Minimax algorithms: convergence and applications

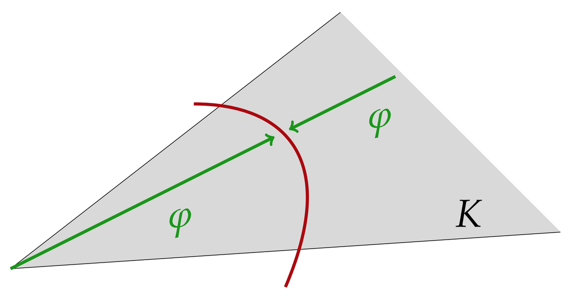
Christophe Troestler
Institute of Mathematics, University of Mons (Belgium)

Mountain pass algorithm with projector [10]

Let X be a Hilbert space, $\mathcal{E} : X \to \mathbb{R}$ a functional and $K \subseteq X$ a closed convex cone pointed at 0 (not necessarily salient). We seek nontrivial critical points of $\mathcal{E}$ lying in K i.e., $u \in K \setminus \{0\}$ such that $\partial\mathcal{E}(u) = 0$.
For all $u \in X$, we denote $P_K(u)$ the projection of u on K, i.e. the unique element of K such that

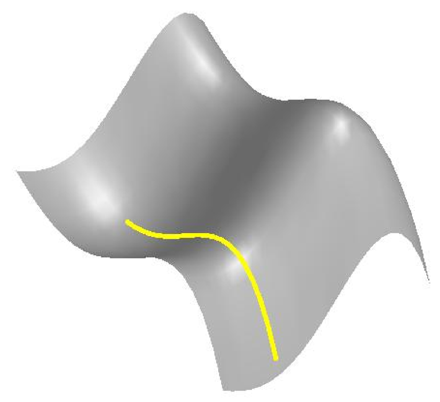
$$\|u - P_K(u)\| = \min_{v \in K} \|u - v\|$$

Definition. A map $\varphi : K \setminus \{0\} \to K \setminus \{0\}$ is said to be a *K-peak selection* for $\mathcal{E}$ iff for every $u \in K \setminus \{0\}$, $\varphi(u)$ is a local maximum of the energy $\mathcal{E}$ on the halfline $\{tu : t > 0\}$ and $\varphi(\lambda u) = \varphi(u)$ for all $\lambda > 0$.



$\mathcal{E} : X \to \mathbb{R}$ has the appropriate "geometry" if

- **(E_1)** $\forall u \in X,\ \mathcal{E}(P_K(u)) \leqslant \mathcal{E}(u)$;
- **(E_2)** there exists a *continuous* K-peak selection $\varphi : K \setminus \{0\} \to K \setminus \{0\}$ for $\mathcal{E}$;
- **(E_3)** $0 \notin \overline{\operatorname{Im}\varphi}$;
- **(E_4)** $\inf\{\mathcal{E}(u) \mid u \in \operatorname{Im}\varphi\} > -\infty$;
- **(E_5)** $\mathcal{E}$ satisfies the Palais-Smale condition i.e., any sequence $(u_n) \subseteq X$ such that $(\mathcal{E}(u_n))$ converges and $\nabla\mathcal{E}(u_n) \to 0$ possesses a convergent subsequence.



Definition. Let $u_0 \in K$ and

$$S_\downarrow(u_0) := \Big\{ s > 0 \,\Big|\, P_K(u_s) \neq 0 \text{ and } \mathcal{E}(\varphi \circ P_K(u_s)) - \mathcal{E}(u_0) < -\frac{s}{2}\|\nabla\mathcal{E}(u_0)\| \Big\} \qquad \text{where} \quad u_s := u_0 - s\frac{\nabla\mathcal{E}(u_0)}{\|\nabla\mathcal{E}(u_0)\|}.$$

The *stepsize set* $S(u_0)$ at u_0 is defined as $S_\downarrow(u_0) \cap \,]\tfrac{1}{2}\sup S_\downarrow(u_0), +\infty[$.

Theorem. The sequence $(u_n)_{n\in\mathbb{N}} \subseteq K$ defined by the following algorithm

$$\begin{cases} \text{Choose } u_0 \in \operatorname{Im}\varphi, \\ \text{If } \nabla\mathcal{E}(u_n) = 0, \text{ then} \\ \quad \text{Stop: } u_n \text{ is a critical point} \\ \text{else} \\ \quad u_{n+1} := \varphi \circ P_K\Big(u_n - s_n\dfrac{\nabla\mathcal{E}(u_n)}{\|\nabla\mathcal{E}(u_n)\|}\Big), \quad \text{where } s_n \in S(u_n) \end{cases}$$

converges up to a subsequence. Any limit point of (u_n) is a critical point of $\mathcal{E}$ in $K \setminus \{0\}$.

