QES systems, invariant spaces and polynomials recursions.

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February 7, 2008

Abstract

Let us denote $\mathcal{V}$, the finite dimensional vector spaces of functions of the form $\psi(x) = p_n(x) + f(x)p_m(x)$ where $p_n(x)$ and $p_m(x)$ are arbitrary polynomials of degree at most n and m in the variable x while $f(x)$ represents a fixed function of x. Conditions on m, n and $f(x)$ are found such that families of linear differential operators exist which preserve $\mathcal{V}$. A special emphasis is accorded to the cases where the set of differential operators represents the envelopping algebra of some abstract algebra. These operators can be transformed into linear matrix valued differential operators. In the second part, such types of operators are considered and a connection is established between their solutions and series of polynomials-valued vectors obeying three terms recurence relations. When the operator is quasi exactly solvable, it possesses a finite dimensional invariant vector space. We study how this property leads to the truncation of the polynomials series.

1 Introduction

Quasi Exactly Solvable (QES) operators are characterized by linear differential operators which preserve a finite-dimensional vector space $\mathcal{V}$ of smooth functions [1]. In the case of operators of one real variable the underlying vector space is often of the form $\mathcal{V} = \mathcal{P}_n$ where $\mathcal{P}_n$ denotes the vector space of polynomials of degree at most n in the variable x. In [2] it is shown that the linear operators preserving $\mathcal{P}_n$ are generated by three basic operators j_-, j_0, j_+ (see Eq.(17) below) which realize the algebra $sl(2, \mathbb{R})$. More general QES operators can then be constructed by considering the elements of the enveloping algebra of these generators, performing a change of variable and/or conjugating the j's with an invertible function, say $g(x)$. The effective invariant space is then the set of functions of the form $g(x)\mathcal{P}_n$.

In this paper we consider a more general situation. Let m, n be two positive integers and let $f(x)$ be a sufficiently derivable function in a domain of the real line. Let $\mathcal{V} = \mathcal{P}_n + f(x)\mathcal{P}_m$ be the vector space of functions of the form $p(x) + f(x)q(x)$ where $p(x) \in \mathcal{P}_n$, $q(x) \in \mathcal{P}_m$.

We want to address the following questions : What are the differential operators which preserve $\mathcal{V}$? and for which choice of $m, n, f(x)$ do these operators posses a relation with the enveloping algebra of some Lie (or "deformed" Lie) algebra ?. This question generalizes the cases of monomials adressed in [3] and more recently in [4] At the moment the question is, to our knowledge, not solved in its generality but we present a few non trivial solutions in the next section.

For all the cases we obtained, the final problem of finding the algebraic modes of the scalar equations leads to 2×2-matrix QES operators preserving the space of couples of polynomials with suitable degree. On the other hand, a few example of systems of QES equations are known and were studied in the past.

One problem that occur in computing the algebraic eigenvalues is to diagonalize the Hamiltonian in a base of the its invariant finite dimensional subspace. In the case of scalar QES equations, this problem can be simplified by a technique presented in [12]. A few years ago, Bender and Dunne [12] have pointed out that for a class of Schroedinger eigenvalue problems, the formal solution constructed for generic values of the spectral parameter E is the generating function of a set of orthogonal polynomials $\{P_n(E)\}$. This property is related to the fact that the equation leads to a three term recursion relation on these polynomials. If the coupling constants of the Schroedinger operators are choosen in such a way that it becomes a quasi exactly solvable operator [13, 14], then the polynomials obey a remarkable property, namely for n larger than a fixed integer N the polynomials $P_n(E)$ factorize in the form $P_n(E) = P_N(E)\tilde{P}_{n-N}(E)$ where P_N is a common factor. The corresponding algebraic eigenvalues associated with the "quasi-exact" property are the roots of $P_N(E)$. The results of [12] was generalized and considered in different frameworks in a series of papers [15, 16, 17, 18, 19, 20].

However, to our knowledge, the approach of [12] has not been applied to systems of quasi exactly solvable equations. Apart from their mathematical interest, such systems appear in a few context, e.g. in the construction of the doubly periodic solutions of the Lamé equation [21] (see e.g. [8]) and in the stability analysis of classical solutions available is some low dimensional field theories [22, 23, 24],.

It is the purpose of this letter to adapt the ideas of Bender and Dunne to systems of coupled Schrodinger equations and to point out that families of polynomials also appear in this context. The eventuality of the orthogonality of these polynomials is still an open question.

The paper is organized as follows. In Sect. 2 we review two examples of well know coupled systems of QES equations, then we discuss the cases of operators preversing vector spaces of the form $p(x) + f(x)q(x)$. The constuction of solutions of the eigenvalue equation $H\psi = E\psi$ in a form of polynomials series is adapted in Sect. 3 for the case of coupled channels. It is shown that, along with [12] the E-dependant coefficients of the series obey three terms recurence relations. Special emphasis is set on the relation between the finite dimensional invariant space of H and the truncation of the series.

