The Inverse Slope Problem and Additive Combinatorics

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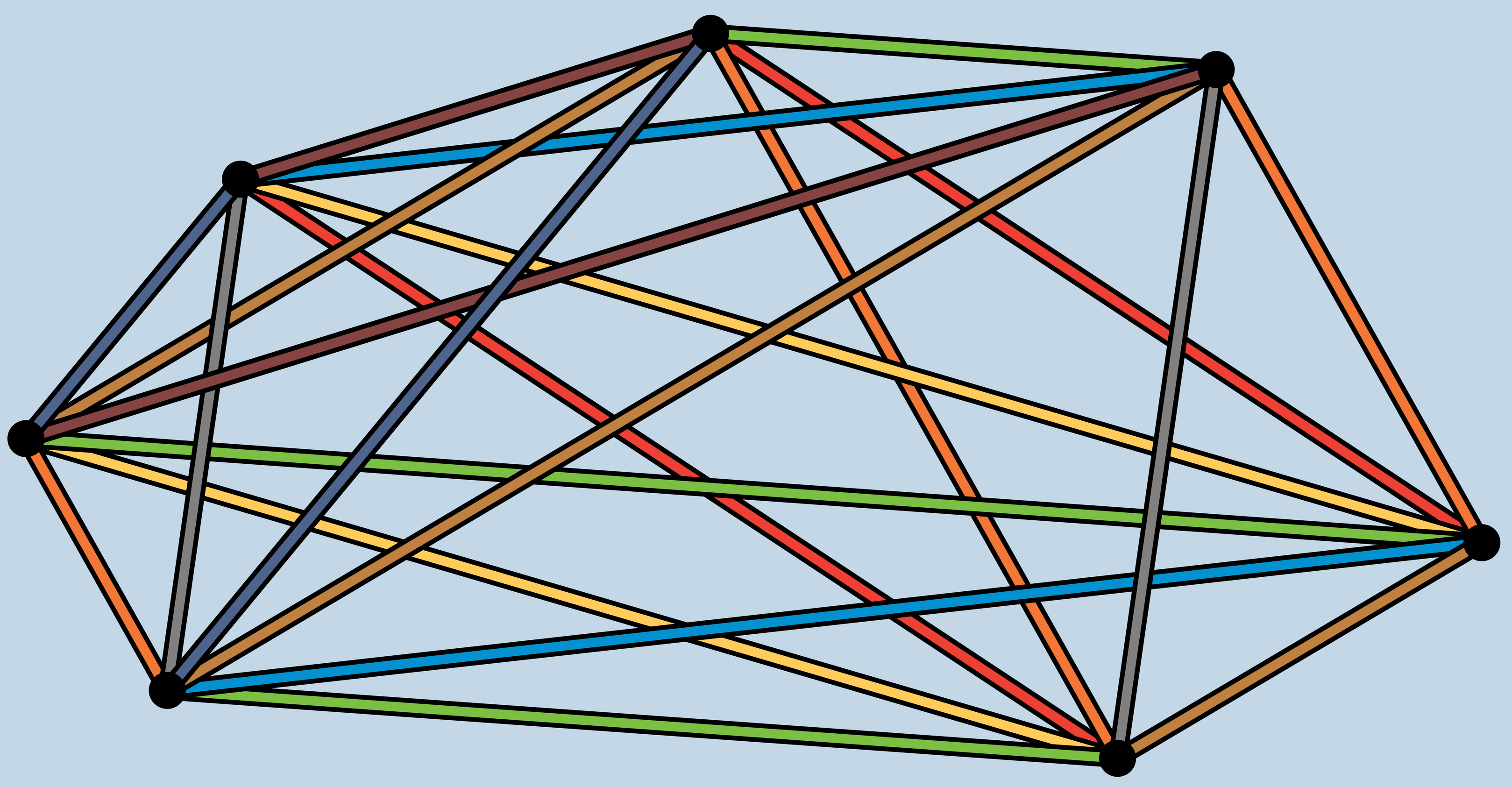
The Inverse Slope Problem

Which point sets in the plane determine ***few slopes***?

Definition

If P is a finite set of points in $\mathbb{R}^2$, we denote by $s(P)$ the *number of distinct slopes* determined by P.



In this example, $|P| = 7$ and $s(P) = 9$.

Minimum number of slopes

We suppose henceforth that P is in *general position* (i.e. no three points of P are collinear).

Observation. *If P is in general position, $s(P) \geq |P|$.*

Equality Case?

