Kaluza-Klein black holes with squashed horizons and $d = 4$ superposed monopoles

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Abstract

We present new solutions of the $d = 5$ Einstein-Yang-Mills theory describing black holes with squashed horizons. These configurations are asymptotically locally flat and have a boundary topology of a fibre bundle $R \times S^1 \hookrightarrow S^2$. In a $d = 4$ picture, they describe black hole solutions with both nonabelian and U(1) magnetic charges.

1 Introduction

The last years have seen an increasing interest in the solutions of Einstein equations involving more than four dimensions. Solutions with a number of compact dimensions, present for $d \geq 5$ spacetime dimensions, are of particular interest, since they exhibit new features that have no analogue in the usual $d = 4$ case. Restricting to the case $d = 5$, the simplest configuration of this type is found by assuming translational symmetry along the extra coordinate direction and corresponds to a uniform vacuum black string with horizon topology $S^2 \times S^1$, approaching asymptotically the four dimensional Minkowski-space times a circle $\mathcal{M}^4 \times S^1$. The Kaluza-Klein (KK) black hole solutions [1] have a S^3 horizon topology and the same asymptotic structure, presenting a dependence on the compact extra dimension.

However, as shown by the Gross-Perry-Sorkin (GPS) monopole solution [2, 3], there are also $d = 5$ asymptotically locally flat configurations, approaching a twisted S^1 bundle over a four dimensional Minkowski spacetime. Black hole solutions with this type of asymptotics enjoyed recently some interest, following the discovery by Ishihara and Matsuno (IM) [4] of a new charged solution in the five dimensional Einstein-Maxwell theory. The horizon of the IM black hole has S^3 topology, and its spacelike infinity is a squashed sphere or S^1 bundle over S^2. The mass and thermodynamics of this solution have been discussed in [5]. A vacuum black hole solution with similar properties has been presented in [6]. As found in [7], the IM solution admits multi-black hole generalizations. KK rotating black hole solutions with squashed horizon in $d = 5$ Einstein gravity are discussed in [8]. Dilatonic generalizations of the IM black holes are studied in [9].

However, these studies restricted to the case of an abelian matter content. At the same time, a number of results obtained in the literature clearly indicates that the solutions of Einstein's equations coupled to nonabelian matter fields possess a much richer structure than the U(1) counterparts. In the five dimensional case, we note that without gravity, the pure Yang-Mills (YM) theory in a flat background admits topologically stable, particle-like and vortex-type solutions obtained by uplifting the $d = 4$ YM instantons and $d = 3$ Yang-Mills-Higgs (YMH) monopoles. However, as found in [10], the particle spectrum become completely destroyed by gravity. Assuming the metric and matter fields to be independent on the extra coordinate, the Einstein-Yang-Mills (EYM) equations present black string solutions [11]. These configurations approache at infinity the $\mathcal{M}^4 \times S^1$ background and possess a nontrivial zero event horizon radius limit [10]. We mention also the EYM particle-like and black hole solutions studied in [12], representing nonabelian generalizations of the Schwarzschild-Tangherlini solution [13]. These configurations are sphericaly symmetric in the five dimensional spacetime and are sustained by higher order terms in the YM hierarchy, approaching at infinity the $d = 5$ Minkowski background $\mathcal{M}^5$.

In this paper we study a different type of $d = 5$ EYM solutions, corresponding to black holes with squashed horizons, generalizing for a SU(2) field the abelian IM solution [4]. These solutions have a number of common features with the abelian counterparts, in particular the event horizons is a squashed S^3, and the asymptotic structure is similar to that of the GPS monopole. However, they possess a complicated structure, with several distinct branches and a maximal allowed value of the event horizon radius.

In a four dimensional picture, these configurations describe black hole solutions in a Einstein-Yang-Mills-Higgs-dilaton-U(1) (EYMHdU(1)) theory, possesing both nonabelian and U(1) magnetic charges.

2 The model

2.1 The action principle

We consider the EYM-SU(2) action with the Gibbons-Hawking boundary term [14],

$$I_5 = \int_{\mathcal{M}} d^5x\sqrt{-g}\Big(\frac{R}{16\pi G} - \frac{1}{2}\mathrm{Tr}\{F_{MN}F^{MN}\}\Big) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}K, \tag{1}$$

where h_{ij} is the induced metric on the boundary $\partial\mathcal{M}$ and K_{ij} is the extrinsic curvature of this boundary, with $K = K_{ij}h^{ij}$. Apart from gravity, (1) contains an SU(2) gauge field $A_M = \frac{1}{2}\tau_a A_M^{(a)}$, with the field strength tensor $F_{MN} = \partial_M A_N - \partial_N A_M - ig\,[A_M, A_M]$ and gauge coupling constant g.

Variation of the action (1) with respect to g^{MN} and A_M leads to the field equations

$$R_{MN} - \frac{1}{2}g_{MN}R = 8\pi G\; T_{MN}, \quad \nabla_M F^{MN} - ig[A_M, FMN] = 0. \tag{2}$$

where the YM stress-energy tensor is

$$T_{MN} = 2\mathrm{Tr}\{F_{MP}F_{NO}g^{PO} - \frac{1}{4}g_{MN}F_{BC}F^{BC}\}. \tag{3}$$

In what follows we will assume that both the matter functions and the metric functions are independent on the extra coordinate x^5. Without any loss of generality, we consider a five-dimensional metric parametrization (with $a = 2/\sqrt{3}$)

$$ds^2 = e^{-a\psi}\gamma_{\mu\nu}dx^\mu dx^\nu + e^{2a\psi}(dx^5 + 2\mathcal{W}_\mu dx^\mu)^2. \tag{4}$$

The four dimensional reduction of the $d = 5$ EYM theory with respect to the Killing vector $\partial/\partial x^5$ has been discussed in [15], $\mathcal{W}_\mu$ corresponding in this picture to a $d = 4$ U(1) potential. For the reduction of the YM action term, a convenient SU(2) ansatz is

$$A = \mathcal{A}_\mu dx^\mu + \Phi(dx^5 + 2\mathcal{W}_\mu dx^\mu), \tag{5}$$

where $\mathcal{A}_\mu$ is a purely four-dimensional nonabelian gauge field potential, while Φ corresponds after the dimensional reduction to a triplet Higgs field.

