

Time Flies When Looking out of the Window: Timed Games with Window Parity Objectives

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Abstract

The *window mechanism* was introduced by Chatterjee et al. to reinforce mean-payoff and total-payoff objectives with time bounds in two-player turn-based games on graphs [17]. It has since proved useful in a variety of settings, including parity objectives in games [14] and both mean-payoff and parity objectives in Markov decision processes [12].

We study *window parity* objectives in timed automata and timed games: given a bound on the window size, a path satisfies such an objective if, in all states along the path, we see a sufficiently small window in which the smallest priority is even. We show that checking that all time-divergent paths of a timed automaton satisfy such a window parity objective can be done in polynomial space, and that the corresponding timed games can be solved in exponential time. This matches the complexity class of timed parity games, while adding the ability to reason about time bounds. We also consider multi-dimensional objectives and show that the complexity class does not increase. To the best of our knowledge, this is the first study of the window mechanism in a real-time setting.

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1 Introduction

Timed automata and games. *Timed automata* [2] are extensions of finite automata with real-valued variables called *clocks*. Clocks increase at the same rate and measure the elapse of time between actions. Transitions are constrained by the values of clocks, and clocks can be reset on transitions. Timed automata are used to model real-time systems [4]. Not all paths of timed automata are meaningful; infinite paths that take a finite amount of time, called *time-convergent* paths, are often disregarded when checking properties of timed automata. Timed automata induce uncountable transition systems. However, many properties can be checked using the region abstraction, which is a finite quotient of the transition system.

Timed automaton games [24], or simply *timed games*, are games played on timed automata: one player represents the system and the other its environment. Players play an infinite amount of rounds: for each round, both players simultaneously present a delay and an action, and the play proceeds according to the fastest move (note that we use paths for automata and plays for games to refer to sequences of consecutive states and transitions). When defining winning conditions for players, convergent plays must be taken in account; we must not allow a player to achieve its objective by forcing convergence but cannot either require a player to force divergence (as it also depends on its opponent). Given an objective as a set of plays, following [20], we declare a play winning for a player if either it is *time-divergent and belongs to the objective*, or it is *time-convergent and the player is not responsible for convergence*.

Parity conditions. The class of ω -regular specifications is widely used (e.g., it can express liveness and safety), and parity conditions are a canonical way of representing them. In (timed) parity games, locations are labeled with a non-negative integer priority and the parity objective is to ensure the smallest priority occurring infinitely often along the path/play is even. Timed games with ω -regular objectives given as parity automata are shown to be solvable in [20]. Furthermore, a reduction from timed parity games to classical turn-based parity games on a graph is established in [19].

Real-timed windows. The parity objective can be reformulated: for all odd priorities seen infinitely often, a smaller even priority must be seen infinitely often. One can see the odd priority as a request and the even one as a response. The parity objective does not specify any *timing constraints* between requests and responses. In applications however, this may not be sufficient: for example, a server should respond to requests in a timely manner.

We revisit the window mechanism introduced by Chatterjee et al. for mean-payoff and total-payoff games [17] and later applied to parity games [14] and to parity and mean-payoff objectives in Markov decision processes [12]: we provide *the first* (to the best of our knowledge) *study of window objectives in the real-time setting*. More precisely, we lift the (resp. direct) fixed window parity objective of [14] to its real-time counterpart, the (resp. direct) *timed window parity objective*, and study it in timed automata and games.

Intuitively, given a non-negative integer bound λ on the window size, the *direct* timed window parity objective requires that *at all* times along a path/play, we see a window of size at most λ such that the smallest priority in this window is even. While time was counted as steps in prior works (all in a discrete setting), *we naturally measure window size using delays* between configurations in real-time models. The (*non-direct*) timed window parity objective is simply a prefix-independent version of the direct one, thus more closely matching the spirit of classical parity: it asks that some suffix satisfies the direct objective.

Contributions. We extend window parity objectives to a dense-time setting, and study both *verification* of timed automata and *realizability* in timed games. We consider adaptations of the *fixed window parity objectives* of [14], where the window size is given as a parameter. We establish that (a) verifying that all time-divergent paths of a timed automaton satisfy a timed window parity specification is PSPACE-complete; and that (b) checking the existence of a winning strategy for a window parity objective in timed games is EXPTIME-complete. These results (Theorem 18) hold for both the direct and prefix-independent variants, and they extend to multi-dimensional objectives, i.e., conjunctions of window parity.

All algorithms are based on a reduction to an *expanded timed automaton* (Definition 4). We establish that, similarly to the discrete case, it suffices to keep track of one window at a time (or one per objective in the multi-dimensional case) instead of all currently open windows, thanks to the so-called *inductive property of windows* (Lemma 3). A window can be summarized using its smallest priority and its current size: we encode the priorities in a window by extending locations with numbers and using an additional clock to measure its size. The (resp. direct) timed window parity objective translates to a co-Büchi (resp. safety) objective on the expanded automaton. Locations to avoid for the co-Büchi (resp. safety) objective indicate a window exceeding the supplied bound without the smallest priority of the window being even — a *bad window*. To check that all time-divergent paths of the expanded automaton satisfy the safety (resp. co-Büchi) objective, we check for the existence of a time-divergent path visiting (resp. infinitely often) an unsafe location using the PSPACE algorithm of [2]. To solve the similarly-constructed expanded game, we use the EXPTIME

algorithm of [20].

Lower bounds (Lemma 17) are established by encoding safety objectives on timed automata as (resp. direct) timed window parity objectives. Checking safety properties over time-divergent paths in timed automata is PSPACE-complete [2] and solving safety timed games is EXPTIME-complete [22].

Comparison. Window variants constitute *conservative approximations* of classical objectives (e.g., [17, 14, 12]), strengthening them by enforcing timing constraints. Complexity-wise, the situation is varied. In one-dimension turn-based games on graphs, window variants [17, 14] provide *polynomial-time alternatives* to the classical objectives, bypassing long-standing complexity barriers. However, in multi-dimension games, their complexity becomes worse than for the original objectives: in particular, fixed window parity games are EXPTIME-complete for multiple dimensions [14]. We show that timed games with multi-dimensional window parity objectives are in the same complexity class as untimed ones, i.e., dense time comes for free.

For classical parity objectives, timed games can be solved in exponential time [19, 20]. The approach of [19] is as follows: from a timed parity game, one builds a corresponding turn-based parity game on a graph, the construction being polynomial in the number of priorities and the size of the region abstraction. We recall that despite recent progress on quasi-polynomial-time algorithms (starting with [16]), no polynomial-time algorithm is known for parity games; the blow-up comes from the number of priorities. Overall, *the two sources of blow-up* — region abstraction and number of priorities — *combine in a single-exponential solution* for timed parity games. We establish that (multi-dimensional) window parity games can be solved in time polynomial in the size of the region abstraction, the number of priorities and the window size, and exponential in the number of dimensions. Thus *even for conjunctions of objectives, we match the complexity class of single parity objectives of timed games, while avoiding the blow-up related to the number of priorities and enforcing time bounds between odd priorities and smaller even priorities via the window mechanism.*

Outline. This work is organized as follows. Section 2 summarizes all prerequisite notions and vocabulary. In Section 3, we introduce the timed window parity objective, compare it to the classical parity objective, and establish some useful properties. Our reduction from window parity objectives to safety or co-Büchi objectives is presented in Section 4: the construction of the expanded timed automaton/game used in the reduction is provided in Section 4.1, Section 4.2 develops mappings between plays of a game and plays of its expansion such that a time-divergent play in one game satisfies the objective if and only if its image satisfies the objective of the other game and Section 4.3 shows how these mappings can be used to transfer winning strategies between a timed game and its expansion. The reduction is extended to the case of multiple window parity objectives in Section 5. Finally, Section 6 presents the complexity.

Related work. In addition to the aforementioned foundational works, the window mechanism has seen diverse extensions and applications: e.g., [5, 3, 11, 15, 23, 26, 8]. Window parity games are strongly linked to the concept of *finitary ω -regular games*: see, e.g., [18], or [14] for a complete list of references. The window mechanism can be used to ensure a certain form of (local) guarantee over paths: different techniques have been considered in stochastic models, notably variance-based [10] or worst-case-based [13, 7] methods. Finally, let us recall that game models provide a useful framework for controller synthesis [25], and that timed

automata have been extended in a number of directions (see, e.g., [9] and references therein); applications of the window mechanism in such richer models could be of interest.

2 Preliminaries

Timed automata. A clock variable, or *clock*, is a real-valued variable. Let C be a set of clocks. A *clock constraint* over C is a conjunction of formulae of the form $x \sim c$ with $x \in C$, $c \in \mathbb{N}$, and $\sim \in \{\leq, \geq, >, <\}$. We write $x = c$ as shorthand for the clock constraint $x \geq c \wedge x \leq c$. Let $\Phi(C)$ denote the set of clock constraints over C .

Let $\mathbb{R}_{\geq 0}$ denote the set of non-negative real numbers. We refer to functions $\nu \in \mathbb{R}_{\geq 0}^C$ as *clock valuations* over C . A clock valuation ν over a set C of clocks satisfies a clock constraint of the form $x \sim c$ if $\nu(x) \sim c$ and a conjunction $g \wedge h$ if it satisfies both g and h . Given a clock constraint g and clock valuation ν , we write $\nu \models g$ if ν satisfies g .

For a clock valuation ν and $d \geq 0$, we let $\nu + d$ be the valuation defined by $(\nu + d)(x) = \nu(x) + d$ for all $x \in C$. For any valuation ν and $D \subseteq C$, we define $\text{reset}_D(\nu)$ to be the valuation agreeing with ν for clocks in $C \setminus D$ and that assigns 0 to clocks in D . We denote by $\mathbf{0}^C$ the zero valuation, assigning 0 to all clocks in C .

A *timed automaton* (TA) is a tuple $(L, \ell_{\text{init}}, C, \Sigma, I, E)$ where L is a finite set of *locations*, $\ell_{\text{init}} \in L$ is an initial location, C a finite set of *clocks* containing a special clock γ which keeps track of the total time elapsed, Σ a finite set of actions, $I: L \rightarrow \Phi(C)$ an *invariant* assignment function and $E \subseteq L \times \Phi(C) \times \Sigma \times 2^{C \setminus \{\gamma\}} \times L$ an edge relation. We only consider deterministic timed automata, i.e., we assume that in any location ℓ , there are no two outgoing edges $(\ell, g_1, a, D_1, \ell_1)$ and $(\ell, g_2, a, D_2, \ell_2)$ sharing the same action such that the conjunction $g_1 \wedge g_2$ is satisfiable. For an edge (ℓ, g, a, D, ℓ') , the clock constraint g is called the *guard* of the edge.

A TA $\mathcal{A} = (L, \ell_{\text{init}}, C, \Sigma, I, E)$ gives rise to an uncountable transition system $\mathcal{T}(\mathcal{A}) = (S, s_{\text{init}}, M, \rightarrow)$ with the state space $S = L \times \mathbb{R}_{\geq 0}^C$, the initial state $s_{\text{init}} = (\ell_{\text{init}}, \mathbf{0}^C)$, set of actions $M = \mathbb{R}_{\geq 0} \times (\Sigma \cup \{\perp\})$ and the transition relation $\rightarrow \subseteq S \times M \times S$ defined as follows: for any action $a \in \Sigma$ and delay $d \geq 0$, we have that $((\ell, \nu), (d, a), (\ell', \nu')) \in \rightarrow$ if and only if there is some edge $(\ell, g, a, D, \ell') \in E$ such that $\nu + d \models g$, $\nu' = \text{reset}_D(\nu + d)$, $\nu + d \models I(\ell)$ and $\nu' \models I(\ell')$; for any delay $d \geq 0$, $((\ell, \nu)(d, \perp), (\ell, \nu + d)) \in \rightarrow$ if $\nu + d \models I(\ell)$. Let us note that the satisfaction set of clock constraints is convex: it is described by an conjunction of inequalities. Whenever $\nu \models I(\ell)$, the above conditions $\nu + d \models I(\ell)$ (the invariant holds after the delay) are equivalent to requiring $\nu + d' \models I(\ell)$ for all $0 \leq d' \leq d$ (the invariant holds at each intermediate time step).

A *move* is any pair in $\mathbb{R}_{\geq 0} \times (\Sigma \cup \{\perp\})$ (i.e., an action in the transition system). For any move $m = (d, a)$ and states $s, s' \in S$, we write $s \xrightarrow{m} s'$ or $s \xrightarrow{d, a} s'$ as shorthand for $(s, m, s') \in \rightarrow$. Moves of the form (d, \perp) are called *delay moves*. We say a move m is enabled in a state s if there is some s' such that $s \xrightarrow{m} s'$. There is at most one successor per move in a state, as we do not allow two guards on edges labeled by the same action to be simultaneously satisfied.

A *path* in a TA \mathcal{A} is a finite or infinite sequence $s_0(d_0, a_0)s_1 \dots \in S(MS)^* \cup (SM)^\omega$ such that for all j , s_j is a state of $\mathcal{T}(\mathcal{A})$ and for all $j > 0$, $s_{j-1} \xrightarrow{d_{j-1}, a_{j-1}} s_j$ is a transition in $\mathcal{T}(\mathcal{A})$. A path is *initial* if $s_0 = s_{\text{init}}$. For clarity, we write $s_0 \xrightarrow{d_0, a_0} s_1 \xrightarrow{d_1, a_1} \dots$ instead of $s_0(d_0, a_0)s_1(d_1, a_1) \dots$.

An infinite path $\pi = (\ell_0, \nu_0) \xrightarrow{d_0, a_0} (\ell_1, \nu_1) \dots$ is *time-divergent* if the sequence $(\nu_j(\gamma))_{j \in \mathbb{N}}$ is not bounded from above. A path which is not time-divergent is called *time-convergent*; time-convergent paths are traditionally ignored in analysis of timed automata [1, 2] as they model unrealistic behavior. This includes ignoring *Zeno paths*, which are time-convergent

paths along which infinitely many actions appear. We write $\text{Paths}(\mathcal{A})$ for the set of paths of \mathcal{A} .

Priorities. A *priority function* is a function $p: L \rightarrow \{0, \dots, d-1\}$ with $d \leq |L|$. We use priority functions to express parity objectives. A *k-dimensional priority function* is a function $p: L \rightarrow \{0, \dots, d-1\}^k$ which assigns vectors of priorities to locations.

Timed games. We consider two player games played on TAs. We refer to the players as player 1 (\mathcal{P}_1) for the system and player 2 (\mathcal{P}_2) for the environment. We use the notion of timed automaton games of [20].

A *timed (automaton) game* (TG) is a tuple $\mathcal{G} = (\mathcal{A}, \Sigma_1, \Sigma_2)$ where $\mathcal{A} = (L, \ell_{\text{init}}, C, \Sigma, I, E)$ is a TA and (Σ_1, Σ_2) is a partition of Σ . We refer to actions in Σ_i as \mathcal{P}_i actions for $i \in \{1, 2\}$.

Recall a move is a pair $(d, a) \in \mathbb{R}_{\geq 0} \times (\Sigma \cup \{\perp\})$. Let S denote the set of states of $\mathcal{T}(\mathcal{A})$. In each state $s = (\ell, \nu) \in S$, the moves available to \mathcal{P}_1 are the elements of the set $M_1(s)$ where

$$M_1(s) = \{(d, a) \in \mathbb{R}_{\geq 0} \times (\Sigma_1 \cup \{\perp\}) \mid \exists s', s \xrightarrow{d, a} s'\}$$

contains moves with \mathcal{P}_1 actions and delay moves that are enabled in s . The set $M_2(s)$ is defined analogously with \mathcal{P}_2 actions. We write M_1 and M_2 for the set of all moves of \mathcal{P}_1 and \mathcal{P}_2 respectively.

At each state s along a play, both players simultaneously select a move $(d^{(1)}, a^{(1)}) \in M_1(s)$ and $(d^{(2)}, a^{(2)}) \in M_2(s)$. Intuitively, the fastest player gets to act and in case of a tie, the move is chosen non-deterministically. This is formalized by the *joint destination function* $\delta: S \times M_1 \times M_2 \rightarrow 2^S$, defined by

$$\delta(s, (d^{(1)}, a^{(1)}), (d^{(2)}, a^{(2)})) = \begin{cases} \{s' \in S \mid s \xrightarrow{d^{(1)}, a^{(1)}} s'\} & \text{if } d^{(1)} < d^{(2)} \\ \{s' \in S \mid s \xrightarrow{d^{(2)}, a^{(2)}} s'\} & \text{if } d^{(1)} > d^{(2)} \\ \{s' \in S \mid s \xrightarrow{d^{(i)}, a^{(i)}} s', i = 1, 2\} & \text{if } d^{(1)} = d^{(2)}. \end{cases}$$

For $m^{(1)} = (d^{(1)}, a^{(1)}) \in M_1$ and $m^{(2)} = (d^{(2)}, a^{(2)}) \in M_2$, we write $\text{delay}(m^{(1)}, m^{(2)}) = \min\{d^{(1)}, d^{(2)}\}$ to denote the delay occurring when \mathcal{P}_1 and \mathcal{P}_2 play $m^{(1)}$ and $m^{(2)}$ respectively.

A play is defined similarly to a path: it is a finite or infinite sequence of the form $s_0(m_0^{(1)}, m_0^{(2)})s_1(m_1^{(1)}, m_1^{(2)})\dots \in S((M_1 \times M_2)S)^* \cup (S(M_1 \times M_2))^\omega$ where for all indices j , $m_j^{(i)} \in M_i(s_j)$ for $i \in \{1, 2\}$, and for $j > 0$, $s_j \in \delta(s_{j-1}, m_{j-1}^{(1)}, m_{j-1}^{(2)})$. A play is *initial* if $s_0 = s_{\text{init}}$. For a finite play $\pi = s_0 \dots s_n$, we set $\text{last}(\pi) = s_n$. For an infinite play $\pi = s_0 \dots$, we write $\pi|_n = s_0(m_0^{(0)}, m_0^{(1)}) \dots s_n$. A play follows a path in the TA, but there need not be a unique path compatible with a play: if along a play, at the n th step, the moves of both players share the same delay and target state, either move can label the n th transition in a matching path.

Similarly to paths, an infinite play $\pi = (\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)}) \dots$ is *time-divergent* if and only if $(\nu_j(\gamma))_{j \in \mathbb{N}}$ is not bounded from above. Otherwise, we say a play is *time-convergent*. We define the following sets: $\text{Plays}(\mathcal{G})$ for the set of plays of \mathcal{G} ; $\text{Plays}_{\text{fin}}(\mathcal{G})$ for the set of finite plays of \mathcal{G} ; $\text{Plays}_\infty(\mathcal{G})$ for the set of time-divergent plays of \mathcal{G} . We also write $\text{Plays}(\mathcal{G}, s)$ to denote plays starting in state s of $\mathcal{T}(\mathcal{A})$.

Strategies. A strategy for \mathcal{P}_i is a function describing which move a player should use based on a play history. Formally, a strategy for \mathcal{P}_i is a function $\sigma_i: \text{Plays}_{\text{fin}}(\mathcal{G}) \rightarrow M_i$ such that for

all $\pi \in \text{Plays}_{f_{in}}(\mathcal{G})$, $\sigma_i(\pi) \in M_i(\text{last}(\pi))$. This last condition requires that each move given by a strategy be enabled in the last state of a play.

A play $s_0(m_0^{(1)}, m_0^{(2)})s_1 \dots$ is said to be consistent with a \mathcal{P}_i -strategy σ_i if for all indices j , $m_j^{(i)} = \sigma_i(\pi|_j)$. Given a \mathcal{P}_i strategy σ_i , we define $\text{Outcome}_i(\sigma_i)$ (resp. $\text{Outcome}_i(\sigma_i, s)$) to be the set of plays (resp. set of plays starting in state s) consistent with σ_i .

