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STUDY OF QUOMMUTATORS OF QUANTUM VARIABLES AND GENERALIZED DERIVATIVES.

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Abstract :

A general deformation of the Heisenberg algebra is introduced with two deformed operators instead of just one. This is generalised to many variables, and permits the simultaneous existence of coherent states, and the transposition of creation operators.

I. Introduction

Let X_i $(i = 1, \ldots, N)$ be a set of quantum variables which obey the quommutation relations

$$X_i * X_j = x_{ij} X_j * X_i \tag{1.1a}$$

where the c-numbers x_{ij} obviously satisfy

$$x_{ij} x_{ji} = 1 \tag{1.1b}$$

and $*$ represents an associative non commutative product. This is not by any means the most general starting point[1], but is the most convenient choice as associativity is automatically satisfied for the X_i. Let D_i $(i = 1, \ldots, N)$ the corresponding quantum derivatives

$$D_i * D_j = d_{ij} D_j * D_i \tag{1.2a}$$

where again

$$d_{ij} d_{ji} = 1 \quad . \tag{1.2b}$$

It is then useful to introduce the operators G_i $(i = 1, \ldots, N)$ defined for each i by

$$G_i = X_i * D_i \tag{1.3}$$

which as a consequence of (1.1) and (1.2) satisfy similar quommutation relations

$$G_i * G_j = g_{ij} G_j * G_i \tag{1.4a}$$

and

$$g_{ij} g_{ji} = 1 \quad . \tag{1.4b}$$

The operators G_i are to be interpreted as quantum dilatation operators analogous to the classical dilatation operator usually defined as $X_i \partial / \partial_{X_i}$.

Since we expect the quommutator of the operator D_i with the corresponding X_i to involve diagonal neutral operators[2] which we call A_i $(i = 1, \ldots, N)$, we write

$$D_i * X_j = v_{ij} X_j * D_i + \delta_{ij} A_i \tag{1.5}$$

where the arbitrary normalisation of the diagonal δ_{ij} term has been supposed to be non zero for all i and has been included in the definition of the A_i.

The motivation for this study, is connected with the potentiality of representing A by something other than the identity operator. This will permit us to overcome a deficiency which has been ignored by much of the recent literature on q-deformations,[3,4] and will enable us to demonstrate the existence of coherent states and at the same time permit transposition of the variables X_i. As far as we are aware, there are only certain particularly simple parameter choices in some recent work on differential calculus[4,5] which currently allow this. A secondary motivation is to write an operator realisation of the symmetric form of the q-derivative, and generalise this to many variables.

We can write after multiplication of (1.5) on the left by X_i

$$G_i * X_j = p_{ij} X_j * G_i + \delta_{ij} X_i * A_i \tag{1.6a}$$

where

$$p_{ij} = v_{ij} x_{ij} \tag{1.6b}$$

Let us define naive scale invariance as follows : X_i scales as a length $[L^1]$, D_i as an inverse length $[L^{-1}]$ and hence G_i and A_i as $[L^0]$. This naive scale invariance would allow the addition of a term $\rho_i X_i$ to the right hand side of (1.6a). We have however shown that the associativity restrictions on the operators lead to $\rho_i = 0$ in a natural way (except for pathological configurations which we have excluded).

In order to construct a complete set of quommutator relations the equations (1.1), (1.4) and (1.6) have to be supplemented by relations between the A_i themselves as well as between the A_i and both the X_j and the G_j.