After Kaluza-Klein reduction with respect to the Killing vector $\partial/\partial x^5$, we find a $d = 4$ gravitating YMH model nontrivially interacting with a dilaton and a U(1) field

$$I_4 = \int d^4x\sqrt{-\gamma}\Big[\tfrac{1}{4\pi G}\big(\tfrac{\mathcal{R}}{4} - \tfrac{1}{2}\nabla_\mu\psi\nabla^\mu\psi - \tfrac{1}{4}e^{3a\psi}G_{\mu\nu}G^{\mu\nu}\big) - \tfrac{1}{2}e^{a\psi}\left(Tr\{\mathcal{F}'_{\mu\nu}\mathcal{F}'^{\mu\nu}\} + e^{-3a\psi}Tr\{D_\mu\Phi D^\mu\Phi\}\right)\Big]. \tag{6}$$

Here $\mathcal{R}$ is the Ricci scalar for the metric $\gamma_{\mu\nu}$, while $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - ig[\mathcal{A}_\mu, \mathcal{A}_\nu]$, $G_{\mu\nu} = \partial_\mu\mathcal{W}_\nu - \partial_\nu\mathcal{W}_\mu$ are the SU(2) and U(1) field strength tensors defined in $d = 4$ and we note $\mathcal{F}'_{\mu\nu} = \mathcal{F}_{\mu\nu} + 2\Phi G_{\mu\nu}$.

2.2 The ansatz and field equations

In this paper we consider a $d=5$ metric ansatz used in previous studies on Kaluza-Klein monopoles

$$ds^2 = -L(r)dt^2 + U(r)dr^2 + B(r)(d\theta^2 + \sin^2\theta d\phi^2) + F(r)(dx^5 + 4n\sin^2\frac{\theta}{2}d\varphi)^2, \tag{7}$$

Here θ and φ are the standard angles parametrizing an S^2 with ranges $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$, n being an arbitrary real constant. This spacetime has an isometry group $SO(3) \times U(1)$. A three dimensional surface $r = const., t = const.$ has the topology of the Hopf bundle, S^1 fiber over S^2 base space. To avoid a Dirac-Misner string singularity, the period of the extra coordinate x^5 is restricted to $8\pi n$.

The SU(2) YM ansatz compatible with the symmetries of the above line element reads

$$A = \frac{1}{2g}\Big\{u(r)\tau_3 dt + w(r)\tau_1 d\theta + (\cot\theta\tau_3 + w(r)\tau_2)\sin\theta d\varphi + H(r)(dx^5 + 4n\sin^2\frac{\theta}{2}d\varphi)\tau_3\Big\}, \tag{8}$$

τ_a corresponding to the Pauli matrices. In this work we will restrict to a $u(r) = 0$ consistent truncation of the above ansatz, the issue of dyonic generalizations being briefly discussed in Section 5. The equations satisfied by the $d=5$ metric functions and gauge potentials are

$$\begin{aligned}
&\tfrac{B'^2}{4B} + \tfrac{F'}{2F}(B' + \tfrac{BL'}{4L}) + \tfrac{L'}{2L}(B' + \tfrac{BF'}{4F}) + U(-1 + \tfrac{n^2F}{B}) - \tfrac{4\pi G}{g^2}(2w'^2 + \tfrac{B}{F}H'^2 - \tfrac{2U}{F}H^2w^2 - \tfrac{U}{B}(w^2-1+2nH)^2) = 0,\\
&F'' - \tfrac{B'}{3B}(\tfrac{FB'}{2B} - 2F' + \tfrac{FL'}{L}) - \tfrac{F'}{2}(\tfrac{U'}{U} + \tfrac{F'}{F} - \tfrac{2L'}{3L}) + \tfrac{2UF(B-7n^2F)}{3B^2}\\
&\quad - \tfrac{8\pi G}{g^2}\left(\tfrac{2F}{3B}w'^2 - \tfrac{5}{3}H'^2 - \tfrac{2U}{B}H^2w^2 + \tfrac{FU}{B^2}(w^2-1+2nH)^2\right) = 0,\\
&B'' + \tfrac{B'}{6}(-\tfrac{3U'}{U} - \tfrac{B'}{B} + \tfrac{F'}{F} + \tfrac{L'}{L}) - \tfrac{BF'L'}{6FL} + \tfrac{2U}{3B}(-2B + 5n^2F) + \tfrac{4\pi G}{g^2}(\tfrac{8}{3}w'^2 - \tfrac{2B}{3F}H'^2 + \tfrac{2U}{B}(w^2-1+2nH)^2) = 0,\\
&L'' - \tfrac{B'}{3B}(\tfrac{LB'}{2B} + \tfrac{LF'}{F} - 2L') - \tfrac{L'}{2}(\tfrac{U'}{U} - \tfrac{2F'}{3F} + \tfrac{L'}{L}) + \tfrac{2UL}{3B} - \tfrac{2n^2UFL}{3B^2}\\
&\quad - \tfrac{8\pi G}{g^2}(\tfrac{2L}{3B}w'^2 + \tfrac{L}{3F}H'^2 + \tfrac{2LU}{BF}H^2w^2 + \tfrac{UL}{B^2}(w^2-1+2nH)^2) = 0,\\
&w'' + w'(\tfrac{F'}{2F} + \tfrac{L'}{2L} - \tfrac{U'}{2U}) - w(\tfrac{U}{F}H^2 + \tfrac{U}{B}(w^2-1+2nH)) = 0,\\
&H'' - H'(\tfrac{F'}{2F} - \tfrac{B'}{B} + \tfrac{U'}{2U} - \tfrac{L'}{2L}) - \tfrac{2nUF}{B^2}(w^2-1+2nH) - \tfrac{2UH}{B}w^2 = 0.
\end{aligned} \tag{9}$$

However, we found more convinient to work with the metric ansatz (4) which allows a direct $d=4$ picture, by taking[1]

$$\gamma_{\mu\nu}dx^\mu dx^\nu = -N(r)\sigma^2(r)dt^2 + \frac{1}{N(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{10}$$

where $N(r) = 1 - 2m(r)/r$. The only nonvanishing component of the $U(1)$ potential is $\mathcal{W}_\varphi = 2n\sin^2\theta/2$, describing a U(1) magnetic monopole. The $d=4$ YM ansatz is $\mathcal{A} = \frac{1}{2}\Big\{w(r)\tau_1 d\theta + (\cot\theta\tau_3 + w(r)\tau_2)\sin\theta d\varphi\Big\}$, while the Higgs field has only one component $\Phi = \frac{1}{2}H\tau_3$.

