

Algorithms and (partial) symmetry for solutions of the non-linear Poisson equation

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*Variational and Topological Methods:
Theory, Applications, Numerical Simulations, and Open Problems*
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Model problem — variational formulation

$$(P) \begin{cases} -\Delta u(x) = f(x, u(x)) & \text{for } x \in \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \Leftrightarrow \partial\mathcal{E}(u) = 0$$

where Ω is an open bounded subset of \mathbb{R}^N and $\mathcal{E} : H_0^1(\Omega) \rightarrow \mathbb{R}$ is given by

$$\mathcal{E}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p} \int_{\Omega} F(x, u(x)) dx$$

where $F(x, u) = \int_0^u f(x, v) dv$ and

$$\partial\mathcal{E}(u)v = \int_{\Omega} \nabla u \nabla v dx - \int_{\Omega} f(x, u(x)) v dx$$

Model $f : f(x, u) = |u|^{p-2}u$.

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Outline

- 1 Positive solutions
- 2 Nodal solutions

1

Positive solutions

- Existence
- Mountain Pass Algorithm
- Guaranteeing Positive solutions
- (Partial) Symmetry & Uniqueness

2

Nodal solutions

Existence of positive solutions

Theorem

*(P) possesses a non-negative solution
(under suitable assumptions on f).*

PROOF INGREDIENTS :

- Mountain pass theorem
- If u is a critical point of \mathcal{E} :
 $\mathcal{E}(u^+) \leq \mathcal{E}(u)$.

REMARKS :

- $u > 0$ in Ω by the maximum principle.
- The projector $u \mapsto u^+$ decreases the slightly modified functional
 $\mathcal{E}_{\text{modif}} : H_0^1(\Omega) \rightarrow \mathbb{R} : u \mapsto \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} F(x, u^+) dx$

Algorithm to compute the solution ?

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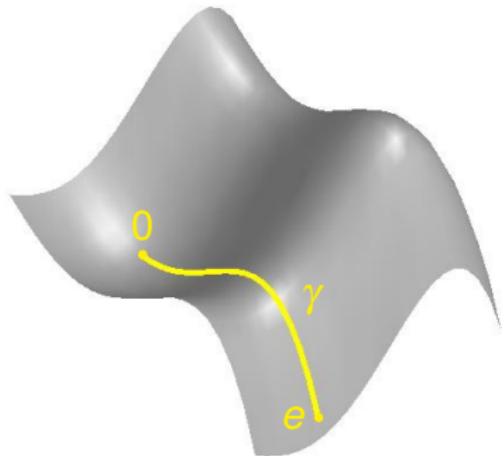
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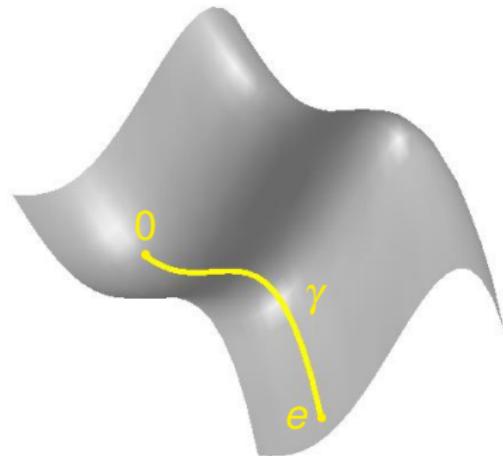
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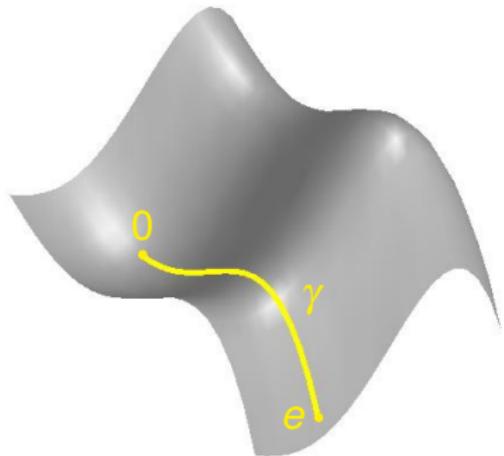
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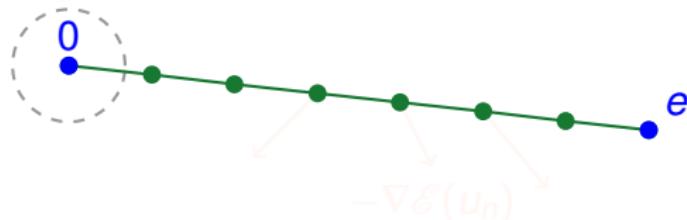
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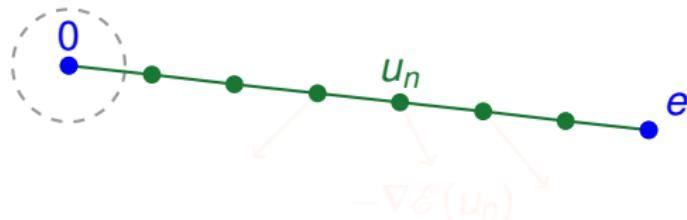


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 $e \in X$ far enough from 0 s.t.
 $\mathcal{E}(e) \leq \mathcal{E}(0)$.