A \mathcal{P}_i -strategy σ_i is *move-independent* if the move it suggests depends only on the sequence of states seen in the play. Formally, σ_i is move-independent if for all finite plays $\pi = s_0(m_0^{(1)}, m_0^{(2)})s_1 \dots s_k$ and $\tilde{\pi} = \tilde{s}_0(\tilde{m}_0^{(1)}, \tilde{m}_0^{(2)})\tilde{s}_1 \dots \tilde{s}_k$ if $s_n = \tilde{s}_n$ for all $n \in \{0, \dots, k\}$, then $\sigma_i(\pi) = \sigma_i(\tilde{\pi})$. We use move-independent strategies in the proof of our reduction to relabel some moves of a play without affecting the suggestions of the strategy.

Objectives. An objective represents the property we desire on paths of a TA or a goal of a player in a TG. Formally, we define an *objective* as a set $\Psi \subseteq \text{Paths}(\mathcal{A})$ of infinite paths (when studying TAs) or a set $\Psi \subseteq \text{Plays}(\mathcal{G})$ of infinite plays (when studying TGs). An objective is *state-based* (resp. *location-based*) if it depends solely on the sequence of states (resp. of locations) in a path or play. Any location-based objective is state-based.

► **Remark 1.** In the sequel, we present objectives exclusively as sets of plays. Definitions for paths are analogous as all the objectives defined hereafter are state-based.

We use the following classical location-based objectives. A *reachability* objective for a set F of locations is the set of plays that pass through a location in F . The complement of a reachability objective is a *safety* objective; given a set F , it is the set of plays that never visit a location in F . A *Büchi* objective for a set F contains all plays that pass through locations in F infinitely often and the complement *co-Büchi* objective consists of plays traversing locations in F finitely often. The *parity* objective for a priority function p over the set of locations requires that the smallest priority seen infinitely often is even.

Fix F a set of locations and p a priority function. The aforementioned objectives are formally defined as follows.

- $\text{Reach}(F) = \{(\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)}) \dots \in \text{Plays}(\mathcal{G}) \mid \exists n, \ell_n \in F\}$;
- $\text{Safe}(F) = \{(\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)}) \dots \in \text{Plays}(\mathcal{G}) \mid \forall n, \ell_n \notin F\}$.
- $\text{Büchi}(F) = \{(\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)}) \dots \in \text{Plays}(\mathcal{G}) \mid \forall j, \exists n \geq j, \ell_n \in F\}$;
- $\text{coBüchi}(F) = \{(\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)}) \dots \in \text{Plays}(\mathcal{G}) \mid \exists j, \forall n \geq j, \ell_n \notin F\}$;
- $\text{Parity}(p) = \{(\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)}) \dots \in \text{Plays}(\mathcal{G}) \mid (\liminf_{n \rightarrow \infty} p(\ell_n)) \bmod 2 = 0\}$.

Winning conditions. In games, we distinguish objectives and *winning conditions*. We adopt the definition of [20]. Let Ψ be an objective. It is desirable to have victory be achieved in a physically meaningful way: for example, it is unrealistic to have a safety objective be achieved by stopping time. This motivates a restriction to time-divergent plays. However, this requires \mathcal{P}_1 to force the divergence of plays, which is not reasonable, as \mathcal{P}_2 can stall using delays with zero time units. Thus we also declare winning time-convergent plays where \mathcal{P}_1 is *blameless*. Let Blameless_1 denote the set of \mathcal{P}_1 -blameless plays, which we define in the following way.

Let $\pi = s_0(m_0^{(1)}, m_0^{(2)})s_1 \dots$ be (possibly finite) play. We say \mathcal{P}_1 is *not responsible* (or not to be blamed) for the transition at step n in π if either $d_n^{(2)} < d_n^{(1)}$ (\mathcal{P}_2 is faster) or $d_n^{(1)} = d_n^{(2)}$ and $s_n \xrightarrow{d_n^{(1)}, a_n^{(1)}} s_{n+1}$ does not hold in $\mathcal{T}(\mathcal{A})$ (\mathcal{P}_2 's move was selected and did not have the same target state as \mathcal{P}_1 's) where $m_n^{(i)} = (d_n^{(i)}, a_n^{(i)})$ for $i \in \{1, 2\}$. The set Blameless_1 is formally defined as the set of infinite plays π such that there is some j such that for all $n \geq j$, \mathcal{P}_1 is not responsible for the transition at step n in π .

Given an objective Ψ , we set the winning condition $WC_1(\Psi)$ for \mathcal{P}_1 to be the set of plays

$$WC_1(\Psi) = (\Psi \cap \text{Plays}_\infty(\mathcal{G})) \cup (\text{Blameless}_1 \setminus \text{Plays}_\infty(\mathcal{G})).$$

Winning conditions for \mathcal{P}_2 are defined by exchanging the roles of the players in the former definition.

We consider that the two players are adversaries and have opposite objectives, Ψ and $\neg\Psi$ (shorthand for $\text{Plays}(\mathcal{G}) \setminus \Psi$). Let us note that there may be plays π such that $\pi \notin WC_1(\Psi)$ and $\pi \notin WC_2(\neg\Psi)$, e.g., any time-convergent play in which neither player is blameless.

A *winning strategy* for \mathcal{P}_i for an objective Ψ from a state s_0 is a strategy σ_i such that $\text{Outcome}_i(\sigma_i, s_0) \subseteq WC_i(\Psi)$. Move-independent strategies are known to suffice for timed automaton games with state-based objectives [20].

Decision problems. We consider two different problems for an objective Ψ . The first is the *verification problem* for Ψ , which asks given a timed automaton whether all *time-divergent initial* paths satisfy the objective. Second is the *realizability problem*, which asks whether in a timed automaton game with objective Ψ , \mathcal{P}_1 has a winning strategy from the initial state.

3 Window objectives

We consider the *fixed window parity* and *direct fixed window parity* problems from [14] and adapt the discrete-time requirements from their initial formulation to dense-time requirements for TAs and TGs. Intuitively, a direct fixed window parity objective for some bound λ requires that at all points along a play or a path, we see a window of size less than λ in which the smallest priority is even. The (non-direct) fixed window parity objective requires that the direct objective holds for some suffix. In the sequel, we drop “fixed” from the name of these objectives.

In this section, we formalize the timed window parity objective in TGs as sets of plays. The definition for paths of TAs is analogous (see Remark 1). First, we define the *timed good window objective*, which formalizes the notion of good windows. Then we introduce the timed window parity objective and its direct variant. We compare these objectives to the parity objective and argue that satisfying a window objective implies satisfying a parity objective, and that window objectives do not coincide with parity objectives in general, via an example. We conclude this section by proving some useful properties of this objective.

For this entire section, we fix a TG $\mathcal{G} = (\mathcal{A}, \Sigma_1, \Sigma_2)$ where $\mathcal{A} = (L, \ell_{\text{init}}, C, \Sigma_1 \cup \Sigma_2, I, E)$, a priority function $p: L \rightarrow \{0, \dots, d-1\}$ and a bound $\lambda \in \mathbb{N} \setminus \{0\}$ on the size of windows.

3.1 Definitions

Good windows. A window objective is based on a notion of good windows. Intuitively, a good window for the parity objective is a fragment of a play in which less than λ time units pass and the smallest priority of the locations appearing in this fragment is even.

The timed good window objective encompasses plays in which there is a good window at the start of the play. We formally define the *timed good window (parity) objective* as the set

$$\begin{aligned} \text{TGW}(p, \lambda) = \{ (\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)}) \dots \in \text{Plays}(\mathcal{G}) \mid \exists j \in \mathbb{N}, \min_{0 \leq k \leq j} p(\ell_k) \bmod 2 = 0 \\ \wedge \nu_j(\gamma) - \nu_0(\gamma) < \lambda \}. \end{aligned}$$

The timed good window objective is a state-based objective.

We introduce some terminology related to windows. Let $\pi = (\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)})(\ell_1, \nu_1) \dots$ be an infinite play. We say that the window opened at step n *closes* at step j if $\min_{n \leq k \leq j} p(\ell_k)$ is even and for all $n \leq j' < j$, $\min_{n \leq k \leq j'} p(\ell_k)$ is odd. Note that, in this case, we must have $\min_{n \leq k \leq j} p(\ell_k) = p(\ell_j)$. In other words, a window closes when an even priority smaller than all other priorities in the window is encountered. The window opened at step n is said to *close immediately* if $p(\ell_n)$ is even.

If a window does not close within λ time units, we refer to it as a *bad* window: the window opened at step n is a bad window if there is some $j^* \geq n$ such that $\nu_{j^*}(\gamma) - \nu_n(\gamma) \geq \lambda$ and for all $j \geq n$, if $\nu_j(\gamma) - \nu_n(\gamma) < \lambda$, then $\min_{n \leq k \leq j} p(\ell_k)$ is odd.

Direct timed window objective. The direct window parity objective in graph games requires that every suffix of the play belongs to the good window objective. To adapt this objective to a dense-time setting, we must express that *at all times, we have a good window*. We require that this property holds not only at states which appear explicitly along plays, but also in the continuum between them (during the delay within a location). To this end, let us introduce a notation for suffixes of play.

Let $\pi = (\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)})(\ell_1, \nu_1) \dots \in \text{Plays}(\mathcal{G})$ be a play. For all $i \in \{1, 2\}$ and all $n \in \mathbb{N}$, write $m_n^{(i)} = (d_n^{(i)}, a_n^{(i)})$ and $d_n = \text{delay}(m_n^{(1)}, m_n^{(2)}) = \nu_{n+1}(\gamma) - \nu_n(\gamma)$. For any $n \in \mathbb{N}$ and $d \in [0, d_n]$, let $\pi_{n \rightarrow}^{+d}$ be the *delayed suffix* of π starting in position n delayed by d time units, defined as

$$\pi_{n \rightarrow}^{+d} = (\ell_n, \nu_n + d)((d_n^{(1)} - d, a_n^{(1)}), (d_n^{(2)} - d, a_n^{(2)}))(\ell_{n+1}, \nu_{n+1})(m_{n+1}^{(1)}, m_{n+1}^{(2)}) \dots$$

If $d = 0$, we write $\pi_{n \rightarrow}$ rather than $\pi_{n \rightarrow}^{+0}$.

Using the notations above, we define the *direct timed window (parity) objective* as the set

$$\text{DTW}(p, \lambda) = \{\pi \in \text{Plays}(\mathcal{G}) \mid \forall n \in \mathbb{N}, \forall d \in [0, d_n], \pi_{n \rightarrow}^{+d} \in \text{TW}(p, \lambda)\}.$$

The direct timed window objective is state-based: the timed good window objective is state-based and the delays d_n are encoded in states (measured by clock γ), thus all conditions in the definition of the direct timed window objective depend only the sequence of states of a play.

A good window for a delayed suffix $\pi_{n \rightarrow}^{+d}$ can be expressed using exclusively indices from the play π . In fact, $\pi_{n \rightarrow}^{+d} \in \text{TW}(p, \lambda)$ if and only if there is some $j \geq n$ such that $\min_{n \leq k \leq j} p(\ell_k)$ is even and $\nu_j(\gamma) - \nu_n(\gamma) - d < \lambda$. We use this characterization to avoid mixing indices from plays π and $\pi_{n \rightarrow}^{+d}$ in proofs.

Timed window objective. We define the *timed window (parity) objective* as a prefix-independent variant of the direct timed window objective. Formally, we let

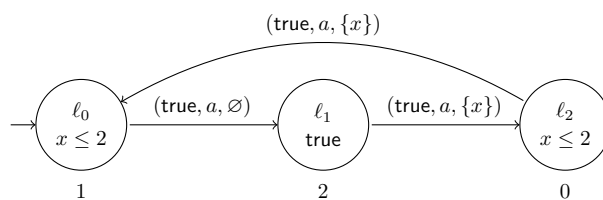
$$\text{TW}(p, \lambda) = \{\pi \in \text{Plays}(\mathcal{G}) \mid \exists n \in \mathbb{N}, \pi_{n \rightarrow} \in \text{DTW}(p, \lambda)\}.$$

The timed window objective requires the direct timed window objective to hold from some point on. This implies that the timed window objective is state-based.

3.2 Comparison with parity objectives

Both the direct and non-direct timed window objectives reinforce the parity objective with time bounds. It can easily be shown that satisfying the direct timed window objective implies satisfying a parity objective. Any odd priority seen along a play in $\text{DTW}(p, \lambda)$ is answered

■ **Figure 1** Timed automaton \mathcal{A} . Edges are labeled with triples guard-action-resets. Priorities are beneath locations. The initial state is denoted by an incoming arrow with no origin



within λ time units by a smaller even priority. Therefore, should any odd priority appear infinitely often, it is followed by a smaller even priority. As the set of priorities is finite, there must be some smaller even priority appearing infinitely often. This in turn implies that the parity objective is fulfilled. Using prefix-independence of the parity objective, we can also conclude that satisfying the non-direct timed window objective implies satisfaction of the parity objective.

However, in some cases, the timed window objectives may not hold even though the parity objective holds. For simplicity, we provide an example on a TA, rather than a TG. Consider the timed automaton \mathcal{A} depicted in Figure 1.

All time-divergent paths of \mathcal{A} satisfy the parity objective. We can classify time-divergent paths in two families: either ℓ_2 is visited infinitely often, or from some point on only delay transitions are taken in ℓ_1 . In the former case, the smallest priority seen infinitely often is 0 and in the latter case, it is 2.

However, there is a path π such that for all window sizes $\lambda \in \mathbb{N} \setminus \{0\}$, π violates the direct and non-direct timed window objectives. Initialize n to 1. This path can be described by the following loop: play action a in ℓ_0 with delay 0, followed by action a with delay n in ℓ_1 and action a in ℓ_2 with delay 0, increase n by 1 and repeat. The window opened in ℓ_0 only closes when location ℓ_2 is entered. At the n -th step of the loop, this window closes after n time units. As we let n increase to infinity, there is no window size λ such that this path satisfies the direct and non-direct timed window objectives for λ .

This example demonstrates the interest of reinforcing parity objectives with time bounds; we can enforce that there is a bounded delay between an odd priority and a smaller even priority in a path.

3.3 Properties of window objectives

We present several properties of the timed window objective. First, we show that we need only check good windows for non-delayed suffixes $\pi_{n \rightarrow}$. Once this property is proven, we move on to the inductive property of windows, which is the crux of the reduction in the next section. This inductive property states that when we close a window in time, all other windows opened in the meantime also close in time.

The definition of the direct timed window objective requires checking uncountably many windows. This can be reduced to a countable number of windows: those opened when entering states appearing along a play. Let us explain why no information is lost through such a restriction. We rely on a timeline-like visual representation given in Figure 2. Consider a window that does not close immediately and is opened in some state of the play delayed by d time units, of the form $(\ell_n, \nu_n + d)$ (depicted by the circle at the start of the bottom line of Figure 2). This implies that the priority of ℓ_n is odd, otherwise this window would close immediately. Assume the window opened at step n closes at step j (illustrated by the

middle line of the figure) in less than λ time units. As the priority of ℓ_n is odd, we must have $j \geq n + 1$ (i.e., the window opened at step n is still open as long as ℓ_n is not left). These lines cover the same locations, i.e., the set of locations appearing along the time-frame given by both the dotted and dashed lines coincide. Thus, the window opened d time units after step n closes in at most $\lambda - d$ time units, at the same time as the window opened at step n .

■ **Figure 2** A timeline representation of a play. Circles with labels indicate entry in a location. The dotted line underneath represents a window opened at step n and closed at step j and the dashed line underneath the window opened d time units after step n .



► **Lemma 2.** Let $\pi = (\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)}) \dots \in \text{Plays}(\mathcal{G})$ and $n \in \mathbb{N}$. Let d_n denote $\text{delay}(m_n^{(1)}, m_n^{(2)})$. Then $\pi_{n \rightarrow} \in \text{TGW}(p, \lambda)$ if and only if for all $d \in [0, d_n]$, $\pi_{n \rightarrow}^{+d} \in \text{TGW}(p, \lambda)$. Furthermore, $\pi \in \text{DTW}(p, \lambda)$ if and only if for all $n \in \mathbb{N}$, $\pi_{n \rightarrow} \in \text{TGW}(p, \lambda)$.

Proof. Assume for all $d \in [0, d_n]$, $\pi_{n \rightarrow}^{+d} \in \text{TGW}(p, \lambda)$ holds. Selecting $d = 0$ yields $\pi_{n \rightarrow} \in \text{TGW}(p, \lambda)$.

Conversely, assume that $\pi_{n \rightarrow} \in \text{TGW}(p, \lambda)$. Let $d \in [0, d_n]$. By definition of timed good window objectives, there is some $j \geq n$ such that $\nu_j(\gamma) - \nu_n(\gamma) < \lambda$ and $\min_{n \leq k \leq j} p(\ell_k)$ is even. The fact that $\pi_{n \rightarrow}^{+d} \in \text{TGW}(p, \lambda)$ follows immediately from the chain of inequalities $\nu_j(\gamma) - \nu_n(\gamma) - d \leq \nu_j(\gamma) - \nu_n(\gamma) < \lambda$.

The last claim of the lemma follows immediately from the first part of the lemma and the definition of direct timed window objectives. ◀

In turn-based games on graphs, window objectives exhibit an inductive property: when a window closes, all subsequently opened windows close (or were closed earlier) [14]. This is also the case for the timed variant. A window closes when an even priority smaller than all priorities seen in the window is encountered. This priority is also smaller than priorities in all windows opened in the meantime, therefore they must close at this point (if they are not yet closed). We state this property only for windows opened at steps along the run and neglect the continuum in between due to Lemma 2.

► **Lemma 3 (Inductive property).** Let $\pi = (\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)})(\ell_1, \nu_1) \dots \in \text{Plays}(\mathcal{G})$. Let $n \in \mathbb{N}$. Assume the window opened at step n closes at step j . Then, for all $n \leq i \leq j$, $\pi_{i \rightarrow} \in \text{TGW}(p, \lambda)$.

Proof. Fix $i \in \{n, \dots, j\}$. The sequence $(\nu_k(\gamma))_{k \in \mathbb{N}}$ is non-decreasing, which implies that $\nu_j(\gamma) - \nu_i(\gamma) \leq \nu_j(\gamma) - \nu_n(\gamma) < \lambda$. It remains to show that $\min_{i \leq k \leq j} p(\ell_k)$ is even. As the window opened at step n closes at step j , we have $\min_{n \leq k \leq j} p(\ell_k) = p(\ell_j)$ and $p(\ell_j)$ is even. We have the inequalities $p(\ell_j) \leq \min_{n \leq k \leq j} p(\ell_k) \leq \min_{i \leq k \leq j} p(\ell_k) \leq p(\ell_j)$; the first follows from above, the second because we take a minimum of a smaller set and the third by definition of minimum. Thus $\min_{i \leq k \leq j} p(\ell_k) = p(\ell_j)$ is even, ending the proof. ◀

It follows from this inductive property that it suffices to keep track of one window at a time when checking whether a play satisfies the (direct) timed window objective.

4 Reduction

We establish in this section that the realizability (resp. verification) problem for the direct/non-direct timed window parity objective can be reduced to the realizability (resp. verification) problem for safety/co-Büchi objectives on an expanded TG (resp. TA). Our reduction uses the same construction of an expanded TA for both the verification and realizability problems. A state of the expanded TA describes the status of a window, allowing the detection of bad windows. This section is divided in three parts.

Firstly, we describe how a TA can be expanded with window-related information. Then we show that time-divergent plays in a TG and its expansion can be related, by constructing two (non-bijective) mappings, in a manner such that a time-divergent play in the base TG satisfies the direct/non-direct timed window parity objective if and only if its related play in the expanded TG satisfies the safety/co-Büchi objective. These results developed for plays are (indirectly) applied to paths in order to show the correctness of the reduction for the verification problem. Thirdly, we establish that the mappings developed in the second part can be leveraged to translate strategies in TGs, and prove that the presented translations preserve winning strategies, proving correctness of the reduction for TGs.

