Nodal properties of eigenfunctions of a generalized buckling problem on balls

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Abstract

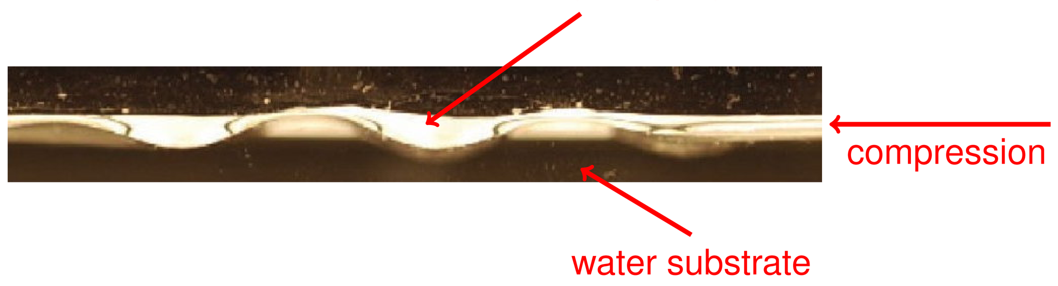
We are interested in the following fourth order eigenvalue problem coming from the buckling of thin films on liquid substrates:

$$\begin{cases} \Delta^2 u + \kappa^2 u = -\lambda \Delta u & \text{in } B_1,\\ u = \partial_r u = 0 & \text{on } \partial B_1, \end{cases}$$

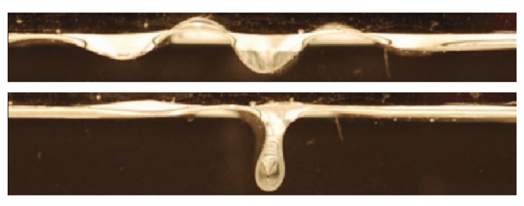
where B_1 is the unit ball in $\mathbb{R}^N$. When $\kappa > 0$ is small, we show that the first eigenvalue is simple and the first eigenfunction, which gives the shape of the film for small displacements, is positive. However, when κ increases, we establish that the first eigenvalue is not always simple and the first eigenfunction may change sign.

1. Physical motivation

Clamped thin elastic membranes supported on a fluid substrate



When the compression is large:



2. Mathematical model

If Ω is the reference domain, the shape of the film after compression is given by the function $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ (which represents the vertical displacement of the film) which minimizes

$$H_0^2(\Omega_\varepsilon) \to \mathbb{R} : v \mapsto \underbrace{\int_{\Omega_\varepsilon} |\Delta v|^2}_{\text{bending}} + \kappa^2 \underbrace{\int_{\Omega_\varepsilon} v^2}_{\text{potential energy}}$$

under the constraint that the membrane can bend but not stretch, thus that its total area does not change.
As $\varepsilon \to 0$, u_ε behaves like u where $u \in H^2(\Omega) \setminus \{0\}$ satisfies (see [2]):

$$\begin{cases} \Delta^2 u + \kappa^2 u = -\lambda \Delta u & \text{in } \Omega,\\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

For the case $\Omega =]-L, L[\subseteq \mathbb{R}$, see [3]. For related works, we refer to [1, 4, 5].

Here we consider the case $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}, N = 2, \kappa > 0$.

3. The spectrum

Proposition 1. *The eigenfunctions of the boundary value problem* (1) *are of the form $u = R(r)\,e^{\mathbf{i}k\theta}$ with $k \in \mathbb{Z}$ and R given by*

$$R(r) := cJ_{|k|}(\alpha r) + dJ_{|k|}\Big(\frac{\kappa}{\alpha} r\Big), \tag{2}$$

where J_k denotes the Bessel function of first kind and $\alpha \neq \sqrt{\kappa}$ is a positive solution to

$$F_k(\alpha) := \frac{\kappa}{\alpha} J_{|k|}(\alpha) J'_{|k|}\Big(\frac{\kappa}{\alpha}\Big) - \alpha J_{|k|}\Big(\frac{\kappa}{\alpha}\Big) J'_{|k|}(\alpha) = 0. \tag{3}$$

The corresponding eigenvalue is $\lambda = \alpha^2 + \kappa^2/\alpha^2$.

