

Wrinkling of thin membranes on fluid substrates

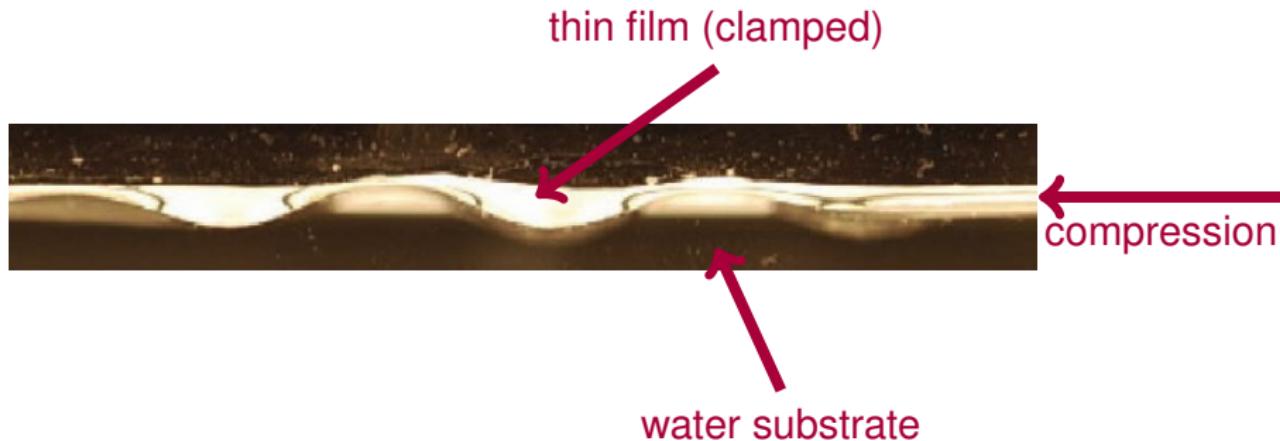
Christophe Troestler

Institut de Mathématique
Université de Mons

UMONS

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The physical problem



Goal: understand the wrinkling of the film.

Physical relevance (example): thin layers of nanoparticles spread on water surfaces to create nanopatterned structures.

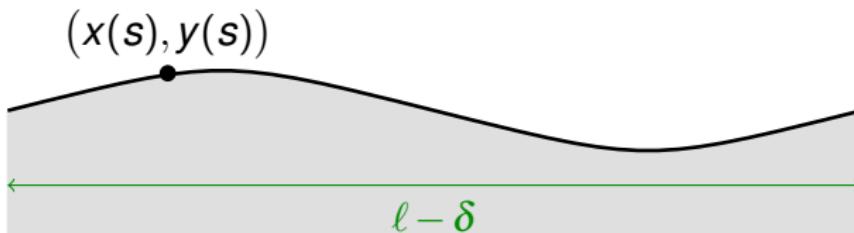
Outline

1 Mathematical model

2 Limit equation for small compression

3 Eigenvalue problem

The mathematical model (1/4)



- Assume translation invariance in the z direction.
- The length of the cuve is denoted ℓ .
- W.l.o.g. the curve $s \mapsto (x(s), y(s))$ is parametrized by its arclength $s \in [-\ell/2, \ell/2]$.
- The film is compressed by a length δ ;
- The film and substrate are homogeneous.

The mathematical model (2/4)



Energies at play:

- energy due to the curvature κ of the film;
- “potential” energy of the substrate.

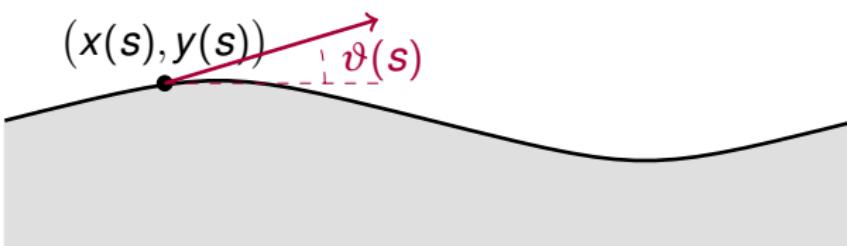
Minimize

$$\mathcal{E}(x, y) = \frac{1}{2} \int_{-\ell/2}^{\ell/2} \kappa^2 ds + \frac{1}{2} K \int_0^{\ell-\delta} y^2 dx$$

(where $K > 0$) under the conditions that the film is clamped:

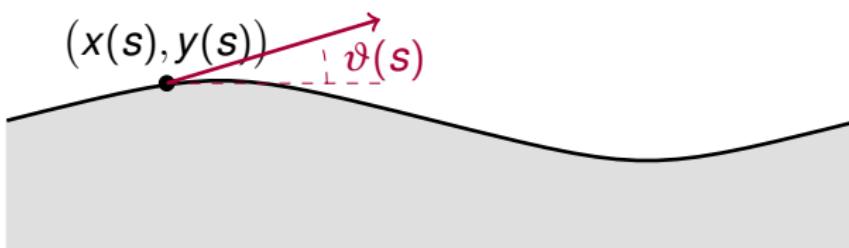
$$\begin{aligned} x(-\ell/2) &= 0, & x(\ell/2) &= \ell - \delta, \\ y(-\ell/2) &= 0, & \partial_s y(-\ell/2) &= 0, & y(\ell/2) &= 0, & \partial_s y(\ell/2) &= 0. \end{aligned}$$

The mathematical model (3/4)



$$(\partial_s x(s), \partial_s y(s)) = (\cos \vartheta(s), \sin \vartheta(s))$$

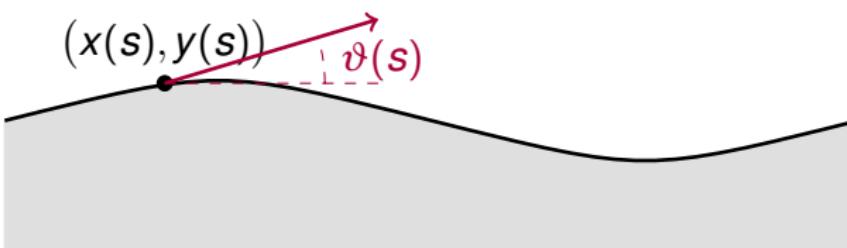
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$$\Rightarrow \kappa(s) = \partial_s \vartheta(s)$$