For this section, we fix a TG $\mathcal{G} = (\mathcal{A}, \Sigma_1, \Sigma_2)$ with TA $\mathcal{A} = (L, \ell_{\text{init}}, C, \Sigma, I, E)$, a priority function p and a bound λ on the size of windows.

4.1 Encoding the objective in an automaton

To solve the verification and realizability problems for the direct/non-direct timed window objective, we rely on a reduction to a safety/co-Büchi objective in an expanded TA. The inductive property (Lemma 3) implies that it suffices to keep track of one window at a time when checking a window objective. Following this, we encode the status of a window in the TA.

A window can be summarized by two characteristics: the lowest priority within it and for how long it has been open. To keep track of the first trait, we encode the lowest priority seen in the current window in locations of the TA. An expanded location is a pair (ℓ, q) where $q \in \{0, \dots, d-1\}$; the number q represents the smallest priority in the window currently under consideration. We say a pair (ℓ, q) is an even (resp. odd) location if q is even (resp. odd). To measure how long a window is opened, we use an additional clock $z \notin C$ that does not appear in \mathcal{A} . This clock is reset whenever a new window opens or a bad window is detected.

The focus of the reduction is over time-divergent plays. Some time-convergent plays may violate a timed good window objective without ever seeing a bad window, e.g., when time does not progress up to the supplied window size. Along time-divergent plays however, the lack of a good window at any point equates to the presence of a bad window. We encode the (resp. direct) timed window objective as a co-Büchi (resp. safety) objective. Locations to avoid in both cases indicate bad windows and are additional expanded locations (ℓ, bad) , referred to as *bad locations*. We introduce two new actions β_1 and β_2 , one per player, for entering and exiting bad locations. While only one additional action for bad locations would suffice to describe the reduction for the verification problem, we use two no matter the considered problem: this enables us to use the same expanded TA construction for both the verification problem and realizability problem.

It remains to discuss how the initial location, edges and invariants of an expanded TA are defined. We discuss edges and invariants for each type of expanded location, starting with even locations, then odd locations and finally bad locations. Each rule we introduce hereafter is followed by an application on an example. We depict the TA of Figure 1 and the

reachable fragment of its expansion in Figure 3 and use these TAs for our example. For this explanation, we use the terminology of TAs (paths) rather than that of TGs (plays).

The initial location of an expanded TA encodes the window opened at the start of an initial path of the original TA. This window contains only a single priority, that is the priority of the initial location of the original TA. Thus, the initial location of the expanded TA is the expanded location $(\ell_{\text{init}}, p(\ell_{\text{init}}))$. In the case of our example, the initial location is ℓ_0 and the priority of ℓ_0 is 1, thus the initial location of the expanded TA is $(\ell_0, 1)$.

Even expanded locations encode windows that are closed and do not need to be monitored anymore. Therefore, the invariant of an even expanded location is unchanged from the invariant of the original location in the original TA. Similarly, we do not add any additional constraints on the edges leaving even expanded locations. Leaving an even expanded location means opening a new window: any edge leaving an even expanded location has an expanded location of the form $(\ell, p(\ell))$ as its target ($p(\ell)$ is the only priority occurring in the new window) and resets z to start measuring the size of the new window. For example, in Figure 3, the edge from $(\ell_2, 0)$ to $(\ell_0, 1)$ of the expanded TA is obtained this way from the edge from ℓ_2 to ℓ_0 in the original TA.

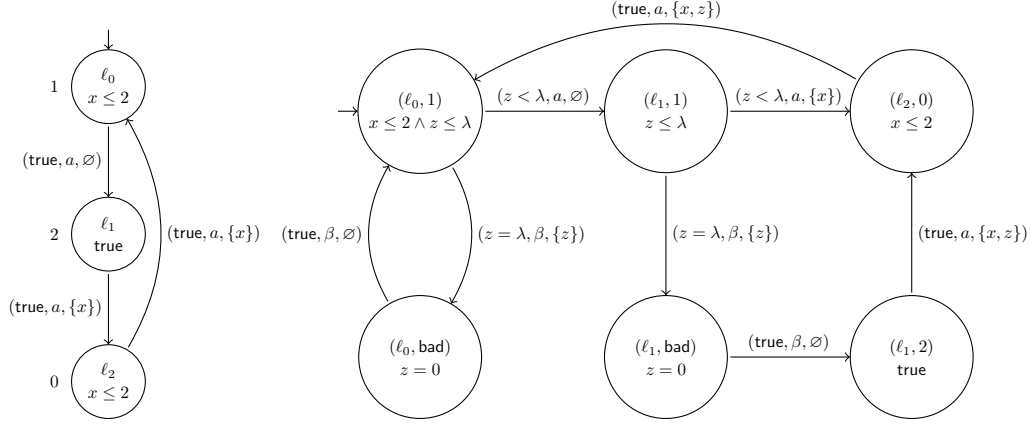
Odd expanded locations represent windows that are still open. The clock z measures how long a window has been opened. If z reaches λ in an odd expanded location, that equates to a bad window in the original TA. In this case, we force time-divergent paths of the expanded TA to visit a bad location. This is done in three steps. We strengthen the invariant of odd expanded locations to prevent z from exceeding λ . We also disable the edges that leave odd expanded locations and do not go to a bad location whenever $z = \lambda$ holds, by reinforcing the guards of such edges by $z < \lambda$. Finally, we include two edges to a bad location (one per additional action β_1 and β_2), that can only be used whenever there is a bad window, i.e., when $z = \lambda$. In the case of our example, if z reaches λ in $(\ell_0, 1)$, we redirect the path to location (ℓ_0, bad) , indicating a window has not closed in time in ℓ_0 . When z reaches λ in $(\ell_0, 1)$, no more non-zero delays are possible, the edge from $(\ell_0, 1)$ to $(\ell_1, 1)$ is disabled and only the edges to (ℓ_0, bad) are enabled.

When leaving an odd expanded location using an edge, assuming we do not go to a bad location, the smallest priority of the window has to be updated. The new smallest priority is the least of the smallest priority of the window prior to traversing the edge and the priority of the target location. In our example for instance, the edge from $(\ell_1, 1)$ to $(\ell_2, 0)$ is derived from the edge from ℓ_1 to ℓ_2 in the original TA. As the priority of ℓ_2 is 0 and is smaller than the current smallest priority of the window encoded by location $(\ell_1, 1)$, the smallest priority of the window is updated to $0 = \min\{1, p(\ell_2)\}$ when traversing the edge. Note that we do not reset z despite the encoded window closing upon entering $(\ell_2, 0)$: the value of z does not matter while in even locations, thus there is no need for a reset when closing the window.

A bad location (ℓ, bad) is entered whenever a bad window is detected while in location ℓ . Bad locations are equipped with the invariant $z = 0$ preventing the passage of time. In this way, for time-divergent paths, a new window is opened immediately after a bad window is detected. For each additional action β_1 and β_2 , we add an edge exiting the bad location. Edges leaving a bad location (ℓ, bad) target the expanded location $(\ell, p(\ell))$; we reopen a window in the location in which a bad window was detected. The clock z is not reset by these edges, as it was reset prior to entering the bad location and the invariant $z = 0$ prevents any non-zero delay in the bad location. For instance, the edges from (ℓ_1, bad) to $(\ell_1, 2)$ in our example represent that when reopening while in location ℓ_1 , the smallest priority of this window is $p(\ell_1) = 2$.

The expansion depends on the priority function p and the bound on the size of windows

■ **Figure 3** The TA of Figure 1 (left) and the reachable fragment of its expansion (right). We write β for actions β_1 and β_2 .



λ . Therefore, we write $\mathcal{A}(p, \lambda)$ for the expansion. The formal definition of $\mathcal{A}(p, \lambda)$ follows.

► **Definition 4.** Given a TA $\mathcal{A} = (L, \ell_{\text{init}}, C, \Sigma, I, E)$, the TA $\mathcal{A}(p, \lambda)$ is defined to be the TA $(L', \ell'_{\text{init}}, C', \Sigma', I', E')$ such that

- $L' = L \times (\{0, \dots, d-1\} \cup \{\text{bad}\})$;
- $\ell'_{\text{init}} = (\ell_{\text{init}}, p(\ell_{\text{init}}))$;
- $C' = C \cup \{z\}$ where $z \notin C$ is a new clock;
- $\Sigma' = \Sigma \cup \{\beta_1, \beta_2\}$ is an expanded set of actions with special actions $\beta_1, \beta_2 \notin \Sigma$ for bad locations;
- $I'(\ell, q) = I(\ell)$ for all $\ell \in L$ and even $q \in \{0, \dots, d-1\}$, $I'(\ell, q) = (I(\ell) \wedge z \leq \lambda)$ for all $\ell \in L$ and odd $q \in \{0, \dots, d-1\}$, and $I'(\ell, \text{bad}) = (z = 0)$ for all $\ell \in L$;
- the set of edges E' of $\mathcal{A}(p, \lambda)$ is the smallest set satisfying the following rules:

- if q is an even number and $(\ell, g, a, D, \ell') \in E$, then

$$((\ell, q), g, a, D \cup \{z\}, (\ell', p(\ell'))) \in E';$$
- if q is an odd number and $(\ell, g, a, D, \ell') \in E$, then

$$((\ell, q), (g \wedge z < \lambda), a, D, (\ell', \min\{q, p(\ell')\})) \in E';$$
- for all locations $\ell \in L$, odd q and $\beta \in \{\beta_1, \beta_2\}$,

$$((\ell, q), (z = \lambda), \beta, \{z\}, (\ell, \text{bad})) \in E' \text{ and } ((\ell, \text{bad}), \text{true}, \beta, \emptyset, (\ell, p(\ell))) \in E'.$$

For a TG $\mathcal{G} = (\mathcal{A}, \Sigma_1, \Sigma_2)$, we set $\mathcal{G}(p, \lambda) = (\mathcal{A}(p, \lambda), \Sigma_1 \cup \{\beta_1\}, \Sigma_2 \cup \{\beta_2\})$.

We write (ℓ, q, \bar{v}) for states of $\mathcal{T}(\mathcal{A}(p, \lambda))$ instead of $((\ell, q), \bar{v})$, for conciseness. The bar over the valuation is a visual indicator of the different domain.

We say that a play $(\ell_0, q_0, \bar{v}_0)(m_0^{(1)}, m_0^{(2)}) \dots$ of $\mathcal{G}(p, \lambda)$ is *well-initialized* if $q_0 = p(\ell_0)$ and $\bar{v}_0(z) = 0$. A well-initialized play can be seen as a play with a window opening at its start. Any initial play in $\mathcal{G}(p, \lambda)$ is well-initialized. Proving statements related to the direct timed window objective only for initial plays (rather than well-initialized plays) is too restrictive to effectively apply them to the timed window objective, as this objective deals with suffixes. We later define expansions of plays as specific well-initialized plays. We write $\text{Bad} = L \times \{\text{bad}\}$ for the set of bad locations.

4.2 Expanding and projecting plays

We prove that any play of \mathcal{G} has an expansion in $\mathcal{G}(p, \lambda)$, and conversely, any play in $\mathcal{G}(p, \lambda)$ projects to a play in \mathcal{G} . This is done by constructing an expansion mapping and a projection mapping, both of which are shown to behave well w.r.t. our objectives (Lemma 6).

► **Remark 5.** Note that we do not construct a bijection between the set of plays of \mathcal{G} and the set of plays of $\mathcal{G}(p, \lambda)$. This cannot be achieved naturally due to the additional information encoded in the expanded automaton, and notably the presence of bad locations. We illustrate this by showing there are some plays of $\mathcal{G}(p, \lambda)$ that are intuitively indistinguishable if seen as plays of \mathcal{G} .

Consider the initial location ℓ_{init} of \mathcal{G} , and assume that its priority is odd and its invariant is true. Consider the initial play $\bar{\pi}_1$ of $\mathcal{G}(p, \lambda)$ where the actions β_i are used by both players with a delay of λ at the start of the play and then only delay moves are taken in the reached bad location, i.e., $\bar{\pi}_1 = (\ell, p(\ell), \mathbf{0}^{C \cup \{z\}})((\lambda, \beta_1), (\lambda, \beta_2))((\ell, \text{bad}, \bar{\nu})((0, \perp), (0, \perp)))^\omega$, where $\bar{\nu}(x) = \lambda$ for all $x \in C$ and $\bar{\nu}(z) = 0$. As the actions β_i and z do not exist in \mathcal{G} , $\bar{\pi}_1$ cannot be discerned from the similar play $\bar{\pi}_2$ of $\mathcal{G}(p, \lambda)$ where instead of using the actions β_i , delay moves were used instead, i.e., $\bar{\pi}_2 = (\ell, p(\ell), \mathbf{0}^{C \cup \{z\}})((\lambda, \perp), (\lambda, \perp))((\ell, p(\ell), \bar{\nu}')((0, \perp), (0, \perp)))^\omega$ with $\bar{\nu}'(x) = \lambda$ for all $x \in C \cup \{z\}$.

This motivates using two mappings instead of a bijection to prove the correctness of our reduction.

Expansion mapping

The expansion mapping $\text{Ex}: \text{Plays}(\mathcal{G}) \rightarrow \text{Plays}(\mathcal{G}(p, \lambda))$ between plays of \mathcal{G} and of $\mathcal{G}(p, \lambda)$ is defined by an inductive construction. We construct expansions step by step. The rough idea is to use the same moves in the play of \mathcal{G} being expanded and in its expansion in $\mathcal{G}(p, \lambda)$, as long as these moves do not allow z to reach λ in an odd location of $\mathcal{G}(p, \lambda)$, i.e., as long as these moves do not make us see a bad window. In fact the construction addresses how to proceed if a move enabled in \mathcal{G} would allow z to reach λ in an odd location. If only one of the two players, say \mathcal{P}_i , suggests a move with a large enough delay for clock z to reach λ , then their adversary \mathcal{P}_{3-i} preempts them and it suffices to replace \mathcal{P}_i 's move by any valid move with a larger delay than \mathcal{P}_{3-i} 's. However, if both players suggest moves with too large a delay, the expanded play goes through a bad location (possibly multiple times) until enough time passes and one of the two players can use their move (with the remaining delay) in the expanded game.

Before presenting a formal construction of the expansion mapping, let us describe the structure of the inductive step of the construction. We number the different cases in the same order as they appear in the upcoming formal definition. Let $\pi = s_0(m_0^{(1)}, m_0^{(2)}) \dots s_{n+1}$ be a play of \mathcal{G} , and assume the expansion of its prefix $\pi_{|n} = s_0(m_0^{(1)}, m_0^{(2)}) \dots s_n$ has already been constructed. We assume inductively that the last states of $\pi_{|n}$ and its expansion $\text{Ex}(\pi_{|n})$ share the same location of the original TG and that their clock valuations agree over C . Furthermore, we also inductively assume the last state of $\text{Ex}(\pi_{|n})$ is not in a bad location. Write $\bar{s} = \text{last}(\text{Ex}(\pi_{|n})) = (\ell, q, \bar{\nu})$ and $s = \text{last}(\pi_{|n}) = (\ell, \nu)$. Denote by $d = \text{delay}(m_n^{(1)}, m_n^{(2)})$ the delay of the last pair of moves of the players in π .

1. If q is even, the same moves are available in s and \bar{s} ; the expansion can be extended using the pair of moves $(m_n^{(1)}, m_n^{(2)})$.
2. If q is odd, the invariant of the expanded location (ℓ, q) prevents z from exceeding λ . We distinguish cases depending on whether the delay d allows z to reach λ .

- a. If $\bar{\nu}(z) + d < \lambda$, then one of the players has offered a move enabled in \bar{s} : this move determines how to extend the play.
- b. Otherwise, $\bar{\nu}(z) + d \geq \lambda$. The construction makes the expansion go through location (ℓ, bad) . When (ℓ, bad) is exited, the path goes to $(\ell, p(\ell))$, the invariant of which depends on the parity of $p(\ell)$. We treat each case differently.
 - i. If $p(\ell)$ is even, then the invariant of $(\ell, p(\ell))$ matches that of ℓ . Once (ℓ, bad) is left, we reason similarly to case 1, using the moves with whatever delay remains.
 - ii. If $p(\ell)$ is odd, it may be required to go to a bad location more than once if the remaining delay after the first visit to the bad location exceeds λ . Once the remaining delay is strictly less than λ , we can operate as in case 2.a.

The formal construction of the expansion mapping follows. The inductive hypothesis in this construction of $\text{Ex} : \text{Plays}(\mathcal{G}) \rightarrow \text{Plays}(\mathcal{G}(p, \lambda))$ is the following: for all finite plays $\pi \in \text{Plays}_{f_{in}}$, using (ℓ, ν) to denote $\text{last}(\pi)$ and $(\ell', q, \bar{\nu})$ to denote $\text{last}(\text{Ex}(\pi))$, then $\ell' = \ell$, $q \neq \text{bad}$ and $\bar{\nu}|_C = \nu$. We proceed by induction on the number of moves along a play.

The base case consists of plays of \mathcal{G} with no moves, i.e., plays in which there is a single state. For any play (ℓ, ν) , we set $\text{Ex}((\ell, \nu))$ to be the play $(\ell, p(\ell), \bar{\nu})$ of $\mathcal{G}(p, \lambda)$ consisting of a single state, where $\bar{\nu}|_C = \nu$ and $\bar{\nu}(z) = 0$. The inductive hypothesis is verified: the states (ℓ, ν) and $(\ell, p(\ell), \bar{\nu})$ share the same location of \mathcal{A} , their clock valuations agree over C and $(\ell, p(\ell))$ is not a bad location.

Next we assume that expansions are defined for all plays with n moves. Fix $\pi = s_0(m_0^{(1)}, m_0^{(2)}) \dots s_{n+1}$ a play with $n + 1$ moves and assume the expansion of its prefix $\pi|_n = s_0(m_0^{(1)}, m_0^{(2)}) \dots s_n$ has already been constructed. Write $s = (\ell, \nu) = \text{last}(\pi|_n)$, $s' = (\ell', \nu') = \text{last}(\pi)$, $\bar{s} = (\ell, q, \bar{\nu}) = \text{last}(\text{Ex}(\pi|_n))$ and for $i \in \{1, 2\}$, $m_n^{(i)} = (d_n^{(i)}, a_n^{(i)})$. We assume w.l.o.g. that $s \xrightarrow{m_n^{(1)}} s'$ holds: the induction step can be done similarly by exchanging the roles of the players if $s \xrightarrow{m_n^{(2)}} s'$ does not hold. This assumption implies \mathcal{P}_1 is faster or as fast as \mathcal{P}_2 . If \mathcal{P}_2 were strictly faster, then $s \xrightarrow{m_n^{(2)}} s'$ would hold, in turn implying that $\nu'(\gamma) = \nu(\gamma) + d_n^{(2)}$ (γ cannot be reset). However, since $s \xrightarrow{m_n^{(1)}} s'$ holds, it follows that $\nu'(\gamma) = \nu(\gamma) + d_n^{(1)}$, contradicting the assumption that \mathcal{P}_2 is faster. In other words, we must have $d_n^{(1)} \leq d_n^{(2)}$. We separate the construction in multiple cases.

1. If q is even, the moves $m_n^{(1)}$ and $m_n^{(2)}$ are enabled in \bar{s} by construction. Indeed, ν and $\bar{\nu}$ agree over C , and we have $I(\ell) = I'((\ell, q))$ and for any outgoing edge $(\ell, g, a, D, \tilde{\ell})$ of ℓ in \mathcal{A} there is an edge $((\ell, q), g, a, D \cup \{z\}, (\tilde{\ell}, p(\tilde{\ell})))$ in $\mathcal{A}(p, \lambda)$ with the same guard and which resets z . We distinguish cases following whether a delay is taken or not.