ALGORITHM

- ➊ Initial path: $\gamma(i) = \frac{i}{N} e$, $i = 0, \dots, N$; $n \leftarrow 0$.
- ➋ compute $\arg \max \{\mathcal{E}(\gamma(i)) : i = 0, \dots, N\}$ and improve it by quadratic interpolation to get $u_n \approx \arg \max \mathcal{E}(\gamma([0, 1]))$.
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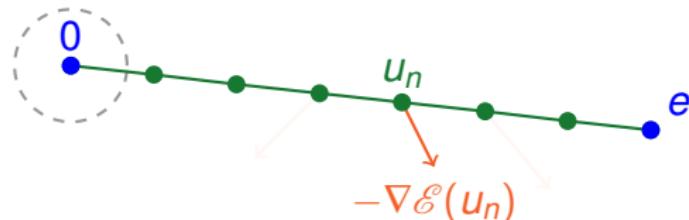


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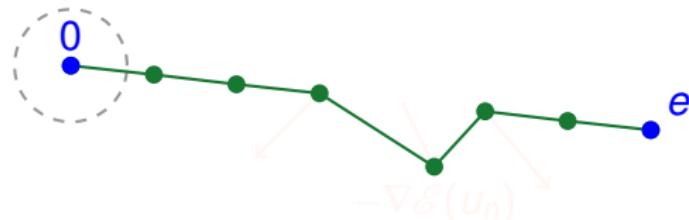


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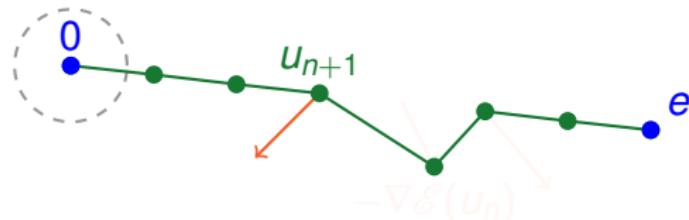


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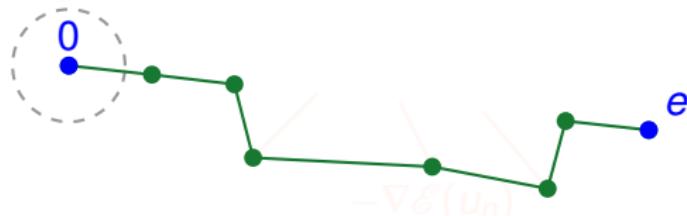


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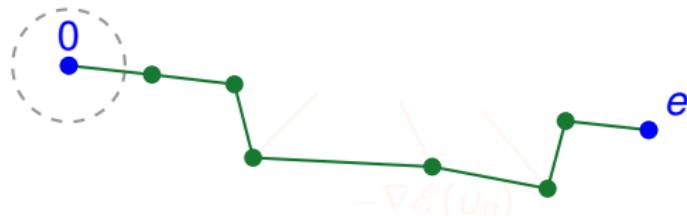


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Convergence?

They try to mimic the MP construction:

$$\Gamma_N = \{\text{piecewise linear path joining } 0 \text{ to } e \text{ with } N \text{ segments}\}$$

$$c_N = \inf_{\gamma \in \Gamma_N} \max_{\gamma([0,1])} \mathcal{E}$$

- $c_N \xrightarrow[N \rightarrow \infty]{} c$ where c is the MP level (easy).
- $u_n \xrightarrow[n \rightarrow \infty]{} u^*$ with u^* being (close to) a critical point for N large?
 - ☞ The elastic string algorithm (Moré & Munson '04).

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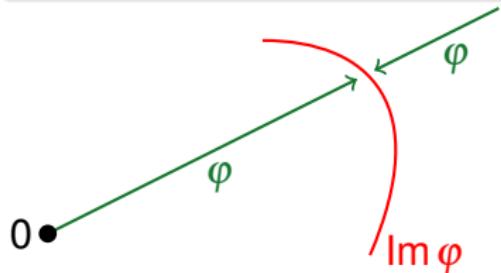
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Algorithm based on the notion of peak selection.

Definition

A function $\varphi : X \setminus \{0\} \rightarrow X \setminus \{0\}$ is a *peak selection* of \mathcal{E} iff, for every $u \in X \setminus \{0\}$,

- $\varphi(u)$ is a local max of \mathcal{E} on $\{tu : t \in]0, +\infty[\}$
- $\forall \lambda > 0, \varphi(\lambda u) = \varphi(u)$



Definition

$\eta : \mathbb{R}_+ \times X \rightarrow X$ is a *descent flow* for $\mathcal{E} \in \mathcal{C}^1(X; \mathbb{R})$ if

- ① η is continuous;
- ② η is a flow : $\eta(0, u) = u$ and $\eta(t, \eta(s, u)) = \eta(t + s, u)$;
- ③ $\forall t \in \mathbb{R}_+, \forall u \in X, \mathcal{E}(\eta(t, u)) \leq \mathcal{E}(u)$.
- ④ $\forall \varepsilon > 0, \exists \delta > 0, \forall u \in X,$

$$\|\partial \mathcal{E}(u)\| \geq \varepsilon \Rightarrow \exists T > 0, \mathcal{E}(\eta(T, u)) \leq \mathcal{E}(u) - \delta T$$

REMARK:

- (2) & (3) $\Rightarrow \forall u, t \mapsto \mathcal{E}(\eta(t, u)) \searrow$

Definition

$$\omega(u) := \left\{ \lim_{n \rightarrow +\infty} \eta(t_n, u) : 0 \leq t_n \nearrow +\infty \right\}$$

RELATION TO CRITICAL POINTS :

- ① $u^* \in \omega(u) \Rightarrow \partial \mathcal{E}(u^*) = 0$
- ② if $c_u := \lim_{t \rightarrow +\infty} \mathcal{E}(\eta(t, u)) > -\infty$, then $\exists t_n \nearrow +\infty$ s.t.

$$\mathcal{E}(\eta(t_n, u)) \rightarrow c_u \quad \text{and} \quad \partial \mathcal{E}(\eta(t_n, u)) \rightarrow 0$$

REMARKS :

- if $c_u > -\infty$ and $(PS)_{c_u}$ holds, then $\omega(u) \neq \emptyset$
- With the definition of the flow above, $\partial \mathcal{E}(u) = 0 \not\Rightarrow \omega(u) = \{u\}$.