Sketch of the proof.
The equation

$$\Delta^2 u + \kappa^2 u = -\lambda \Delta u \tag{4}$$

can be written under the form

$$(\Delta + \alpha^2)(\Delta + \beta^2)u = 0,$$

with $\alpha\beta = \kappa$ and $\alpha^2 + \beta^2 = \lambda$.

Using the ansatz $u(r, \theta) = R(r)\,e^{\mathbf{i}k\theta}$ with $k \in \mathbb{Z}$, we have

$$\begin{aligned} (\Delta + \alpha^2)(R(r)\,e^{\mathbf{i}k\theta}) = 0 &\Leftrightarrow (r\partial_r)^2 R + \alpha^2 r^2 R = k^2 R\\ &\Leftrightarrow R(r) = cJ_{|k|}(\alpha r). \end{aligned}$$

Hence a solution to (4) is in the form (2), with $c, d \in \mathbb{C}$. The boundary conditions in $r = 1$ give a 2×2 system:

$$\begin{cases} c\,J_{|k|}(\alpha) + d\,J_{|k|}(\frac{\kappa}{\alpha}) = 0,\\ c\,\alpha J'_{|k|}(\alpha) + d\,\frac{\kappa}{\alpha} J'_{|k|}(\frac{\kappa}{\alpha}) = 0. \end{cases} \tag{5}$$

That leads to (3). □

4. Roots of F_k

Theorem 1. *For all $k \in \mathbb{N}$ and $\kappa > 0$, there exists an increasing sequence $\alpha_{k,\ell} = \alpha_{k,\ell}(\kappa) > 0$, with $\ell \in \mathbb{Z}$, solutions to* (3) *s.t.*

$$\begin{gathered} \forall \ell \geq 0, \qquad \alpha_{k,-\ell} = \frac{\kappa}{\alpha_{k,\ell}},\\ \alpha_{k,0} = \sqrt{\kappa} \quad \text{and,} \quad \forall \ell > 0, \quad \alpha_{k,\ell} > \sqrt{\kappa} > \alpha_{k,-\ell},\\ \alpha_{k,\ell} \to +\infty \text{ if } \ell \to +\infty,\\ \alpha_{k,\ell} \to 0 \text{ if } \ell \to -\infty. \end{gathered}$$

Sketch of the proof. By the asymptotic behaviour of the Bessel functions, we have

$$F_k(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\kappa^k}{2^k\, k!\, \alpha^{k-1/2}} \Big(\cos(\alpha - \tfrac{2k+3}{4}\pi) + o(1)\Big) \text{ as } \alpha \to +\infty.$$

□

Lemma 1 (Intersection points)**.** *$\alpha \neq \sqrt{\kappa}$ is a solution to $F_k(\alpha) = 0$ and $F_{k+1}(\alpha) = 0$ iff there exist m and n s.t.*

- $\alpha = j_{k,n}$, $\kappa/\alpha = j_{k,m}$ *and* $\kappa = j_{k,m}\, j_{k,n}$, *or*
- $\alpha = j_{k+1,n}$, $\kappa/\alpha = j_{k+1,m}$ *and* $\kappa = j_{k+1,m}\, j_{k+1,n}$.