Lemma (Computational deformation lemma). *If $\nabla\mathcal{E}(u_0) \neq 0$ then there exists some $s_0 > 0$ such that for any $s \in \,]0, s_0[$*

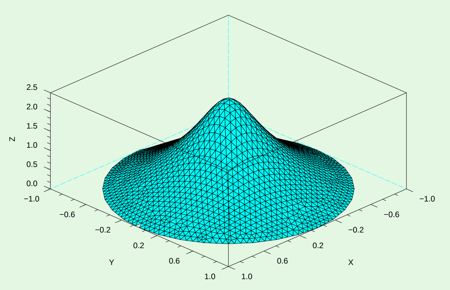
$$\mathcal{E}(\varphi \circ P_K(u_s)) - \mathcal{E}(u_0) < -\tfrac{1}{2}s\|\nabla\mathcal{E}(u_0)\| \qquad \textit{where} \quad u_s := u_0 - s\frac{\nabla\mathcal{E}(u_0)}{\|\nabla\mathcal{E}(u_0)\|}.$$

Lemma. *If $u_0 \in \operatorname{Im}\varphi$ is such that $\nabla\mathcal{E}(u_0) \neq 0$, then there exists a neighborhood V of u_0 and a positive s_0 such that $S(u) \subseteq [s_0, +\infty[$ for all $u \in V \cap \operatorname{Im}\varphi$.*

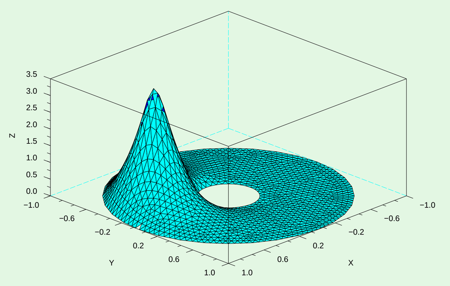
Examples of cones and projectors

- $K = \{u \in H^1_0(\Omega) \mid u \geqslant 0\}$. Beware: $P_K(u) \neq u^+$. In 1D, for the norm $\|u\| := (\int_\Omega |u'|^2)^{1/2}$, one has $P_K(u) = u - \operatorname{conv} u$
- $K = \{u : \mathbb{R} \to \mathbb{R} \mid u \text{ is non-decreasing}\}$. The associated projector is $P_K(u) = t \mapsto \int_0^t (u'(s))^+ \,\mathrm{d}s$.
- $K = \{u \mid \forall g \in G,\ \forall x \in \mathbb{R}^N,\ u(gx) = u(x)\}$ where G is a group acting on $\mathbb{R}^N$.

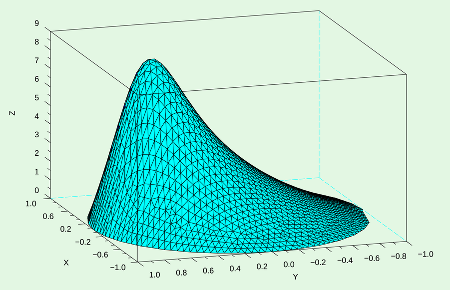
Symmetries of ground states



$-\Delta u = |u|^{p-2}u$ on a ball with Dirichlet boundary conditions. The solutions must be radially symmetric (thanks to the "moving plane" method [4]).

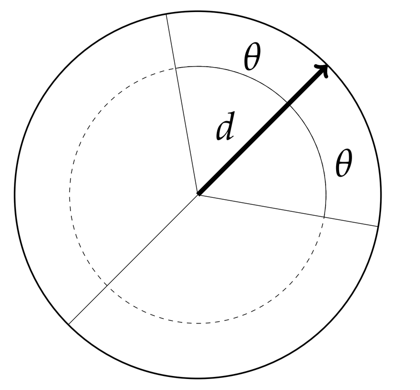


$-\Delta u = |u|^{p-2}u$ on an annulus with Dirichlet boundary conditions. When $p \to 2^*$, least energy solutions u are non-radial [3]. They are nonetheless foliated Schwarz symmetric.



$-\Delta u = |x|^\alpha |u|^{p-2}u$ on a ball with Dirichlet boundary conditions. When α is "large", least energy solutions are not radial [8]. They are foliated Schwarz symmetric [9].