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Basin of attraction

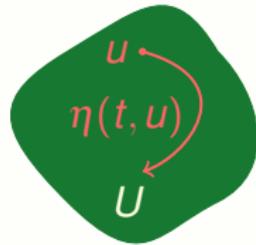
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A set $U \subseteq X$ is (forward) invariant iff $\eta(t, U) \subseteq U$ for all $t \geq 0$.

Definition

The *basin of attraction* of a forward invariant set $U \subseteq X$ is

$$\mathcal{A}(U) := \{u \in X : \exists t \geq 0, \eta(t, u) \in U\}$$



$\mathcal{A}(U)$

EXAMPLE : $\mathcal{E}^{< c} := \{u \in X : \mathcal{E}(u) < c\}$ is flow invariant for any c (for any descent flow).

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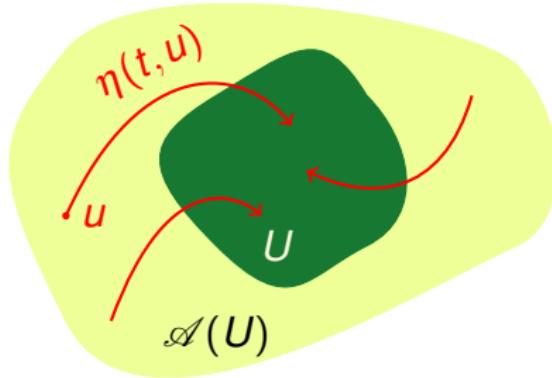
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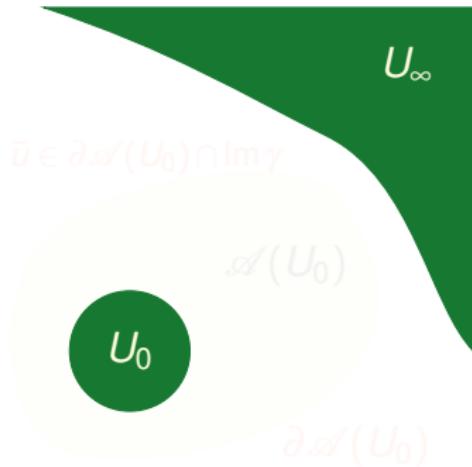
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Mountain pass through flows



Let $\varepsilon > 0$ be small.

$$\mathcal{E}^{<\varepsilon} = U_0 \cup U_\infty$$

$\mathcal{A}(U_0)$ basin of attraction of U_0 is an open set

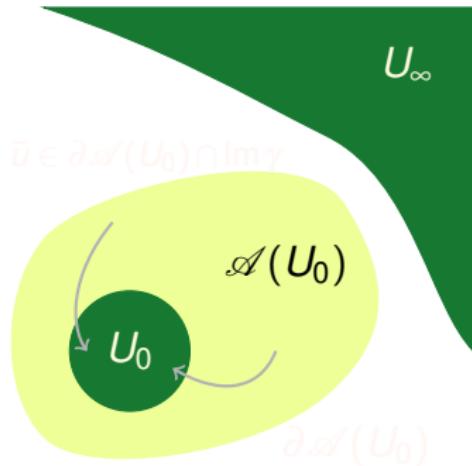
$\partial \mathcal{A}(U_0)$ is forward invariant

$$\inf_{\partial \mathcal{A}(U_0)} \mathcal{E} \geq \varepsilon > -\infty$$

The forward orbit $\{\eta(t, u) : t \geq 0\}$ of any $u \in \partial \mathcal{A}(U_0)$ contains a PS sequence.

Algorithm to locate such a \bar{u} ?