2 Systems of QES equations

In this section, we review the most know 2×2-matrix QES equations and we study a class of scalar QES equations which lead, after a suitable algebra, to matrix operators acting on spaces of polynomials.

2.1 Polynomial potential

In this subsection, we study the Hamiltonian

$$H(y) = -\frac{d^2}{dy^2}\mathbb{1}_2 + \mathrm{M}_6(\mathrm{y}) \tag{1}$$

where $M_6(y)$ is a 2×2 hermitian matrix of the form

$$M_6(y) = \{4p_2^2y^6 + 8p_1p_2y^4 + (4p_1^2 - 8mp_2 + 2(1-2\epsilon)p_2)y^2\}\mathbb{1}_2$$

$$+ \quad (8p_2y^2 + 4p_1)\sigma_3 - 8mp_2\kappa_0\sigma_1 \tag{2}$$

where σ_1, σ_3 are the Pauli matrices, p_2, p_1, κ_0 are free real parameters and m is an integer.

It is known [9, 10] that after the standard "gauge transformation" of $H(y)$ with a factor

$$\phi(y) = y^\epsilon \exp -\{\frac{p_2}{2}y^4 + p_1y^2\} \tag{3}$$

and the change of variable $x = y^2$, the new operator $\hat{H}(x)$

$$\hat{H}(x) = \phi^{-1}(x)H(y)\phi(x) \mid_{y=\sqrt{x}} \tag{4}$$

can be set in a form suitable for acting on polynomials. However, the invariant space is revealed only after the supplementary transformation [9]

$$\tilde{H}(x) = P^{-1}\hat{H}(x)P \quad , \quad P = 1\!\!\mathrm{I}_2 + \frac{\kappa_0}{2}\partial_{\mathrm{x}}(\sigma_1 + \imath\sigma_2) \tag{5}$$

is performed. After a calculation, we find

$$\tilde{H}(x) = -(4xd_x^2 + 2d_x)1\!\!\mathrm{I} + 8\mathrm{pm}\kappa_0^2\sigma_3\mathrm{d_x} \tag{6}$$

$$+8p\mathrm{diag}(J_+(m-2), J_+(m)) + 4\kappa_0(1 + 2mp\kappa^2)\sigma_+d_x^2 - 8mp\kappa_0\sigma_- \tag{7}$$

which manifestly preserves the space of couples of polynomials of the form $(p_{m-2}(x), q_m(x))^t$. Here we choose $\epsilon = 0$ and $p_1 = 0$ for simplicity. If the parameter ϵ is choosen as an arbitrary real number, then the initial potential M_6 acquires a supplementary term of the form $\epsilon(\epsilon - 1)/y^2$.

2.2 Lamé type potential.

As a second example, we consider the family of operators

$$H(z) = -\frac{d^2}{dz^2} + \left[\begin{array}{cc} Ak^2\mathrm{sn}^2 + \delta(1+k^2)/2 & 2\theta k\mathrm{cn}\ \mathrm{dn} \\ 2\theta k\mathrm{cn}\ \mathrm{dn} & Ck^2\mathrm{sn}^2 - \delta(1+k^2)/2 \end{array} \right] \tag{8}$$

where A, C, δ, θ are constants while $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ respectively abbreviate the Jacobi elliptic functions of argument z and modulus k [21]

$$\mathrm{sn}(z,k) \quad , \quad \mathrm{cn}(z,k) \quad , \quad \mathrm{dn}(z,k) \quad . \tag{9}$$

These functions are periodic with period $4K(k), 4K(k), 2K(k)$ respectively ($K(k)$ is the complete elliptic integral of the first type). The above hamiltonian is therefore to be considered on the Hilbert space of periodic functions on $[0, 4K(k)]$. For completeness, we mention the properties of the Jacobi functions which are needed in the calculations

$$\mathrm{cn}^2 + \mathrm{sn}^2 = 1 \quad , \quad \mathrm{dn}^2 + k^2 \mathrm{sn}^2 = 1 \tag{10}$$

$$\frac{d}{dz}\mathrm{sn} = \mathrm{cn}\ \mathrm{dn} \quad , \quad \frac{d}{dz}\mathrm{cn} = -\mathrm{sn}\ \mathrm{dn} \quad , \quad \frac{d}{dz}\mathrm{dn} = -k^2 \mathrm{sn}\ \mathrm{cn} \tag{11}$$

The relevant change of variable which eliminates the transcendental functions sn, cn, dn from (8) in favor of algebraic expressions is (for k fixed)

$$x = \mathrm{sn}^2(z, k) \tag{12}$$

In particular the second derivative term in (8) becomes

$$\frac{d^2}{dz^2} = 4x(1-x)(1-k^2x)\frac{d^2}{d^2x} + 2(3k^2x^2 - 2(1+k^2)x + 1)\frac{d}{dx} \tag{13}$$

Several possibilities of extracting prefactors then lead to equivalent forms of (8), say $\tilde{H}(x)$, which are matrix operators build with the derivative d/dx and polynomial coefficients in x. The requirement that $\tilde{H}(x)$ preserves a finite dimensional vector space leads to two possible sets of values for A, C, θ (see [9] for details). Here we will discuss the set determined by $A = 4m^2 + 6m + 3 - \delta$
$C = 4m^2 + 6m + 3 + \delta$
$\theta = \frac{1}{2}[(4m+3)^2 - \delta^2]^{\frac{1}{2}}$
The parameter δ remains free, and also k which fixes the period of the potential. The results can be generalized easily to the other set [9].