Foliated Schwarz symmetry



A function $u : \mathbb{R}^N \to \mathbb{R} : x \mapsto u(x)$ is *foliated Schwarz symmetric* if it only depends on

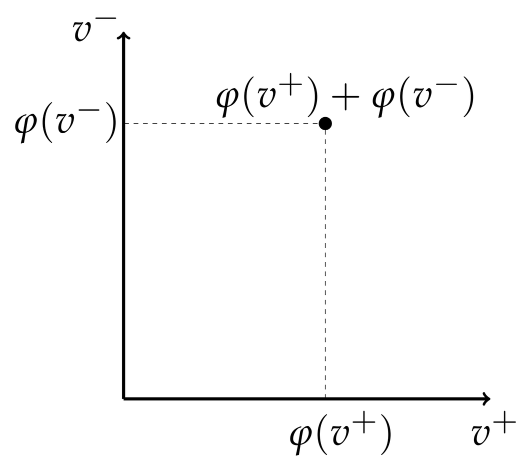
$$r = |x| \text{ and } \theta = \arccos\Big(\frac{x}{|x|} \cdot d\Big),$$

for a certain unit vector d, and u is non-increasing in θ.

Nodal algorithm [7]

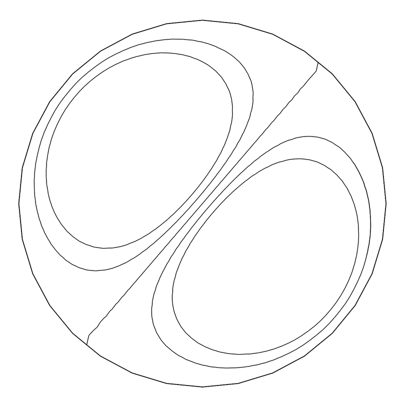
$$\begin{cases} \text{Choose a sign changing } v_0 \text{ and set } u_0 := \varphi(v_0^+) + \varphi(v_0^-) \\ \text{If } \nabla\mathcal{E}(u_n) = 0, \text{ then} \\ \quad \text{Stop: } u_n \text{ is a critical point} \\ \text{else} \\ \quad u_{n+1} := \varphi(v^+) + \varphi(v^-) \quad \text{where } v = u_n - s_n\dfrac{\nabla\mathcal{E}(u_n)}{\|\nabla\mathcal{E}(u_n)\|} \text{ and } s_n \text{ is a "good" step size} \end{cases}$$

where $v = v^+ + v^-$ with $v^+ \geqslant 0$ and $v^- \leqslant 0$.



Despite working reasonably well in practice, this algorithm is not proved convergent.

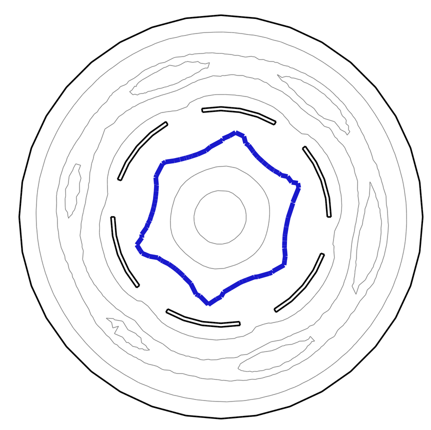
Nodal line

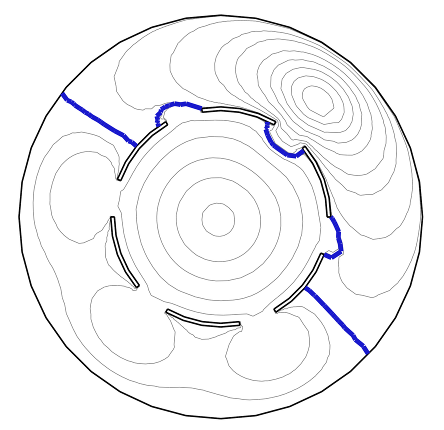


Level curves of "the" least energy nodal solution of $-\Delta u = |u|^{p-2}u$ on a ball with Dirichlet boundary conditions. When $p \to 2$, the nodal line is a diameter [6]. It is widely believed that it is still the case for large p but it is not proved at the moment.

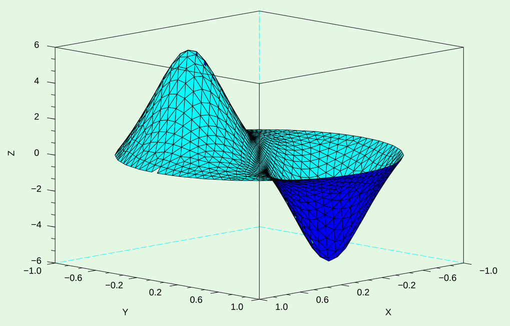
When the domain is convex and $p \xrightarrow{>} 2$, nodal lines of least energy nodal solutions touch the boundary [5]. It is conjectured to still be true for simply connected domains.