For this choice, the equations (9) take a simpler form

$$\begin{aligned}
m' &= \tfrac{1}{2}r^2N\psi'^2 + e^{3a\psi}\tfrac{n^2}{2r^2} + \tfrac{4\pi G}{g^2}\left(e^{a\psi}(Nw'^2 + \tfrac{(w^2-1+2nH)^2}{2r^2}) + e^{-2a\psi}(\tfrac{1}{2}r^2NH'^2 + H^2w^2)\right),\\
\tfrac{\sigma'}{\sigma} &= \tfrac{8\pi G}{g^2 r}(e^{a\psi}w'^2 + e^{-2a\psi}\tfrac{r^2H'^2}{2}) + r\psi'^2,\\
(\sigma Nr^2\psi')' &= \sigma\left\{\tfrac{2n^2e^{3a\psi}}{ar^2} + \tfrac{4\pi G}{g^2}a\left[e^{a\psi}(Nw'^2 + \tfrac{(w^2-1+2nH)^2}{2r^2}) - 2e^{-2a\psi}(\tfrac{1}{2}r^2NH'^2 + H^2w^2)\right]\right\},\\
(e^{a\psi}\sigma Nw')' &= \sigma w(\tfrac{e^{a\psi}(w^2-1+2nH)}{r^2} + e^{-2a\psi}H^2),\\
(e^{-2a\psi}\sigma r^2NH')' &= 2\sigma(e^{-2a\psi}Hw^2 + \tfrac{ne^{a\psi}(w^2-1+2nH)}{r^2}).
\end{aligned} \tag{11}$$

One can see that for $n \neq 0$, one cannot consistently set $H = 0$ unless $\omega = \pm 1$, *i.e.* no gauge field. Thus, it is the Higgs field which supports the interaction of the four dimensional YM field with the U(1) monopole.

[1] The correspondence between the $d=4$ and $d=5$ pictures results straightforward from (4), (5). One finds *e.g.* $L = e^{-a\psi}N\sigma^2$, $F = e^{2a\psi}$, $U = e^{-a\psi}/N$, $B = e^{-a\psi}r^2$.

2.3 Known solutions

Solutions of the equations (9) are already known in a few particular cases. The vacuum black version of the GPS monopole presented in [6] is found for a pure gauge nonabelian configuration $w(r) = \pm 1, \quad H(r) = 0$ and has a metric form

$$ds^2 = -(1-\frac{2S}{r})dt^2 + (1+\frac{2p}{r})(\frac{dr^2}{1-\frac{2S}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)) + \frac{1}{1+\frac{2p}{r}}(dx^5 + 4n\sin^2\frac{\theta}{2}d\varphi)^2, \tag{12}$$

where $p = -S/2 + \sqrt{n^2 + S^2/4}$, the GPS monopole solution being recovered for $S = 0$. After KK reduction with respect to the Killing vector $\partial/\partial x^5$, one finds $d = 4$ magnetically charged black hole solutions.

The embedded U(1) IM configuration [4] corresponds to

$$\begin{aligned} ds^2 = & \frac{r(r+r_0)}{(r-r_-)(r-r_+)}dr^2 + r(r+r_0)(d\theta^2 + \sin^2\theta d\varphi^2) - \frac{(r-r_-)(r-r_+)}{r^2}dt^2 \\ & + \frac{r(r+r_-)(r+r_+)}{4n^2(r+r_0)}(dx^5 + 4n\sin^2\frac{\theta}{2}d\varphi)^2, \quad A = \pm\frac{1}{4r}\sqrt{\frac{3r_+r_-}{\pi G}}\tau_3 dt. \end{aligned} \tag{13}$$

It may be worth noting that when viewed in a four dimensional perspective, the IM solution describes black holes in a theory with two U(1) fields which couple different to the dilaton[2]. One of these fields has an electric charge and the other corresponds to a Dirac monopole.

For $n = 0$, the system (9) presents nonabelian black string solutions originally found in [11]. In the limit of zero event horizon radius, the nonabelian vortices discussed in [10] are recovered. No exact solutions with reasonable asymptotics are available analytically in this case and the field equations have to be solved numerically. A detailed analysis of the properties of the black string solutions have been presented in [16]. It has been found that the pattern of solutions is very similar to that observed for non-abelian vortices, depending crucially on the value of the gravitational coupling constant $\alpha = \sqrt{4\pi G}H_0/g$ (with H_0 the asymptotic value of the $H-$function). Several branches of solutions exist and the extend of the branches in α gets smaller and smaller for successive branches.

2.4 Boundary conditions

In this work we will consider black hole solutions of the system (9), with an event horizon located at $r = r_h > 0$. As $r \to r_h$, we have $N(r_h) = 0$, while the other functions stay finite and nonzero. The formal power series expansion near the event horizon in terms of the functions which enters the $d = 4$ picture is

$$\begin{aligned} m(r) &= \frac{r_h}{2} + m'_h(r-r_h) + O(r-r_h)^2, \quad \sigma(r) = \sigma_h + \sigma'_h(r-r_h) + O(r-r_h)^2, \\ \psi(r) &= \psi_h + \psi'_h(r-r_h) + O(r-r_h)^2, \quad H(r) = H_h + H'_h(r-r_h) + O(r-r_h)^2, \\ w(r) &= w_h + w'_h(r-r_h) + O(r-r_h)^2, \end{aligned} \tag{14}$$