If $m_n^{(1)}$ is a delay move, we set $\text{Ex}(\pi) = \text{Ex}(\pi|_n)(m_n^{(1)}, m_n^{(2)})(\ell, q, \bar{\nu} + d)$. This play is well-defined: q is even, thus ℓ and (ℓ, q) share the same invariant and support the same delay moves. The inductive hypothesis is verified: $\ell' = \ell$ because a transition labeled by a delay move was taken, $q \neq \text{bad}$ and $\nu' = \nu + d = \bar{\nu}|_C + d$.

Otherwise, $m_n^{(1)}$ is not a delay move and is associated with an edge of the TA. We set $\text{Ex}(\pi) = \text{Ex}(\pi|_n)(m_n^{(1)}, m_n^{(2)})(\ell', p(\ell'), \bar{\nu}')$ with $\bar{\nu}'|_C = \nu'$ and $\bar{\nu}'(z) = 0$. This is a well-defined play, owing to the edges recalled above. It is not difficult to verify that the inductive hypothesis is satisfied.

Note that z is reset in the second case. It may be the case that \mathcal{P}_2 is not responsible for the last transition in the expansion despite being responsible for the last transition of π . In other words, it is possible that both $s \xrightarrow{m_n^{(1)}} s'$ and $s \xrightarrow{m_n^{(2)}} s'$ hold, but that $\bar{s} \xrightarrow{m_n^{(2)}} \text{last}(\text{Ex}(\pi))$ does not hold. This occurs whenever the moves of both players share

the same target state in the base TG but one player uses a delay move and the other a move with an action. We choose to have \mathcal{P}_1 's move be responsible for the transition in the expansion in this case. This choice is for technical reasons related to blamelessness.

2. If q is odd, one or both of the moves $m_n^{(1)}$ or $m_n^{(2)}$ may not be enabled in \bar{s} due to the different invariant. Recall that $I'((\ell, q)) = (I(\ell) \wedge z \leq \lambda)$ and for any outgoing edge $(\ell, g, a, D, \tilde{\ell})$ of ℓ in \mathcal{A} there is an edge $((\ell, q), (g \wedge z < \lambda), a, D, (\tilde{\ell}, \min\{q, p(\tilde{\ell})\}))$ in $\mathcal{A}(p, \lambda)$. If $a \in \Sigma$, a move (t, a) is disabled in state \bar{s} if $\bar{\nu}(z) + t \geq \lambda$. We reason as follows, depending on whether a delay of d allows clock z to reach λ from the state \bar{s} .

- a. Assume $\bar{\nu}(z) + d < \lambda$. Then $m_n^{(1)}$ is enabled in \bar{s} (recall we assume \mathcal{P}_1 is responsible for the last transition of π). To ensure the \mathcal{P}_2 -selected move in the expansion is enabled in \bar{s} , we alter $m_n^{(2)}$ if its delay is too large: let $\tilde{m}_n^{(2)} = m_n^{(2)}$ if $\bar{\nu}(z) + d_n^{(2)} < \lambda$ and $\tilde{m}_n^{(2)} = (\lambda - \bar{\nu}(z), \beta_2)$ otherwise.

If $m_n^{(1)}$ is a delay move, define $\text{Ex}(\pi) = \text{Ex}(\pi|_n)(m_n^{(1)}, \tilde{m}_n^{(2)})(\ell, q, \bar{\nu} + d)$. This is a well-defined play: $\bar{\nu} + d \models I'((\ell, q))$ holds because $I'((\ell, q)) = (I(\ell) \wedge z \leq \lambda)$, ν and $\bar{\nu}$ coincide on C and the move $m_n^{(1)}$ is available in s , and $\bar{\nu}(z) + d < \lambda$. Otherwise, if $m_n^{(1)}$ is not a delay move, define $\text{Ex}(\pi) = \text{Ex}(\pi|_n)(m_n^{(1)}, \tilde{m}_n^{(2)})(\ell', \min\{q, p(\ell')\}, \bar{\nu}'$ where $\bar{\nu}'|_C = \nu'$ and $\bar{\nu}'(z) = \bar{\nu}(z) + d$. By definition of the edges recalled above, and because ν and $\bar{\nu}$ coincide on C and the move $m_n^{(1)}$ is available in s , we conclude that $\text{Ex}(\pi)$ is a well-defined play. In either case, the inductive hypothesis is satisfied.

- b. Otherwise, assume $\bar{\nu}(z) + d \geq \lambda$. In this case, a bad location appears along $\text{Ex}(\pi)$. Denote by $t = \lambda - \bar{\nu}(z)$ the time left before the current window becomes a bad window. For any non-negative real r , we write $\bar{\nu}_r = \text{reset}_{\{z\}}(\bar{\nu} + r)$ for the clock valuation obtained by shifting $\bar{\nu}$ by r time units and then resetting z , $b_r^{(i)}$ for the move (r, β_i) and $m_n^{(i)} - r$ for $(d_n^{(i)} - r, a_n^{(i)})$, i.e., the move $m_n^{(i)}$ with a delay shortened by r time units.

Recall, when (ℓ, bad) is left, by definition of edges of $\mathcal{A}(p, \lambda)$, the location $(\ell, p(\ell))$ is entered. Depending on the parity of $p(\ell)$, the invariant of $(\ell, p(\ell))$ is different. Thus, there are two cases to consider: $p(\ell)$ is even and $p(\ell)$ is odd.

- i. If $p(\ell)$ is even, we set $\text{Ex}(\pi)$ to be the play

$$\text{Ex}(\pi|_n)(b_t^{(1)}, b_t^{(2)})(\ell, \text{bad}, \bar{\nu}_t)(b_0^{(1)}, b_0^{(2)})(\ell, p(\ell), \bar{\nu}_t)(m_n^{(1)} - t, m_n^{(2)} - t)(\ell', p(\ell'), \bar{\nu}'), \quad (1)$$

where $\bar{\nu}'|_C = \nu'$ and $\bar{\nu}'(z) = 0$ if $a_n^{(1)} \in \Sigma_1$ and $\bar{\nu}'(z) = d - t$ otherwise (z is reset if \mathcal{P}_1 's action is not a delay). We obtain this expansion by first using actions β_i with the delay t to enter a bad location, then the actions β_i immediately again to exit the bad location and finally use the original moves of the players, but with an offset of t times units (t time units passed before entering the bad location). This expansion is a play of $\mathcal{G}(p, \lambda)$: the moves $b_t^{(1)}$ and $b_t^{(2)}$ are enabled in $(\ell, q, \bar{\nu})$ as $\bar{\nu} + t \models I'((\ell, q))$ (because $\nu + d \models I(\ell)$ as $m_n^{(1)}$ is enabled in (ℓ, ν) , $\bar{\nu}(z) + t = \lambda$ and $I'((\ell, q)) = (I(\ell) \wedge z \leq \lambda)$), and lead to the state $(\ell, \text{bad}, \bar{\nu}_t)$ (recall edges entering bad locations reset z). The moves $b_0^{(1)}$ and $b_0^{(2)}$ are enabled in $(\ell, \text{bad}, \bar{\nu}_t)$ due to the edges $((\ell, \text{bad}), \text{true}, \beta_i, \emptyset, (\ell, p(\ell)))$ for $i \in \{1, 2\}$ of $\mathcal{A}(p, \lambda)$. One can argue the moves $m_n^{(1)} - t$ and $m_n^{(2)} - t$ are enabled in $(\ell, p(\ell), \bar{\nu}_t)$ using the same arguments as in case 1. The inductive hypothesis is preserved in this case.

- ii. Whenever $p(\ell)$ is odd, the invariant of $(\ell, p(\ell))$ implies $z \leq \lambda$. Let μ denote the integral part of $\frac{\bar{\nu}(z) + d}{\lambda}$; μ represents the number of bad windows we detect with our construction during delay d . In a nutshell, we divide the single step in the original TG into $2\mu + 1$ steps in the expansion: we enter and exit the bad location

μ times and finally play the original move in the end with the remaining delay. It may be necessary to modify \mathcal{P}_2 's move due to the invariant of the odd location $(\ell, p(\ell))$ implying $z \leq \lambda$ and the guard of its outgoing edges labeled by actions in Σ implying $z < \lambda$. To this end, let $\tilde{m}_n^{(2)} = m_n^{(2)}$ if $d_n^{(2)} - t < \mu\lambda$ and $\tilde{m}_n^{(2)} = (\mu\lambda + t, \beta_2)$ otherwise. If $\mu = 1$, we get an expansion similar to the previous case. We set $\text{Ex}(\pi)$ to be

$$\text{Ex}(\pi|_n)(b_t^{(1)}, b_t^{(2)})(\ell, \text{bad}, \bar{\nu}_t)(b_0^{(1)}, b_0^{(2)})(\ell, p(\ell), \bar{\nu}_t)(m_n^{(1)} - t, \tilde{m}_n^{(2)} - t)(\ell', q', \bar{\nu}'),$$

where $q' = \min\{p(\ell), p(\ell')\}$, $\bar{\nu}'_C = \nu'$ and $\bar{\nu}'(z) = d - t$. That is indeed a play of $\mathcal{G}(p, \lambda)$: moves $b_r^{(i)}$ ($i \in \{1, 2\}$ and $r \in \{0, t\}$) are enabled for the same reasons as the previous case and the other moves are valid for the same reasons as in case 2.a: $d - t = d - \lambda + \bar{\nu}(z) < \lambda$ because $\mu = 1$. For $\mu \geq 2$, we define $\text{Ex}(\pi)$ as

$$\left. \begin{array}{l} \text{Ex}(\pi|_n)(b_t^{(1)}, b_t^{(2)})(\ell, \text{bad}, \bar{\nu}_t)(b_0^{(1)}, b_0^{(2)}) \\ (\ell, p(\ell), \bar{\nu}_t)(b_\lambda^{(1)}, b_\lambda^{(2)})(\ell, \text{bad}, \bar{\nu}_{t+\lambda})(b_0^{(1)}, b_0^{(2)}) \\ \vdots \\ (\ell, p(\ell), \bar{\nu}_{t+(\mu-2)\lambda})(b_\lambda^{(1)}, b_\lambda^{(2)})(\ell, \text{bad}, \bar{\nu}_{t+(\mu-1)\lambda})(b_0^{(1)}, b_0^{(2)}) \\ (\ell, p(\ell), \bar{\nu}_{t'}) (m_n^{(1)} - t', \tilde{m}_n^{(2)} - t')(\ell', q', \bar{\nu}'), \end{array} \right\} \mu - 1 \text{ steps}$$

where $q' = \min\{p(\ell), p(\ell')\}$, $t' = t + (\mu - 1)\lambda$ represents the time spent repeatedly entering and exiting the bad location, $\bar{\nu}'_C = \nu'$ and $\bar{\nu}'(z) = d - t'$. This is a well-defined play of $\mathcal{G}(p, \lambda)$ for the reasons argued above. Extending the expansion this way preserves the inductive hypothesis.

This construction generalizes to infinite plays. In some cases, an expansion of a finite play may contain more steps. However, this is only the case when a bad location appears in an expansion. If a finite play and its expansion share the same number of steps, we say that they are *coherent*. The expansion mapping preserves time-convergence and divergence for infinite paths: the sum of delays are identical in a play and its expansion.

The behavior of the expansion mapping w.r.t. suffixes and well-initialized plays is of interest for studying the connection between the non-direct timed window objective on the base TG and the co-Büchi objective on the expanded TG. Given a (finite or infinite) play $\pi = (\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)}) \dots$, a suffix of $\text{Ex}(\pi)$ is not necessarily the expansion of a suffix of π . However, any well-initialized suffix of $\text{Ex}(\pi)$ can be shown to be the expansion of a delayed suffix $\pi_{n \rightarrow}^{+d}$ for some $n \in \mathbb{N}$ and $d \in [0, \text{delay}(m_n^{(1)}, m_n^{(2)})]$. A well-initialized suffix of an expansion starts whenever an even location is left through an edge (case 1) or a bad location is left through an edge (case 2.b). In the former case, the suffix of the expansion is the expansion of a suffix $\pi_{n \rightarrow}$ by construction (the last state of the expansion, viewed as a play, is well-initialized). In the latter case, a delayed suffix may be required. This is observable with equation (1) (and is similar in other cases involving bad locations): the suffix $(\ell, p(\ell), \bar{\nu}_t)(m_n^{(1)} - t, m_n^{(2)} - t)(\ell', p(\ell'), \bar{\nu}')$ of the expansion under construction is the expansion of the shifted suffix $\pi_{n \rightarrow}^{+t}$ of the finite play under consideration and is well-initialized.

Projection mapping

The counterpart to the expansion mapping is the projection mapping $\text{Pr}: \text{Plays}(\mathcal{G}(p, \lambda)) \rightarrow \text{Plays}(\mathcal{G})$. The projection mapping removes window information in any play in $\mathcal{G}(p, \lambda)$ to obtain a play in \mathcal{G} . Any action β_1 or β_2 is replaced by the action \perp . Formally, we define the projection mapping over finite and infinite plays as follows.

For any (finite or infinite) play $\bar{\pi} = (\ell_0, q_0, \bar{\nu}_0)((d_0^{(1)}, a_0^{(1)}), (d_0^{(2)}, a_0^{(2)})) \dots \in \text{Plays}(\mathcal{G}(p, \lambda))$, we set $\text{Pr}(\bar{\pi})$ to be the sequence $\pi = (\ell_0, (\bar{\nu}_0)_C)(\tilde{m}_0^{(1)}, \tilde{m}_0^{(2)}) \dots$ where for all $i \in \{1, 2\}$ and all j , $\tilde{m}_j^{(i)} = (d_j^{(i)}, a_j^{(i)})$ if $a_j^{(i)} \notin \{\beta_1, \beta_2\}$ and $\tilde{m}_j^{(i)} = (d_j^{(i)}, \perp)$ otherwise. This sequence is indeed a well-defined play: any move (d, a) enabled in a state $\bar{s}_j = (\ell_j, q_j, \bar{\nu}_j)$ such that $a \notin \{\beta_1, \beta_2\}$ is enabled in $s_j = (\ell_j, (\bar{\nu}_j)_C)$. If the expanded location of \bar{s}_j is bad, the only such move is $(0, \perp)$. If it is not bad, guards of outgoing edges and invariants in $\mathcal{A}(p, \lambda)$ are either the same or strengthened from their counterpart in \mathcal{A} , i.e., if the constraints are verified in the expanded TA, they must be in the original one. Furthermore, as edges unrelated to bad states are derived from the edges of the original TA, this ensures that $s_j \in \delta(s_{j-1}, \tilde{m}_{j-1}^{(1)}, \tilde{m}_{j-1}^{(2)})$ for all $j > 0$ (where δ is the joint destination function).

The projection mapping preserves time-divergence. Unlike the expansion mapping, projecting a finite play does not alter the amount of moves. This mapping respects suffixes: for all finite plays $\bar{\pi}$ and (finite or infinite) plays $\bar{\pi}'$ of $\mathcal{G}(p, \lambda)$ such that $\bar{\pi}((d^{(1)}, a^{(1)}), (d^{(2)}, a^{(2)}))\bar{\pi}'$ is a well-defined play, we have $\text{Pr}(\bar{\pi}((d^{(1)}, a^{(1)}), (d^{(2)}, a^{(2)}))\bar{\pi}') = \text{Pr}(\bar{\pi})(\tilde{m}^{(1)}, \tilde{m}^{(2)})\text{Pr}(\bar{\pi}')$, where the move $\tilde{m}^{(i)}$ is $(d^{(i)}, a^{(i)})$ if $a^{(i)} \notin \{\beta_1, \beta_2\}$ and $\tilde{m}^{(i)} = (d^{(i)}, \perp)$ otherwise. We refer to this property as *suffix compatibility*.

Objective preservation

We now establish the main theorem of this section: a play of \mathcal{G} satisfies the (resp. direct) timed window objective if and only if its expansion satisfies the co-Büchi (resp. safety) objective over bad locations; and a play of $\mathcal{G}(p, \lambda)$ satisfies the co-Büchi (resp. safety) objective over bad locations if and only if its projection satisfies the (resp. direct) timed window objective.

► **Lemma 6.** *The following assertions hold. For all time-divergent plays $\pi \in \text{Plays}_\infty(\mathcal{G})$:*

- A.1.** $\pi \in \text{DTW}(p, \lambda)$ if and only if $\text{Ex}(\pi) \in \text{Safe}(\text{Bad})$;
- A.2.** $\pi \in \text{TW}(p, \lambda)$ if and only if $\text{Ex}(\pi) \in \text{coBüchi}(\text{Bad})$.

For all well-initialized time-divergent plays $\bar{\pi} \in \text{Plays}_\infty(\mathcal{G}(p, \lambda))$:

- B.1.** $\bar{\pi} \in \text{Safe}(\text{Bad})$ if and only if $\text{Pr}(\bar{\pi}) \in \text{DTW}(p, \lambda)$;
- B.2.** $\bar{\pi} \in \text{coBüchi}(\text{Bad})$ if and only if $\text{Pr}(\bar{\pi}) \in \text{TW}(p, \lambda)$.

The form of this result is due to the lack of a bijection between the sets of plays of a TG and its expansion (Remark 5). This lemma essentially follows from the construction of $\mathcal{A}(p, \lambda)$ and the definitions of the expansion and projection mappings.

Proof. We start with **Item A.1**. Fix a time-divergent play $\pi = (\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)}) \dots \in \text{Plays}_\infty(\mathcal{G})$ and write $\bar{\pi} = (\ell'_0, q_0, \bar{\nu}_0)(\tilde{m}_0^{(1)}, \tilde{m}_0^{(2)}) \dots$ for its expansion. Assume π satisfies the direct timed window objective. We establish that $\bar{\pi}$ is safe and all of its prefixes are coherent with their expansion using an inductive argument. Intuitively, for the first step, we show that if the window opened at step 0 closes at step n , then the n th location of $\bar{\pi}$ is even and is reached in less than λ time units.

Since $\pi \in \text{TGW}(p, \lambda)$, there is some n_0 such that $\nu_{n_0}(\gamma) - \nu_0(\gamma) < \lambda$, $\min_{0 \leq k < j} p(\ell_k)$ is odd for all $0 \leq j < n_0$ and $\min_{0 \leq k \leq n_0} p(\ell_k)$ is even. Since $\bar{\nu}_0(z) = 0$, $\bar{\pi}$ and π are coherent up to step n_0 as z does not reach λ (i.e., no bad locations can occur up to step n_0). By construction of the expansion mapping (see case 2 of the construction), $q_{n_0} = \min_{0 \leq k \leq n_0} p(\ell_k) = p(\ell_{n_0})$ and $p(\ell_{n_0})$ is even. From there, there are two possibilities: either location ℓ_{n_0} is never left (only delays are taken from there) or after some delay moves, location ℓ_{n_0} is exited. In the first case, the safety objective is trivially satisfied as the location $(\ell_{n_0}, p(\ell_{n_0}))$ is never left in the expansion and no bad locations were visited beforehand. Otherwise, some edge is traversed

(for the first time since step n_0) at some step j_0 . Then, $\bar{\nu}_{j_0+1}(z) = 0$, $q_{j_0+1} = p(\ell_{j_0+1})$ and no bad locations appear in the first j_0 steps of $\bar{\pi}$.

We can repeat the first argument in position $j_0 + 1$ because $\pi \in \text{DTW}(p, \lambda)$ and thus $\pi_{j_0+1 \rightarrow} \in \text{TGW}(p, \lambda)$. We conclude there is some $n_1 \geq j_0 + 1$ such that $q_n \neq \text{bad}$ for $j_0 + 1 \leq n \leq n_1$ and q_{n_1} is even. Once more, we separate cases following if ℓ_{n_1} is left through an edge. Iterating this argument shows that no bad locations appear along $\bar{\pi}$.

Conversely, assume π does not satisfy the direct window objective. Let j_0 be the smallest index j such that $\pi_{j \rightarrow} \notin \text{TGW}(p, \lambda)$ (Lemma 2). We can argue, using a similar inductive argument as above, that π and $\bar{\pi}$ are coherent up to step j_0 , no bad locations occur up to step j_0 , and that $q_{j_0} = p(\ell_{j_0})$ and $\bar{\nu}_{j_0}(z) = 0$. In the sequel, we argue that a bad location is entered using the fact that there is no good window at step j_0 .