Corresponding to these values, four invariant vector spaces are available. We will study only one of them, refering, again, to [9] for the three others.

In order to present the change of variable we conveniently define

$$R = \frac{4m + 3 - \delta}{4m + 3 + \delta}, \tag{14}$$

We have then

$$\mathcal{V} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{cn}\ \mathrm{dn} \end{pmatrix} \begin{pmatrix} 1 & \kappa x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{P}(m) \\ \mathcal{P}(m) \end{pmatrix} \quad , \quad \kappa^2 = k^2 R \tag{15}$$

After performing the change of variable and the change of function, the components of the operator (8) take the following form

$$\begin{aligned}
\tilde{H}_{11} &= -4xk^2(D+m+\frac{1}{2})(D-m)+(k^2+1)(4D^2+\frac{\delta}{2})-2(1+2D)d \ , \\
\tilde{H}_{12} &= 4x\kappa(k^2+1)(D-m)+\kappa(-8D+\delta+4m+1) \ , \\
\tilde{H}_{21} &= \kappa(\delta+4m+3) \ , \\
\tilde{H}_{22} &= -4xk^2(D+m+\frac{5}{2})(D-m) \\
&+ (k^2+1)(4D^2+2D+1-\frac{\delta}{2})-2(1+2D)d \ , \qquad (16)
\end{aligned}$$

where $d \equiv d/dx$ and $D = xd$. The operator $\tilde{H}$ obviously preserves $(\mathcal{P}(m), (\mathcal{P}(m))$.

2.3 Scalar QES operators preserving $P_n \oplus fP_m$

In this subsection, we investigate [25] the linear operators preserving vectors spaces of the form $P_n \oplus fP_m$ and we construct several forms of the continuous function $f(x)$, together with the associated values of the integers m, n for which the operators of interest are the envelopping algebra of some Lie (or deformed Lie) Algebra.

2.3.1 Case $f(x) = 0$

This is off course the well known case of [1, 2]. The relevant operators read

$$j_+(n) = x(x\frac{d}{dx}-n) \ , \ j_0(n) = (x\frac{d}{dx}-\frac{n}{2}) \ , \ j_- = \frac{d}{dx} \qquad (17)$$

and represent the three generators of $sl(2,\mathbb{R})$. Most of known one-dimensional QES equations are build with these operators. For later convenience, we further define a family of equivalent realizations of $sl(2,\mathbb{R})$ by means of the conjugated operators $k_\epsilon(a) \equiv x^a j_\epsilon x^{-a}$ for $\epsilon = +, 0, -$ and a is a real number.

2.3.2 Case $f(x) = x^a$

The general cases of vector spaces constructed over monomials was first adressed in [3] and the particular subcase $f(x) = x^a$ was reconsidered recently [4]. The corresponding vector space was denoted $V^{(1)}$ in [4]; here we will reconsider this case and extend the discussion of the operators which leave it invariant. For later convenience, it is usefull to introduce more precise notations, setting $\mathcal{P}_n \equiv \mathcal{P}(n,x)$ and

$$V^{(1)} \equiv V^{(1)}(N,s,a,x) = \mathcal{P}(n,x) + x^a\mathcal{P}(m,x)$$

$$
\begin{aligned}
&= \text{span}\{1, x, x^2, \ldots, x^n; x^a, x^{a+1}, \ldots, x^{a+m}\} \\
&= V_1^{(1)} \oplus V_2^{(1)}
\end{aligned} \tag{18}
$$

in passing, note that the notations of [4] are $n = s$ and $m = N - s - 2$.

The vector space above is clearly constructed as the direct sum of two subspaces. As pointed out in [3, 4] three independent, second order differential operators can be constructed which preserve the vector space $V^{(1)}$. Writing these operators in the form

$$
\begin{aligned}
J_+ &= x(x\frac{d}{dx} - n)(x\frac{d}{dx} - (m+a)) \\
J_0 &= (x\frac{d}{dx} - \frac{m+n+1}{2}) \\
J_- &= (x\frac{d}{dx} + 1 - a)\frac{d}{dx}
\end{aligned} \tag{19}
$$

makes it obvious that they preserve $V^{(1)}$.

These operators close under the commutator into a polynomial deformation of the $sl(2, \mathbb{R})$ algebra:

$$
\begin{aligned}
[J_0, J_\pm] &= \pm J_\pm \\
[J_+, J_-] &= \alpha J_0^3 + \beta J_0^2 + \gamma J_0 + \delta
\end{aligned} \tag{20}
$$

where $\alpha, \beta, \gamma, \delta$ are constants given in [4].