where

$$\begin{aligned} m'_h &= \frac{n^2e^{3a\psi_h}}{2r_h^2} + \frac{4\pi G}{g^2}\left(e^{a\psi_h}\frac{(w_h^2-1+2nH_h)^2}{2r_h^2} + e^{-2a\psi_h}H_h^2w_h^2\right), \\ \psi'_h &= \frac{1}{r_h^2N'_h}\left\{\frac{2n^2e^{3a\psi_h}}{ar_h^2} + a\frac{4\pi G}{g^2}\left(e^{a\psi_h}\frac{n(w_h^2-1+2nH_h)^2}{2r_h^2} - 2e^{-2a\psi_h}H_h^2w_h^2\right)\right\}, \\ w'_h &= \frac{w_h}{N'_h}\left(\frac{(w_h^2-1+2nH_h)}{r_h^2} + e^{-3a\psi_h}H_h^2\right), \quad H'_h = \frac{2}{r_h^2N'_h}\left(H_hw_h^2 + e^{3a\psi_h}\frac{n(w_h^2-1+2nH_h)}{r_h^2}\right), \\ \sigma'_h &= \sigma_h\left(r_h\psi_h'^2 + \frac{8\pi G}{g^2r_h}(e^{a\psi_h}w_h'^2 + e^{-2a\psi_h}\frac{r_h^2H_h'^2}{2})\right), \end{aligned} \tag{15}$$

[2] The corresponding $d = 4$ action principle can be read from (6) by taking $\Phi = 0$ and an abelian subgroup of SU(2).

with $\sigma_h,\ w_h,\ H_h,\ \ \psi_h$ being arbitrary parameters, while $N'_h = (1-2m'_h)/r_h$.

We are interested in solutions of the $d=5$ EYM equations whose metric asymptotic structure is similar to that of the GPS monopole, *i.e.* with a line element as $r \to \infty$

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + (dx^5 + 4n\sin^2\frac{\theta}{2}d\varphi)^2. \tag{16}$$

The analysis of the field equations as $r \to \infty$ for this metric asymptotics gives

$$m(r) = M + \frac{m_1}{r} + \dots,\quad \sigma(r) = 1 + \frac{s_2}{r^2} + \dots,\quad \psi(r) = \frac{\psi_1}{r} + \frac{\psi_2}{r^2} + \dots, \tag{17}$$
$$H(r) = H_0 + \frac{h_1}{r} + \frac{h_2}{r^2} + \dots,\quad w(r) = c_2 e^{-H_0 r} + \dots,$$

where

$$\psi_2 = \frac{3}{4}an^2 + \psi_1 M + \frac{\pi a G}{g^2}((1-2H_0 n)^2 - 2h_1^2),\quad h_2 = h_1(a\psi_1 + M) + n(2H_0 n - 1),$$
$$m_1 = -\frac{1}{2}(\psi_1^2 + n^2) - \frac{2a\pi G}{g^2}(h_1^2 + (1-2H_0 n)^2),\quad s_2 = -\frac{\phi_1^2}{2} - \frac{2\pi G}{g^2}h_1^2,$$

with $M,\ \psi_1,\ H_0,\ h_1$ and c_2 real constants.

After dimensional reduction, the $d=5$ EYM configurations become four dimensional spherically symmetric black hole solutions of the EYMHdU(1) model (6), the event horizon still being located at $r=r_h$. They will possess a unit nonabelian magnetic charge and a U(1) magnetic charge $Q_m = 2n$, approaching asymptotically the $\mathcal{M}^4$ background. The constant M in (17) corresponds to the total mass of the $d=4$ configurations, while ψ_1 gives the dilaton charge.

3 Numerical solutions

Although an analytic or approximate solution of the equations (11) appears to be intractable, here we present arguments for the existence of nontrivial solutions which smoothly interpolate between the asymptotic expansions (14), (17). We restrict our integration to the region outside the event horizon[3]. $n \neq 0$ generalizations are found for any black string EYM configuration by slowly increasing the parameter n (since the transformation $n \to -n$ leaves the field equations unchanged except for the sign of the H, we consider here only positive values of n).

To find dimensionless quantities with the right asymptotics, we use the observation that the equations are left invariant by the following rescaling

$$r \to e^{a\phi_0/2}H_0\ r,\quad H \to e^{a\phi_0}H/H_0,\quad m \to m\ e^{a\phi_0/2}H_0,\quad n \to n\ e^{-a\phi_0}H_0. \tag{18}$$

where H_0 is the asymptotic value of $H(r)$ and ϕ_0 an arbitrary constant. Thus, similar to the black string case, the constant of the theory are absorbed into the coupling constant $\alpha = \sqrt{4\pi G}H_0/g$, which is an input parameter. One can take advantage of the this double rescaling and set in the numerical analysis $H(\infty) = 1$ and $\psi(\infty) = 0$ without loosing any generality.

As expected, these configurations have many features in common with the black string solutions discussed in [11]; they also present new features that we will pointed out in the discussion.

All solutions we found have a positive asymptotic value of the metric function $m(r)$. The gauge functions $\omega(r)$ and $H(r)$ interpolates monotonically between some constant values on the event horizon and zero respectively one at infinity, without presenting any local extremum (see Figure 1). For all $r_h > 0$, the dilaton function takes a finite value at the event horizon. The profiles of two typical profiles are presented in Figure 1.

[3]To integrate the equations, we used the differential equation solver COLSYS which involves a Newton-Raphson method [17].