The negation of $\text{TGW}(p, \lambda)$ yields that for all $j \geq j_0$, if $\nu_j(\gamma) - \nu_{j_0}(\gamma) < \lambda$, then $\min_{j_0 \leq k \leq j} p(\ell_k)$ is odd. There is some j such that $\nu_j(\gamma) - \nu_{j_0}(\gamma) \geq \lambda$ as π is assumed to be time-divergent. Write j_1 for the smallest such j . As j_1 is minimal, $\nu_{j_1-1}(\gamma) - \nu_{j_0}(\gamma) < \lambda$ holds. It follows from the above that q_{j_1-1} is odd as $q_{j_1-1} = \min_{j_0 \leq k \leq j_1} p(\ell_k)$ and that there are no resets of z between steps j_0 and $j_1 - 1$ (resets of z require an even location or a bad location), hence $\bar{\nu}_{j_1-1}(z) = \nu_{j_1-1}(\gamma) - \nu_{j_0}(\gamma)$. The delay $d_{j_1-1} = \text{delay}(m_{j_1-1}^{(1)}, m_{j_1-1}^{(2)})$ is such that $\bar{\nu}_{j_1-1}(z) + d_{j_1-1} = \nu_{j_1}(\gamma) - \nu_{j_0}(\gamma) \geq \lambda$; the definition of the expansion function redirects $\bar{\pi}$ to a bad location (case 2.b of the expansion definition). This shows that $\bar{\pi}$ does not satisfy $\text{Safe}(\text{Bad})$.

Let us move on to **Item A.2**. Fix a time-divergent play $\pi = (\ell_0, \nu_0)(m_0^{(1)}, m_0^{(2)}) \dots \in \text{Plays}_\infty(\mathcal{G})$ and write $\bar{\pi} = (\ell'_0, q_0, \bar{\nu}_0)(\bar{m}_0^{(1)}, \bar{m}_0^{(2)}) \dots$ for its expansion. Assume $\bar{\pi}$ does not satisfy the objective $\text{coBüchi}(\text{Bad})$, i.e., there are infinitely many occurrences of bad locations along $\bar{\pi}$. We establish that $\pi \notin \text{TW}(p, \lambda)$, using Item A.1.

It suffices to show that for infinitely many n , there is some d such that $\pi_{n \rightarrow}^{+d} \notin \text{DTW}(p, \lambda)$. Indeed, Lemma 2 and this last assertion imply that for infinitely many n , $\pi_{n \rightarrow} \notin \text{DTW}(p, \lambda)$, hence, there are infinitely many bad windows along π . In turn, this property ensures no suffix of π satisfies the direct timed window objective, i.e., $\pi \notin \text{TW}(p, \lambda)$.

Recall a well-initialized suffix of $\bar{\pi}$ is the expansion of some $\pi_{n \rightarrow}^{+d}$ and a well-initialized suffix always follows after a bad location. Bad locations are assumed to occur infinitely often along $\bar{\pi}$, therefore $\bar{\pi}$ has infinitely many well-initialized suffixes. It follows from $\bar{\pi} \in \text{coBüchi}(\text{Bad})$ that there are infinitely many well-initialized suffixes of $\bar{\pi}$ that violate $\text{Safe}(\text{Bad})$. Therefore, there are infinitely many n such that $\text{Ex}(\pi_{n \rightarrow}^{+d})$ does not satisfy $\text{Safe}(\text{Bad})$ for some d , because each step of π induces finitely many visits to a bad location in the expansion by construction (in other words, there are finitely many well-initialized suffixes of $\bar{\pi}$ per step in π). From Item A.1, there are infinitely many n such that $\pi_{n \rightarrow}^{+d}$ does not satisfy the direct window objective for some d . This ends this direction of the proof.

Conversely, assume $\bar{\pi}$ satisfies the objective $\text{coBüchi}(\text{Bad})$. If $\bar{\pi}$ is safe, then it follows from Item A.1 that $\pi \in \text{DTW}(p, \lambda) \subseteq \text{TW}(p, \lambda)$. If $\bar{\pi}$ is unsafe, there is a well-initialized suffix of $\bar{\pi}$ that is safe, i.e., the suffix of $\bar{\pi}$ starts after the last occurrence of a bad location. This suffix is of the form $\text{Ex}(\pi_{n \rightarrow}^{+d})$ for some n and d . By Item A.1, $\pi_{n \rightarrow}^{+d} \in \text{DTW}(p, \lambda)$, which implies that $\pi_{(n+1) \rightarrow} \in \text{DTW}(p, \lambda)$. Thus, we have $\pi \in \text{TW}(p, \lambda)$, which ends the proof of this item.

Let us proceed to **Item B.1**. Let $\bar{\pi} = (\ell_0, q_0, \bar{\nu}_0)(m_0^{(1)}, m_0^{(2)}) \dots \in \text{Plays}_\infty(\mathcal{G}(p, \lambda))$ be a time-divergent well-initialized play and write π for its projection. Assume $\bar{\pi}$ satisfies the objective $\text{Safe}(\text{Bad})$ and let us prove that $\pi \in \text{DTW}(p, \lambda)$. It suffices to establish that for all $n \in \mathbb{N}$, $\pi_{n \rightarrow} \in \text{TGW}(p, \lambda)$ (Lemma 2). First, we argue there is an even location along $\bar{\pi}$. Then we establish that when the first even location is entered in $\bar{\pi}$, the window opened at

step 0 in π closes. We conclude, using the inductive property of windows and an inductive argument, that π satisfies the direct window objective.

Safety and divergence of $\bar{\pi}$ ensure that an even location appears along $\bar{\pi}$. Assume that there are no even locations along $\bar{\pi}$. Then every location appearing in $\bar{\pi}$ must be odd by safety of $\bar{\pi}$. Thus z cannot be reset as it requires exiting an even location or entering a bad location. The invariant $z \leq \lambda$ of odd locations would prevent time-divergence of $\bar{\pi}$, which would be contradictory. Thus, there must be some even location along $\bar{\pi}$.

Let n_0 denote the smallest index n such that q_n is even. We establish that the window opened at step 0 in π closes at step n_0 . It must hold that $\bar{v}_{n_0}(z) = \bar{v}_{n_0}(\gamma) - \bar{v}_0(\gamma) < \lambda$, as the outgoing edges of odd locations that do not target bad locations have guards implying $z < \lambda$ and do not reset z . Furthermore, q_n is odd for all $n < n_0$ and $q_{n_0} = \min_{0 \leq k \leq n_0} p(\ell_k)$, by definition of the edge relation of $\mathcal{A}(p, \lambda)$ and since $q_0 = p(\ell_0)$. This proves that the window opened at step 0 closes at step n_0 in less than λ time units. It follows from the inductive property of windows (Lemma 3) that for all $0 \leq n \leq n_0$, $\pi_{n \rightarrow} \in \text{TW}(\lambda, p)$.

As q_n is odd for $n < n_0$, we have $p(\ell_{n_0}) = q_{n_0}$. There are two possibilities for $\bar{\pi}$: either the even location $(\ell_{n_0}, p(\ell_{n_0}))$ is never left or there is some j_0 such that at step j_0 , the expanded location is exited through an edge (via the pair of moves $(m_{j_0}^{(1)}, m_{j_0}^{(2)})$). In the first case, only delays are taken in the location ℓ_{n_0} in π , the priority of which is even, yielding $\pi_{n \rightarrow} \in \text{TW}(p, \lambda)$ for all $n \geq n_0$ (all windows after step n_0 close immediately), and combining this with the previous paragraph implies $\pi \in \text{DTW}(p, \lambda)$. In the latter case, we have $\pi_{n \rightarrow} \in \text{TW}(p, \lambda)$ for $n_0 \leq n \leq j_0$ (similarly to the former case, only delays are taken in ℓ_{n_0} in π up to step j_0), and $\bar{v}_{j_0+1}(z) = 0$ and $q_{j_0+1} = p(\ell_{j_0+1})$ as edges leaving even locations reset z and lead to locations of the form $(\ell', p(\ell'))$. Repeating the previous arguments from position $j_0 + 1$ ($\bar{\pi}_{j_0+1 \rightarrow}$ is a time-divergent well-initialized suffix of $\bar{\pi}$), one can find some n_1 such that $q_{n_1} = p(\ell_{n_1})$ is even and for all $j_0 + 1 \leq n \leq n_1$, $\pi_{n \rightarrow} \in \text{TW}(p, \lambda)$. Once more, we split in cases following whether any edge is traversed in location $(\ell_{n_1}, p(\ell_{n_1}))$. It follows from an induction that $\pi \in \text{DTW}(p, \lambda)$.

Assume now that $\bar{\pi}$ does not satisfy the safety objective. There is some smallest n such that $q_n = \text{bad}$. Let j_0 be 0 if z was never reset before position n or the greatest $j < n$ such that the pair of moves $(m_{j-1}^{(1)}, m_{j-1}^{(2)})$ induces a reset of z otherwise. We have $q_{j_0} = p(\ell_{j_0})$ and $\bar{v}_{j_0}(z) = 0$: if $j_0 = 0$, this is due to $\bar{\pi}$ being well-initialized and if $j_0 > 0$, this follows from n being the smallest index of a bad location and from the definition of edges in $\mathcal{A}(p, \lambda)$; edges that reset z have as their target expanded locations of the form $(\ell, p(\ell))$ or (ℓ, bad) . We argue that $\pi_{j_0 \rightarrow}$ does not satisfy the timed good window objective. There cannot be any even locations between positions j_0 and n ; there are no edges to bad locations in even locations and edges leaving even locations reset z . Thus, for all $j_0 \leq j < n$, $q_j = \min_{j_0 \leq k \leq j} p(\ell_k)$ and q_j is odd. Bad locations can only be entered when z reaches λ in an odd location: the window opened at step j_0 does not close in time. Therefore, $\pi \notin \text{DTW}(p, \lambda)$.

For **Item B.2**, we use compatibility of the projection mapping with suffixes. Fix a divergent well-initialized play $\bar{\pi} \in \text{Plays}_\infty(\mathcal{G}(p, \lambda))$ and let π be its projection. Assume $\bar{\pi}$ satisfies the objective $\text{coBüchi}(\text{Bad})$. If $\bar{\pi}$ is safe, we have the property by **Item B.1**. Assume some bad location appears along $\bar{\pi}$. As the co-Büchi objective is satisfied, this location is left and the suffix following this exit is well-initialized. As there are finitely many bad locations along $\bar{\pi}$, it follows $\bar{\pi}$ has a well-initialized suffix satisfying $\text{Safe}(\text{Bad})$. From the compatibility of projections with suffixes and **Item B.1**, π has a suffix satisfying the direct timed window objective, hence π satisfies the timed window objective.

Conversely, assume $\bar{\pi}$ does not satisfy $\text{coBüchi}(\text{Bad})$, i.e., there are infinitely many occurrences of bad locations along $\bar{\pi}$. Divergence ensures any bad location is eventually left

through an edge, yielding a well-initialized suffix. From Item B.1 and suffix compatibility of the projection mapping, it follows that π has infinitely many suffixes that do not satisfy the direct timed window objective. This implies that π does not satisfy the timed window objective, ending the proof. \blacktriangleleft

This lemma completely disregards plays that are not well-initialized. This is not an issue, as any play starting in the initial state of the TG $\mathcal{G}(p, \lambda)$ is well-initialized. Indeed, if ℓ_0 is the initial location of \mathcal{A} , then $(\ell_0, p(\ell_0))$ is the initial location of $\mathcal{A}(p, \lambda)$, and thus $(\ell_0, p(\ell_0), \mathbf{0}^{C \cup \{z\}})$ is the initial state of $\mathcal{T}(\mathcal{A}(p, \lambda))$. Any initial play of \mathcal{G} expands to an initial play of $\mathcal{G}(p, \lambda)$ and any initial play of $\mathcal{G}(p, \lambda)$ projects to an initial play of \mathcal{G} .

The previous result can be leveraged to prove that the verification problem for the (resp. direct) timed window objective on \mathcal{A} can be reduced to the verification problem for the co-Büchi (resp. safety) objective on $\mathcal{A}(p, \lambda)$.

► Theorem 7. *Let $\mathcal{A} = (L, \ell_{\text{init}}, C, \Sigma, I, E)$ be a TA, p a priority function and $\lambda \in \mathbb{N} \setminus \{0\}$. All time-divergent paths of \mathcal{A} satisfy the (resp. direct) timed window objective if and only if all time-divergent paths of $\mathcal{A}(p, \lambda)$ satisfy the co-Büchi (resp. safety) objective over bad locations.*

Proof. We show that if there is a time divergent path of \mathcal{A} that violates the (resp. direct) timed window objective, then there is a time-divergent path of $\mathcal{A}(p, \lambda)$ that violates the co-Büchi (resp. safety) objective for bad locations. The other direction is proven using similar arguments and the projection mapping rather than the expansion mapping.

Assume there is a time-divergent initial path $\pi = s_0 \xrightarrow{d_0, a_0} s_1 \dots$ of \mathcal{A} that does not satisfy the (resp. direct) timed window objective. Consider the TG $\mathcal{G} = (\mathcal{A}, \emptyset, \Sigma)$. The sequence $\pi' = s_0((d_0, \perp), (d_0, a_0))s_1 \dots$ is a time-divergent initial play of \mathcal{G} as π is a time-divergent initial path of \mathcal{A} . Furthermore, π' does not satisfy the (resp. direct) timed window objective as it shares the same sequence of states as π . By Lemma 6, the time-divergent play $\text{Ex}(\pi')$ of $\mathcal{G}(p, \lambda)$ does not satisfy the co-Büchi (resp. safety) objective over bad locations. There is some path of the TA $\mathcal{A}(p, \lambda)$ that shares the same sequence of states as $\text{Ex}(\pi')$. This path is time-divergent and does not satisfy the co-Büchi (resp. safety) objective over bad locations, ending the proof. \blacktriangleleft

4.3 Translating strategies

In this section, we present how strategies can be translated from the base game to the expanded game and vice-versa, using the expansion and projection mappings. We restrict our attention to move-independent strategies, as this subclass of strategies suffices for state-based objectives [20].

We open the section with a binary classification of time-convergent plays of the expanded TG, useful to prove how our translations affect blamelessness of outcomes. Then we proceed to our translations. We define how a strategy of $\mathcal{G}(p, \lambda)$ can be translated to a strategy of \mathcal{G} using the expansion mapping. Then we define a translation of strategies of \mathcal{G} to strategies of $\mathcal{G}(p, \lambda)$, using the projection mapping. Each translation definition is accompanied by a technical result that establishes a connection between outcomes of a translated strategy and outcomes of the original strategy, through the projection or expansion mapping. It follows from these technical results that the translation of a winning strategy in one game is a winning strategy in the other.

Recall that given a winning strategy of \mathcal{P}_1 , all of its time-convergent outcomes are \mathcal{P}_1 -blameless. Therefore, when translating a winning strategy from one game to the other,

we must argue that all time-convergent outcomes of the obtained strategy are \mathcal{P}_1 -blameless. To argue how our translations preserve this property, let us introduce a binary classification of time-convergent plays of $\mathcal{G}(p, \lambda)$. We argue that any time-convergent play of the expanded TG either remains in a bad location from some point on or visits finitely many bad locations. Whenever a bad location is left, a subsequent visit to a bad location requires at least λ time units to elapse. It is thus impossible to visit bad locations infinitely often without either remaining in a bad location from some point onward or having time diverge. We now formalize this property and present its proof.

► **Proposition 8.** *Let $\bar{\pi} = (\ell_0, q_0, \bar{v}_0)(m_0^{(1)}, m_0^{(2)}) \cdots$ be an infinite play of $\mathcal{G}(p, \lambda)$. If $\bar{\pi}$ is time-convergent, then $\bar{\pi} \in \text{coBüchi}(L' \setminus \text{Bad}) \uplus \text{coBüchi}(\text{Bad})$.*

Proof. Assume towards a contradiction that neither $\bar{\pi} \in \text{coBüchi}(L' \setminus \text{Bad})$ nor $\bar{\pi} \in \text{coBüchi}(\text{Bad})$ hold. It follows that there are infinitely many q_n such that $q_n = \text{bad}$ and infinitely many j such that $q_j \neq \text{bad}$. Consider some n such that $q_n = \text{bad}$. There is some $j > n$ such that $q_j \neq \text{bad}$ and some $n' > j$ such that $q_{n'} = \text{bad}$. We argue that at least λ time units pass between steps n and n' , i.e., $\bar{v}_{n'}(\gamma) - \bar{v}_n(\gamma) \geq \lambda$. Once this is established, iterating this argument establishes divergence of $\bar{\pi}$, reaching a contradiction. The invariant of (ℓ_n, bad) ensures that $\bar{v}_n(z) = 0$. The bad location (ℓ_n, q_n) is exited at some stage to reach (ℓ_j, q_j) and the guards of edges to the location $(\ell_{n'}, q_{n'})$ are $z = \lambda$. It follows that at least λ time units must elapse between steps n and n' . ◀

We now move on to our translations. We translate \mathcal{P}_1 -strategies of the expanded TG to \mathcal{P}_1 -strategies of the original TG using the expansion mapping and by replacing any occurrences of β_1 by \perp . The fact that translating a winning strategy of the expanded TG this way yields a winning strategy of the original TG is not straightforward: when we translate a strategy $\bar{\sigma}$ of $\mathcal{G}(p, \lambda)$ to a strategy σ of \mathcal{G} , the expansion of an outcome π of σ may not be consistent with $\bar{\sigma}$, preventing the direct use of Lemma 6. Indeed the definition of the expansion mapping may impose moves (d, β_1) in $\text{Ex}(\pi)$ where $\bar{\sigma}$ would suggest (d, \perp) . However, we can mitigate this issue by constructing another play $\bar{\pi}$ in parallel that is consistent with $\bar{\sigma}$ and shares the same sequence of states as $\text{Ex}(\pi)$. We leverage the non-deterministic behavior of tie-breaking and move-independence of $\bar{\sigma}$ to ensure consistency of $\bar{\pi}$ with $\bar{\sigma}$, by changing the moves of \mathcal{P}_2 on $\bar{\pi}$ comparatively to $\text{Ex}(\pi)$. We also prove that if $\bar{\pi}$ is \mathcal{P}_1 -blameless and time-convergent, then π also is \mathcal{P}_1 -blameless.

► **Lemma 9.** *Let $\bar{\sigma}$ be a move-independent strategy of \mathcal{P}_1 in $\mathcal{G}(p, \lambda)$. Let σ be the \mathcal{P}_1 -strategy in \mathcal{G} defined by*

$$\sigma(\pi) = \begin{cases} \bar{\sigma}(\text{Ex}(\pi)) & \text{if } \bar{\sigma}(\text{Ex}(\pi)) \notin \mathbb{R}_{\geq 0} \times \{\beta_1\} \\ (d, \perp) & \text{if } \bar{\sigma}(\text{Ex}(\pi)) = (d, \beta_1) \end{cases}$$

for all finite plays $\pi \in \text{Plays}_{\text{fin}}(\mathcal{G})$. For all $\pi \in \text{Outcome}_1(\sigma)$, there is a play $\bar{\pi} \in \text{Outcome}_1(\bar{\sigma})$ such that $\bar{\pi}$ shares the same sequence of states as $\text{Ex}(\pi)$ and such that if $\bar{\pi}$ is time-convergent and \mathcal{P}_1 -blameless, then π is \mathcal{P}_1 -blameless.

Proof. Fix an infinite play $\pi \in \text{Outcome}_1(\sigma)$. We construct $\bar{\pi}$ inductively: at step $n \in \mathbb{N}$ of the construction, we assume that we have constructed a finite play $\bar{\pi}_n \in \text{Plays}_{\text{fin}}(\mathcal{G}(p, \lambda))$ such that $\bar{\pi}_n$ shares the same sequence of states as $\text{Ex}(\pi|_n)$ and $\bar{\pi}_n$ is consistent with $\bar{\sigma}$.