The mathematical model (3/4)

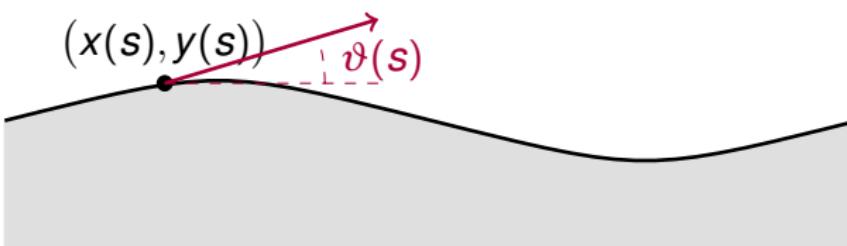


$$(\partial_s x(s), \partial_s y(s)) = (\cos \vartheta(s), \sin \vartheta(s))$$

$$\Rightarrow \kappa(s) = \partial_s \vartheta(s)$$

$$\Rightarrow y(s) = \int_{-\ell/2}^s \sin \vartheta(s) ds \quad (y(-\ell/2) = 0)$$

The mathematical model (3/4)



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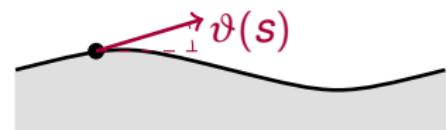
$$\Rightarrow \kappa(s) = \partial_s \vartheta(s)$$

$$\Rightarrow y(s) = \int_{-\ell/2}^s \sin \vartheta(s) ds \quad (y(-\ell/2) = 0)$$

$$\Rightarrow dx = \cos \vartheta(s) ds$$

The mathematical model (4/4)

Minimize



$$\mathcal{E}(v) = \frac{1}{2} \int_{-\ell/2}^{\ell/2} (\partial_s v)^2 ds + \frac{1}{2} K \int_{-\ell/2}^{\ell/2} y^2 \cos v(s) ds$$

(where $y(s) := \int_{-\ell/2}^s \sin v(s) ds$)

subject to

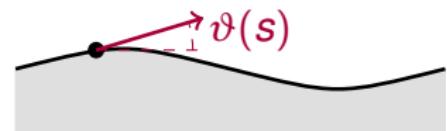
$$v(-\ell/2) = 0 = v(\ell/2)$$

$$\int_{-\ell/2}^{\ell/2} \cos v(s) ds = \ell - \delta$$

$$\int_{-\ell/2}^{\ell/2} \sin v(s) ds = 0.$$

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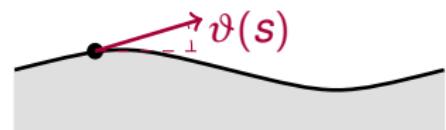
$$v(-\ell/2) = 0 = v(\ell/2) \quad \Rightarrow v \in H_0^1(-\ell/2, \ell/2)$$

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$$\int_{-\ell/2}^{\ell/2} \cos v(s) ds = \ell - \delta \quad \Leftrightarrow \int_{-\ell/2}^{\ell/2} 1 - \cos v(s) ds = \delta$$

$$\int_{-\ell/2}^{\ell/2} \sin v(s) ds = 0.$$

Existence of a minimizer

A constrained minimizer exists for

$$\mathcal{E} : H_0^1 \rightarrow \mathbb{R} : \vartheta \mapsto \frac{1}{2} \int_{-\ell/2}^{\ell/2} (\partial_s \vartheta)^2 ds + \frac{1}{2} K \int_{-\ell/2}^{\ell/2} y^2 \cos \vartheta(s) ds$$

by the direct method of the calculus of variations because

- \mathcal{E} is w.l.s.c. and coercive:

$$|y(s)| \leq \int_{-\ell/2}^s |\sin \vartheta| ds \leq s + \ell/2 \leq \ell \quad \Rightarrow \quad \mathcal{E}(\vartheta) \geq \frac{1}{2} \|\vartheta\|_{H_0^1}^2 - \frac{1}{2} K \ell$$

- The constraints are weakly continuous:

$$\vartheta \mapsto \int_{-\ell/2}^{\ell/2} 1 - \cos \vartheta(s) ds, \quad \vartheta \mapsto \int_{-\ell/2}^{\ell/2} \sin \vartheta(s) ds.$$

Small compressions, $\delta \rightarrow 0$

Let ϑ_δ be a minimizer for the compression $\delta > 0$.

$\vartheta_\delta \rightarrow 0$ in H_0^1 .

Using

$$\delta = \int 1 - \cos \vartheta_\delta(s) ds \approx \frac{1}{2} |\vartheta_\delta|_{L^2}^2$$

one eventually obtains that $\|\vartheta_\delta\|_{H_0^1} = O(\sqrt{\delta})$.

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➡ equation for ϑ^* ?

Limit problem (1/3)

Euler-Lagrange equation: for all $\varphi \in \mathcal{C}_c^\infty$,

$$\begin{aligned} \int \partial_s \vartheta_\delta \partial_s \varphi \, ds + \frac{1}{2} K \int 2y \langle \partial_\vartheta y, \varphi \rangle \cos \vartheta_\delta - y^2 \sin \vartheta_\delta \varphi \, ds \\ = \lambda_\delta \int \sin \vartheta_\delta \varphi + \mu_\delta \int \cos \vartheta_\delta \varphi \end{aligned}$$

where

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where

$$\langle \partial_\vartheta y, \varphi \rangle = \int_{-\ell/2}^{-} \cos \vartheta_\delta \varphi \, ds$$

and the Lagrange multipliers satisfy

$$\lambda_\delta = O(1) \quad \text{and} \quad \mu_\delta = O(\sqrt{\delta}).$$

Up to subsequences: $\lambda_\delta \rightarrow \lambda^*$ and $\mu_\delta / \sqrt{\delta} \rightarrow \mu^*$.