For these quommutation relations we propose

$$A_i * A_j = a_{ij} A_j * A_i \tag{1.7a}$$

$$a_{ij} a_{ji} = 1 \tag{1.7b}$$

i.e. of a form analogous to (1.1), (1.4) and

$$A_i * X_j = q_{ij} X_j * A_i + \delta_{ij} \alpha_i X_i * G_i \tag{1.8}$$

$$G_i * A_j = r_{ij} A_j * G_i + \delta_{ij} (\beta_i A_i^2 + \gamma_i G_i^2) \tag{1.9}$$

Let us state again that in (1.8) we could have added a term $\sigma_i X_i$ and in (1.9) a term $\lambda_i A_i + \mu_i G_i + \nu_i$ in agreement with the naive scale invariance and the idea that all terms with degree less or equal to two should be included in these relations. But all these terms turn out to be zero $\sigma_i = \lambda_i = \mu_i = \nu_i = 0$, in a natural way, upon the application of the associativity requirements. Here again it would be possible to generalise, by intermixing terms with different indices as is done[5–8] , but we take the simplest choice. As we shall see this will permit us to change the basis to simplify the algebra. What we have is a generalisation of previous work[9], with the incorporation of the A_i operators. Once the form (1.8–9) is obtained both A_i and G_i can still be renormalized (rescaled) by the same factor.

The rules given above enables one to rewrite any product of operators in what we shall call a normal order i.e.

- the G_i at the right of all the other operators (X_i and A_i)
- the A_i at the right of the X_i
- within the operators of the same name, the operators are ordered in, say, decreasing order of their indices.

There is a however a subtle point connected to (1.9). If one takes that equation for $i = j$, i.e.

$$G_i * A_i = r_{ii} A_i * G_i + \beta_i A_i^2 + \gamma_i G_i^2 \tag{1.10}$$

and multiplies it by A_i on the right, one obtains, dropping the index i as in (2.3) below, using (1.10) repeatedly and after rearrangement of the terms

$$G * A^2 = \beta\gamma G * A^2 + \beta(1+r)A^3 + \gamma^2(1+r)G^3 + r\gamma(1+r)A * G^2 + r(r+\beta\gamma)A^2 * G \tag{1.11}$$

All the terms in the right hand side, except the first one are normal ordered. There is thus a condition to be able to write $G * A^2$ in normal order : namely that

$$\beta\gamma \neq 1 \tag{1.12}$$

which we shall call the "normal ordering condition". In this case

$$G * A^2 = \frac{1}{(1-\beta\gamma)}\big(\beta(1+r)A^3 + \gamma^2(1+r)G^3 + r\gamma(1+r)A * G^2 + r(r+\beta\gamma)A^2 * G\big) \tag{1.13}$$

The condition (1.12), essential to the consistency of the definition of the normal product, will be systematically imposed in the next section.

In section 2 we solve, in general, the associativity requirements which, as we know, are not automatic for quommutation relations of the form given above. In section 3 we outline definitions of symmetric quantum derivatives which give representations of one of the solutions obtained in section 2.

II. Solutions of the associativity requirements.

We present here the general restrictions on the parameters which follow from the braiding relations which ensure that any product of operators can be rewritten in an unambiguous unique way as a normal product. Let us note that we have excluded pathological solutions such that $x_{ij} = 0$, $p_{ij} = 0$, ...

After some lengthy computations one finds that associativity for the products

$$\begin{array}{cc} A * X * X, & G * X * X, \\ G * A * A, & A * A * X, \\ G * G * X, & G * G * A \end{array} \tag{2.1}$$

for all the indices i, j, k require very simple equalities between the non-diagonal coefficients

$$r_{ij} = a_{ij} = g_{ij} \tag{2.2a}$$

$$i \neq j$$

$$q_{ij} = p_{ij} \tag{2.2b}$$

while the coefficients of the other possible terms in the right hand sides, i.e. the linear $(\lambda, \sigma, \mu, \rho)$ terms as well as the constant term ν, are forced, for non pathological solutions, to be zero.

The treatment of the diagonal case is somewhat more complicated.

Indeed the relations imposed by associativity which remain to be checked are those for $(G_i * A_i) * X_i = G_i * (A_i * X_i)$, i.e. within one family only. Dropping the index i one has to find the restrictions on the parameters for the set of quommutators

$$G * X = pX * G + X * A \tag{2.3a}$$
$$A * X = qX * A + \alpha X * G \tag{2.3b}$$
$$G * A = rA * G + \beta A^2 + \gamma G^2 \tag{2.3c}$$

with $p = p_{ii}$, $\alpha = \alpha_i$, ...