Initially, we set $\bar{\pi}_0 = \text{Ex}(\pi|_0)$. This is a play consisting of only one state and with no moves. It shares the same sequence of states as $\text{Ex}(\pi|_0)$ by construction and is consistent with $\bar{\sigma}$.

Now assume by induction that we have constructed a finite play $\bar{\pi}_n$ that shares the same sequence of states as $\text{Ex}(\pi_{|n})$ and is consistent with $\bar{\sigma}$. Let $(d^{(1)}, a^{(1)}) = \sigma(\pi_{|n})$. Let $m^{(2)} = (d^{(2)}, a^{(2)})$ be a move of \mathcal{P}_2 and s be a state of $\mathcal{T}(\mathcal{A})$ such that $\pi_{|n+1} = \pi_{|n}(\sigma(\pi_{|n}), m^{(2)})s$. Write $\text{last}(\bar{\pi}_n) = (\ell, q, \bar{\nu})$. We construct $\bar{\pi}_{n+1}$ so that it shares the same sequence of moves as $\text{Ex}(\pi_{|n+1})$ and so it is consistent with $\bar{\sigma}$.

Recall that an expansion never ends in a bad location. The proof is divided in different cases. Case 1 is when $\text{last}(\bar{\pi}_n)$ is in an even location. Case 2 is when $\text{last}(\bar{\pi}_n)$ is in an odd location. We further split case 2 in four sub-cases:

- 2.a $\bar{\nu}(z) + d^{(1)} < \lambda$;
- 2.b $\bar{\nu}(z) + d^{(2)} < \lambda$ and $\bar{\nu}(z) + d^{(1)} = \lambda$;
- 2.c $d^{(2)} \geq d^{(1)} = \lambda - \bar{\nu}(z)$ and \mathcal{P}_1 is responsible for the last transition;
- 2.d $d^{(2)} \geq d^{(1)} = \lambda - \bar{\nu}(z)$ and \mathcal{P}_1 is not responsible for the last transition (this implies $d^{(2)} = d^{(1)}$).

These four sub-cases are disjoint. We show that they cover all possibilities before moving on to the remainder of the construction. Assume $\text{last}(\bar{\pi}_n)$ is in an odd location. The inequality $\bar{\nu}(z) + d^{(1)} \leq \lambda$ holds due to $d^{(1)}$ being the delay of the move $\sigma(\pi_{|n})$ and this delay being the same as the delay of $\bar{\sigma}(\text{Ex}(\pi_{|n}))$ (this follows from the definition of σ) and because $\text{last}(\text{Ex}(\pi_{|n})) = \text{last}(\bar{\pi}_n)$ is an odd location with an invariant implying $z \leq \lambda$. We now describe how the construction proceeds in each case.

Case 1. Assume $\text{last}(\bar{\pi}_n) = \text{last}(\text{Ex}(\pi_{|n}))$ is in an even location. Then the same moves are enabled in $\text{last}(\bar{\pi}_n)$ and $\text{last}(\pi_{|n})$. We have $\text{Ex}(\pi_{|n+1}) = \text{Ex}(\pi_{|n})(\sigma(\pi_{|n}), m^{(2)})\bar{s}$ for some state \bar{s} of $\mathcal{T}(\mathcal{A}(p, \lambda))$ given by the definition of Ex . We let $\bar{\pi}_{n+1}$ be $\bar{\pi}_n(\bar{\sigma}(\bar{\pi}_n), m^{(2)})\bar{s}$. This is a well-defined play: $\text{Ex}(\pi_{|n})(\sigma(\pi_{|n}), m^{(2)})\bar{s}$ being a play implies $\bar{s} \in \delta(\text{last}(\text{Ex}(\pi_{|n})), \sigma(\pi_{|n}), m^{(2)})$ and $\bar{\pi}_n$ and $\text{Ex}(\pi_{|n})$ share the same sequence of states, hence $\text{last}(\bar{\pi}_n) = \text{last}(\text{Ex}(\pi_{|n}))$. Furthermore, by move-independence of $\bar{\sigma}$ and definition of σ , $\sigma(\pi_{|n}) = \bar{\sigma}(\text{Ex}(\pi_{|n})) = \bar{\sigma}(\bar{\pi}_n)$. The play $\bar{\pi}_{n+1}$ is consistent with $\bar{\sigma}$ by construction.

Case 2. Now assume that $\text{last}(\bar{\pi}_n) = (\ell, q, \bar{\nu})$ is in an odd location. Recall that we must have $\bar{\nu}(z) + d^{(1)} \leq \lambda$.

Sub-case 2.a. Assume $\bar{\nu}(z) + d^{(1)} < \lambda$. Then $\text{Ex}(\pi_{|n+1}) = \text{Ex}(\pi_{|n})(\sigma(\pi_{|n}), \tilde{m}^{(2)})\bar{s}$ for some state \bar{s} of $\mathcal{T}(\mathcal{A}(p, \lambda))$ given by the definition of the expansion mapping and where $\tilde{m}^{(2)} = (\lambda - \bar{\nu}(z), \beta_2)$ if $\bar{\nu}(z) + d^{(2)} \geq \lambda$ and $\tilde{m}^{(2)} = m^{(2)}$ otherwise. We define $\bar{\pi}_{n+1}$ to be the play $\bar{\pi}_n(\bar{\sigma}(\bar{\pi}_n), \tilde{m}^{(2)})\bar{s}$, which is a well-defined play sharing the sequence of states of $\text{Ex}(\pi_{|n+1})$ and consistent with $\bar{\sigma}$ for the same reasons as case 1.

Sub-case 2.b. Assume $\bar{\nu}(z) + d^{(2)} < \lambda$ and $\bar{\nu}(z) + d^{(1)} = \lambda$. Then the move of \mathcal{P}_1 is changed in the expansion to $(d^{(1)}, \beta_1)$: we have $\text{Ex}(\pi_{|n+1}) = \text{Ex}(\pi_{|n})((d^{(1)}, \beta_1), m^{(2)})\bar{s}$ for some state \bar{s} of $\mathcal{T}(\mathcal{A}(p, \lambda))$. It follows from \mathcal{P}_2 preempting \mathcal{P}_1 that $\text{last}(\text{Ex}(\pi_{|n})) \xrightarrow{m^{(2)}} \bar{s}$. By move-independence of $\bar{\sigma}$ and definition of σ , the delay of the moves $\sigma(\pi_{|n})$ and $\bar{\sigma}(\text{Ex}(\pi_{|n})) = \bar{\sigma}(\bar{\pi}_n)$ match. Let $\bar{\pi}_{n+1}$ be $\bar{\pi}_n(\bar{\sigma}(\bar{\pi}_n), m^{(2)})\bar{s}$. Thus $\bar{\pi}_{n+1}$ is a well-defined play sharing the same sequence of states as $\text{Ex}(\pi_{|n+1})$ and consistent with $\bar{\sigma}$.

Sub-cases 2.c. and 2.d. For the two remaining cases, assume that $\bar{\nu}(z) + d^{(1)} = \lambda$. Thus, $a^{(1)} = \perp$: the only \mathcal{P}_1 actions available in state $(\ell, q, \bar{\nu} + d^{(1)})$ are β_1 and \perp as $I'((\ell, q))$ implies $z \leq \lambda$ and edges leaving (ℓ, q) with actions other than β_1 and β_2 have guards requiring $z < \lambda$. We write $\bar{\nu}_{d^{(1)}} = \text{reset}_{\{z\}}(\bar{\nu} + d^{(1)})$ and for $i \in \{1, 2\}$ and $t \geq 0$, we write $b_t^{(i)} = (t, \beta_i)$ in the following.

Sub-case 2.c. If $d^{(2)} \geq d^{(1)}$ and \mathcal{P}_1 is responsible for the last transition of $\pi_{|n+1}$, then $\text{Ex}(\pi_{|n+1})$ is of the form

$$\text{Ex}(\pi_{|n})(b_{d^{(1)}}^{(1)}, b_{d^{(1)}}^{(2)})(\ell, \text{bad}, \bar{\nu}_{d^{(1)}})(b_0^{(1)}, b_0^{(2)})(\ell, p(\ell), \bar{\nu}_{d^{(1)}})((0, \perp), \tilde{m}^{(2)})(\ell, p(\ell), \bar{\nu}_{d^{(1)}})$$

for some $\tilde{m}^{(2)}$ given by the expansion definition. We construct $\bar{\pi}_{n+1}$ in three steps. The sequence $\bar{\pi}_n^+ = \bar{\pi}_n(\bar{\sigma}(\bar{\pi}_n), b_{d^{(1)}}^{(2)})(\ell, \text{bad}, \bar{\nu}_{d^{(1)}})$ is a well-defined play: both appended moves share the same delay because the delay $d^{(1)}$ of the move $\sigma(\pi|_n)$ is that of $\bar{\sigma}(\text{Ex}(\pi|_n)) = \bar{\sigma}(\bar{\pi}_n)$ by definition of σ and move-independence of $\bar{\sigma}$, and $\text{last}(\bar{\pi}_n) \xrightarrow{b_{d^{(1)}}^{(1)}} (\ell, \text{bad}, \bar{\nu}_{d^{(1)}})$ holds. Its extension $\bar{\pi}_n^{++} = \bar{\pi}_n^+(\bar{\sigma}(\bar{\pi}_n^+), b_0^{(2)})(\ell, p(\ell), \bar{\nu}_{d^{(1)}})$ is also a well-defined play as $\bar{\sigma}(\bar{\pi}_n^+)$ must have a delay of zero due to the invariant of bad locations enforcing $z = 0$. We define $\bar{\pi}_{n+1}$ to be the play $\bar{\pi}_n^{++}(\bar{\sigma}(\bar{\pi}_n^{++}), (0, \perp))(\ell, p(\ell), \bar{\nu}_{d^{(1)}})$. The sequence $\bar{\pi}_{n+1}$ is a play: the move $(0, \perp)$ is available in any state and performing it does not change the state, and it cannot be outsped. By construction, $\bar{\pi}_{n+1}$ shares the same sequence of states as $\text{Ex}(\pi|_{n+1})$ and is consistent with $\bar{\sigma}$.

Sub-case 2.d. Assume now that $d^{(2)} = d^{(1)}$ and that \mathcal{P}_1 is not responsible for the last transition. Then $\text{Ex}(\pi|_{n+1})$ is of the form

$$\text{Ex}(\pi|_n)(b_{d^{(1)}}^{(1)}, b_{d^{(1)}}^{(2)})(\ell, \text{bad}, \bar{\nu}_{d^{(1)}})(b_0^{(1)}, b_0^{(2)})(\ell, p(\ell), \bar{\nu}_{d^{(1)}})((0, \perp), (0, a^{(2)}))\bar{s}$$

for some state \bar{s} of $\mathcal{T}(\mathcal{A}(p, \lambda))$. Like in case 2.c., we extend $\bar{\pi}_n$ using the same states and changing the moves along the above to ensure consistency with $\bar{\sigma}$. Let $\bar{\pi}_n^+$ and $\bar{\pi}_n^{++}$ be defined identically to case 2.c. We define $\bar{\pi}_{n+1}$ to be $\bar{\pi}_n^{++}(\bar{\sigma}(\bar{\pi}_n^{++}), (0, a^{(2)}))\bar{s}$. The sequence $\bar{\pi}_{n+1}$ is a well-defined play (the last transition depends on \mathcal{P}_2 's move), is consistent with $\bar{\sigma}$ and shares the same sequence of states as $\text{Ex}(\pi|_{n+1})$.

The inductive construction above yields a play $\bar{\pi} \in \text{Outcome}_1(\bar{\sigma})$ that shares the same sequence of states as $\text{Ex}(\pi)$. It remains to show that if $\bar{\pi}$ is time-convergent and \mathcal{P}_1 -blameless, then π is \mathcal{P}_1 -blameless. Assume that $\bar{\pi}$ is time-convergent and \mathcal{P}_1 -blameless. By Proposition 8, either $\bar{\pi}$ does not leave a bad location from some point on or satisfies $\text{coBüchi}(\text{Bad})$. An expansion always exits a bad location the step following entry, therefore $\bar{\pi}$ is necessarily of the second kind, as it shares its sequence of states with $\text{Ex}(\pi)$. Thus, from some point on, the inductive construction of $\bar{\pi}$ is done using cases 1, 2.a or 2.b. Furthermore, from some point on, \mathcal{P}_1 's moves are no longer responsible for transitions in $\bar{\pi}$. In cases 1 and 2.a, \mathcal{P}_1 is responsible for the added transition in $\bar{\pi}_n$ if and only if \mathcal{P}_1 is responsible for the last transition of $\pi|_{n+1}$ (by construction of the expansion mapping). In case 2.b, \mathcal{P}_2 is strictly faster in both plays. Therefore, if from some point on \mathcal{P}_1 is no longer responsible for transitions in $\bar{\pi}$, then from some point on \mathcal{P}_1 is no longer responsible for transitions in π , i.e., π is \mathcal{P}_1 -blameless. ◀

We can also translate \mathcal{P}_1 -strategies defined on \mathcal{G} to \mathcal{P}_1 -strategies on $\mathcal{G}(p, \lambda)$ using the projection mapping. Translating strategies this way must be done with care: we must consider the case where a move suggested by the strategy in \mathcal{G} requires too long a delay to be played in the expanded TG $\mathcal{G}(p, \lambda)$. In this case, the suggested move is replaced by (d, β_1) for a suitable delay d . By construction, the translated strategy always suggests the move $(0, \beta_1)$ when the play ends in a bad location.

Similarly to the first translation, when deriving a strategy $\bar{\sigma}$ of $\mathcal{G}(p, \lambda)$ by translating a strategy σ of \mathcal{G} , the projection of an outcome $\bar{\pi}$ of $\bar{\sigma}$ may not be consistent with σ . However, we show that there is a play π consistent with σ that shares the same sequence of states as $\text{Pr}(\bar{\pi})$, by using techniques similar to those used to prove Lemma 9. Analogously to Lemma 9, we establish that time-convergence and \mathcal{P}_1 -blamelessness of π imply \mathcal{P}_1 -blamelessness of $\bar{\pi}$.

► **Lemma 10.** *Let σ be a move-independent strategy of \mathcal{P}_1 in \mathcal{G} . Let $\bar{\pi}$ be a finite play in*

$\mathcal{G}(p, \lambda)$ and let (ℓ, q, \bar{v}) denote $\text{last}(\bar{\pi})$ and $(d^{(1)}, a^{(1)}) = \sigma(\text{Pr}(\bar{\pi}))$. We set:

$$\bar{\sigma}(\bar{\pi}) = \begin{cases} (\lambda - \bar{v}(z), \beta_1) & \text{if } q \bmod 2 = 1 \wedge \bar{v}(z) + d^{(1)} \geq \lambda \\ (0, \beta_1) & \text{if } q = \text{bad} \\ \sigma(\text{Pr}(\bar{\pi})) & \text{otherwise.} \end{cases}$$

For all $\bar{\pi} \in \text{Outcome}_1(\bar{\sigma})$, there is a play $\pi \in \text{Outcome}_1(\sigma)$ such that π shares the same sequence of states as $\text{Pr}(\bar{\pi})$ and if π is time-convergent and \mathcal{P}_1 -blameless, then $\bar{\pi}$ is \mathcal{P}_1 -blameless.

Proof. Fix $\bar{\pi} \in \text{Outcome}_1(\bar{\sigma})$. We construct the sought play π by induction. We assume that at step $n \in \mathbb{N}$ of the construction, we have constructed $\pi_n \in \text{Plays}_{\bar{f}_n}(\mathcal{G})$ that shares the same sequence of states as $\text{Pr}(\bar{\pi}|_n)$, and such that π_n is consistent with σ .

The base case is straightforward. We set $\pi_0 = \text{Pr}(\bar{\pi}|_0)$. Now, assume that we have constructed a finite play $\pi_n \in \text{Plays}_{\bar{f}_n}(\mathcal{G})$ as described above and let us construct π_{n+1} . Let $m^{(2)} = (d^{(2)}, a^{(2)})$ be a move of \mathcal{P}_2 and \bar{s} be a state of $\mathcal{T}(\mathcal{A}(p, \lambda))$ such that $\bar{\pi}|_{n+1} = \bar{\pi}|_n(\bar{\sigma}(\bar{\pi}|_n), m^{(2)})\bar{s}$. Write $(d^{(1)}, a^{(1)}) = \bar{\sigma}(\bar{\pi}|_n)$, $(\ell, q, \bar{v}) = \text{last}(\bar{\pi}|_n)$ and $(\ell', q', \bar{v}') = \bar{s}$.

We discuss different cases: 1. q is an even number, 2. $q = \text{bad}$ and 3. q is an odd number. The third case is divided in three disjoint sub-cases: 3.a. $q' = \text{bad}$; 3.b. $q' \neq \text{bad}$ and $a^{(1)} \neq \beta_1$; 3.c. $q' \neq \text{bad}$ and $a^{(1)} = \beta_1$.

Case 1. Assume q is an even number. Then the same moves are enabled in $\text{last}(\bar{\pi}|_n)$ and $\text{last}(\text{Pr}(\bar{\pi}|_n))$. By definition of $\bar{\sigma}$ and move-independence of σ , $\bar{\sigma}(\bar{\pi}|_n) = \sigma(\text{Pr}(\bar{\pi}|_n)) = \sigma(\pi_n)$. We have $\text{Pr}(\bar{\pi}|_{n+1}) = \text{Pr}(\bar{\pi}|_n)(\bar{\sigma}(\bar{\pi}|_n), m^{(2)})(\ell', \bar{v}'_C)$. We define π_{n+1} as the play $\pi_n(\sigma(\pi_n), m^{(2)})(\ell', \bar{v}'_C)$. This is a well-defined play consistent with σ that shares the same sequence of states as $\text{Pr}(\bar{\pi}|_{n+1})$, due to the fact that $\text{Pr}(\bar{\pi}|_n)$ and π_n share the same sequence of states and $\sigma(\pi_n) = \bar{\sigma}(\bar{\pi}|_n)$.

Case 2. Assume $q = \text{bad}$. We have $\ell = \ell'$, $\bar{v} = \bar{v}'$. Indeed, the only \mathcal{P}_i moves available in a bad location are $(0, \perp)$ and $(0, \beta_i)$. The play either proceeds using an edge to $(\ell, p(\ell))$ or stays in (ℓ, bad) . The values of clocks do not change in both of these cases. By definition of the projection mapping, $\text{Pr}(\bar{\pi}|_{n+1}) = \text{Pr}(\bar{\pi}|_n)((0, \perp), (0, \perp))(\ell, \bar{v}|_C)$. We let π_{n+1} be $\pi_n(\sigma(\pi_n), (0, \perp))(\ell, \bar{v}|_C)$: π_{n+1} is a well-defined play (the last transition is valid by the move $(0, \perp)$ of \mathcal{P}_2), is consistent with σ and has the same sequence of states as $\text{Pr}(\bar{\pi}|_{n+1})$.

Case 3. Assume q is an odd number. We study three sub-cases: a. $q' = \text{bad}$; b. $q' \neq \text{bad}$ and $a^{(1)} = \beta_1$; c. $q' \neq \text{bad}$ and $a^{(1)} \neq \beta_1$. They are disjoint and cover all possibilities.