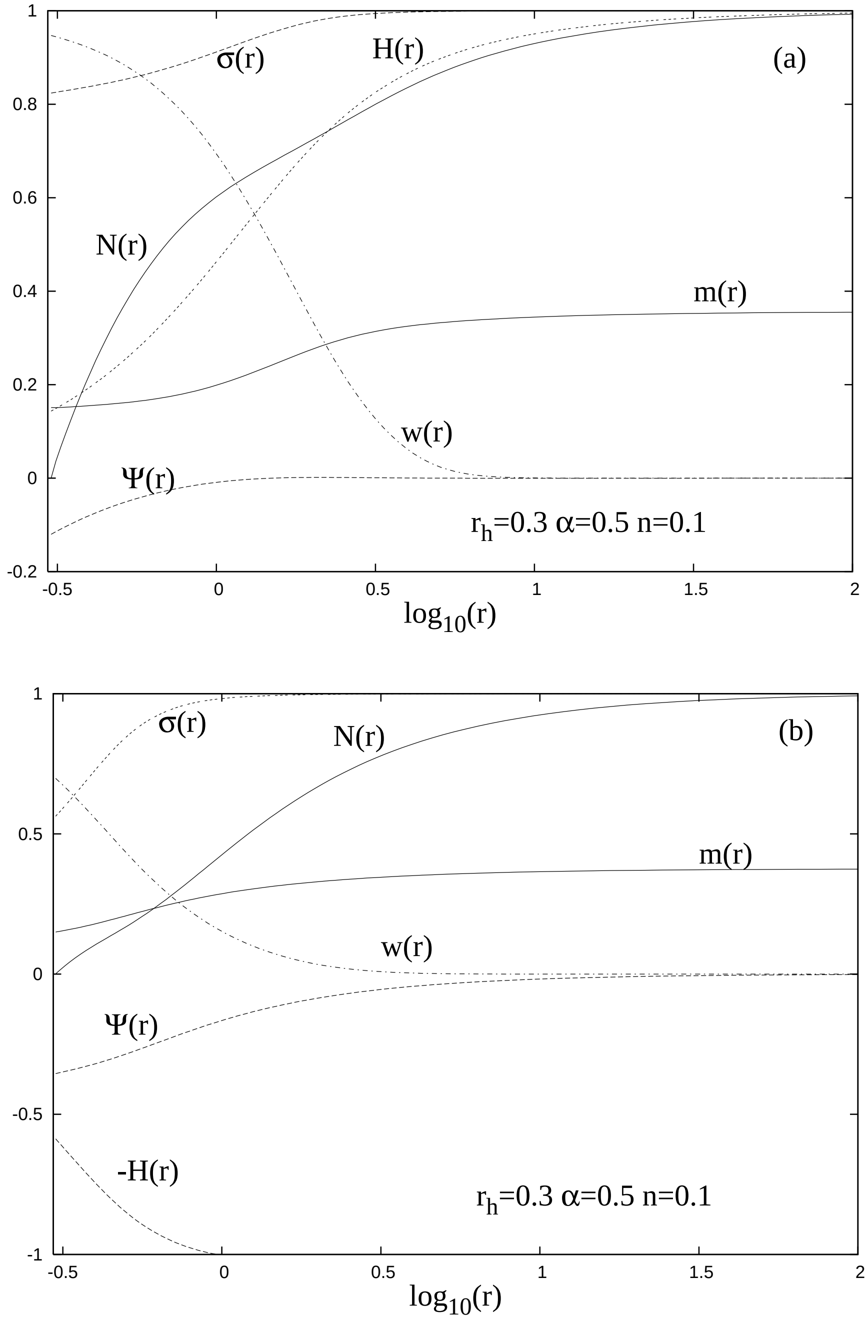

Figure 1. The profiles of the functions N, m, σ, w, H, ψ are represented for typical first (Figure 1a) and second branch (Figure 1b) solutions with $\alpha = 0.5,\ r_h = 0.3,\ n = 0.1$.

In the near-horizon region, the area of the squashed spheres is proportional to r^3. Thus, similar to the IM case, there is a region such that the solution will behave as a five-dimensional black hole for observers in that region. Far away from the event horizon, the metric function $g_{55} \equiv e^{2a\psi}$ is almost constant and the spacetime is effectively four dimensional.

The complete classification of the solutions in the space of parameters $(\alpha,\ n,\ r_h)$ is a considerable task which is not aimed in this paper. Instead, we analyzed in details a few particular classes of solutions which, hopefully, reflect all relevant properties of the general pattern. It is important to remember that for $n = 0$ the EYM equations admit several branches of black string solutions, the number of them depending on r_h and of α [11].

For simplicity, we have studied mainly solutions in the region $\alpha = 0.5$ where it is known that two branches