There are four disconnected cases which we now give explicitely

Case A

$$\begin{aligned} G * X &= pX * G + X * A \\ A * X &= qX * A + pqX * G \\ G * A &= rA * G + \beta A^2 + (q - qr - q^2\beta)G^2 \end{aligned} \tag{2.4Aa}$$

with the normal ordering condition (1.12)

$$\beta(q - qr - q^2\beta) \neq 1 \tag{2.4Ab}$$

Case B

$$\begin{aligned} G * X &= pX * G + X * A \\ A * X &= qX * A + \alpha X * G \\ G * A &= (1 + \beta p - \beta q)A * G + \beta A^2 - \alpha\beta G^2 \end{aligned} \tag{2.4Ba}$$

with the restriction

$$\alpha - pq \neq 0 \tag{2.4Bb}$$

and with the normal ordering condition (1.12)

$$\alpha\beta^2 + 1 \neq 0 \tag{2.4Bc}$$

Case C

$$\begin{aligned} G * X &= pX * G + X * A \\ A * X &= -pX * A + \alpha X * G \\ G * A &= -A * G + \beta A^2 + (\alpha\beta - 2p)G^2 \end{aligned} \tag{2.4Ca}$$

with the restrictions

$$\begin{aligned}\beta p + 1 &\neq 0 \\ \alpha + p^2 &\neq 0\end{aligned} \tag{2.4Cb}$$

and with the normal ordering condition (1.12)

$$\beta(\alpha\beta - 2p) \neq 1 \tag{2.4Cc}$$

Case D

$$\begin{aligned}G * X &= pX * G + X * A \\ A * X &= -pX * A - (2p^2 + p\gamma)X * G \\ G * A &= -A * G - \frac{1}{p}A^2 + \gamma G^2\end{aligned} \tag{2.4Da}$$

with the restriction

$$p + \gamma \neq 0 \tag{2.4Db}$$

while the normal ordering condition (1.12) coincides here with (2.8b).

Obviously, the situation is somewhat complicated by the fact that for each i, the solution can be chosen arbitrarily to belong to one of the four cases (A, B, C, D) above.

The usual Heisenberg algebra belongs to class B.

It may be interesting to ask the question whether, by a suitable linear change of basic operators, the set of quommutators, which is written in (2.3) with underlying physical motivations, cannot be brought in a simpler canonical position. Let us first stress that X plays a special role and that we are thus restricted to consider linear combinations of G and A only. Let G' (or A') be given by

$$G' = vG + wA \tag{2.5}$$

It will fulfill the equation

$$G'X = p'XG' \tag{2.6}$$

provided p' and $\lambda = \frac{w}{v}$ satisfy the equations

$$\begin{aligned}p'^2 - (p+q)p' + (pq - \alpha) &= 0 \\ \alpha\lambda^2 + (p-q)\lambda - 1 &= 0\end{aligned} \tag{2.7}$$

If there are two different roots, say p' and q', to (2.7) i.e. if the discriminant Δ which is the same for both equations is non zero

$$\Delta \equiv (p-q)^2 + 4\alpha \neq 0 \tag{2.8}$$

we see that (2.3) can be brought to the canonical form

$$G' * X = p'X * G' \tag{2.9a}$$

$$A' * X = q'X * A' \tag{2.9b}$$

$$s'G' * A' = r'A' * G' + \beta' A'^2 + \gamma' G'^2 \tag{2.9c}$$

where the last equation is the most general quadratic combination of naïve zero degree. If we suppose that s' is non zero, it can be renormalised to 1.

If $\alpha = pq$, i.e. case A above, one of roots of (2.7), q' say, is zero.