Limit problem (2/3)

$$\begin{aligned} \int \frac{\partial_s \vartheta_\delta}{\sqrt{\delta}} \partial_s \varphi \, ds + \tfrac{1}{2} K \int 2 \frac{y}{\sqrt{\delta}} \langle \partial_\vartheta y, \varphi \rangle \cos \vartheta_\delta - y^2 \frac{\sin \vartheta_\delta}{\sqrt{\delta}} \varphi \, ds \\ = \lambda_\delta \int \frac{\sin \vartheta_\delta}{\sqrt{\delta}} \varphi + \frac{\mu_\delta}{\sqrt{\delta}} \int \cos \vartheta_\delta \varphi \end{aligned}$$

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At the limit $\delta \rightarrow 0$, that gives

$$\int \partial_s \vartheta^* \partial_s \varphi$$

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$$\int \partial_s \vartheta^* \partial_s \varphi + \frac{1}{2} K \int 2 \left(\int_{-\ell/2}^{-} \vartheta^* \, ds \right) \left(\int_{-\ell/2}^{-} \varphi \, ds \right)$$

using

$$\frac{y}{\sqrt{\delta}} = \int_{-\ell/2}^{-} \frac{\sin \vartheta_\delta}{\sqrt{\delta}} \, ds \xrightarrow[\delta \rightarrow 0]{} \int_{-\ell/2}^{-} \vartheta^* \, ds,$$

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Limit problem (3/3)

This weak formulation

$$\int \partial_s \vartheta^* \partial_s \varphi + K \int \left(\int_{-\ell/2}^{-} \vartheta^* ds \right) \left(\int_{-\ell/2}^{-} \varphi ds \right) = \lambda^* \int \vartheta^* \varphi + \mu^* \int \varphi.$$

correspond to the differential equation

$$-\partial_s^2 \vartheta^* - K \int_{-\ell/2}^s \left(\int_{-\ell/2}^{-} \vartheta^* \right) = \lambda^* \vartheta^* + \mu^* - \underbrace{K \int_{-\ell/2}^{\ell/2} \int_{-\ell/2}^{-} \vartheta^*}_{\text{constant term}}$$

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Let $u := \int_{-\ell/2}^{-} \vartheta^*$ and differentiate the equation to obtain:

$$\partial_s^4 u + Ku = -\lambda^* \partial_s^2 u.$$

The clamped boundary conditions hold:

$$u(-\ell/2) = 0, \quad \partial_s u(-\ell/2) = 0, \quad u(\ell/2) = 0, \quad \partial_s u(\ell/2) = 0.$$

Eigenvalue problem

The value λ^* in

$$\partial_s^4 u + Ku = -\lambda^* \partial_s^2 u$$

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is proportional to the “limit energy”, thus

$\lambda^* = \lambda_1$ is the first eigenvalue.

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Standard: there exist a countable number of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \xrightarrow{n} +\infty.$$

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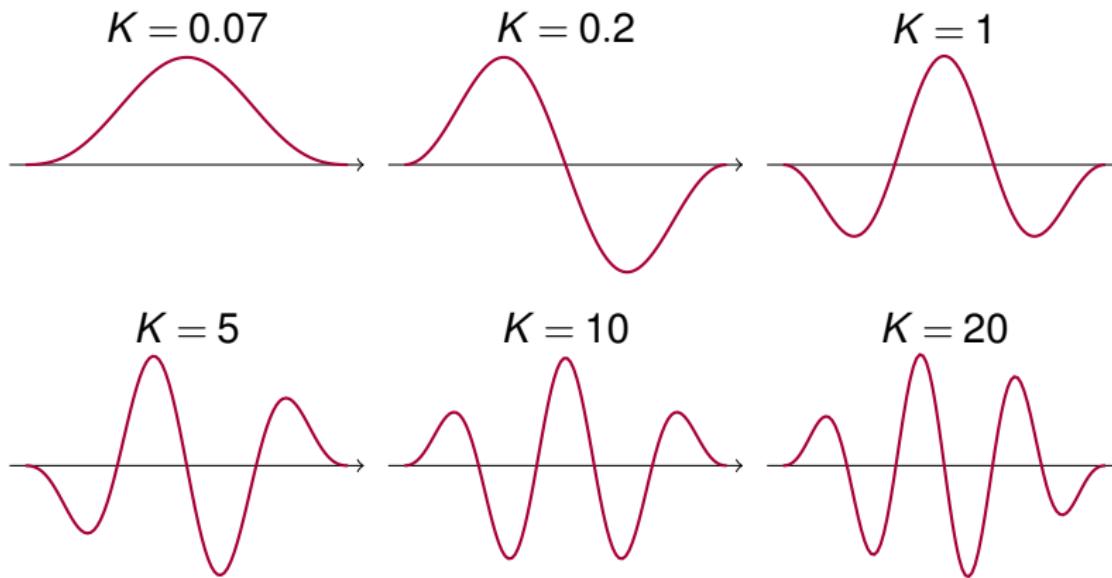
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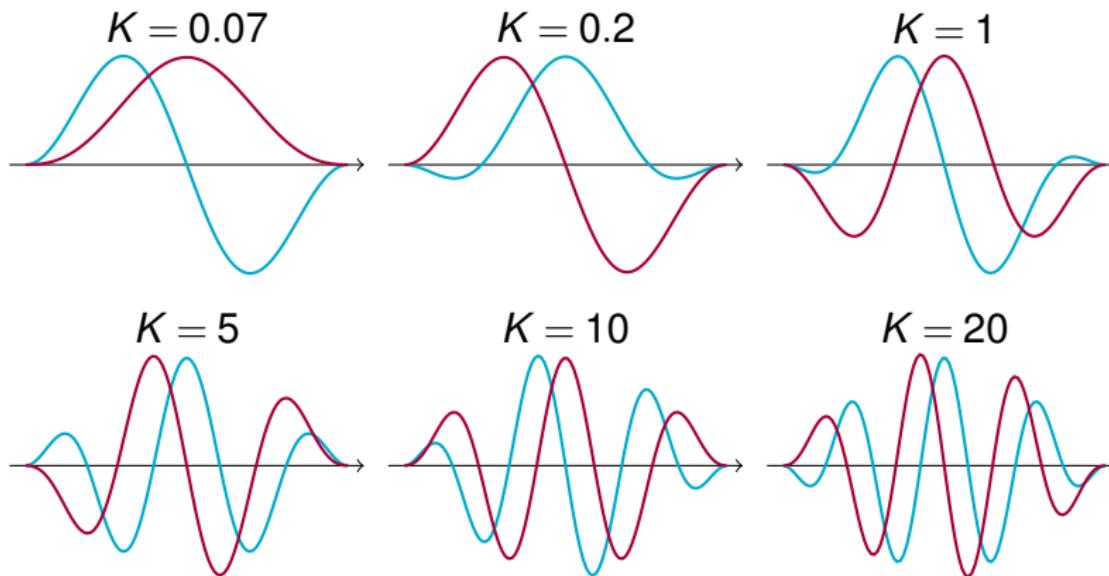
⇒ shape of the first eigenfunction u as K varies.