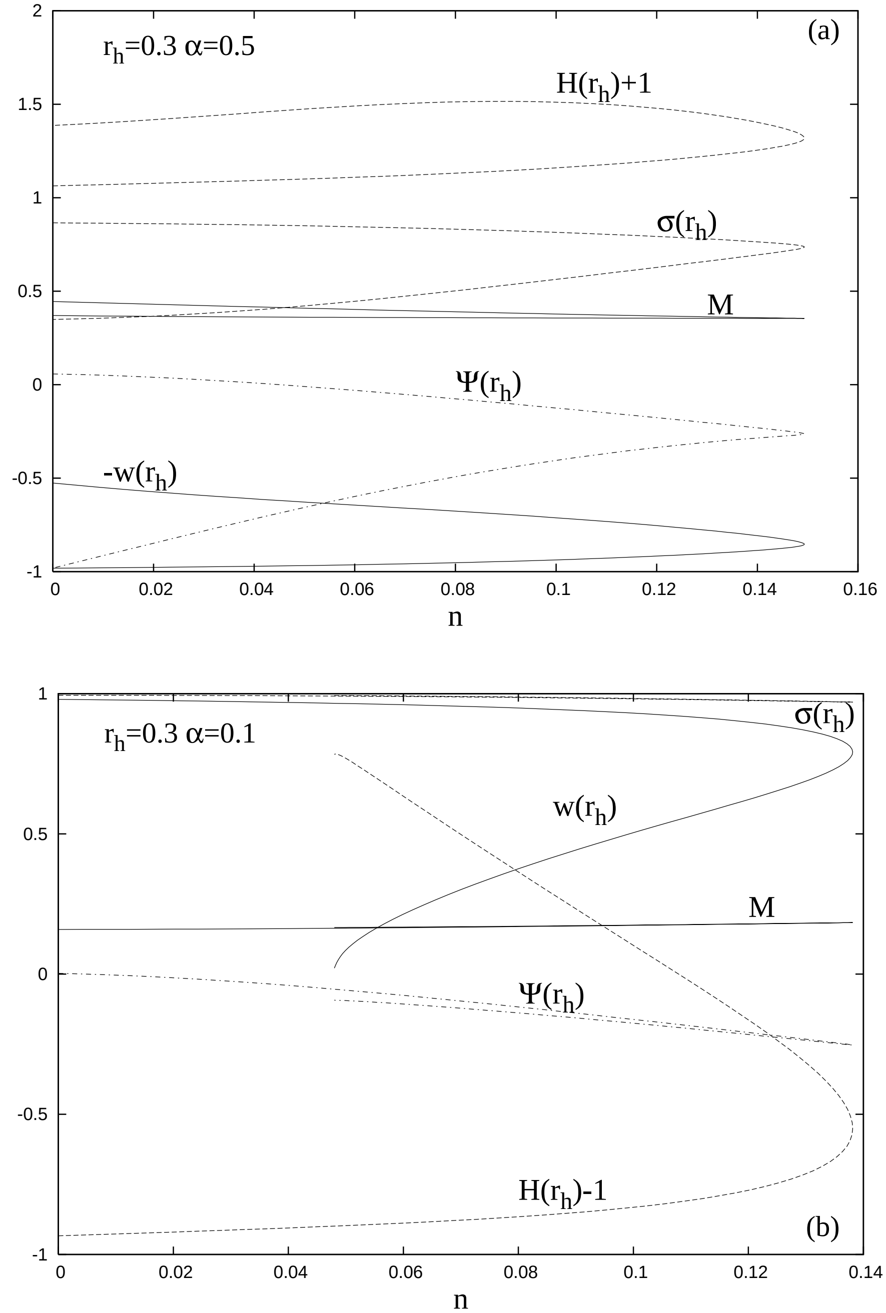

Figure 2. Several relevant quantities are plotted as a function of n for $r_h = 0.3$ and two different values of α.

of black string solutions exist. The $\alpha = 0.5$ branches of $n = 0$ black string solutions exist for $0 < r_h < r_{h,max}$ with $r_{h,max} \approx 0.65$. In the limit $r_h \to 0$, the solutions converge to regular solutions first constructed in [10]. The regular solutions have $N(0) = 1$ so that the convergence is not pointlike at $r = 0$. Accordingly, the surface gravity κ of the black string (given by $\kappa = \sigma(r_h)N'(r_h)/2$) becomes infinite for $r_h \to 0$; all other parameters, *e.g.* the dilaton and its derivative remain finite. We expect the features of $n \neq 0$ solutions we will discuss for $\alpha = 0.5$ to be qualitatively the same when more than two branches of solutions occur.

We have tried to understand the domain of existence of these solutions when the parameter n varies while the other parameters are kept fixed. It turns out that (at least in the region for $\alpha = 0.5$) the two black string solutions existing for $n = 0$ get slowly deformed for $n > 0$, forming two different branches extending in n. These two distinct branches join at a maximal value of n. Our numerical analysis, strongly suggests that there are no solutions for n larger than this maximal value. This phenomenon is illustrated on Figure 2a where a few parameters characterizing these solutions are reported as functions of n. With this choice of

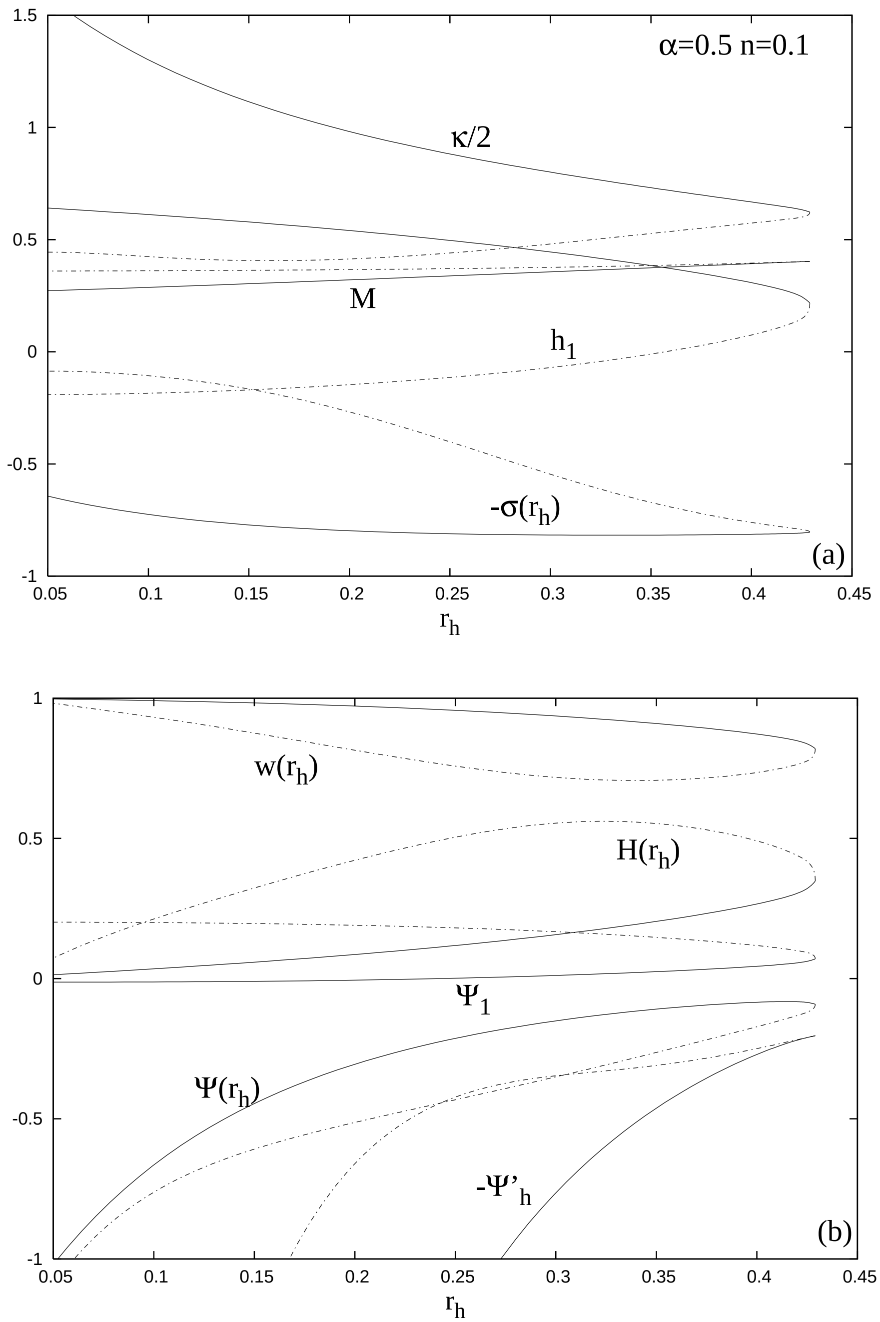

Figure 3. The dependence of solution properties on the value of the event horizon radius r_h is plotted for $\alpha = 0.5$, $n = 0.1$. The dotted line indicates the higher mass branch solutions. $\kappa = \sigma(r_h)N'(r_h)/2$ is the surface gravity of the black hole solutions.