The associativity requirements for (2.9) are (provided that we suppose that s' and r' are not both zero and remembering that $q' \neq p'$)

$$p'\gamma' = 0 \tag{2.9d}$$

$$q'\beta' = 0 \tag{2.9e}$$

Since at least one of the roots is non-zero, say p', (2.9d) implies that

$$\gamma' = 0 \tag{2.9f}$$

and then (2.9e) leads to two cases

$$Case\ A' \quad : \ q' = 0, \ \gamma' = 0 \tag{2.9g}$$

$$Case\ B' \quad : \ \beta' = 0, \ \gamma' = 0 \tag{2.9h}$$

When the two roots of (2.7) are equal (i.e. if Δ of (2.8) is zero and $p' = (p+q)/2$), the system can be brought to the form

$$GX = p'X * G + XA' \tag{2.10a}$$

$$A' * X = p'X * A' \tag{2.10b}$$

$$s'G * A' = r'A' * G + \beta' A'^2 + \gamma' G^2 \tag{2.10c}$$

where

$$A' = A + \frac{p-q}{2} G \tag{2.10d}$$

$$p' = \frac{p+q}{2} \tag{2.10e}$$

$$s' = 1 - \frac{\beta(q-p)}{2} \tag{2.10f}$$

$$r' = r + \frac{\beta(q-p)}{2} \tag{2.10g}$$

$$\beta' = \beta \tag{2.10h}$$

$$\gamma' = \gamma + \frac{\beta(p-q)^2}{4} + \frac{(q-p)(r-1)}{2} \tag{2.10i}$$

The form of (2.10a), (2.10b) is somewhat similar to that of the well known deformation of the Heisenberg commutation relations of Biedenharn[10] and Macfarlane[11], with the formal identification of $a^\dagger$ with X, $a^\dagger a = N$ with G and q^{-N} with A. Of course in their scheme, the operators A and G are not independent, as they are for us.

If $s' \neq 0$ it can be renormalized to 1 and the associativity requirements for (2.10) are

$$\gamma' = 0 \tag{2.10j}$$

$$p'(s' - r') = 0 \tag{2.10k}$$

leading to two cases again.

$$Case\ C' \quad : \ p' = 0, \ \gamma' = 0 \tag{2.10l}$$

$$Case\ D' \quad : \ r' = s', \ \gamma' = 0 \tag{2.10m}$$

If $s' = 0$, r' can be renormalized to -1 and the associativity requirements lead to

$$p'\gamma' = 0 \tag{2.10n}$$

$$p' - \gamma' = 0 \tag{2.10o}$$

This leads to the rather uninteresting case

$$Case\ E' \quad : \ p' = 0, \ \gamma' = 0 \tag{2.10p}$$

It is obvious that the associativity conditions were easier to write in the new basis but we first presented the results in the old one as it is more physical. The change of basis facilitates, in certain cases, the discussion of representations.

Let us also note that the conditions (2.2) are exactly those which allow the above redefinitions in terms of G' and A' in a coherent way between different indices.

III. Symmetric q-derivatives.

We now present a particular case of associative operators which are suited to define a symmetric q-derivative.

Let, as above, the X_i be a set of quantum variables. For a function $f(x_i)$ of one variable alone, let us define

$$(X_i f)(x_i) = x_i f(x_i) \tag{3.1a}$$

$$(D_i f)(x_i) = \frac{f(p_i x_i) - f(x_i/p_i)}{x_i(p_i - 1/p_i)} \tag{3.1b}$$

as the symmetrical q-derivative.

As a consequence the operator G_i defined in (1.3) has the following action

$$(G_i f)(x_i) = \frac{f(p_i x_i) - f(x_i/p_i)}{(p_i - 1/p_i)} \tag{3.2}$$

The operator A_i defined by

$$D_i * X_i = p_{ii} X_i * D_i + A_i \tag{3.3}$$

has still some arbitrariness but can also be chosen, in a unique way, to have a symmetric action

$$(A_i f)(x_i) = \frac{f(p_i x_i) + f(x_i/p_i)}{2} \tag{3.4}$$

From these definitions the quommutation relations within the i family can be deduced

$$\begin{aligned} G_i * X_i &= p_{ii} X_i * G_i + X_i * A_i \\ A_i * X_i &= p_{ii} X_i * A_i + (p_{ii}^2 - 1) X_i * G_i \\ G_i * A_i &= A_i * G_i \end{aligned} \tag{3.5}$$

where

$$p_{ii} = \frac{p_i + 1/p_i}{2} \tag{3.6}$$

They correspond to Case B above with $q = p$, $\alpha = p^2 - 1$ and $\beta = 0$, a particularly simple case.