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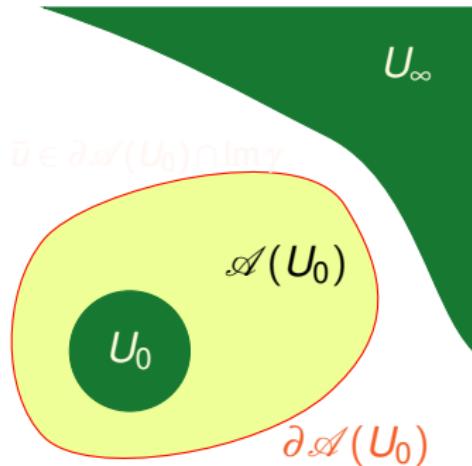
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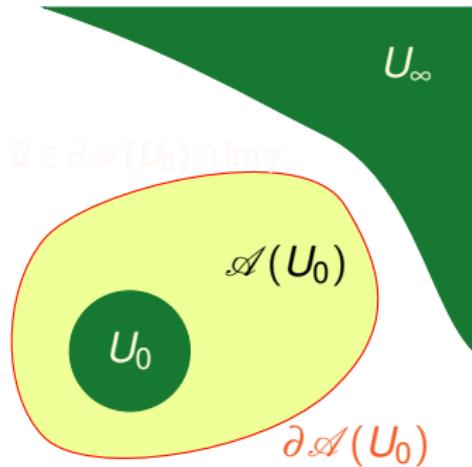
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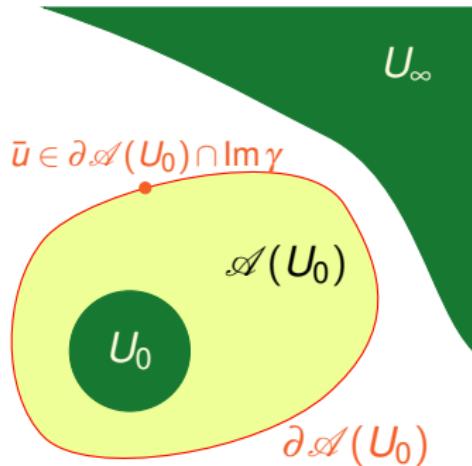
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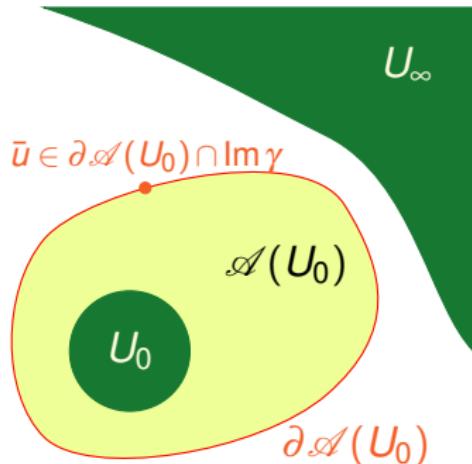
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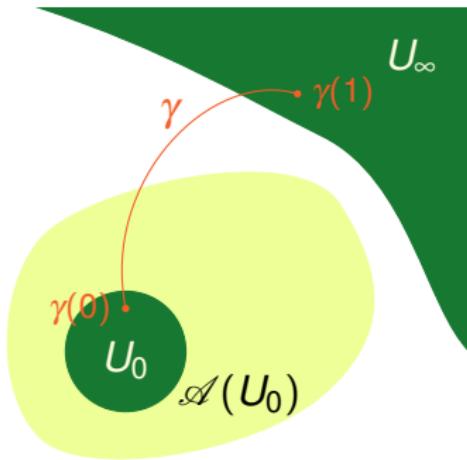
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Algorithm to locate such a \bar{u} ?

Bisection Algorithm



Take a path $\gamma \in \mathcal{C}([0, 1]; X)$ joining U_0 to U_∞ .

Bisection algorithm

$s_0 \leftarrow 0$ and $s_1 \leftarrow 1$

loop

let $s_{\text{mid}} = \frac{1}{2}(s_0 + s_1)$

if $\gamma(s_{\text{mid}}) \in \mathcal{A}(U_0)$ then

$s_0 \leftarrow s_{\text{mid}}$

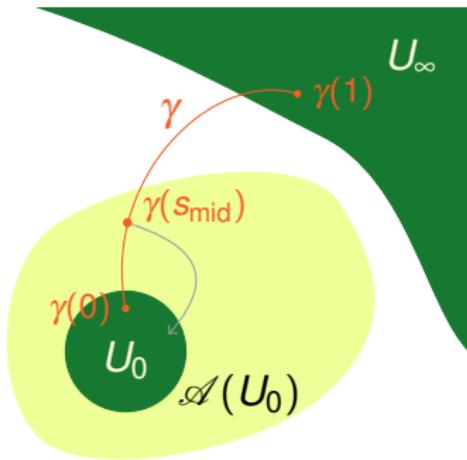
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Algorithm to locate a point in $\omega(\bar{u})$?

Bisection Algorithm



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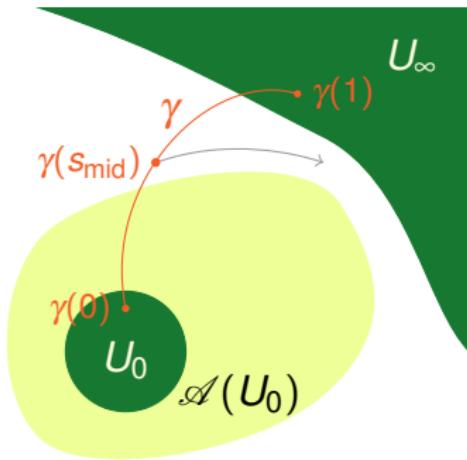
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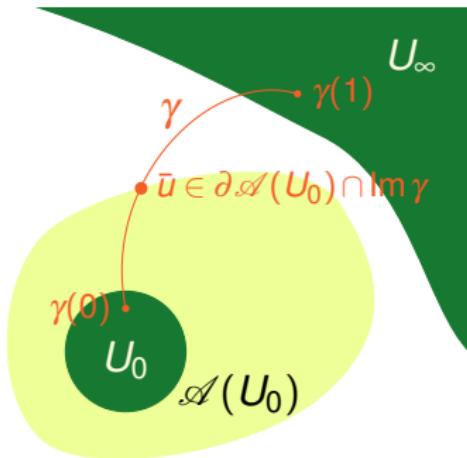
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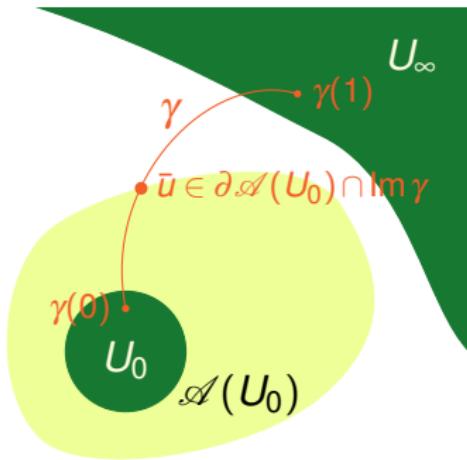
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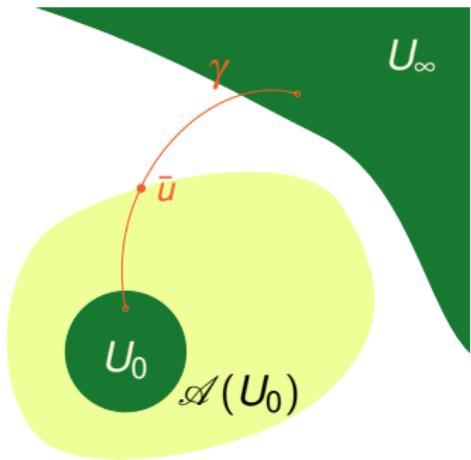
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Bisection Mountain Pass Algorithm