Numerical experiments ($\ell = 10$)



— u , first eigenfunction (λ_1)

Numerical experiments ($\ell = 10$)



— u , first eigenfunction (λ_1)

— second eigenfunction (λ_2)

Explicit solutions (1/3)

$$u(s) = a_1 \sin(\mu s) + a_2 \cos(\mu s) + a_3 \sin(\nu s) + a_4 \cos(\nu s)$$

where, w.l.o.g. $\mu \geqslant \nu > 0$, and

$$K = \mu^2 \nu^2 \quad \text{and} \quad \lambda_i = \mu^2 + \nu^2$$

Explicit solutions (1/3)

$$u(s) = a_1 \sin(\mu s) + a_2 \cos(\mu s) + a_3 \sin(v s) + a_4 \cos(v s)$$

where, w.l.o.g. $\mu \geq v > 0$, and

$$K = \mu^2 v^2 \quad \text{and} \quad \lambda_i = \mu^2 + v^2 > 2\sqrt{K}.$$

u satisfies the boundary conditions iff

$$(\mu - v)^2 \cos(\ell(\mu + v)) - (\mu + v)^2 \cos(\ell(\mu - v)) + 4\mu v = 0$$

i.e.

$$\begin{aligned} \text{or } & \mu \sin\left(\frac{\ell}{2}\mu\right) \cos\left(\frac{\ell}{2}v\right) - v \sin\left(\frac{\ell}{2}v\right) \cos\left(\frac{\ell}{2}\mu\right) = 0, \\ & \mu \cos\left(\frac{\ell}{2}\mu\right) \sin\left(\frac{\ell}{2}v\right) - v \cos\left(\frac{\ell}{2}v\right) \sin\left(\frac{\ell}{2}\mu\right) = 0. \end{aligned}$$

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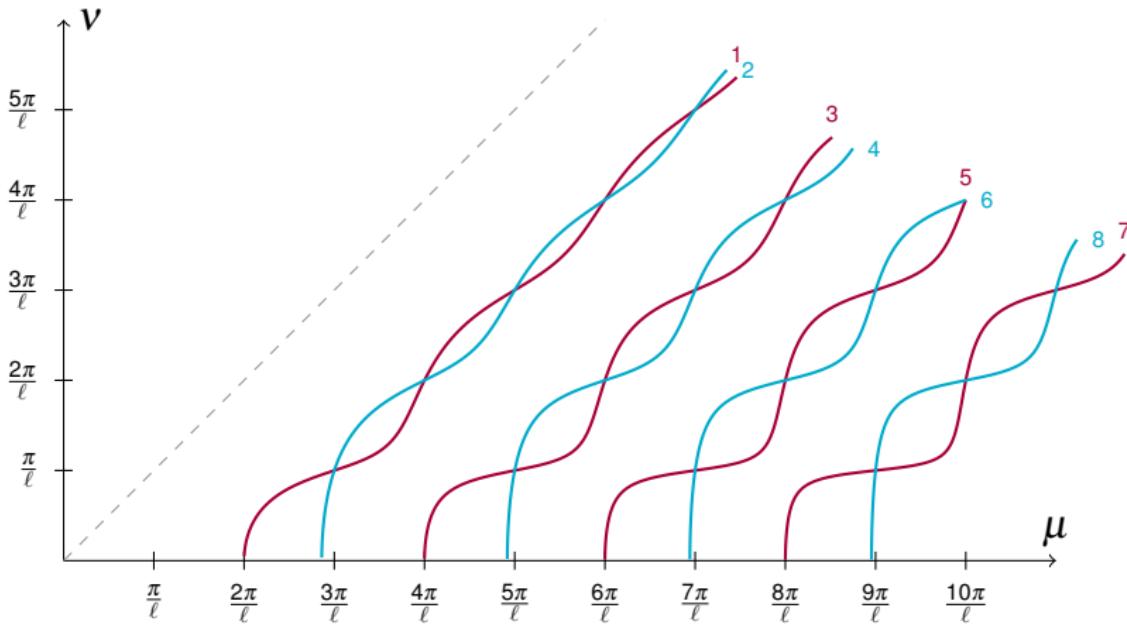
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Explicit solutions (2/3)