Clearly the operators (19) leave separately invariant two vector spaces $V_1^{(1)}$ and $V_1^{(1)}$ entering in (18). In the language of representations they act reducibly on $V^{(1)}$. However, operators can be constructed which preserve $V^{(1)}$ while mixing the two subspaces. The form of these supplementary operators is different according to the fact that the number a is an integer or not; we now adress these two cases separately.

Case $a \in \mathbb{R}$

In order to construct the operators which mix $V_1^{(1)}$ and $V_2^{(1)}$, we first define

$$
\begin{aligned}
K &= (D-n)(D-n+1)\ldots D \quad , \quad D \equiv x\frac{d}{dx} \\
K' &= (D-m-a)(D-m-a+1)\ldots(D-a)
\end{aligned} \tag{21}
$$

which belong to the kernals of the subvector spaces $\mathcal{P}_n$ and $x^a\mathcal{P}_m$ of $V^{(1)}$ respectively. Notice that the products $j_\epsilon \tilde{K}$ and $k_\epsilon(a)K$ (with $\epsilon = 0, \pm$) also preserve the vector space. For generic values of m, n these operators contain more than second derivatives and, as so, they were not considered in [4].

In order to construct the operators which mix the two vector subspaces entering in $\mathcal{V}$, we first have to construct the operators which transform a generic element of

$\mathcal{P}_m$ into an element of $\mathcal{P}_n$ and vice-versa. In [5] it is shown that these operators are of the form

$$q_\alpha \quad = x^\alpha \quad , \quad \alpha = 0, 1, \ldots, \Delta \tag{22}$$

$$\overline{q}_\alpha \quad = \prod_{j=0}^{\alpha-1} (D - (p + 1 - \Delta) - j)(\frac{d}{dx})^{\Delta - \alpha} \tag{23}$$

where $\Delta \equiv |m - n|$, $p \equiv$max$\{m, n\}$

The operators preserving $\mathcal{V}$ while exchanging the two subspaces can finally be constructed by means of

$$Q_\alpha = q_\alpha x^{-a} K \quad , \quad \overline{Q}_\alpha = x^a \overline{q}_\alpha K' \quad , \quad \alpha = 0, 1, \ldots, \Delta. \tag{24}$$

Here we assumed $n \leq m$, the case $n \geq m$ is obtained by exchanging q_α with $\overline{q}_\alpha$ in the formula above.

It can be checked easily that Q_α, transform a vector of the form $p_n + x^a q_m$ into a vector of the form $\tilde{q}_n \in \mathcal{P}_n$ while $\overline{Q}_\alpha$ transforms the same vector into a vector of the form $x^a \tilde{p}_m \in x^a \mathcal{P}_m$.

The generators constructed above are in one to one correspondance with the 2×2 matrix generators preserving the direct sum of vector spaces $\mathcal{P}_m \oplus \mathcal{P}_n$ classified in [5] although their form is quite different (the same notation is nevertheless used). The commutation relations (defining a normal order) which the generators fullfill is also drastically different as we shall discuss now. First of all it can be easily checked that all products of operators Q (and separately of $\overline{Q}$) belong to the kernal of the full space $V^{(1)}$, so we can write

$$Q_\alpha Q_\beta = \overline{Q}_\alpha \overline{Q}_\beta = 0 \tag{25}$$

which suggests that the operators Q's and the $\overline{Q}$'s play the role of fermionic generators, in contrast to the J's which are bosonic (note that the same distinction holds in the case [5]).

From now on, we assume $n = m$ in this section (the evaluation of the commutators for generic values of m, n is straightforward but leads to even more involved expressions) and suppress the superflous index α on the the fermionic operators. The commutation relations between fermionic and bosonic generators leads to

$$\begin{aligned} [Q, J_-] &= (2a - n - 1) j_- Q \, , \qquad [Q, J_+] = (2a + n + 1) j_+ Q \\ [\overline{Q}, J_-] &= -(2a + n + 1) k_-(a) \overline{Q} \, , \quad [\overline{Q}, J_+] = -(2a - n - 1) k_+(a) \overline{Q} \end{aligned} \tag{26}$$

where the $j_\pm$ and $k_\pm(a)$ are defined in (17). These relations define a normal order but we notice that the right hand side are not linear expressions of the generators choosen as basic elements. We also have

$$[Q, D] = (D + a)Q \quad , \quad [\overline{Q}, D] = (D - a)\overline{Q} \tag{27}$$

This is to be contrasted with the problem studied in [5] where, for the case $\Delta = 0$, the Q (and the $\overline{Q}$) commute with the three bosonic generators, forming finally an sl(2)$\times$ sl(2) algebra. Here we see that the bosonic operators J and fermionic operators $Q, \overline{Q}$ do not close linearly under the commutator. The commutators involve in fact extra factors which can be expressed in terms of the operators j or $k(a)$ acting on the appropriate subspace $\mathcal{P}_m$. This defines a normal order among the basic generators but makes the underlying algebraic stucture (if any) non linear. For completeness, we also mention that the anti-commutator $\{Q, \overline{Q}\}$ is a polynomial in J_0.