The connection between the operators coresponding to two different indices i and j still depend on four a priori independent parameters x_{ij}, g_{ij}, p_{ij} and p_{ji}. However the consistency of the quommutator (1.7) when applied to the function $f \equiv 1$ implies $a_{ij} = 1$ i.e. $g_{ij} = 1$ since through (3.4) the action of A_i on 1 is 1 for any i.

Let us note also that a exponential can be easily defined as a solution, say for one variable x only, of

$$DE(x) = E(x) \tag{3.7}$$

For the symmetrical q-derivative it reads

$$E(x) = \sum_{n=0}^{\infty} a_n x^n \tag{3.8}$$

where

$$\begin{aligned} a_0 &= 1, \qquad a_1 = 1 \\ a_n &= \frac{1}{\prod_{k=1}^{n-1} [k]_p} \qquad n > 1. \end{aligned} \tag{3.9}$$

Here

$$[j]_{q_i} = \frac{q_i^j - q_i^{-j}}{q_i - q_i^{-1}}. \tag{3.10}$$

In order to generalise the q-exponential to many variables it is necessary to choose a particular set of values for the parameters p_{ij} etc. This choice is

$$\begin{aligned} p_{ij} = q_{ij} =& \frac{(q_i + q_i^{-1})}{2}, \quad \forall j, \\ \alpha_i =& \frac{(q_i - q_i^{-1})^2}{4}, \\ r_{ij} =& 1, \;\; \gamma = 0. \end{aligned} \tag{3.11}$$

It allows $x_i x_j$ to be interchanged; it turns out that

$$(q_j + q_j^{-1}) x_i x_j = (q_i + q_i^{-1}) x_j x_i \quad \forall i, j \tag{3.12}$$

may be imposed without affecting the q-exponential. This equation allows all monomials in $x_i, x_j, \ldots$ to be ordered alphabetically.

The general q-exponential is given by the simultaneous solution of the equations

$$D_i E(x_1, x_2, \ldots x_N) = E(x_1, x_2, \ldots x_N). \tag{3.13}$$

The general term in the expression $E(x_1, x_2, x_3)$ for example, is given by

$$\prod_j^a \frac{1}{[j]_{q_1}} \prod_k^b \frac{1}{[k]_{q_2}} \prod_l^c \frac{1}{[l]_{q_3}} x_1^a x_2^b x_3^c \Big(\frac{2}{[2]_{q_1}}\Big)^{a(b+c)} \Big(\frac{2}{[2]_{q_1}}\Big)^{bc}. \tag{3.14}$$

The generalisation to an arbitrary number of variables is obvious. Thus this choice of parameters allows a simultaneous construction of a generalised q-exponential, or coherent state, and a set of permutable creation operators. This last feature is admittedly absent in the work of Greenberg[3] and in most of the literature on coherent states by implication.

IV. General q-derivatives.

A slight generalisation of the preceding section can be obtained as follows in the case of one variable only.

Suppose we define the X as before on a function $f(x)$

$$(Xf)(x) = xf(x) \tag{4.1}$$

and try to construct the action of G as

$$(Gf)(x) = \sum_{k=1}^{M} \chi_k f(\lambda_k x) \tag{4.2}$$

Equation (2.3a) is then a simple definition of A

$$(Af)(x) = \sum_{k=1}^{M} \chi_k(\lambda_k - p) f(\lambda_k x) \tag{4.3}$$

Equation (2.3b) is then a consistency equation which reads

$$\sum_{k=1}^{M} \chi_k \left(\lambda_k^2 - (p+q)\lambda_k + qp - \alpha\right) f(\lambda_k x) = 0. \tag{4.4}$$

Since the $f(\lambda_k x)$ are independent for sufficiently general choices of $f(x)$ (4.4) implies that

$$\lambda_k^2 - (p+q)\lambda_k + qp - \alpha \tag{4.5}$$

for every k (equation (2.7) again). But since p, q and α don't depend on k and since also a second degree equation has only two solutions, there are at most two allowed values of λ_k. Hence $M = 2$ and the two λ's are given in terms of the three free parameters p, q and α.