Bisection MP algorithm

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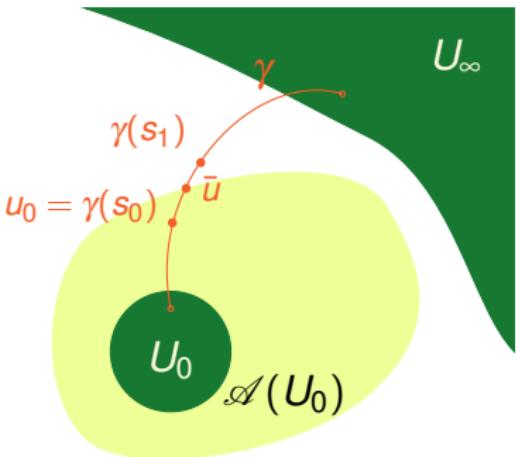
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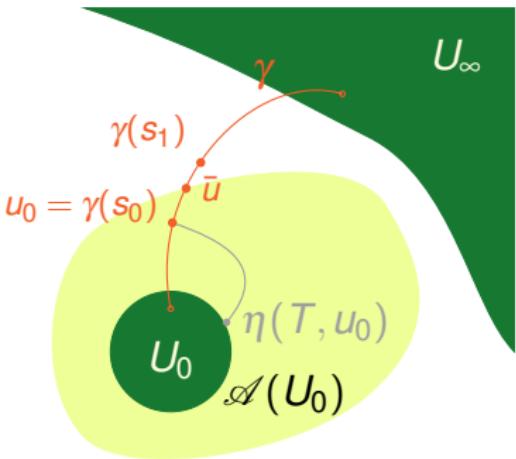
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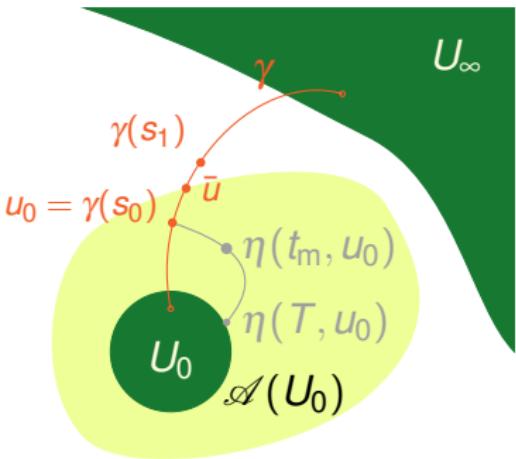
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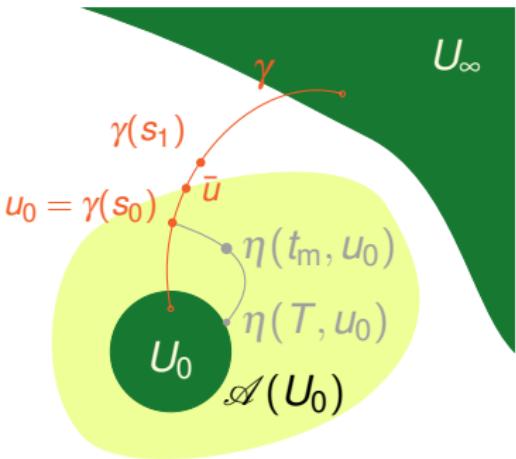
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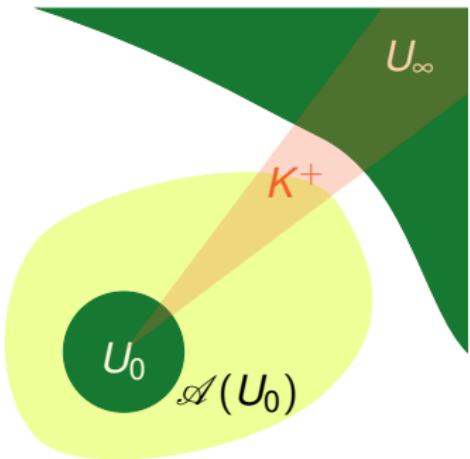
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Positive solutions



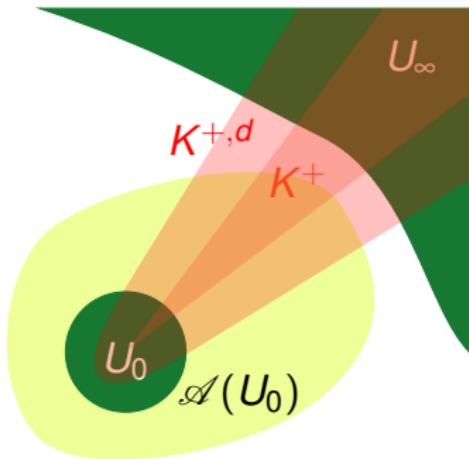
K^+ cone of non-negative functions

For $d > 0$ small enough,

$$K^{+,d} := \{u \in X : \text{dist}(u, K^+) < d\}$$

is flow invariant and every nontrivial solution in $K^{+,d}$ is positive.