We start with **case 3.a.** Assume that $q' = \text{bad}$. Then $d^{(1)} = d^{(2)} = \lambda - \bar{v}(z)$ holds: first, to enter a bad location from an odd location, z must reach λ , thus $d^{(i)} \geq \lambda - \bar{v}(z)$ for $i \in \{1, 2\}$; second, the invariant of odd locations implies $z \leq \lambda$, thus no delay greater than $\lambda - \bar{v}(z)$ can be played by either player in the state $\text{last}(\bar{\pi}|_n) = (\ell, q, \bar{v})$. It follows from $d^{(1)} = \lambda - \bar{v}(z)$ that the delay of the move $\sigma(\pi_n) = \sigma(\text{Pr}(\bar{\pi}|_n))$ must be greater than or equal to $d^{(1)}$ (by line 1 of the definition of $\bar{\sigma}$). We have $\text{Pr}(\bar{\pi}|_{n+1}) = \text{Pr}(\bar{\pi}|_n)((d^{(1)}, \perp), (d^{(2)}, \perp))(\ell', \bar{v}'_C)$. We define π_{n+1} to be $\pi_n(\sigma(\pi_n), (d^{(2)}, \perp))(\ell', \bar{v}'_C)$. The sequence π_{n+1} is a well-defined play consistent with σ ; the last step is valid due to the delay of $\sigma(\pi_n)$ being more than $d^{(2)} = d^{(1)}$. By construction, π_{n+1} has the same sequence of states as $\text{Pr}(\bar{\pi}|_{n+1})$.

Assume for **cases 3.b and 3.c** that $q' \neq \text{bad}$. We start with **case 3.b.** Assume $a^{(1)} = \beta_1$. Then $d^{(1)} = \lambda - \bar{v}(z) \geq d^{(2)}$ because $m^{(2)}$ is a \mathcal{P}_2 move enabled in the state (ℓ, q, \bar{v}) which is constrained by the invariant $z \leq \lambda$. As $q' \neq \text{bad}$, necessarily $a^{(2)} \neq \beta_2$ and \mathcal{P}_2 is responsible for the last transition of $\bar{\pi}|_{n+1}$. We have $\text{Pr}(\bar{\pi}|_{n+1}) = \text{Pr}(\bar{\pi}|_n)((d^{(1)}, \perp), m^{(2)})(\ell', \bar{v}'_C)$. In this case, we set $\pi_{n+1} = \pi_n(\sigma(\pi_n), m^{(2)})(\ell', \bar{v}'_C)$. This is a well-defined play, as $d^{(2)} \leq d^{(1)}$ and $d^{(1)}$ is smaller than the delay suggested by $\sigma(\pi_n) = \sigma(\text{Pr}(\bar{\pi}|_n))$ by definition of $\bar{\sigma}$.

For **case 3.c**, assume $a^{(1)} \neq \beta_1$. Then $(d^{(1)}, a^{(1)}) = \sigma(\text{Pr}(\bar{\pi}_{|n}))$ by definition of σ . To avoid distinguishing cases following whether $a^{(2)} = \beta_2$, let $\tilde{a}^{(2)}$ denote \perp if $a^{(2)} = \beta_2$ and $a^{(2)}$ otherwise. We have $\text{Pr}(\bar{\pi}_{|n+1}) = \text{Pr}(\bar{\pi}_{|n})(\sigma(\text{Pr}(\bar{\pi}_{|n})), (d^{(2)}, \tilde{a}^{(2)}))(\ell', \bar{v}'_C)$. We let π_{n+1} be the sequence $\pi_n(\sigma(\pi_n), (d^{(2)}, \tilde{a}^{(2)}))(\ell', \bar{v}'_C)$. It is a well-defined play, consistent with σ and sharing the same sequence of states as $\text{Pr}(\bar{\pi}_{|n+1})$. These last statements follow from $\text{Pr}(\bar{\pi}_{|n})$ and π_n sharing the same sequence of states and move-independence of σ ensuring $\sigma(\text{Pr}(\bar{\pi}_{|n})) = \sigma(\pi_n)$.

The previous inductive construction shows the existence of a play $\pi \in \text{Outcome}_1(\sigma)$ such that $\text{Pr}(\bar{\pi})$ and π share the same sequence of states. We now argue that if the constructed play π is time-convergent and \mathcal{P}_1 -blameless, then $\bar{\pi}$ is also \mathcal{P}_1 -blameless.

Assume that π is time-convergent and \mathcal{P}_1 -blameless. It follows that $\bar{\pi}$ is also time-convergent because π and $\text{Pr}(\bar{\pi})$ share the same sequence of states (recall the projection mapping preserves time-convergence). By Proposition 8, either $\bar{\pi}$ does not leave a bad location from some point on or satisfies $\text{coBüchi}(\text{Bad})$. If from some point on $\bar{\pi}$ does not leave a bad location, then it is \mathcal{P}_1 -blameless by construction of $\bar{\sigma}$: in a bad location, $\bar{\sigma}$ always suggests a move that exits the bad location. Assume now that $\bar{\sigma}$ satisfies $\text{coBüchi}(\text{Bad})$. Then, from some point on, the construction of π proceeds following cases 1, 3.b and 3.c of the inductive construction. In cases 1 and 3.c, the player responsible for the transition that is added is the same in both games. In case 3.b, \mathcal{P}_1 is not responsible for the last transition of $\bar{\pi}_{|n+1}$. Therefore, if from some point on \mathcal{P}_1 is no longer responsible for transitions in π , then from some point on, \mathcal{P}_1 is no longer responsible for transitions in $\bar{\pi}$, i.e., $\bar{\pi}$ is \mathcal{P}_1 -blameless. ◀

The translations of strategies described in Lemma 9 (resp. Lemma 10) establish a relation associating to any outcome of a translated strategy, an outcome of the original strategy which shares states with the expansion (resp. projection) of this outcome. Recall that all the objectives we have considered are state-based, and therefore move-independent strategies suffice for these objectives. Using the relations developed in Lemma 9 and Lemma 10, and Lemma 6, we can establish that translating a winning strategy of \mathcal{G} yields a winning strategy of $\mathcal{G}(p, \lambda)$, and vice-versa. From this, we can conclude that the realizability problem on TGs with (resp. direct) timed window objectives can be reduced to the realizability problem on TGs with co-Büchi (resp. safety) objectives.

► **Theorem 11.** *Let s_{init} be the initial state of \mathcal{G} and \bar{s}_{init} be the initial state of $\mathcal{G}(p, \lambda)$. There is a winning strategy σ for \mathcal{P}_1 for the objective $\text{TW}(p, \lambda)$ (resp. $\text{DTW}(p, \lambda)$) from s_{init} in \mathcal{G} if and only if there is a winning strategy $\bar{\sigma}$ for \mathcal{P}_1 for the objective $\text{coBüchi}(\text{Bad})$ (resp. $\text{Safe}(\text{Bad})$) from \bar{s}_{init} in $\mathcal{G}(p, \lambda)$.*

Proof. Let $(\Psi, \Psi(p, \lambda)) \in \{(\text{TW}(p, \lambda), \text{coBüchi}(\text{Bad})), (\text{DTW}(p, \lambda), \text{Safe}(\text{Bad}))\}$.

Assume \mathcal{P}_1 has a winning strategy σ for the objective Ψ in \mathcal{G} from s_{init} . We assume this strategy to be move-independent because Ψ is a state-based objective. By Lemma 10, there is a strategy $\bar{\sigma}$ such that for any play $\bar{\pi} \in \text{Outcome}_1(\bar{\sigma}, \bar{s}_{\text{init}})$, there is a play $\pi \in \text{Outcome}_1(\sigma, s_{\text{init}})$ such that the sequence of states of ρ and $\text{Pr}(\bar{\pi})$ coincide, and if π is time-convergent and \mathcal{P}_1 -blameless, then $\bar{\pi}$ is \mathcal{P}_1 -blameless.

Fix $\bar{\pi} \in \text{Outcome}_1(\bar{\sigma}, \bar{s}_{\text{init}})$ and let π as above. We establish that $\bar{\pi} \in \text{WC}_1(\Psi(p, \lambda))$. Because σ is winning, $\pi \in \text{WC}_1(\Psi)$. First, assume $\bar{\pi}$ is time-divergent. Then π is also time-divergent. It follows that $\pi \in \Psi$ and since Ψ is state-based, $\text{Pr}(\bar{\pi}) \in \Psi$. Lemma 6 ensures that $\bar{\pi} \in \Psi(p, \lambda)$. Assume now that $\bar{\pi}$ is time-convergent. Then π is also time-convergent. Because σ is winning, π is \mathcal{P}_1 -blameless. This implies \mathcal{P}_1 -blamelessness of $\bar{\pi}$. Thus, we have $\bar{\pi} \in \text{WC}_1(\Psi(p, \lambda))$ and have shown that $\bar{\sigma}$ is winning.

Conversely, assume \mathcal{P}_1 has a winning strategy $\bar{\sigma}$ in $\mathcal{G}(p, \lambda)$ for objective $\Psi(p, \lambda)$ from \bar{s}_{init} . We assume that the strategy $\bar{\sigma}$ is move-independent. We show that the strategy σ defined in Lemma 9 is winning in \mathcal{G} from s_{init} . Fix $\pi \in \text{Outcome}_1(\sigma, s_{\text{init}})$. Let $\bar{\pi} \in \text{Outcome}_1(\bar{\sigma}, \bar{s}_{\text{init}})$ be such that the sequence of states of $\bar{\pi}$ coincides with that of $\text{Ex}(\pi)$, and that if $\bar{\pi}$ is time-convergent and \mathcal{P}_1 -blameless, then π is also blameless. We prove that $\pi \in \text{WC}_1(\Psi)$ and distinguish cases following time-divergence of π . Assume π is time-divergent, then so is $\bar{\pi}$. The play $\bar{\pi}$ is consistent with a winning strategy and time-divergent, thus $\bar{\pi} \in \Psi(p, \lambda)$. Safety/co-Büchi objectives are state-based, and hence $\bar{\pi} \in \Psi(p, \lambda)$ if and only if $\text{Ex}(\pi) \in \Psi(p, \lambda)$. It follows from Lemma 6 that $\pi \in \Psi$. Now assume that π is time-convergent. Then, $\bar{\pi}$ is time-convergent thus \mathcal{P}_1 -blameless, therefore π is also \mathcal{P}_1 -blameless. We have proven that $\pi \in \text{WC}_1(\Psi)$, which ends the proof. \blacktriangleleft

5 Multi-dimensional objectives

We can generalize the former reduction to conjunctions of (resp. direct) timed window parity objectives. Fix a TG $\mathcal{G} = (\mathcal{A}, \Sigma_1, \Sigma_2)$ with set of locations L . We consider a k -dimensional priority function $p: L \rightarrow \{0, \dots, d-1\}^k$. We write $p_i: L \rightarrow \{0, \dots, d-1\}$ for each component function. Fix $\lambda = (\lambda_1, \dots, \lambda_k) \in (\mathbb{N} \setminus \{0\})^k$ a vector of bounds on window sizes.

Generalized (resp. direct) timed window objectives are defined as conjunctions of (resp. direct) timed window objectives. Formally, the *generalized timed window (parity) objective* is defined as

$$\text{GTW}(p, \lambda) = \bigcap_{1 \leq i \leq k} \text{TW}(p_i, \lambda_i)$$

and the *generalized direct timed window (parity) objective* as

$$\text{GDTW}(p, \lambda) = \bigcap_{1 \leq i \leq k} \text{DTW}(p_i, \lambda_i).$$

The verification and realizability problems for these objectives can be solved using a similar construction to the single-dimensional case. The inductive property (Lemma 3) ensures only one window needs to be monitored at a time on each dimension. We adapt the construction of the expanded TA to keep track of several windows at once.

Expanded locations are labeled with vectors $q = (q_1, \dots, q_k)$ where q_i represents the smallest priority in the current window on dimension i for each i . Besides these, for each location ℓ , there is an expanded location marked by *bad*. We do not keep track of the dimension on which a bad window was seen: we only consider one kind of bad location. To measure the size of a window in each dimension, we introduce k new clocks $z_1, \dots, z_k \notin C$. For any $q \in \{0, \dots, d-1\}^k$, denote by O_q the set $\{1 \leq i \leq k \mid q_i \bmod 2 = 1\}$ of indices of components that are odd.

Updates of the vector of priorities are independent between dimensions and follow the same logic as the simpler case. In the single-dimensional case, an edge leaving an even location (ℓ, q) leads to a location of the form $(\ell', p(\ell'))$ and an edge leaving an odd location (ℓ, q) leads to a location of the form $(\ell', \min\{q, p(\ell')\})$. The behavior of edges is similar in this case. The update function up generalizes the handling of updates. Let q be a vector of current priorities and ℓ be a location of \mathcal{A} . We set $\text{up}(q, \ell)$ to be the vector q' such that $q'_i = \min\{q_i, p_i(\ell)\}$ if $i \in O_q$ and $q'_i = p_i(\ell)$ otherwise.

For any non-bad location (ℓ, q) and edge (ℓ, g, a, D, ℓ') leaving location ℓ , there is a matching edge in the expanded TA. The target location of this edge is $(\ell', \text{up}(q, \ell'))$. Its

guard is a strengthened version of g : it is disabled if a bad window is detected on some dimension by adding a conjunct $z_i < \lambda_i$ for each $i \in O_q$ to the guard of the edge. In the same manner that odd locations were equipped with a strengthened invariant of the form $I(\ell) \wedge z \leq \lambda$, expanded locations (ℓ, q) are equipped with an invariant that is the conjunction of $I(\ell)$ with the conjunction over O_q of $z_i \leq \lambda_i$.

It remains to discuss how bad locations are generalized. They have the invariant $z_1 = 0$ (z_1 is chosen arbitrarily) and for each $\beta \in \{\beta_1, \beta_2\}$ and location ℓ , there is an edge $((\ell, \text{bad}), \text{true}, \beta, \emptyset, (\ell, p(\ell)))$. In the single-dimensional case, each odd location had an edge to a bad location with a guard $z = \lambda$. Analogously, for any location (ℓ, q) with O_q non-empty, there are edges to (ℓ, bad) . There are two such edges (one for each $\beta \in \{\beta_1, \beta_2\}$) per dimension in O_q . Recall that there cannot be two edges (ℓ, g, a, D, ℓ') and (ℓ, h, a, D', ℓ'') with $g \wedge h$ satisfiable. This prevents adding $2 \cdot |O_q|$ edges guarded by $z_i = \lambda_i$. We do not need more than $2 \cdot |O_q|$ edges however. For instance, instead of having one edge guarded by $z_1 = \lambda_1$ and another guarded by $z_2 = \lambda_2$, we can replace the guard of the second edge by $z_1 < \lambda_1 \wedge z_2 = \lambda_2$ to ensure the guards are incompatible. Upon entry to a bad location, all additional clocks z_1, \dots, z_k are reset, no matter the dimension on which a bad window was detected.

The formal definition of $\mathcal{A}(p, \lambda)$ follows.

► **Definition 12.** *Given a TA $\mathcal{A} = (L, \ell_{\text{init}}, C, \Sigma, I, E)$, $\mathcal{A}(p, \lambda)$ is defined to be the TA $(L', \ell'_{\text{init}}, C', \Sigma', I', E')$ such that*

- $L' = L \times (\{0, \dots, d-1\}^k \cup \{\text{bad}\})$;
- $\ell'_{\text{init}} = (\ell_{\text{init}}, p(\ell_{\text{init}}))$;
- $C' = C \cup \{z_1, \dots, z_k\}$ where $z_1, \dots, z_k \notin C$;
- $\Sigma' = \Sigma \cup \{\beta_1, \beta_2\}$ where $\beta_1, \beta_2 \notin \Sigma$;
- $I'(\ell, q) = (I(\ell) \wedge \bigwedge_{i \in O_q} (z_i \leq \lambda_i))$ for all $\ell \in L$, $q \in \{0, \dots, d-1\}^k$, and $I'(\ell, \text{bad}) = (z_1 = 0)$ for all $\ell \in L$;
- the set of edges E' of $\mathcal{A}(p, \lambda)$ is the smallest set that satisfies the following rules:
 - if $q \neq \text{bad}$ and $(\ell, g, a, D, \ell') \in E$, then

$$\left((\ell, q), g \wedge \bigwedge_{i \in O_q} (z_i < \lambda_i), a, D \cup \{z_i \mid i \notin O_q\}, (\ell', \text{up}(q, \ell')) \right) \in E';$$

- for all $\ell \in L$, $q \neq \text{bad}$, $i \in O_q$ and $\beta \in \{\beta_1, \beta_2\}$, $((\ell, q), \varphi_i, \beta, \{z_1, \dots, z_k\}, (\ell, \text{bad})) \in E'$ where the clock condition φ_i is defined by

$$\varphi_i := (z_i = \lambda_i) \wedge \bigwedge_{\substack{1 \leq j < i \\ j \in O_q}} (z_j < \lambda_j);$$

- for all $\ell \in L$ and $\beta \in \{\beta_1, \beta_2\}$, $((\ell, \text{bad}), \text{true}, \beta, \emptyset, (\ell, p(\ell))) \in E'$.

For a TG $\mathcal{G} = (\mathcal{A}, \Sigma_1, \Sigma_2)$, we set $\mathcal{G}(p, \lambda) = (\mathcal{A}(p, \lambda), \Sigma_1 \cup \{\beta_1\}, \Sigma_2 \cup \{\beta_2\})$.

The generalized (resp. direct) timed window objective on a TA \mathcal{A} or TG \mathcal{G} translates to a co-Büchi (resp. safety) objective on the expansion $\mathcal{A}(p, \lambda)$ and $\mathcal{G}(p, \lambda)$ respectively. Let $\text{Bad} = L \times \{\text{bad}\}$. We can derive the following theorems by adapting the proofs of the single-dimensional case (Theorems 7 and 11). The only nuance is that there are several dimensions to be handled in parallel. However, different dimensions minimally interact with one another. The only way a dimension may affect the others is by resetting them in the expansion (when a bad location is visited), and that would entail a bad window was detected on one dimension.

► **Theorem 13.** *All time-divergent paths of \mathcal{A} satisfy the generalized (resp. direct) timed window objective if and only if all time-divergent paths of $\mathcal{A}(p, \lambda)$ satisfy the co-Büchi (resp. safety) objective over bad locations.*

► **Theorem 14.** *There is a winning strategy σ for \mathcal{P}_1 for the objective GTW(p, λ) (resp. GDTW(p, λ)) from the initial state of \mathcal{G} if and only if there is a winning strategy $\bar{\sigma}$ for \mathcal{P}_1 for the objective coBüchi(Bad) (resp. Safe(Bad)) from the initial state of $\mathcal{G}(p, \lambda)$.*

6 Algorithms and complexity

This section presents algorithms for solving the verification and realizability problems for the generalized (resp. direct) timed window parity objective. We establish lower bounds that match the complexity class of our algorithms. We consider the general multi-dimensional setting and we denote by k the number of timed window parity objectives under consideration.

6.1 Algorithms

We use the construction presented in the previous sections to solve both the realizability problem and the verification problem for the (resp. direct) timed window objective. In both cases, we invoke known sub-algorithms. For games, we rely on a general algorithm for solving TGs with ω -regular location-based objectives. For automata, we use an algorithm for the emptiness problem of timed automata.

First, we analyze the time required to construct the expanded TA with respect to the inputs to the problem.

► **Lemma 15.** *Let $\mathcal{A} = (L, \ell_{\text{init}}, C, \Sigma, I, E)$ be a TA. Let $p: L \rightarrow \{0, \dots, d-1\}^k$ be a k -dimensional priority function and $\lambda \in (\mathbb{N} \setminus \{0\})^k$ be a vector of bounds on window sizes. The expanded TA $\mathcal{A}(p, \lambda) = (L', \ell'_{\text{init}}, C', \Sigma', I', E')$ can be computed in time exponential in k and polynomial in the size of L , the size of E , the size of C , d , the length of the encoding of the clock constraints of \mathcal{A} and in the encoding of λ .*

Proof. We analyze each component of $\mathcal{A}(p, \lambda)$. First, we study the size of the set of expanded locations. The set of expanded locations L' is given by $L' = L \times (\{0, \dots, d-1\}^k \cup \{\text{bad}\})$. Therefore, $|L'| = |L| \cdot (d^k + 1)$. The set of clocks C' contains $|C| + k$ clocks.