the parameters ($\alpha = 0.5,\ r_h = 0.3$) the two branches terminate at $n \simeq 0.1515$.

However, for small values of α (typically $\alpha < 0.25$) only one black string exists for a given r_h [11]. The numerical analysis shows that these black strings are deformed by n, forming a branch which terminates at a maximal value of n (*e.g.* $n \simeq 0.138$ for $\alpha = 0.1$ and $r_h = 0.3$). Then we manage to construct a second branch of solutions, terminating at the same value, say $n = n_{max}$. This second branch, however exists only on an interval $n \in [n_{cr}, n_{max}]$ (with $n_{cr} \approx 0.05$ in our case, see Figure 2b). In the limit $n \to n_{cr}$, the solution converges to a configuration with $w(r) = 0$, *i.e.* the Yang-Mills field become a Wu-Yang monopole. The function $H(r)$ remains non trivial and, surprisingly, it has $H(r_h) > 1$ and decreases monotonically to its asymptotic value $H(\infty) = 1$. The second branch solutions have no counterparts in the $n = 0$ case.

Different from the U(1) case, the EYM configurations exist for a limited range of the event horizon radius only. The analysis of the behaviour of the solutions in the limit $r_h \to 0$ is clearly interesting. For $n > 0$, the situation deeply contrasts from the case $n = 0$, where a $d = 4$ regular limiting solution is available. Indeed, our numerical analysis strongly suggest that in the limit $r_h \to 0$ the $d = 4$ black hole solution converge to a singular configuration characterized by a diverging value of $\psi(0)$ and of $\psi'(0)$. This agrees with the physical intuition that no nonsingular particle-like Dirac monopole can exist. However, these configurations are globally regular in a five-dimensional picture, describing SU(2) generalizations of the GPS monopoles. $r = 0$ corresponds here to the origin of the coordinate system and is a regular point.

These features are illustrated on Figure 3 where the parameters α and n are fixed, while the horizon radius r_h is varied. The occurence of two branches of solutions terminating at a maximal value of r_h is clearly visible on that plot; in this case we have $r_{h,max} \approx 0.429$. The numerical construction of the solutions turns out to be difficult for small values of r_h but the results reported on Figure 3 suggest that $w(r_h) \to 1$ and $H(r_h) \to 0$ in the limit $r_h \to 0$ for both branches. This indicates that the non Abelian character of the solution persists in the $r_h \to 0$ limit.

The limiting $d = 5$ particle-like solutions are found by solving directly the system (9), with the boundary conditions $L(0) = L_0 > 0,\ U(0) = 1,\ F(0) \sim O(r^2)$, where we fix the metric gauge by taking $B(r) = r^2$. The boundary conditions satisfied by gauge field potentials are $w(0) = 1$, $H(0) = 0$. The asymptotics at infinity of the particle-like solutions are similar to those of the black hole counterparts. This behaviour strongly contrasts with that of the globally regular vortices of [10] or the particle-like solutions in [12]. A systematic study of the particle-like solutions emerging in the $r_h \to 0$ limit will be presented elsewhere.

4 The mass of the $d = 5$ solutions

The construction of the conserved quantities for solutions with the asymptotic structure (16) is an interesting problem. At a conceptual level, the background subtraction method is not entirely satisfactorly, since it relies on the introduction of a spacetime which is auxiliarly to the problem and is not obvious in this case.

Inspired by the AdS results [18], Kraus *et al.* have proposed in [19] a counterterm in a five-dimensional asymptotically flat spacetime with boundary topology $R \times S^3$ or $R^2 \times S^2$. By taking the variation of the action plus this counterterm part with respect to the boundary metric h_{ij}, one finds the boundary stress-energy tensor [20], and then define the conserved charge associated with some Killing vector of the boundary metric (see [21] for further work in this direction).

The computation of the action and conserved charges of a Kaluza-Klein monopole within this approach has been done by Mann and Stelea [22], who have proposed a counterterm expression[4]

$$I_{ct} = \frac{1}{8\pi G}\int d^4x\sqrt{-h}\sqrt{2\mathsf{R}}\,, \tag{19}$$

where R is the Ricci scalar of the induced metric on the boundary. With this counterterm, the boundary stress-energy tensor is found to be

$$T_{ij} = \frac{1}{8\pi G}\left(K_{ij} - Kh_{ij} - \Psi(\mathsf{R}_{ij} - \mathsf{R}h_{ij}) - h_{ij}h^{kl}\Psi_{;kl} + \Psi_{;ij}\right)\,, \tag{20}$$

where K is the trace of extrinsic curvature K_{ij} of the boundary, and $\Psi = \sqrt{2/\mathsf{R}}$. If the boundary geometry has an isometry generated by a Killing vector ξ^i, then $T_{ij}\xi^j$ is divergence free, from which it follows that the quantity

$$\mathcal{Q} = \int_\Sigma d\Sigma_i T^i{}_j\xi^j, \tag{21}$$

associated with a closed surface Σ, is conserved.

[4]In this Section we do not take the rescaling (18).