Positive solutions



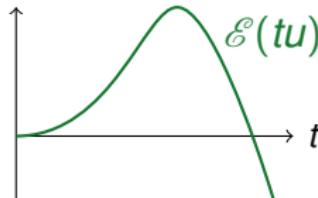
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Work to do

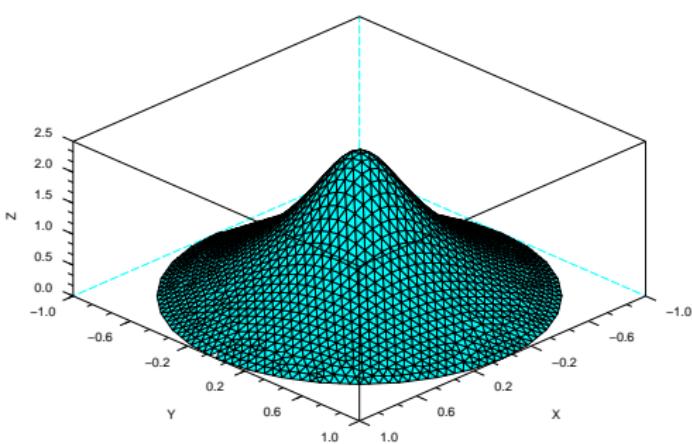
- ① Show that the algorithm still converges if the flow is only **computed approximately**.
- ② Work out sufficient conditions for the decidability of $x \in \mathcal{A}(U)$ (e.g. does it at least work under the “single peak” assumption?).



Positive solution for the “power nonlinearity”

$$\begin{cases} -\Delta u = \lambda u + u^{p-1} & \text{in } B(0,1) \\ u = 0 & \text{on } \partial B(0,1) \\ u > 0 & \text{in } B(0,1) \end{cases}$$

where $-\infty < \lambda < \lambda_1(B(0,1))$.



Gidas-Ni-Nirenberg ('79)
moving planes $\Rightarrow u$ is radial
 $\Rightarrow u$ is unique

Numerical experiment

Let $\mu_2(p) \geq 0$ be the second eigenvalue of

$$-\Delta \varphi - \lambda \varphi - (p-1)u^{p-2}\varphi = 0$$

with D.B.C. Is $\mu_2(p) > 0$? Is

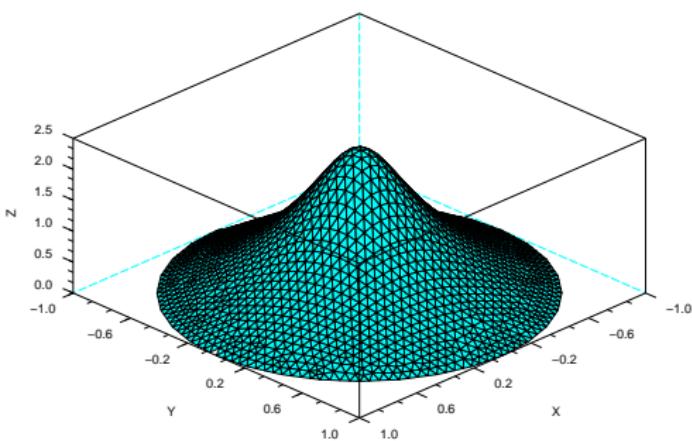
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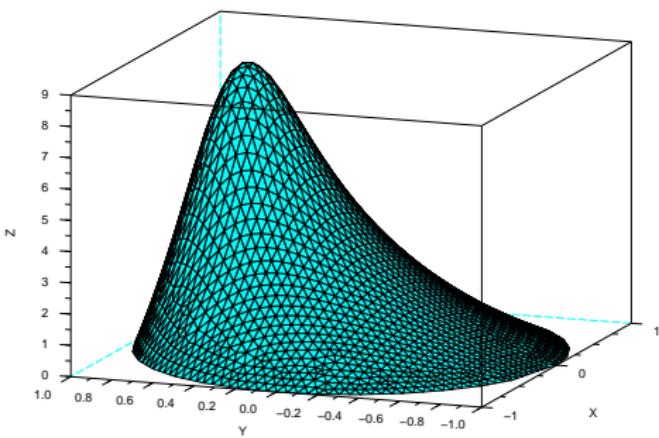
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Positive solution of Henon's equation

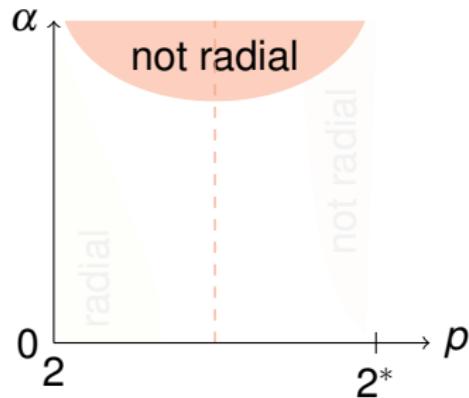
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$\forall p \in]2, 2^*[$, $\exists \alpha^* > 0$, $\forall \alpha > \alpha^*$, least energy solutions are not radial.