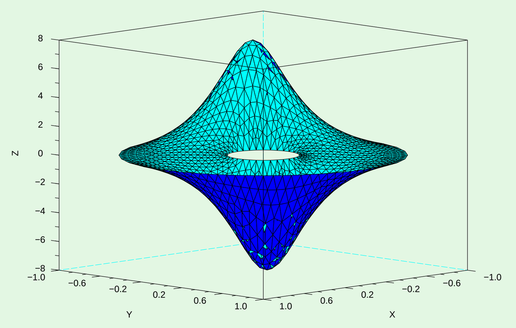
Some condition on the domain is nonetheless required as we show that the nodal line of the of least energy nodal solution of $-\Delta u = |u|^{p-2}u$ on the domain Ω below, $u = 0$ on $\partial\Omega$, does *not* touch $\partial\Omega$ for p close to 2 [5]. This does not seem to persist as p increases.





Symmetries of least energy nodal solutions





Least energy nodal solutions of $-\Delta u = |u|^{p-2}u$ with Dirichlet boundary conditions are never radial, even on a ball [1]. They are foliated Schwarz symmetric [2] and, for $p \xrightarrow{>} 2$, are odd in the direction d [6] (d is the unit vector of the Schwarz symmetry).

References

[1] A. Aftalion and F. Pacella. Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains. *C.R.A.S.*, 339:339–344, 2004.

[2] T. Bartsch, T. Weth, and M. Willem. Partial symmetry of least energy nodal solutions to some variational problems. *Journal d'Analyse Mathématique*, 96(1):1–18, 2005.

[3] H. Brezis and L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(4):437–477, 1983.

[4] B. Gidas, W. M. Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phy.*, 68:209–243, 1979.

[5] C. Grumiau and C. Troestler. Nodal line structure of least energy nodal solutions for lane-emden problem. *submitted to C.R.A.S.*, 2009.

[6] C. Grumiau and C. Troestler. Oddness of least energy nodal solution on radial domains. *To appear in Elec. Journal of differential equations*, 2009.

[7] J. M. Neuberger. A numerical method for finding sign-changing solutions of superlinear dirichlet problems. *Nonlinear World*, 4(1):73–83, 1997.

[8] D. Smets, J. Su, and M. Willem. Non-radial ground states for the Hénon equation. *Comm. Contemp. Math.*, 4:467–480, 2002.

[9] D. Smets and M. Willem. Partial symmetry and asymptotic behavior for some elliptic variational problems. *Calc. Var. Partial Differential Equations*, 18:57–75, 2003.

[10] N. Tacheny and C. Troestler. A mountain pass algorithm with projector. *submitted to J. Comput. Appl. Math.*, 2009.

Perspectives & future work

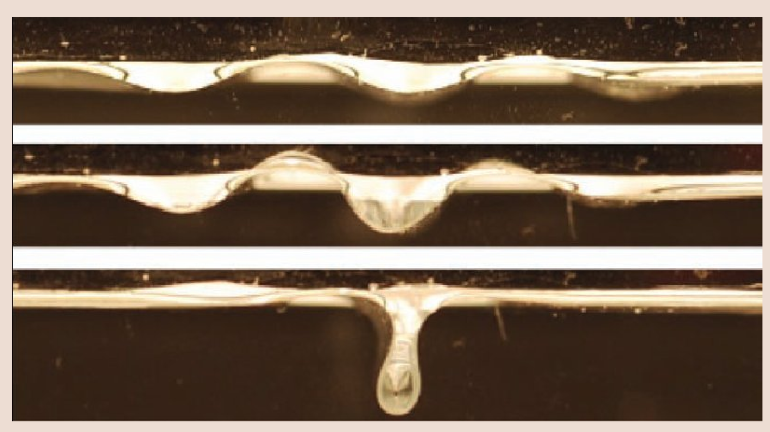
Symmetry breaking curves

Symmetries, uniqueness, and multiplicity of solutions for Neumann boundary conditions

Wrinkling shapes of thin elastic films on water or polymer substrates

Minimize

$$\mathcal{E} : X \to \mathbb{R} : \gamma \mapsto \underbrace{\tfrac{1}{2}B\int_0^L \|\partial_s^2\gamma(s)\|^2 \,\mathrm{d}s}_{\text{bending energy}} + \underbrace{\tfrac{1}{2}K\int_0^L (\gamma_2(s))^2 \partial_s\gamma_1(s)\,\mathrm{d}s}_{\text{potential energy}}$$



where B and K are constants and X is the space of configurations

$$X = \{\gamma \in H^2(]0, L[; \mathbb{R}^2) \mid \forall s \in \,]0, L[,\ \|\partial_s\gamma(s)\| = 1 \text{ et } \gamma(0) = (0,0),\ \partial_s\gamma(0) = (1,0),\ \gamma(L) = (L - \delta, 0),\ \partial_s\gamma(L) = (1,0)\}.$$