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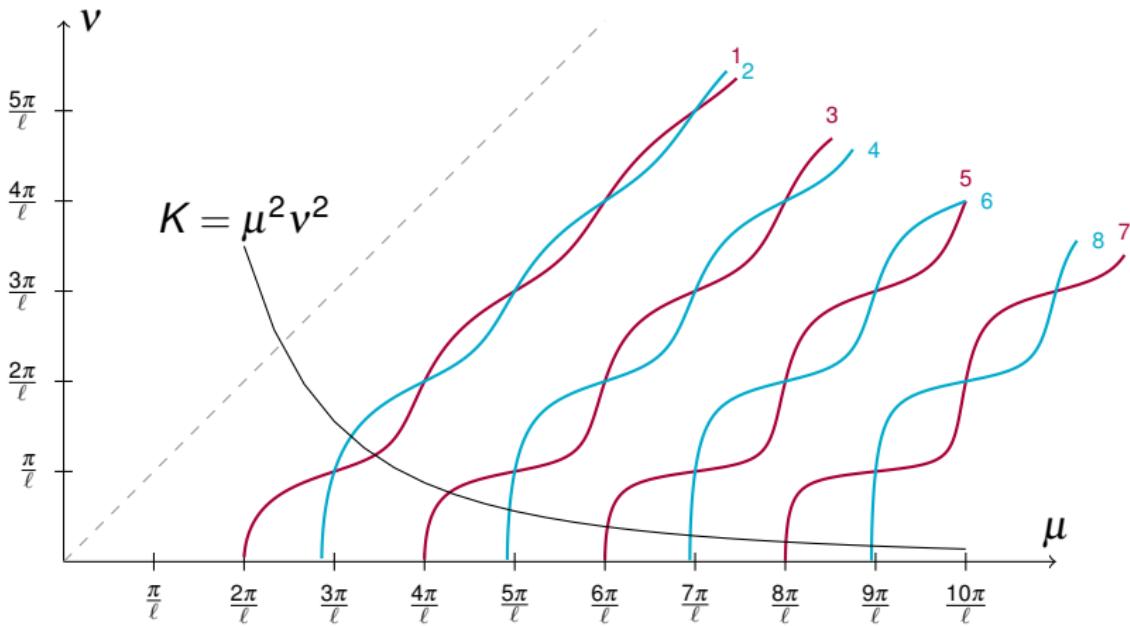
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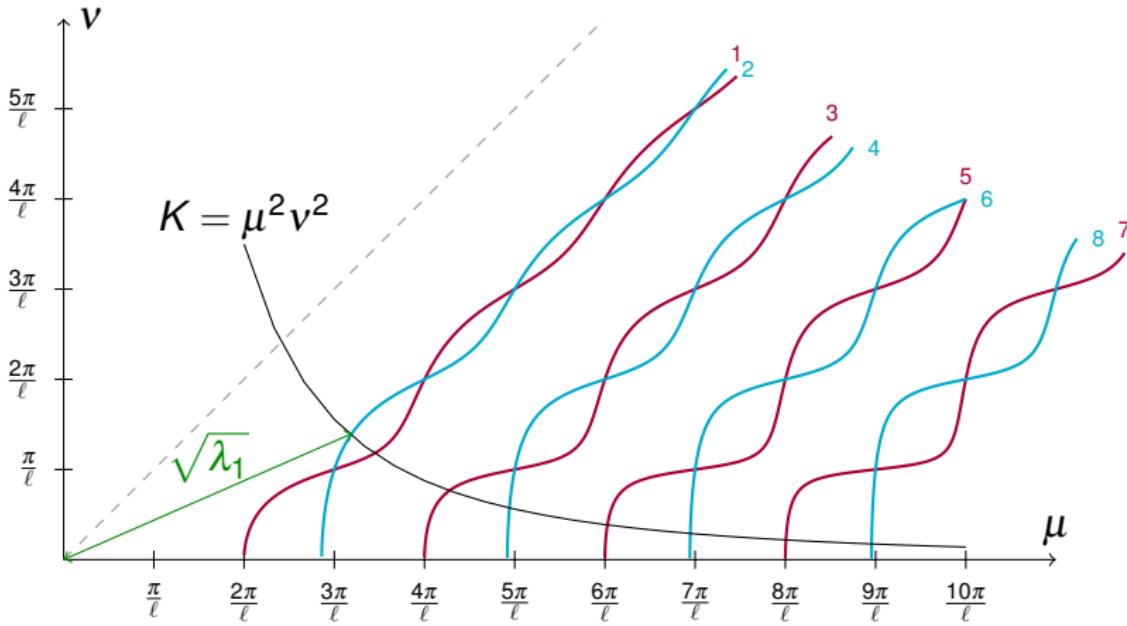
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Explicit solutions (3/3) — degenerate cases

Explicit solutions allow to compute values of K for which the eigenspace of λ_1 has dimension 2:

$$\dim \text{eigenspace}(\lambda_1) = 2 \Leftrightarrow \exists i \in \mathbb{N}^{>0}, K = i^2(i+2)^2 \frac{\pi^4}{\ell^4}.$$

In this case

$$\mu = (i+2) \frac{\pi}{\ell} \quad \text{and} \quad v = i \frac{\pi}{\ell}.$$

Explicit solutions (3/3) — degenerate cases

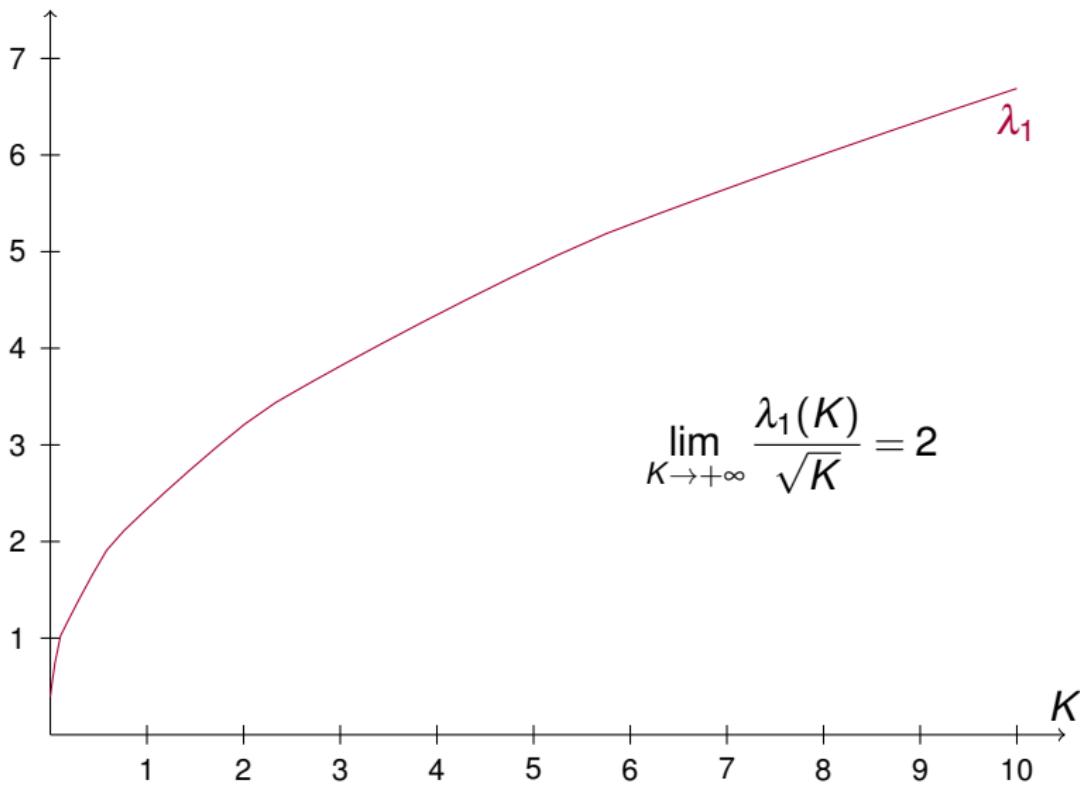
Explicit solutions allow to compute values of K for which the eigenspace of λ_1 has dimension 2:

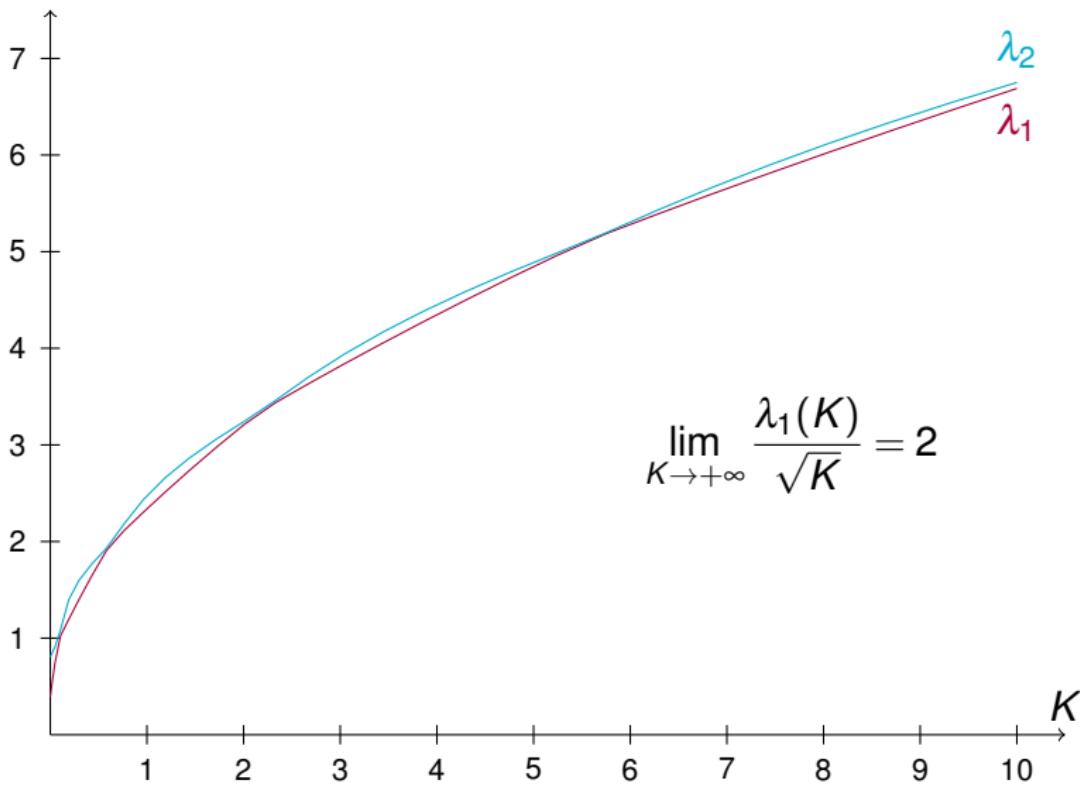
$$\dim \text{eigenspace}(\lambda_1) = 2 \Leftrightarrow \exists i \in \mathbb{N}^{>0}, K = i^2(i+2)^2 \frac{\pi^4}{\ell^4}.$$

In this case

$$\mu = (i+2)\frac{\pi}{\ell} \quad \text{and} \quad v = i\frac{\pi}{\ell}.$$

⇒ number of nodes of u as a function of K ?

$\lambda_1(K), K \rightarrow \infty$ 

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Large number of oscillations

Let $(u_K)_K$ be a family of 1st eigenfunctions of $\partial_s^4 u + Ku = -\lambda_1(K) \partial_s^2 u$.

$$v_K(t) := u_K(K^{-1/4}t + s_K) \quad \text{where } \partial_t^2 v_K(0) = |\partial_t^2 v_K|_{L^\infty} = 1.$$

v_K satisfies the equation:

$$\partial_s^4 v_K + v_K = -\frac{\lambda_1(K)}{\sqrt{K}} \partial_s^2 v_K, \quad \text{on }]K^{1/4}(-\frac{1}{2}\ell - s_K), K^{1/4}(\frac{1}{2}\ell - s_K)[.$$

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The sequence (v_K) converges (up to a subsequence) to v^* in $H^4(I)$ for each interval I , where v^* satisfies

$$\partial_s^4 v^* + v^* = -2\partial_s^2 v^* \quad \text{on } \mathbb{R} \text{ or a half line, with } \partial_s^2 v^*(0) = |\partial_s^2 v^*|_{L^\infty} = 1.$$

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$\Rightarrow v^*(t) = \cos(t) \Rightarrow u_K$ has many oscillations for K large.

Counting zeros (1/3)

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All zeros in $]-\ell/2, \ell/2[$ of a first eigenfunction u are simple.

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PROOF. Suppose u as a double zero in $]-\ell/2, \ell/2[$. Then $u = u_1 + u_2$ with $u_1, u_2 \in H^2 \setminus \{0\}$. Let $Q(u) := \frac{1}{2} \int (\partial^2 u)^2 + Ku^2 ds$.

$$\frac{\alpha_1^2 Q(u_1) + \alpha_2^2 Q(u_2)}{\alpha_1^2 |\partial u_1|_2^2 + \alpha_2^2 |\partial u_2|_2^2} \geq \min \left\{ \frac{Q(u_1)}{|\partial u_1|_2^2}, \frac{Q(u_2)}{|\partial u_2|_2^2} \right\}$$

and equality requires $\alpha_1 = 0$ or $\alpha_2 = 0$.