Case $a \in \mathbb{N}$

Let us consider the case $a \equiv k \in \mathbb{N}_0$, with $n \leq k$ and assume for definiteness $m - k \geq n$. Operators that preserve $V^{(1)}$ while exchanging some monomials of the subspace $V_1^{(1)}$ with some of $V_2^{(1)}$ (and vice versa) can be expressed as follows :

$$W_+ = x^k \prod_{j=0}^{k-1} (D - k - m + j) \tag{28}$$

$$W_- = \frac{1}{x^k} \prod_{j=0}^{n} (D - j) \prod_{i=1}^{k-n-1} (D - k - n - i) \tag{29}$$

These operators are both of order k, W_+ is of degree k while W_- is of degree $-k$. When acting on the monomial of Eq.(18), W_+ transforms the $n + 1$ monomials of $V_1^{(1)}$ into the first $n+1$ monomials of $V_2^{(1)}$ and annihilates the k monomials of highest degrees in $V_2^{(1)}$. To the contrary W_- annihilates the $n + 1$ monomials of $V_1^{(1)}$ and shifts the $n+1$ monomials of lowest degrees of $V_2^{(1)}$ into $V_1^{(1)}$. Operators of the same type performing higher jumps can be constructed in a straighforward way; they are characterized by a higher order and higher degrees but we will not present them here.

The two particular cases $k = n + 1$ and $k = 2$ can be further commented. In the case $k = n + 1$ the space $V^{(1)}$ is just $\mathcal{P}_{m+n+1}$ and the operators W_+, W_- can be rewritten as

$$W_+ = (j_+(m + n + 1))^{n+1} \quad , \quad W_- = (j_-)^{n+1} \tag{30}$$

where $j_\pm$ are defined in (17). Setting $k = 2$ (and $n = 0$ otherwise we fall on the case just mentionned), we see that the operators $W_\pm$ become second order and coincide with the operators noted $T_2^{(+2)}$, $T_2^{(-2)}$ in Sect. 4 of the recent preprint [11]. With our notation they read

$$W_+ = x^2(D - (m + 2))(D - (m + 1)) \quad , \quad W_- = x^{-2} D(D - 3) \tag{31}$$

A natural question which come out is to study whether the non-linear algebra (20) is extended in a nice way by the supplementary operators $W_\pm$ and their higher order counterparts. So far, we have not found any interesting extended structure. For example, for $k = 2, n = 0$, we computed :

$$\begin{aligned} [W_+, J_+] &= -2x^3(D - (m+2))(D - (m+1))(D - m) \\ [W_+, J_-] &= -6xD(D - (m+2))(D - \frac{2}{3}(m+2)) \end{aligned} \tag{32}$$

which just show that the commutators close within the envelopping algebra of the $V^{(1)}$ preserving operators but would need more investigation to be confirmed as an abstract algebraic stucture.

We end up this section by mentionning that the two other vector spaces constructed in [4] and denoted $V^{(a-1)}$ and $V^{(a)}$ can in fact be related to $V^{(1)}$ by means of the following relations :

$$V^{(a-1)}(x) = V^{(1)}(N, s = 0, \frac{1}{a-1}, x^{a-1}) \tag{33}$$

$$V^{(a)}(x) = V^{(1)}(N, s, \frac{1}{a}, x^a) \tag{34}$$

Off course the operators preserving them can be obtained from the operators above (19) after a suitable change of variable and the results above can easily be extended to these vector spaces.

2.3.3 Case $f(x) = \sqrt{p_2(x)}$, $m = n - 1$

Here, $p_2(x)$ denotes a polynomial of degree 2 in x, we take it in the canonical form $p_2(x) = (1-x)(1-\lambda x)$. In the case $m = n-1$, three basic operators can be contructed which preserve $\mathcal{V}$ in the case $\lambda = -1$; they are of the form

$$\begin{aligned} S_1 &= nx + p_2\frac{d}{dx} \\ S_2 &= \sqrt{p_2}(nx - x\frac{d}{dx}) \\ S_3 &= \sqrt{p_2}(\frac{d}{dx}) \end{aligned} \tag{35}$$

and obey the commutation relations of so(3). The operator corresponding to$\lambda \neq -1$ can be constructed from the three given above by a suitable affine transformation of the variable x.The family of operators preserving $\mathcal{V}$ is in this case the enveloping algebra of the Lie algebra of SO(3) in the realization above. Two particular cases are worth to be pointed out :

- $\lambda = -1$

  Using the variable $x = \cos(\phi)$, the vector space $\mathcal{V}$ can be re-expressed is the form

  $$\mathcal{V} = \mathrm{span}\{\cos(n\phi), \sin(n\phi), \cos((n-1)\phi), \sin((n-1)\phi), \ldots\} \tag{36}$$

  and the operators S_a above can be expressed in terms of trigonometric functions. Exemples of QES equations of this type were studied in [6] in relation with spin systems.