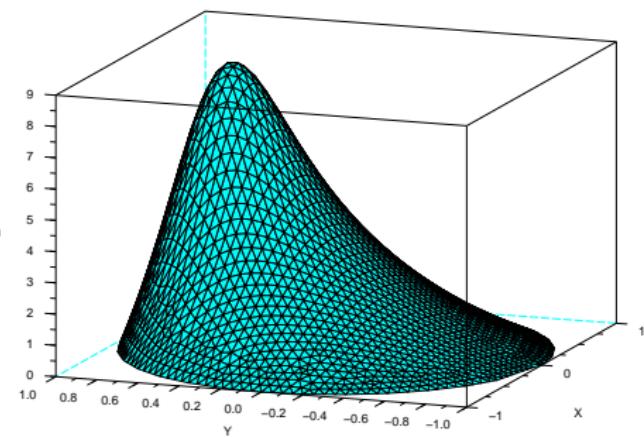
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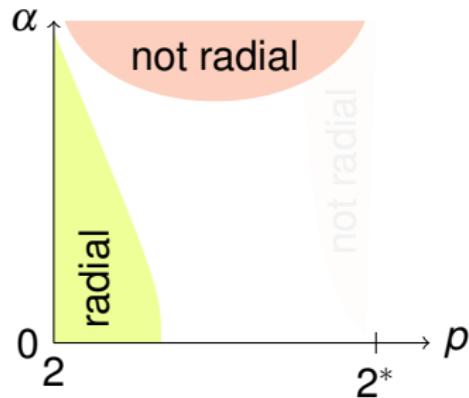
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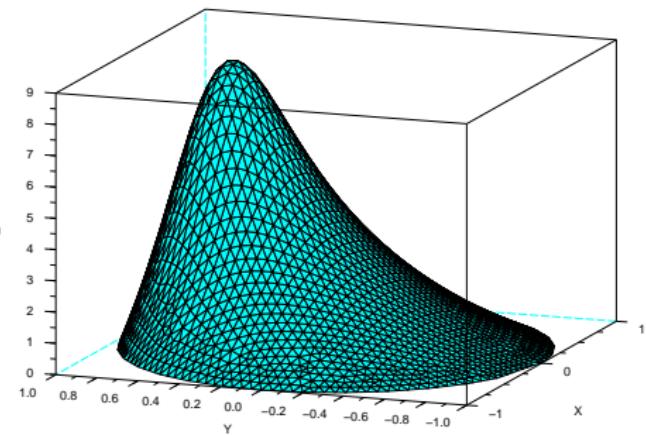
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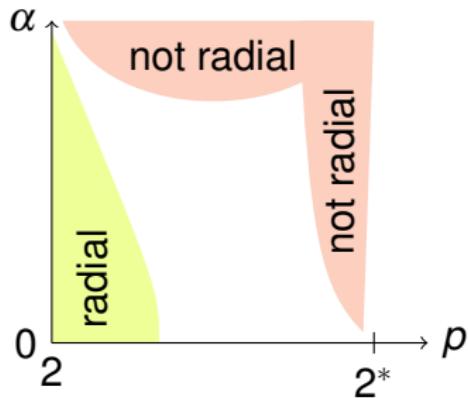
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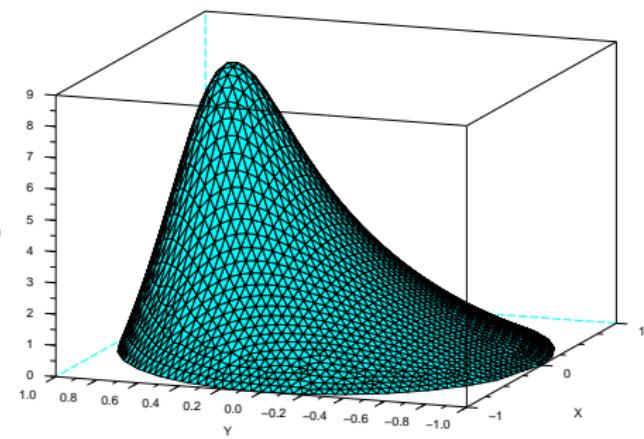
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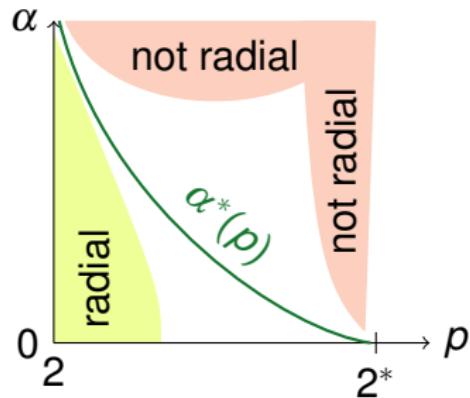
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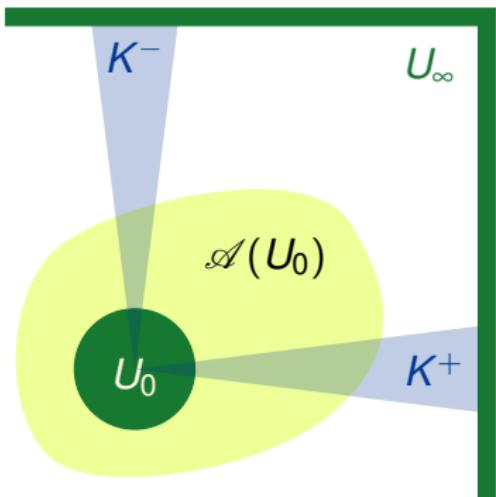
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1 Positive solutions