Then $\lambda_1 = \frac{Q(u)}{|\partial u|_2^2} > \min \left\{ \frac{Q(u_1)}{|\partial u_1|_2^2}, \frac{Q(u_2)}{|\partial u_2|_2^2} \right\}$.



Counting zeros (2/3)

Proposition

$\dim \text{eigenspace}(\lambda_1) = 1 \Leftrightarrow \exists u, \partial^2 u(-\ell/2) \neq 0 \Leftrightarrow \exists u, \partial^2 u(\ell/2) \neq 0.$

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PROOF. (\Leftarrow) Suppose on the contrary $\dim \text{eigenspace}(\lambda_1) = 2$. Then for each value for $(\partial^2 u(-\ell/2), \partial^3 u(-\ell/2)) \in \mathbb{R}^2$ there exists a unique eigenfunction. In particular for $(\partial^2 u(-\ell/2), \partial^3 u(-\ell/2)) = (0, 1)$.

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(\Rightarrow) Suppose on the contrary that $\partial^2 u(-\ell/2) = 0$. Since u is even or odd $\partial^2 u(\ell/2) = 0$. Thus $\partial u \in H^2$ and is again an eigenfunction.

Therefore

$$\partial_s u = \alpha u, \quad \alpha \in \mathbb{R},$$

which forbids the boundary conditions to be satisfied. □

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New zeros for first eigenfunctions can only occur when λ_1 is degenerate and can only come through the boundary.

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⇒ Creation of zeros and change of parity at degenerate K ?

Behavior when λ_1 is double (1/3)

$\lambda_{1,\text{even}}$ = the first eigenvalue on the subspace of even functions,

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$\lambda_{1,\text{even}}(K)$ and $\lambda_{1,\text{odd}}(K)$ are always simple.

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Consequence: $\lambda_{1,\text{odd}}(K)$ and $\lambda_{1,\text{even}}(K)$ are differentiable w.r.t. K .

Behavior when λ_1 is double (2/3)

Proposition

For all $K > 0$, $\lambda_1(K) = \min\{\lambda_{1,\text{even}}(K), \lambda_{1,\text{odd}}(K)\}$.

PROOF. (\leq) Obvious.

(\geq) There always exists an even or odd 1st eigenfunction.



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Proposition

$\lambda_1(K)$ is degenerate $\Leftrightarrow \lambda_{1,\text{even}}(K) = \lambda_{1,\text{odd}}(K)$.

PROOF. (\Leftarrow) Obvious.

(\Rightarrow) Let u be an even eigenfunction. Let $v \perp u$ be another eigenfunction. Let $w(s) := v(s) - v(-s)$. $w \perp u$ because u is even. $w \neq 0$ otherwise v is even, contradicting the simplicity of $\lambda_{1,\text{even}}(K)$. □

Behavior when λ_1 is double (3/3)

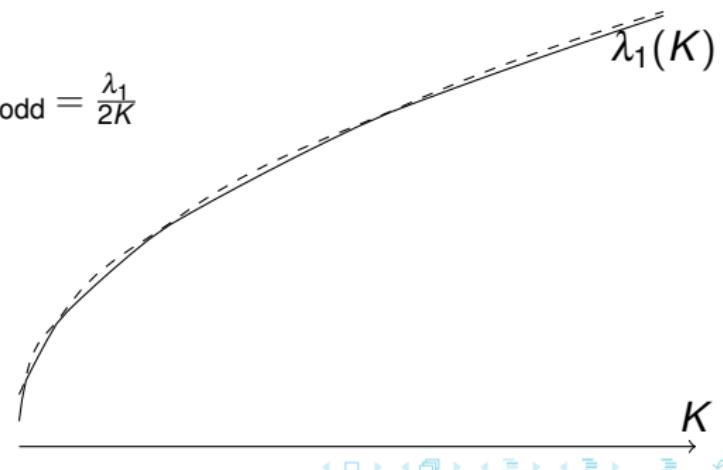
If $K = i^2(i+2)^2 \frac{\pi^4}{\ell^4}$ with $i \in \mathbb{N}^{>0}$,

- if i is odd:

- ▶ $\partial_K \lambda_{1,\text{even}} = \frac{\lambda_1}{2K} > \partial_K \lambda_{1,\text{odd}} = \frac{2}{\lambda_1}$
- ▶ $\partial^2 u_{\text{even}}(\pm \ell/2) = 0$
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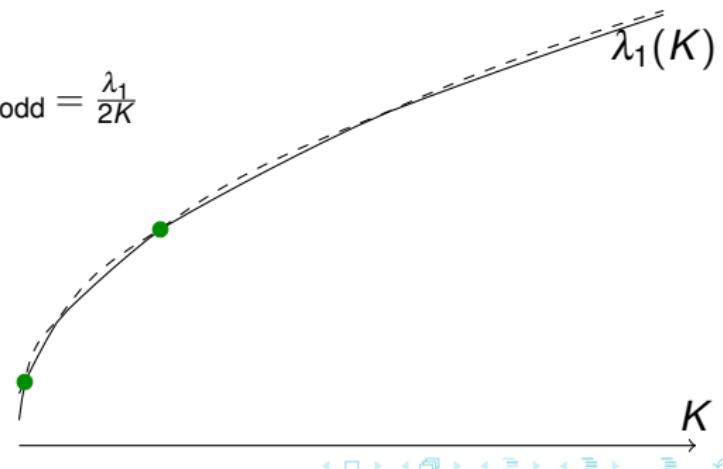
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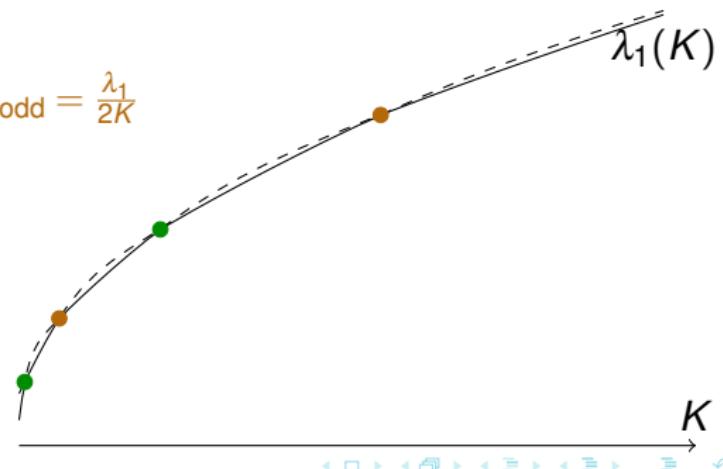
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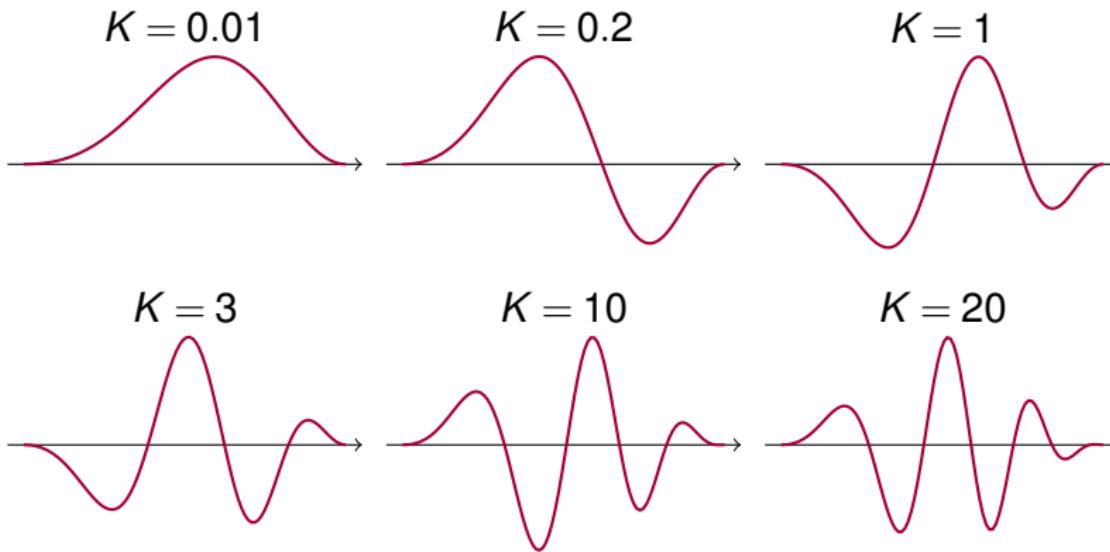
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Perspectives

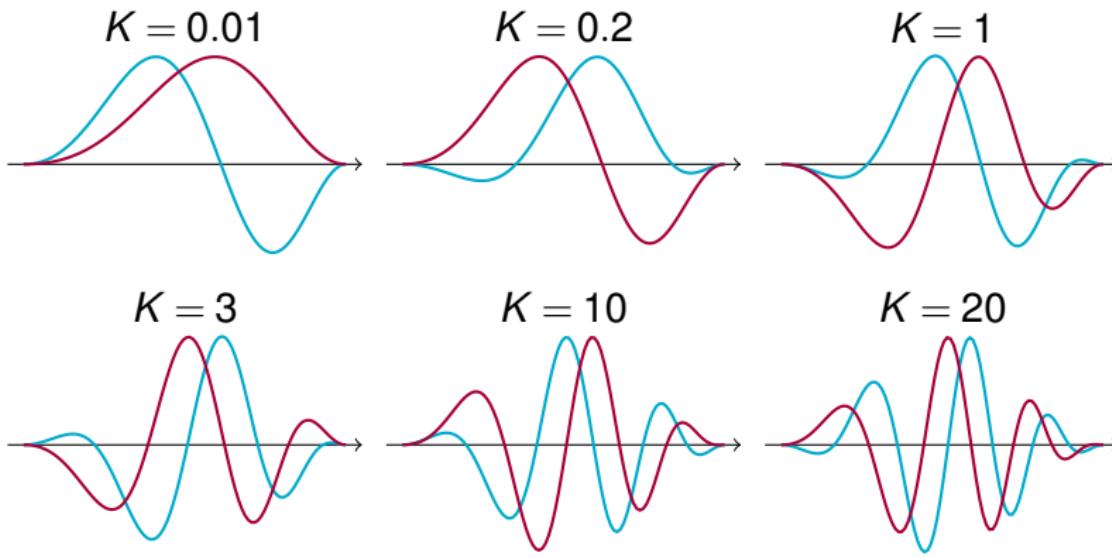
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— first eigenfunction (λ_1)

Perspectives

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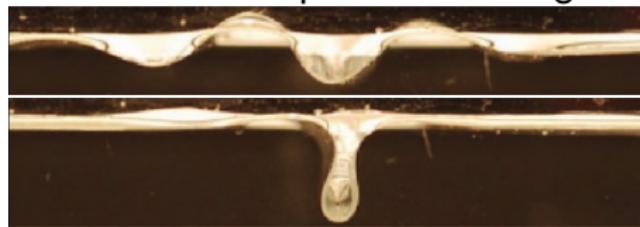


— first eigenfunction (λ_1)

— second eigenfunction (λ_2)

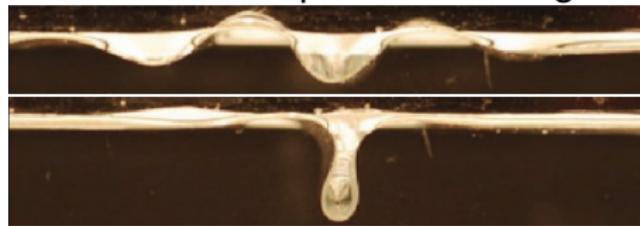
Perspectives

When the compression is larger:



Perspectives

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Thank you for your attention!