An ***affinely regular polygon*** is the image of the vertex set of a regular polygon under an affine transformation.

Theorem (Jamison, 1984)**.** *If P is in general position, then $s(P) = P$ iff P is an affinely regular polygon.*

Jamison's Conjecture (1984)

Gives a *complete description* of the structure of P in the near-equality case.

Conjecture (Jamison, modified)**.** *If P is in general position and*

$$|P| \leq s(P) < \frac{3}{2}|P|,$$

then P is contained in an affinely regular $s(P)$-gon.

My Contribution

Theorem (P.)**.** *Jamison's conjecture is true for*

- *(2018) $s(P) = |P| + 1$;*
- *(2021) $s(P) \leq |P| + c\sqrt[6]{|P|}$, for some $c > 0$.*

The latter result follows from a deep structure theorem of Ben Green and Terence Tao.

Additive Combinatorics

Compare Jamison's conjecture with the following fundamental result in additive combinatorics.

Proposition. *Let A be a subset of an abelian group G such that $|A \dot{+} A| < \frac{3}{2}|A|$. Then A is contained in a coset of a subgroup of G of size $|A \dot{+} A|$.*

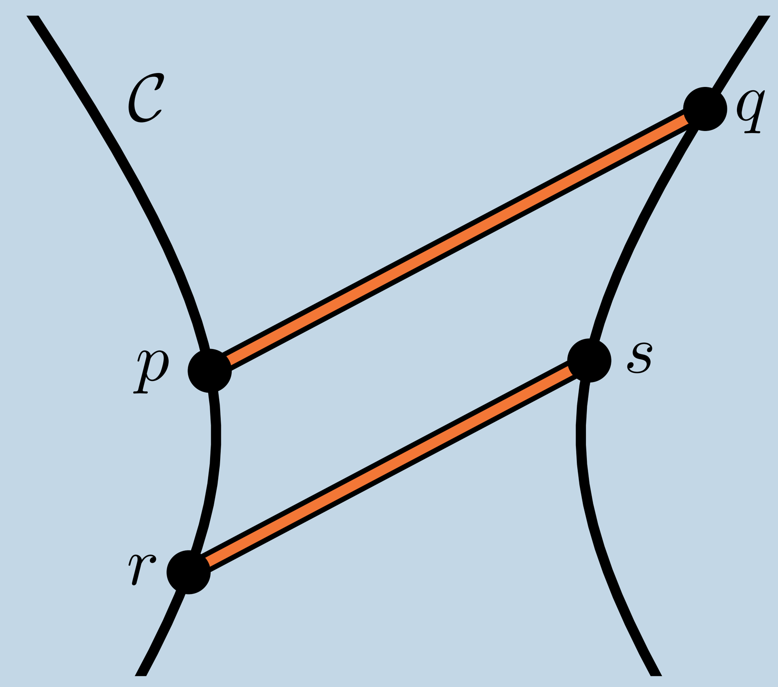
Here $A \dot{+} A$ denotes the ***restricted sumset***

$$A \dot{+} A := \{a + b \mid a, b \in A,\ a \neq b\}.$$

Group Law on a Conic

The bridge between additive combinatorics and Jamison's conjecture is the *group law on a conic.*

Proposition. *Let $\mathcal{C}$ be an irreducible conic. There is an **abelian group law** $+$ on the points of $\mathcal{C}$ with the property that $p + q = r + s$ if and only if the lines pq and rs are parallel.*



If P is a subset of a conic $\mathcal{C}$, there is a full equivalence:

Discrete Geometry	Additive Combinatorics
P is an affinely regular polygon	P is a coset of a finite subgroup of $\mathcal{C}$
the number of slopes $s(P)$	size of restricted sumset $\lvert P \dot{+} P \rvert$

In the regime $s(P) \leq C|P|$

Conjecture (Elekes, 1999)**.** *For all $m \geq 6$ and $C \geq 1$, there is some n_0 such that every set P with*

$$|P| \geq n_0 \quad \text{and} \quad s(P) \leq C|P|$$

*contains **m points** that lie **on a common conic**.*

Elekes' conjecture is still open, even for $m = 6$.

Conclusion

Central difficulty. To show that P as in Jamison's conjecture is indeed *contained in a conic.*

Possible future work. Approach Jamison's conjecture by adapting the techniques of Green and Tao: a combination of *algebraic topology*, *algebraic geometry* and *additive combinatorics.*

References

All available at https://arxiv.org/pdf/1811.01055.