- $\lambda = k^2$ Using the variable $x = \mathrm{sn}(z,k)$, (with $\mathrm{sn}(z,k)$ denoting the Jacobi elliptic function of modulus k, $0 \le k^2 \le 1$), and considering the Lamé equation :

  $$-\frac{d^2\psi}{dz^2} + N(N+1)k^2\mathrm{sn}^2(z,k)\psi = E\psi \tag{37}$$

  It in known (see e.g. [21]) that doubly periodic solutions exist if N is a semi integer. If $N = (2n+1)/2$ these solutions are of the form

  $$\psi(z) = \sqrt{\mathrm{cn}(z,k) + \mathrm{dn}(z,k)}(p_n(x) + \mathrm{cn}(z,k)\mathrm{dn}(z,k)p_{n-1}(x)) \tag{38}$$

  where $\mathrm{cn}(z,k), \mathrm{dn(z,k)}$ denote the other Jacobi elliptic functions. The second factor of this expression is exactly an element of the vector space under consideration. The relations between the doubly periodic solutions of the Lamé equation and QES operators was pointed out in [8].

2.3.4 Case $f(x) = \sqrt{(1-x)/(1-\lambda x)}$

For this form of $f(x)$, the vector space $\mathcal{V}$ is preserved by the operators $\tilde{S}_a$ with

$$\tilde{S}_a = \frac{1}{\sqrt{1-\lambda x}} S_a \sqrt{1-\lambda x}$$

provided $m = n$.

Two cases are worth considering, in complete paralelism with Sect. 2.3.3:

- $\lambda = -1$. Using the new variable $x = \cos\phi$, and using the identity $\tan(\phi/2) = \sqrt{(1-x)/(1+x)}$ the vector space $\mathcal{V}$ can be reexpressed is the form

  $$\mathcal{V} = \mathrm{span}\{\cos(\frac{2n+1}{2}\phi), \sin(\frac{2n+1}{2}\phi), \cos(\frac{2n-1}{2}\phi), \sin(\frac{2n-1}{2}\phi), \ldots\} \tag{39}$$

  and exemples of QES operators having solutions in this vector space are presented in [6].

- $\lambda = k^2$. Again, in this case, the variable $x = \mathrm{sn}(z,k)$ is usefull and the doubly periodic solutions of the Lamé equation (8) corresponding to $N = (2n+3)/2$ of the form [8]

$$\psi(z) = \sqrt{\mathrm{cn}(z,k) + \mathrm{dn}(z,k)}(\mathrm{cn}(z,k)p_n(x) + \mathrm{dn}(z,k)q_n(x)) \tag{40}$$

provide examples of QES solutions constructed in the space under consideration.

2.4 Relations with matrix operators

The scalar operators constructed in the previous subsection are equivalent to matrix valued differential operators acting on polynomial-valued vector. This observation is rather trivial but we illustrate the statement by mean of the case of Sect. 2.3.3.

Let us now reconsider the spaces of the form $\mathcal{V} = p_n + f(x)q_m$ invariant through the symmetries S_1, S_2 and S_3 introduced above. A closed link can be obviously made between the later operators with matricials one which can be expressed in terms of the usual generators $J_\pm(n)$ and J_0 of the $sl(2,\mathbb{R})$ Lie algebra.
We obtain for$f(x) = \sqrt{(1-x)(1-\lambda x)}$,$m = n-1$

$$\tilde{S}_1 = \begin{pmatrix} J_- - J_+(n) & 0 \\ 0 & J_- - J_+(n-1) \end{pmatrix}$$

$$\tilde{S}_2 = \begin{pmatrix} 0 & -J_0(n/2) - x^2 J_+(n-1) \\ -J_0(n/2) & 0 \end{pmatrix}$$

$$\tilde{S}_3 = \begin{pmatrix} 0 & J_- - x(D+1) \\ J_- & 0 \end{pmatrix}$$

3 Recurence relations

In this section, we will adapt the formulation of the QES solution in terms of recurence relations [12] to the case of matrix operator. Let us assume that a $2*2$ matrix hamiltonian H preserves a finite dimensional vector space. Looking for solutions of the form

$$\psi(y) = \psi_0(y) \sum_{n=0}^{\infty} P_n(E) y^n \quad , \quad P_n(E) \equiv \begin{pmatrix} p_n(E) \\ q_n(E) \end{pmatrix} \tag{41}$$

We will see that the eigenvalue equation $H\psi = E\psi$ leads to the a system of three terms recursion relations for the vectors $P_n(E)$ of the form:

$$P_{n+1} = (E\mathbb{1} + \mathrm{A})\mathrm{P_n} + \mathrm{B}\mathrm{P_{n-1}}$$

where A and B are matrices depending on n but independant of the energy E. Moreover, B is diagonal and posseses a zero eigenvalue for some specific value of n. We will see how this property, together with the form of the recurence relations lead to a truncation of the formal series (41).