Using the asymptotic expression (17) for the metric functions $m(r)$, $\sigma(r)$ and $\psi(r)$ we find the boundary stress-energy tensor of the EYM solutions[5]

$$
\begin{aligned}
8\pi G T^{t}_{\ t} &= \frac{2M}{r^2}+O(1/r^3), \quad 8\pi G T^{5}_{\ 5}=\frac{2M-3a\psi_1}{2r^2}+O(1/r^3), \\
8\pi G T^{\varphi}_{\varphi} &= 8\pi G T^{\theta}_{\theta}=-\frac{M+2m_1+n^2+4s_2}{2r^3}+O(1/r^4), \quad 8\pi G T^{5}_{\ \varphi}=\frac{2n(2M-3a\psi_1)\sin^2\theta/2}{r^2}+O(1/r^3).
\end{aligned}
\tag{22}
$$

The solutions' mass is the conserved charge associated with the Killing vector $\partial/\partial t$ of the boundary metric

$$
\mathcal{M}=\frac{8\pi M n}{G}. \tag{23}
$$

As usual, a positive-definite metric is found by considering in (7) the analytical continuation $t\to it$. In this case, the absence of conical singularities at the root r_h of the function $N(r)$ imposes a periodicity in the Euclidean time coordinate

$$
\beta=\frac{4\pi}{N'(r_h)\sigma(r_h)}\ , \tag{24}
$$

the Hawking temperature being $T_H=1/\beta$.

One can also prove that although the horizon of these black holes is deformed, their entropy still obeys the area formula. One starts by evaluating the classical tree-level action I_5 [14], where the R volume term is replaced with $2R^t_t-16\pi G T^t_t$. For our purely magnetic ansatz, the term T^t_t exactly cancels the matter field lagrangean in the bulk action $L_m=-1/2g^2Tr(F_{MN}F^{MN})$ (see also the general discussion in [23]). To evaluate the integral of R^t_t one uses the Killing identity $\nabla^M\nabla_N\zeta_M=R_{NM}\zeta^M$, for the Killing vector $\zeta^M=\delta^M_t$. As expected, the counterterm action (19) regularizes also the action (1) and, upon application of the Gibbs-Duhem relation $S=\beta\mathcal{M}-I_5$ we find the entropy $S=8\pi^2 n r_h^2$, which is one quarter of the event horizon area.

5 Further remarks

Black objects in $d=5$ dimensions have a much richer spectrum of horizon topology than the four dimensional solutions. The black hole solutions with an S^3 horizon topology and approaching asymptotically a twisted S^1 bundle over a four dimensional Minkowski spacetime are a particularly interesting case. For such solutions, the spacetime behaves as a five-dimensional black hole near the horizon, while the dimensional reduction to four is realized in the far region.

In this work we have analysed the basic properties of this type of solutions in EYM-SU(2) theory. We have found that despite the existence of a number of similarities to the U(1) IM solution, the nonabelian configurations exhibit some new qualitative features, in particular a complicated branch structure and a different zero event horizon radius limit. ¿From a four dimensional perpective, these solutions correspond to dilatonic-Reissner-Nordström black holes sitting inside the center of a nonabelian monopole.

It would be interesting to consider the axially symmetric generalizations of these configurations, with a $d=5$ ansatz presenting a nontrivial dependence on the azimuthal coordinate θ. The static $d=4$ solutions will also be axially symmetric, with a U(1) field possessing both a U(1) magnetic monopole charge and a magnetic dipole moment.

Following [16], new interesting $d=4$ solutions can be found by boosting the $d=5$ solutions in the (x^5,t)-plane, $x^5=\cosh\beta\ U+\sinh\beta\ T$, $\ t=\sinh\beta\ U+\cosh\beta\ T$, where β is an arbitrary parameter. The dimensional reduction of the $d=5$ EYM configurations along the U-direction provides new solutions of the EYMHd-U(1) model (6) [16]. The resulting $d=4$ line element reads

$$
\bar{\gamma}_{\mu\nu}dx^\mu dx^\nu=-e^{a(\psi-\bar{\psi})}N\sigma^2(dT-4n\sinh\beta\sin^2\frac{\theta}{2}d\varphi)^2+e^{a(\bar{\psi}-\psi)}\left(\frac{dr^2}{N}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right), \tag{25}
$$

[5]Note that a different counterterm choice leads to a different expression of the T^θ_θ, T^φ_φ components of the boundary stress tensor, but the same action and conserved charges.

where $\bar{\psi} = \psi + \frac{1}{2a}\log(\cosh^2\beta - \sinh^2\beta e^{-3a\psi}N\sigma^2)$ is the new dilaton field. The new $U(1)$ field has both electric and magnetic components

$$\bar{\mathcal{W}} = \frac{2ne^{2a\psi}\cosh\beta\ \sin^2\frac{\theta}{2}}{e^{2a\psi}\cosh^2\beta - e^{-a\psi}N\sigma^2\sinh^2\beta}d\varphi + \frac{1}{2}\frac{(e^{2a\psi} - e^{-a\psi}N\sigma^2)\sinh\beta\cosh\beta}{e^{2a\psi}\cosh^2\beta - e^{-a\psi}N\sigma^2\sinh^2\beta}dT. \tag{26}$$

The $\mathcal{A}_r$ and $\mathcal{A}_\theta$ components of the $d = 4$ SU(2) gauge field are not affected, while

$$\bar{A_\varphi} = A_\varphi - 4\Phi n\sin^2\frac{\theta}{2}\frac{e^{-2a\psi}N\sigma^2\sinh^2\beta}{e^{2a\psi}\cosh^2\beta - e^{-a\psi}N\sigma^2\sinh^2\beta}\ ,\quad \bar{A_T} = \Phi\sinh\beta\frac{e^{-a\psi}N\sigma^2}{e^{2a\psi}\cosh^2\beta - e^{-a\psi}N\sigma^2\sinh^2\beta}, \tag{27}$$

the new Higgs field being $\bar{\Phi} = \Phi\cosh\beta$. Different from the seed solution, these configurations possess a nut-charge $\bar{n} = n\sinh\beta$, and are dilatonic-U(1) generalizations of the "nutty dyons" discussed in [24], representing the nonabelian version of the Brill solution [25]. The gauge fields possess in this case both electric and magnetic charges.

A systematic discussion of these aspects, together with higher winding number generalizations of the black hole solutions and particle-like configurations will be presented elsewhere.

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