][r_2]}$ in (17).

In full generality, for every factorisation $r = r_1 r_2 \cdots r_k$ with $r_1 \ge r_2 \ge \cdots \ge r_k > 1$, we can consider graphs having a group of symmetries isomorphic to $\mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_k}$ acting transitively on the r vertices. A convenient labeling of the vertices is given by $v^{i_1,\ldots,i_k} := \sigma_1^{i_1} \sigma_2^{i_2} \ldots \sigma_k^{i_k}(v_0)$ where the tuples $i_1, \ldots, i_k \in \{1, \ldots, r_1\} \times \cdots \times \{1, \ldots, r_k\}$ and the $\sigma_A, A = 1, \ldots, k$, are generators of $\mathbb{Z}_{r_A}$, cyclic permutations of order r_A. Then the distance matrices have elements

$$d(v^{i_1,\ldots,i_k}, v^{j_1,\ldots,j_k}) = d(v_0, v^{j_1-i_1,\ldots,j_k-i_k}) = d_{j_1-i_1,\ldots,j_k-i_k} \quad , \quad i_A < j_A \, , \; A = 1, \ldots, k. \qquad (18)$$

If we have a graph function f for this matrix, with the property that $f(\sigma S) = f(S)$ for all sets of vertices S and all $\sigma \in \mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_k}$, then the corresponding graph is democratic.

If we wish to represent the distance matrix $M^{[r]}_{ij}$ in matrix form, corresponding to (16) or (17), then the labeling of the vertices by tuples turns out to be a nuisance, because we would have to straighten the indices as in the $k = 2$ case. Instead, it is more natural to replace the vector space $\mathbb{R}^r$ by a tensor product $\mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_k}$, with standard basis indexed by precisely the tuples above. The generators of the symmetry group then take the form

$$P^{[A]} = \mathbb{1}^{[1]} \otimes \mathbb{1}^{[2]} \otimes \cdots \otimes p^{[A]} \otimes \cdots \otimes \mathbb{1}^{[k]}. \tag{19}$$

where $\mathbb{1}^{[B]}$ is the $r_B \times r_B$ unit matrix and $p^{[A]}$ are the permutation matrices of order r_A, which permute the indices i_A cyclically and thus induce an $r \times r$ permutation on the indices i. As in the $k = 2$ case above, there corresponds to every factorisation $r = r_1 r_2 \cdots r_k$ a distance matrix invariant under all the permutations $P^{[A]}$, their powers and their products.

If r has m prime factors, $r = s_1 \cdots s_m$, with all the s_i's different, the number of inequivalent democratic graphs with r vertices depends only on m and corresponds to the number of ways a set with n elements can be partitioned into disjoint, non-empty subsets. This is precisely the m-th Bell number B_m, which is given by the formula

$$B_{m+1} = \sum_{k=0}^{m} \binom{m}{k} B_k \quad , \quad B_0 = 1 \ . \tag{20}$$

If some of the prime factors s_i are equal, the partitions in different subsets leading to the same set of r_A's yield equivalent graphs. We shall give further details and examples elsewhere [DNW].

References

[DNW] C. Devchand, J. Nuyts, G. Weingart, Special democratic graphs and forms, to appear.

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