Conversely, if the two values of λ are given $\lambda_1 = \lambda$, $\lambda_2 = \mu$, the condition (4.5) gives the restrictions

$$p + q = \lambda + \mu \tag{4.6}$$

$$pq - \alpha = \lambda\mu \tag{4.7}$$

So that there are altogether again three free parameters, say λ, μ and p. The remaining ones q and α being fixed by (4.6) and (4.7).

Finally A and G commute

$$G * A = A * G. \tag{4.8}$$

All these equation finally generate a representation of case B with $\beta = 0$ but the other parameters are free. This representation can be extended in a natural way when there is more than one variable x_i. Obviously these variables are then quantum variables and have to fulfill quommutation relations in agreement with (1.1a).

The representations discussed so far are those acting on an infinite dimensional function space.

V. Representations with G and A diagonal.

Let us look now for representations such that both G and A are diagonal and hence commute.

Suppose we start with a vector $\mid 0 >$, eigenvector of G and A with eigenvalue g_0 and a_0 respectively

$$\begin{aligned} G \mid 0 > &= g_0 \mid 0 > \\ A \mid 0 > &= a_0 \mid 0 > \end{aligned} \tag{5.1}$$

Let us define

$$| n >= X^n \mid 0 > \tag{5.2}$$

Then

$$\begin{aligned} G \mid n > &= g_n \mid n > \\ A \mid n > &= a_n \mid n > \end{aligned} \tag{5.3}$$

With the vector v_n defined by

$$v_n = \begin{pmatrix} g_n \\ a_n \end{pmatrix} \tag{5.4}$$

and the matrix M defined by

$$M = \begin{pmatrix} p & 1 \\ \alpha & q \end{pmatrix} \tag{5.4}$$

it is easy to prove that

$$v_n = M v_{n-1} = M^n v_0 \tag{5.5}$$

This obviously depends on the precise form of the matrix M and of its eigenvalues, obviously equation (2.7) again.

Let us now make the following important remark. Since both G and A are diagonal, they commute for this representation. But this commutation is **not** a qualgebra relation. What has to hold is (2.5c) which has still to be satisfied at every stage of the procedure. It reads in the general case

$$C_n \equiv \beta a_n^2 + \gamma g_n^2 + (r-1) a_n g_n = 0 \tag{5.6}$$

More precisely if, say a_n and g_n are chosen in such a way that $C_n = 0$ there is a condition on the free parameters to garantee that C_{n+1} be zero. Either

$$\begin{aligned} &p^2\beta\gamma - pqr^2 + 2pqr + 2pq\beta\gamma - pq - pr\gamma + pr\alpha\beta + p\gamma - p\alpha\beta + q^2\beta\gamma \\ &+ qr\gamma - qr\alpha\beta - q\gamma + q\alpha\beta + \gamma^2 - 2\alpha\beta\gamma + \alpha^2\beta^2 = 0 \end{aligned} \tag{5.7a}$$

or

$$\begin{aligned} &p^2\gamma\beta - 2pq\beta\gamma - pr\gamma + pr\alpha\beta + p\gamma - p\beta\alpha + q^2\beta\gamma \\ &+ qr\gamma - qr\alpha\beta - q\gamma + q\alpha\beta - r^2\alpha + 2r\alpha + \gamma^2 + 2\alpha\beta\gamma + \alpha^2\beta^2 - \alpha = 0 \end{aligned} \tag{5.7b}$$

It is amusing to note that the product of the two expressions is always identicallly zero in the four allowed cases $A - D$. This fact shows that this representation always exists. Let us stress again that the fact that G and A commute is a simple artefact of the representation. It is analogous for example to the fact that, if the Pauli matrices are 2-dimensional representations of $SU(2)$, the fact that $\sigma_1\sigma_2 = i\sigma_3$ is a simple artefact which has nothing to do with the basic commutation relations of the $SU(2)$ algebra.