3.1 Polynomial potential

The solutions of the equation

$$\tilde{H}(x)\Psi = E\psi$$

can be expressed in terms of formal series in the variable $x = y^2$ of the form
$\psi(x) = \exp(-p_2 x^2/4)\sum_{n=0}^{\infty} P_n(E)x^n$
We can see after an algebra that the corresponding recurence relation has the form

$$C_n \begin{pmatrix} p_{n+1} \\ q_{n+2} \end{pmatrix} = E \begin{pmatrix} p_n \\ q_{n+1} \end{pmatrix} + B_n \begin{pmatrix} p_{n-1} \\ q_n \end{pmatrix} \tag{42}$$

where

$$B_n = 8p \begin{pmatrix} n-m-1 & 0 \\ 0 & n-m \end{pmatrix} \tag{43}$$

The form of C_n can be obtained in a straightforward way but is irrelevant for our calculation.This equation can be solved recursively and it turns out that q_0 and q_1 are arbitrary parameters. From the structure of matrix B_n,it turns out that p_m,p_{m+1} and q_{m+2} can be expressed as linear combination of p_{m-1} and q_{m+1}. Therefore,considering the equations

$$p_{m-1} = 0, q_{m+1} = 0$$

as a linear system in the free parameters q_0,q_1, we obtain a polynomial condition on the energy say $P(E)$. Those energies that fullfill $P(E) = 0$ leads to a truncated series and therefore correspond to the quasi-exactly solvable solutions of the initial system.

3.2 Lame type of potential

After some calculus,we obtain a solution of the form

$$\psi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(2n+1)} \mathbf{P}_n(E) x^n \tag{44}$$

Here,the matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} -(k^2+1)(4n^2+\frac{\delta}{2}) & -\kappa(-8n+\delta+4m+1) \\ -\kappa(\delta+4m+3) & -(k^2+1)(4n^2+2n+1-\frac{\delta}{2}) \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 4k^2(n-m-1)(n+m-\frac{1}{2}) & 0 \\ 0 & 4k^2(n-m-1)(n+m+\frac{3}{2}) \end{pmatrix}$$

The "initial" conditions write

$$P_1 = \begin{pmatrix} E-\frac{\delta}{2}(1+k^2) & -\kappa(\delta+4m+1) \\ -\kappa(\delta+4m+3) & E-4(1+k^2)(1-\frac{\delta}{2}) \end{pmatrix} P_0$$

the vector P_0 being arbitrary.

We immediately observe that for $n = m+1$, the Matrix B vanish identically (its two eigenvalues are zero). Therefore all the polynomials of the form P_{m+j}(j=2,3,...) depend linearly of P_{m+1}. As a consequence, solving the system of two equations $P_{m+1} = 0$ as function of the two arbitrary constants entering in P_0 leads to a truncation of the series and a corresponding set of algebraic eigenvalues. This finally leads to a factorization of all the polynomials P_{m+j}(j=3,4,...) in terms of P_{j+1}.

3.3 Bose-Hubbard Model

In the above sections, we illustrated some new aspects of the formulation of the QES property in terms of recursive polynomials. Here we would like to present another exemple where still a new feature of this formulation is revealed. It is connected to the Bose-Hubbard (BH) model. It can be described by the Schrodinger equation as

$$[-\frac{d^2}{dx^2} + V(x)]\psi(x) = E_1\psi \tag{45}$$

where $V(x) = \frac{1}{\gamma}\cosh^2(\sqrt{\gamma}x) - (n+1)\cosh(\sqrt{\gamma}x) - \frac{1}{\gamma} - \gamma\frac{n}{2}(\frac{n}{2}-1)$ and $\frac{\gamma}{2}$ is the coefficient of the term $a^\dagger a^\dagger aa$ term in the BH Hamiltonian of Ref. [30, 31].

The Eq. (45) can be written as

$$[-\frac{d^2}{dx^2} + (\frac{1}{\alpha}\cosh\alpha x - M)^2]\psi = E\psi \tag{46}$$

where, $\alpha = \sqrt{\gamma},\ E = E_1 + E_0,\ E_0 = \frac{(n+1)^2}{4}\alpha^2 + \frac{1}{\alpha^2} + \frac{\alpha}{4}n(n-2),\ M = \frac{(n+1)}{2}\alpha$

Now to see the orthogonal polynomial associated with this model we substitute,

$$\psi(x) = \exp(-\frac{1}{\alpha^2}\cosh\alpha x)\phi(x) \tag{47}$$

in the eq. (46) to obtain,

$$\phi'' - \frac{2}{\alpha}\sinh\alpha x\ \phi' + [E - \frac{1}{\alpha^2} - M^2 + (\frac{2M}{\alpha} - \frac{1}{\alpha^2})\cosh\alpha x]\phi = 0 \tag{48}$$