2 Nodal solutions

- Existence & Nodal algorithm
- Symmetry

Existence of nodal solutions



Liu-Sun ('01)

$$\mathcal{E}^{<\varepsilon} = U_0 \cup U_\infty$$

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K^- cone of non-positive functions

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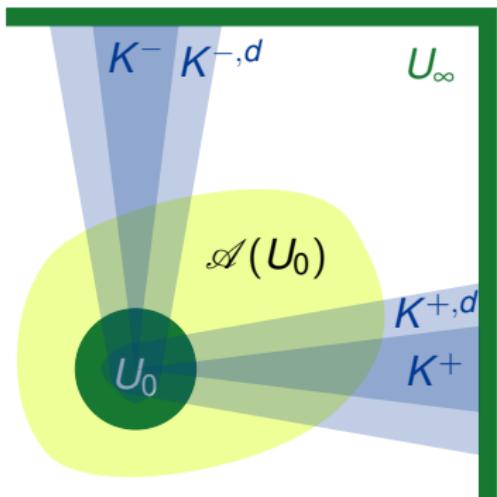
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Nodal algorithm

Can this idea be made effective?

Algorithm that is proved convergent.

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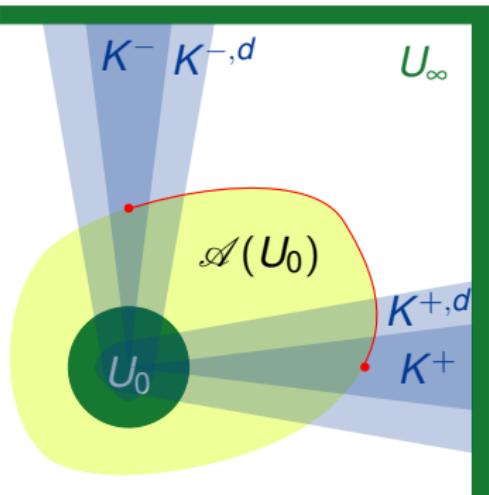
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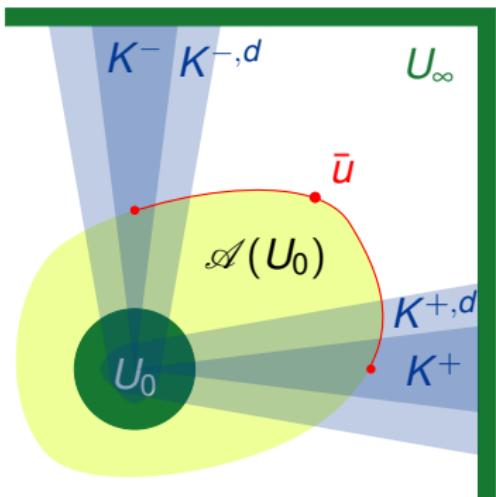


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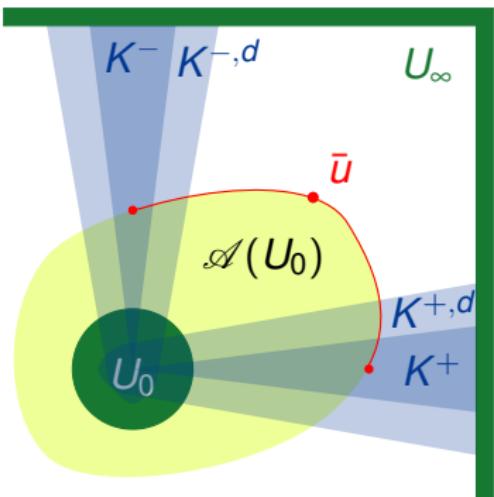
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Examples of nodal solutions with small Morse index

Theorem (Pacella-Weth '07)

Let u be a solution of

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with Morse index $\leq N$. Then u is foliated Schwarz symmetric (under the assumption that $u \in \mathbb{R} \mapsto \partial_u f(|x|, u)$ is convex).

Numerical experiment

Does there exist solutions with Morse index $\leq N$ besides the three obvious ones?

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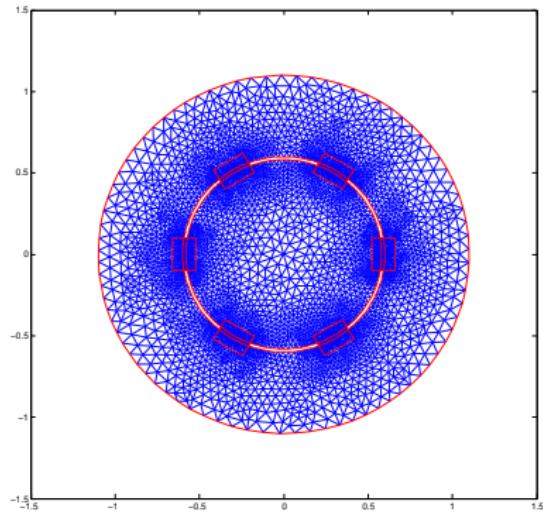
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