VI. Representations where X has an eigenvector.

In the preceding paragraph we have chosen to present the case where both G and A are diagonal, as they are simple. However we believe that the physically interesting representations rather correspond to cases where X has an eigenvector $\mid 0>$ with eigenvalue x_0

$$X \mid 0 >= x_0 \mid 0 > \tag{6.1}$$

It is now more convenient to re-introduce the linear combinations in terms of which the quommutation relations simplify to (2.9).

An infinite set of states $\mid k, l >,\ k = 1, \ldots, l;\ l = 1, \ldots$ can then be constructed through

$$\mid k, l >= A'^k G'^l \mid 0 >, \quad 0 \le k \le l \tag{6.2}$$

The action of the operators in this infinite dimensional space in the case can then be found easily through the equations

$$\begin{aligned} X \mid k, l > &= x_0 q'^{-k} p'^{-} l \mid k, l > \\ A' \mid k, l > &= \mid k+1, k > \\ G' \mid k, l > &= \mid k, l+1 > \end{aligned} \tag{6.3}$$

When G' and A' do not commute then the last of these equations requires modification. Using (2.9c), applied to the states $\mid l, 0 >, \mid l+1, 1 >$,

$$\begin{aligned} G' \mid 1, k >=& r \mid 1, k+1 > + \beta \mid 2, k > + \gamma \mid 0, k+2 > \\ G' \mid 2, k >=& \frac{1}{(1-\beta\gamma)} \big(\beta(1+r) \mid 3, k > + \gamma^2 (1+r) \mid 0, k+3 > \\ & + r\gamma(1+r) \mid 1, k+2 > + r(r+\beta\gamma) \mid 2, k+1 > \big) \end{aligned} \tag{6.4}$$

The last of these equations also follows from (1.13) applied to $\mid l, 0 >$. The general result will follow upon iteration.

VII. Finite dimensional representations.

Finite dimensional representations of operators are always useful to consider. We restrict ourselves to situations where $p \neq 0$, $q \neq 0$ and $r \neq 0$.

One dimensional representations.

First, the one dimensional representations are obviously trivial but we present them in order to be able to make the remark following (7.2) below.

When X is represented by 1 by naïve rescaling and $G = 1$ also.

Then

$$\begin{aligned} X &= 1 \\ G &= 1 \\ A &= (1-p) \end{aligned} \tag{7.1a}$$

We can apparently solve directly for the parameters of (2.3), without going through the four cases (A-D) one by one.

The parameters are restricted by the two relations

$$\alpha = (1-p)(1-q) \tag{7.1b}$$
$$\gamma = (1-p)(1-r-\beta+p\beta) \tag{7.1c}$$

When $X = 0$ then

$$\begin{aligned} X &= 0 \\ G &= g \\ A &= a \end{aligned} \tag{7.2a}$$

and the numbers g and a must satisfy

$$(1-r)ga - \beta a^2 - \gamma g^2 = 0 \tag{7.2b}$$

It may appear strange at first sight that the representation (7.1) (or (7.2)) looks more general than any of the cases (A–D) above. The justification for this fact is as follows. To derive the conditions of associativity we have supposed, rightly, that, once put in normal order, any product of the starting operators X, G, A, for up to three operators in the product, is linearly independent of any other. This is clearly not true for $X = G = 1$. Hence there are apparently more solutions to the associativity requirements. These extra solutions should obviously be rejected as they are not bona fide representations of the abstract quommutators.

Two dimensional representations.

Henceforth we will restrict ourselves to representations where not all the operators are represented by diagonal matrices, i.e. irreducible representations.