The Eq.(48) further can be reduced to,

$$\alpha^2 z(z+2)\phi'' + [\alpha^2(z+1) - 2z^2 - 4z]\ \phi' + [E - \frac{1}{\alpha^2} - M^2 + (\frac{2M}{\alpha} - \frac{1}{\alpha^2})(z+1)]\phi = 0 \tag{49}$$

where $z = \cosh\alpha x - 1$ and primes indicate derivatives with respect to z. Now this equation has regular singular point at $z = 0$, therefore we seek a solution of the form,

$$\phi(z) = z^s f(z) \tag{50}$$

By substituting this in Eq. (49) and putting the coefficient s of the term $z^{s-1}f(z)$ equal to zero, we obtain the indicial equation for s as $2s^2 = s$. This implies s can be either 0 or $\frac{1}{2}$. The differential equation in terms of f can be written as,

$$\begin{aligned} & \alpha^2[(z+2)^2 - 2(z+2)]\ f'' + [-(z+2)^2 + (z+2)\{\alpha^2(2s+1) + 2\} - \alpha^2]\ f' \\ + & \ [E - M^2 + \alpha^2 s^2 - \frac{2M}{\alpha} + (\frac{2M}{\alpha} - \frac{1}{\alpha^2})(z+2)]\ f = 0 \end{aligned} \tag{51}$$

We further substitute

$$f = \sum_n \frac{R_n(E)}{n!}(\frac{z+2}{2})^{\frac{n}{2}} \tag{52}$$

in the Eq. 51 to obtain the three term recursion as

$$\begin{aligned} \frac{\alpha^2}{4}R_{n+2}(E) &= R_n(E)[E + \frac{n^2\alpha^2}{4} + sn\alpha^2 + n + s^2 - M^2 - \frac{2M}{\alpha}] \\ &+ \ R_{n-2}(E)n(n-1)[\frac{2M}{\alpha} - \frac{1}{\alpha^2} - n - 2] \end{aligned} \tag{53}$$

provided $2s^2 = s$. Thus we have two sets of independent solutions: the even states (i.e., states with even number of nodes) for $s = 0$ and the odd states for $s = \frac{1}{2}$. Note that unlike Bender-Dunne (or most other QES) cases, s is not contained in the potential and this is perhaps related to the fact that for any integer value of $\tilde{M}$ with

$$\tilde{M} \equiv \frac{2\alpha M - 1}{\alpha^2}$$

the QES solutions corresponding to both even and odd states are obtained. Also from Eq, 53 we observe that the even and odd polynomials, $R_n(E)$ do not mix with each other and hence we have two separate three-term recursion relations depending on whether n is odd or even. In particular, it is easily shown that three term recursion relations corresponding to the even and odd n cases , respectively are given by, $n \geq 1$

$$\begin{aligned}
\frac{\alpha^2}{4}P_n(E) &= P_{n-1}(E)\ [E+\alpha^2(n^2-2n+1+2ns-2s)+2n-2+s^2-M^2-\frac{2M}{\alpha}] \\
&+ 2P_{n-2}(E)\ (n-1)(2n-3)[\frac{2M}{\alpha}-\frac{1}{\alpha^2}-2n] \qquad (54) \\
\frac{\alpha^2}{4}Q_n(E) &= Q_{n-1}(E)\ [E+\alpha^2(n^2-n+\frac{1}{4}+2ns-2s)+2n-1+s^2-M^2-\frac{2M}{\alpha}] \\
&+ 2Q_{n-2}(E)\ (n-1)(2n-1)[\frac{2M}{\alpha}-\frac{1}{\alpha^2}-2n-1] \qquad (55)
\end{aligned}$$

with $P_0(E) = 1$ $Q_0(E) = 1$. These recursion relations generate a set of monic polynomials and forms separately the complete set orthogonal polynomials. These odd and even polynomials satisfied the factorization properties of the Bender-Dunne polynomials. It is easily seen that when $\tilde{M}$ is a positive integer, exact solutions for first $\tilde{M}$ levels are obtained . In particular, if $\tilde{M}$ is odd (even) integer, then solutions with even number of nodes ($s = 0$) are obtained when the coefficient of $P_{(n-2)}$ ($Q_{(n-2)}$ vanishes. Similarly if $\tilde{M}$ is odd (even) integer, the solution with odd number of nodes ($s = \frac{1}{2}$) are obtained when the coefficient of $Q_{(n-2)}$ ($P_{(n-2}$) vanishes. Further for $\tilde{M}$ even (say $2k+2,\ k = 0, 1, 2\cdots$), half of levels, i.e. $k+1$ levels are obtained each from the zeros of the orthogonal polynomials $P_{k+1}(E)$ and $Q_{k+1}(E)$. On the other hand , when $\tilde{M}$ is odd (say $2k+1,\ k = 0, 1, 2\cdots$) then $k+1$ and k levels are obtained from the zeros of the orthogonal polynomials $P_{k+1}(E)$ and $Q_k(E)$, respectively.

4 Concluding remarks

The examples of operators presented above give evidences of the difficulty to classify the coupled-channel (or matrix) QES Schrodinger equations. The way of constructing the QES potential M_6 in Sect. 2 further provides a clear link between the approaches [26] and [28] to this mathematical problem; we hope that this note will motivate further investigations of it. We also presented a few new aspects of the recurence relations for polynomials associated to QES operators. Especially we formulated it for matrix operators. An open question is to show that these vector-valued polynomial are orthogonal with respect to an appropriate measure. We failed to find the matrix counterpart of the orthogonality theorem [32] for vector-valued polynomials.

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