To determine a bound on the size of the set of edges, we base ourselves on the rules that define E' . For each edge $(\ell, g, a, D, \ell') \in E$ and $q \in \{0, \dots, d-1\}^k$, there is an edge in E' exiting location (ℓ, q) . There are $|E| \cdot d^k$ edges obtained this way. Furthermore, for each non-bad expanded location (ℓ, q) , there are up to $2k$ edges to the location (ℓ, bad) . There are at most $2k \cdot |L| \cdot d^k$ edges obtained by this rule. Finally, it remains to count the edges that leave bad locations. There are two such edges per bad location, totaling to $2 \cdot |L|$. Overall, an upper bound on the size of E' is given by $|E| \cdot d^k + 2k \cdot |L| \cdot d^k + 2 \cdot |L|$.

It remains to study the total length of the encoding of the clock constraints of $\mathcal{A}(p, \lambda)$. We have shown above that L' and E' are of size exponential in k and polynomial in $|L|$, $|E|$ and d . There are as many clock constraints in $\mathcal{A}(p, \lambda)$ as there are locations and edges. Therefore, it suffices to show that the encoding of each individual clock constraint of $\mathcal{A}(p, \lambda)$ is polynomial in the total length of the encoding of the clock constraints of \mathcal{A} , k and the encoding of λ to end the proof.

A clock constraint of $\mathcal{A}(p, \lambda)$ is either derived from a clock constraint of \mathcal{A} , the invariant of a bad location or the guard of an edge to or from a bad location. Clock constraints derived from \mathcal{A} are either unchanged, or they are obtained by reinforcing a clock constraint

of \mathcal{A} with conjuncts $z_i \leq \lambda_i$ in the case of invariants or with a conjuncts $z_i < \lambda_i$ in the case of guards. At most, we extend clock constraints of \mathcal{A} with k conjuncts that can be encoded linearly w.r.t. the encoding of the λ_i . The invariant of bad locations is $z_1 = 0$ and is therefore constant in length. The guards of edges exiting bad locations are **true** and are also constant in length. Finally, guards of edges to bad locations are conjunctions of the form $z_{i_1} < \lambda_{i_1} \wedge \dots \wedge z_{i_{j-1}} < \lambda_{i_{j-1}} \wedge z_{i_j} = \lambda_{i_j}$, which can be encoded linearly w.r.t. k and the encoding of the λ_i . \blacktriangleleft

Timed games. To solve the realizability problem, we rely on the algorithm from [20]. To use the algorithm of [20], we introduce deterministic parity automata. For all finite alphabets A , deterministic parity automata can represent all ω -regular subsets of A^ω . We use deterministic parity automata to encode location-based objectives (the set of locations of the studied timed automaton serves as the alphabet of the parity automaton).

Let A be a finite non-empty alphabet. A *deterministic parity automaton* (DPA) of order m is a tuple $H = (Q, q_{\text{init}}, A, \delta, \Omega)$, where Q is a finite set of states, $q_{\text{init}} \in Q$ is the initial state, $\delta: Q \times A \rightarrow Q$ is the transition function and $\Omega: Q \rightarrow \{0, \dots, 2m-1\}$ is a function assigning a priority to each state of the DPA. An execution of H on an infinite word $a_0 a_1 \dots \in A^\omega$ is an infinite sequence of states $q_0 q_1 \dots \in Q^\omega$ that starts in the initial state of H , i.e., $q_0 = q_{\text{init}}$ and such that for all $i \in \mathbb{N}$, $q_{i+1} = \delta(q_i, a_i)$, i.e., each step of the execution is performed by reading a letter of the input word. An infinite word $w \in A^\omega$ is accepted by H if there is an execution $q_0 q_1 \dots \in Q^\omega$ such that the smallest priority appearing infinitely often along the execution is even, i.e. if $(\liminf_{n \rightarrow \infty} \Omega(q_i)) \bmod 2 = 0$. A DPA is *total* if its transition function is total, i.e., for all $q \in Q$ and $a \in A$, $\delta(q, a)$ is defined.

The algorithm of [20] checks the existence of a \mathcal{P}_1 -winning strategy in a TG $\mathcal{G} = (\mathcal{A}, \Sigma_1, \Sigma_2)$ where the set of locations of \mathcal{A} is L and the set of clocks is C with a location objective specified by a total DPA H with set of states Q and of order m in time

$$\mathcal{O} \left(\left(|L|^2 \cdot |C|! \cdot 2^{|C|} \cdot \prod_{x \in C} (2c_x + 1) \cdot |Q| \cdot m \right)^{m+2} \right), \quad (2)$$

where c_x is the largest constant to which x is compared to in the clock constraints of \mathcal{A} .

We now show that the realizability problem for generalized (resp. direct) timed window objectives is in EXPTIME using the algorithm described above. This essentially boils down to specifying DPAs that encode safety and co-Büchi objectives and using these along with the algorithm of [20] to check the existence of a \mathcal{P}_1 -winning strategy in $\mathcal{G}(p, \lambda)$.

Proof. Fix a TG $\mathcal{G} = (\mathcal{A}, \Sigma_1, \Sigma_2)$ and let L denote the set of locations of \mathcal{A} . Let p be a multi-dimensional priority function and $\lambda \in (\mathbb{N} \setminus \{0\})^k$ be a vector of window sizes. Let L' denote the set of locations of $\mathcal{A}(p, \lambda)$. We describe total DPAs $H_{\text{Safe}(\text{Bad})}$ and $H_{\text{coBüchi}(\text{Bad})}$ for the objectives **Safe**(Bad) and **coBüchi**(Bad) in the TG $\mathcal{G}(p, \lambda)$.

We encode the safety objective using a DPA with two states. Intuitively, the initial state is never left as long as no bad location is read. If the DPA reads a bad location, it moves to a sink state, representing that the safety objective was violated. The initial state is given an even priority and the sink state an odd priority so that accepting executions are those that read sequences of locations that respect the safety objective. Formally, let $H_{\text{Safe}(\text{Bad})}$ be the DPA $(Q, q_{\text{init}}, L', \delta, \Omega)$ where $Q = \{q_{\text{init}}, q_{\text{bad}}\}$, $\Omega(q_{\text{init}}) = 0$ and $\Omega(q_{\text{bad}}) = 1$, and the transition function is defined by $\delta(q_{\text{init}}, (\ell, q)) = q_{\text{init}}$ for all $(\ell, q) \in L' \setminus \text{Bad}$, $\delta(q_{\text{init}}, (\ell, \text{bad})) = q_{\text{bad}}$ for all $\ell \in L$ and $\delta(q_{\text{bad}}, (\ell, q)) = q_{\text{bad}}$ for all $(\ell, q) \in L'$. This DPA is total by construction.

The co-Büchi objective is also encoded by a DPA with two states. The first state q_{init} is entered every time a non-bad location is read by the DPA and the second state q_{bad} whenever a bad location is read. Runs that visit q_{bad} infinitely often violate the co-Büchi objective, therefore we give q_{bad} an odd priority smaller than the even priority of q_{init} . Formally, let $H_{\text{coBüchi}(\text{Bad})}$ be the DPA $(Q, q_{\text{init}}, L', \delta, \Omega)$ where $Q = \{q_{\text{init}}, q_{\text{bad}}\}$, $\Omega(q_{\text{init}}) = 2$ and $\Omega(q_{\text{bad}}) = 1$, and the transition function is defined by $\delta(q, (\ell, q)) = q_{\text{init}}$ for all $(\ell, q) \in L' \setminus \text{Bad}$ and $q \in \{q_{\text{init}}, q_{\text{bad}}\}$, and $\delta(q, (\ell, \text{bad})) = q_{\text{bad}}$ for all $\ell \in L$ and $q \in \{q_{\text{init}}, q_{\text{bad}}\}$. This deterministic parity automaton is total by construction.

By Theorem 14, it suffices to check the existence of a winning strategy for \mathcal{P}_1 in the expanded TG $\mathcal{G}(p, \lambda)$ to answer the realizability problem over \mathcal{G} . By Lemma 15, the construction of the TG $\mathcal{G}(p, \lambda)$ from \mathcal{G} takes exponential time w.r.t. to the inputs to the problem.

Recall that there are $|L| \cdot (d^k + 1)$ locations in $\mathcal{A}(p, \lambda)$. We have shown that the objective $\text{Safe}(\text{Bad})$ (resp. $\text{coBüchi}(\text{Bad})$) can be encoded using a DPA with two states and order at most 2. Combining this with equation (2), it follows that we can check the existence of a \mathcal{P}_1 -winning strategy in $\mathcal{G}(p, \lambda)$ for a safety or co-Büchi objective in time

$$\mathcal{O} \left(\left(|L|^2 \cdot (d^k + 1)^2 \cdot (|C| + k)! \cdot 2^{|C|+k} \prod_{x \in C} (2c_x + 1) \cdot \prod_{1 \leq i \leq k} (2\lambda_i + 1) \right)^4 \right), \quad (3)$$

where c_x is the largest constant to which x is compared to in the clock constraints of \mathcal{A} .

Overall, $\mathcal{G}(p, \lambda)$ can be constructed in exponential time and the existence of a \mathcal{P}_1 -winning strategy in $\mathcal{G}(p, \lambda)$ can be checked in exponential time, establishing EXPTIME-membership of the realizability problem for (resp. direct) timed window objectives. \blacktriangleleft

Timed automata. We describe a polynomial space algorithm for the verification problem for generalized (resp. direct) timed window objectives derived from our reduction. First, we remark that to check that all time-divergent paths of a TA satisfy a conjunction of objectives, we can proceed one objective at a time: for any family of objectives, there is some time-divergent path that does not satisfy the conjunction of objectives in the family if and only if, for some objective from the family, there is some time-divergent path that does not satisfy it. This contrasts with TGs, in which \mathcal{P}_1 may have a winning strategy for each individual objective but is unable to satisfy their conjunction.

Following the observation above, we establish membership in PSPACE of the verification problem for the generalized (resp. direct) timed window objective in two steps. First, we argue that the verification problem is in PSPACE for the single-dimensional (resp. direct) timed window parity objective. Then, in the multi-dimensional setting, we use the single-dimensional case as an oracle to check satisfaction of a generalized objective one dimension at a time.

\blacktriangleright **Lemma 16.** *The generalized (direct) direct timed window verification problem is in PSPACE.*

Proof. Fix a TA $\mathcal{A} = (L, \ell_{\text{init}}, C, \Sigma, I, E)$, a priority function $p: L \rightarrow \{0, \dots, d-1\}$ and a bound $\lambda \in \mathbb{N} \setminus \{0\}$ on window sizes. By Theorem 7, the verification problem on \mathcal{A} for the (resp. direct) timed window objective can be reduced to the verification problem on $\mathcal{A}(p, \lambda)$ for the co-Büchi objective $\text{coBüchi}(\text{Bad})$ (resp. the safety objective $\text{Safe}(\text{Bad})$). The complement of a co-Büchi (resp. safety) objective is a Büchi (resp. reachability) objective. Therefore, there is a time-divergent path of \mathcal{A} that does not satisfy the (resp. direct) timed

window objective if and only if there is some time-divergent path in $\mathcal{A}(p, \lambda)$ that satisfies Büchi(Bad) (resp. Reach(Bad)).

Our algorithm for the verification problem on \mathcal{A} for the (resp. direct) timed window objective proceeds as follows: construct $\mathcal{A}(p, \lambda)$ and check if there is some time-divergent path in $\mathcal{A}(p, \lambda)$ satisfying Büchi(Bad) (resp. Reach(Bad)) and return no if that is the case.

This algorithm is in polynomial space. By Lemma 15, as we have fixed $k = 1$, $\mathcal{A}(p, \lambda)$ can be computed in time polynomial in d , the sizes of L , E , C , the length of the encoding of the clock constraints of \mathcal{A} and the encoding of λ . In other words, the verification problem for the (resp. direct) timed window objective can be reduced to checking the existence of a time-divergent path of a TA satisfying a Büchi (resp. reachability objective) in polynomial time. Checking the existence of a time-divergent path satisfying a Büchi or reachability objective can be done in polynomial space [2]. Thus, the verification problem for the (one-dimensional) (resp. direct) timed window objective can be solved in polynomial space.

For the k -dimensional case, the previous algorithm can be used for each individual component, to check that all dimensions satisfy their respective objective. Complexity-wise, this shows the multidimensional problem belongs in $\mathsf{P}^{\mathsf{PSPACE}}$ (with an oracle in PSPACE for the single-dimensional case). Because, $\mathsf{P}^{\mathsf{PSPACE}} = \mathsf{PSPACE}$ [6], this proves that the generalized (resp. direct) timed window verification problem is in PSPACE . ◀

6.2 Lower bounds

We have presented algorithms which share the same complexity class for one or multiple dimensions. In this section, we establish that our bounds are tight. It suffices to show hardness for the single-dimensional problems, as they are subsumed by the k -dimensional case.

The verification and realizability problems for timed window objectives can be shown to be at least as hard as the verification and realizability problems for safety objectives. The safety verification problem is PSPACE -complete [2]. The realizability problem for safety objectives is $\mathsf{EXPTIME}$ -complete (this follows from $\mathsf{EXPTIME}$ -completeness of the safety control problem [22]). The same construction is used for the verification and realizability problem. Given a timed automaton, we expand it so as to encode in locations whether the safety objective was violated at some point. We assign an even priority to locations that indicate the safety objective never was violated and an odd priority to the other locations: as long as the safety objective is not violated, windows close immediately and as soon as it is violated, it no longer is possible to close windows.

► **Lemma 17.** *The verification (resp. realizability) problem for the (direct) timed window objective is PSPACE -hard (resp. $\mathsf{EXPTIME}$ -hard).*

Proof. Fix a TA $\mathcal{A} = (L, \ell_{\text{init}}, C, \Sigma, I, E)$ and a set of unsafe locations $F \subseteq L$. We construct a TA \mathcal{A}' and a priority function p such that all time-divergent initial paths of \mathcal{A} satisfy $\text{Safe}(F)$ if and only if all time-divergent initial paths of \mathcal{A}' satisfy $\text{TW}(p, 1)$ (resp. $\text{DTW}(p, 1)$) (the choice of 1 as the bound of the window size is arbitrary, the provided reduction functions for any bound on the size of windows). We encode the safety objective in a TA \mathcal{A}' as a (resp. direct) timed window objective.

We expand locations of \mathcal{A} with a Boolean representing whether F was visited. Formally, let $\mathcal{A}' = (L', \ell'_{\text{init}}, C, \Sigma, I', E')$ be the TA with $L' = L \times \{0, 1\}$, $\ell'_{\text{init}} = (\ell_{\text{init}}, 0)$ if $\ell_{\text{init}} \notin F$ and $\ell'_{\text{init}} = (\ell_{\text{init}}, 1)$ otherwise, $I'((\ell, b)) = I(\ell)$ for all $\ell \in L$ and $b \in \{0, 1\}$, and the set of edges E' is the smallest set satisfying, for each edge $(\ell, g, a, D, \ell') \in E$: $((\ell, 0), g, a, D, (\ell', 0)) \in E'$

if $\ell' \notin F$; $((\ell, 0), g, a, D, (\ell', 1)) \in E'$ if $\ell' \in F$; and $((\ell, 1), g, a, D, (\ell', 1)) \in E'$. The priority function p defined over locations of \mathcal{A}' is defined by $p((\ell, b)) = b$.

There is a natural bijection f between the set of initial paths of \mathcal{A} and the set of initial paths of \mathcal{A}' . An initial path $\pi = (\ell_0, \nu_0) \xrightarrow{m_0} (\ell_1, \nu_1) \dots$ of \mathcal{A} is mapped via f to the initial path $\bar{\pi} = ((\ell_0, b_0), \nu_0) \xrightarrow{m_0} ((\ell_1, b_1), \nu_1) \dots$ of \mathcal{A}' , where the sequence $(b_n)_{n \in \mathbb{N}}$ is $(0)_{n \in \mathbb{N}}$ if for all $n \in \mathbb{N}$, $\ell_n \notin F$ and otherwise, $b_n = 0$ for all $n < n^*$ and $b_n = 1$ for all $n \geq n^*$, where n^* denotes $\min\{n \in \mathbb{N} \mid \ell_n \in F\}$. This mapping is well-defined and bijective: the same moves are enabled in a state (ℓ, ν) of $\mathcal{T}(\mathcal{A})$ and in the state $((\ell, b), \nu)$ of $\mathcal{T}(\mathcal{A}')$ for all $b \in \{0, 1\}$. Furthermore, for all $b \in \{0, 1\}$ and all moves m enabled in (ℓ, ν) , $(\ell, \nu) \xrightarrow{m} (\ell', \nu')$ holds if and only if there is some b' such that $((\ell, b), \nu) \xrightarrow{m} ((\ell', b'), \nu')$

The bijection f preserves time-divergence. Furthermore, a path π of \mathcal{A} satisfies $\text{Safe}(F)$ if and only if $f(\pi)$ does not visit a location of the form $(\ell, 1)$. The initial paths of \mathcal{A}' that visit a location of the form $(\ell, 1)$ are exactly those that do not satisfy the (resp. direct) timed window objective: once such a location is entered, it is no longer possible to close windows as the set of locations $L \times \{1\}$ cannot be left by construction. Therefore, there is a time-divergent path of \mathcal{A} that does not satisfy $\text{Safe}(F)$ if and only if there is a time-divergent path of \mathcal{A}' that does not satisfy $\text{TW}(p, 1)$ (resp. $\text{DTW}(p, 1)$). Furthermore, the TA \mathcal{A}' has the same set of clocks as \mathcal{A} and twice as many locations and edges as \mathcal{A} , and the overall length of the encoding of the clock constraints of \mathcal{A}' is double that of \mathcal{A} . This shows that the safety verification problem can be reduced in polynomial time to the (resp. direct) timed window objective. This establishes PSPACE-hardness of the verification problem for (resp. direct) timed window objectives.

The same construction can be used for the realizability problem. There is an analogous mapping for initial plays. This mapping can be used to establish a bijection between the restriction of strategies over initial paths in the safety TG and the (resp. direct) timed window TG. It follows that the realizability problem for safety objectives can be reduced to the realizability problem for (resp. direct) timed window objectives in polynomial time, establishing EXPTIME-hardness of the realizability problem for (resp. direct) timed window objectives. \blacktriangleleft

6.3 Wrap-up

We summarize our complexity results in the following theorem.

► **Theorem 18.** *The verification problem for the (direct) generalized timed window parity objective is PSPACE-complete and the realizability problem for the (direct) generalized timed window parity objective is EXPTIME-complete.*

We conclude with a comparison of TGs with parity objectives and TGs with (resp. direct) timed window objectives. It was shown in [19] that TGs with parity objectives can be reduced to turn-based parity games on graphs. Their solution is as follows: to check if a \mathcal{P}_1 -winning strategy exists in a TG $\mathcal{G} = (\mathcal{A}, \Sigma_1, \Sigma_2)$ with the objective $\text{Parity}(p)$, they construct a turn-based game on a graph with $256 \cdot |S_{\text{Reg}}| \cdot |C| \cdot d$ states and $d + 2$ priorities, where S_{Reg} denotes the set of states of the region abstraction of \mathcal{A} , the size of which is bounded by $|L| \cdot |C|! \cdot 2^{|C|} \cdot \prod_{x \in C} (2c_x + 1)$. Despite recent progress on quasi-polynomial-time algorithms for parity games [16], there are no known polynomial-time algorithms, and, in many techniques, the blow-up is due to the number of priorities. Therefore, the complexity of checking the existence of a \mathcal{P}_1 -winning strategy in the TG \mathcal{G} for a parity objective suffers from a blow-up related to the number of priorities and the size of the region abstraction.

In contrast, our solution for TGs with (resp. direct) timed window objectives does not suffer from a blow-up due to the number of priorities: for a single dimension, the complexity given in (3) is polynomial in the size of the set of states of the region abstraction and in λ . This shows that adding time bounds to parity objectives in TGs comes for free complexity-wise.

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36 Timed Games with Window Parity Objectives

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