By performing a general change of basis in the two-dimensional space upon which the operators act and by using a suitable rescaling of the naive length $[L]$ unit, it is always to bring the operator X in one of the following well-known canonical positions :

a) $X = 1_2$ is the 2-dimensional unit matrix
b) $X = \mathrm{diag}(1, x)$ where $x \neq 0$ and $x \neq 1$
c) $X = \mathrm{diag}(1, 0)$
d) $X = \sigma_+$ where σ_+ is the Pauli matrix with only non-zero element $\sigma_+(1, 2) = 1$
e) $X = 1_2 + x\sigma_+$ where $x \neq 0$.

The full discussion is rather rich. Indeed, the allowed representations depend often on particular and more detailed relations between the parameters than those which define the four cases (A–D) above. We have thus chosen not to present them though some of them are quite interesting.

Finite representations with G and A diagonal.

Using the results of section V, finite dimensional representations of the qualgebras can be constructed. Indeed, if we suppose that there exists a positive integer P such that (see (5.2))

$$\mid P >=\mid 0 > \tag{7.3}$$

the space on which the representation acts becomes $P-$dimensional. In order to reproduce the same eigenvalues for G and A, one must have $v_P = v_0$. Consequently one needs (see (5.5))

$$M^P = 1 \tag{7.4}$$

For this to be the case, the two eigenvalues of M i.e. p' and q' have to be $P-$roots of unity. We find once again the occurrence of the important "roots of unity" which play a central role in qualgebras.

In the transformed basis where M is diagonalized, X is essentially a matrix of cyclic permutation.

Other representations can be constructed from these by using suitable direct sums or direct products of representations. An example using direct products, usually equivalent to one of the general type constructed above, or reducible to direct sum of them is as follows : If a and b in are prime numbers, then a representation by $ab \times ab$ dimensional matrices in the changed basis is given by

$$\begin{aligned} G =&\text{diag}\{1, p', p'^2, \dots p'^{a-1}\} \otimes \text{diag}\{1, 1, 1, \dots 1\} \\ A =&\text{diag}\{1, 1, \dots 1\} \otimes \text{diag}\{1, q', q'^2 \dots q'^{b-1}\} \\ X =&P_{a\times a} \otimes P_{b\times b} \end{aligned} \tag{7.5}$$

Here $p'^a = 1,\ q'^b = 1$ and $P_{a\times a}$ and $P_{b\times b}$ are matrix representations of cyclic permutations of a and b objects respectively. Here again G and A commute but, as explained above this is a simple artefact of the representation of a more general qualgebra.

VIII. Conclusion.

Using a natural set of a priori quommutation relations between quantum variables X_i, quantum derivatives D_i or better the related quantum dilatation operators $G_i = X_i * D_i$ we have been led to introduce the corresponding neutral operators A_i, in order to give an algebraic realisation of symmetric q-differentiation. We have outlined all the possible choices allowed by the associativity requirements (or braiding relations) within our basic restriction (1.1a), that the transposition of two quantum variables does not intoduce any other operators. We have shown that, within a given i (the i family) and taking into account the normal ordering condition, there are four disconnected cases, (2.4A-D). The relations beween two families i and j depend, due to (2.2), on four arbitrary parameters $x_{ij} = 1/x_{ji}$, $g_{ij} = 1/g_{ji}$, p_{ij} and p_{ji}. We have presented some representations of our abstract qualgebras with one X, one G and one A only both in finite and infinite dimensional spaces. Having constructed all 2-dimensional representations, we have realized that the space of representation is apparently very rich.

A particular example of a representation for these operators and their action has been shown explicitly for what we have called symmetric or general quantum derivatives. It allows $x_i x_j$ to be interchanged and at the same time permits the construction of a multivariable q-exponential; explicitly the result is obtained that

$$(q_j + q_j^{-1}) x_i x_j = (q_i + q_i^{-1}) x_j x_i \quad \forall i, j \tag{8.2}$$

may be imposed without affecting the q-exponential. This may have some bearing upon attempts to apply quantum groups to quantum optics. We hope to elaborate on those points in the near future.

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