Sketch of the proof. We have

$$\begin{aligned} F_k(\alpha) &= \alpha J_k\Big(\frac{\kappa}{\alpha}\Big) J_{k+1}(\alpha) - \frac{\kappa}{\alpha} J_k(\alpha) J_{k+1}\Big(\frac{\kappa}{\alpha}\Big).\\ F_{k+1}(\alpha) &= \frac{\kappa}{\alpha} J_k\Big(\frac{\kappa}{\alpha}\Big) J_{k+1}(\alpha) - \alpha J_k(\alpha) J_{k+1}\Big(\frac{\kappa}{\alpha}\Big). \end{aligned}$$

If $F_k(\alpha) = 0$, we obtain

$$F_{k+1}(\alpha) = \frac{\kappa^2 - \alpha^4}{\kappa\alpha} J_{k+1}(\alpha) J_k\Big(\frac{\kappa}{\alpha}\Big).$$

□

Proposition 2. *For all $\kappa > 0$, we have*

$$\bar{\alpha}(\kappa) := \min\{\alpha_{k,\ell}(\kappa) \mid k \in \mathbb{N}, \ell \geq 1\} = \min\{\alpha_{0,1}(\kappa), \alpha_{1,1}(\kappa)\}.$$

Moreover, the first eigenvalue $\lambda_1(\kappa)$ is given by $\lambda_1(\kappa) = \bar{\alpha}^2(\kappa) + \kappa^2/\bar{\alpha}^2(\kappa)$.

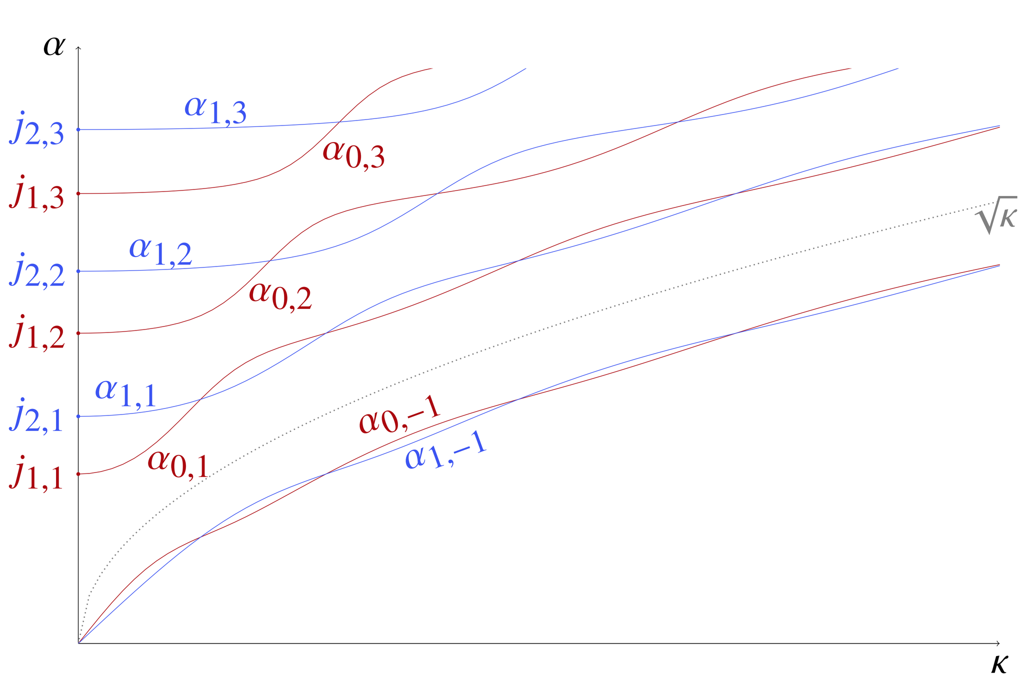


Figure 1: *Some roots of F_0, F_1*

5. Structure of the first eigenspace

Theorem 2. *Denote $R_{k,\ell}$ a function defined by equation* (2) *with (c, d) a non-trivial solution of* (5) *and $\alpha = \alpha_{k,\ell}$ with $\alpha_{k,\ell}$ given by Theorem 1.*

- *If $\kappa \in [0, j_{0,1}j_{0,2}[$, the first eigenvalue is simple and is given by $\lambda_1(\kappa) = \alpha_{0,1}^2(\kappa) + \kappa^2/\alpha_{0,1}^2(\kappa)$ and the eigenfunctions φ_1 are radial, one-signed and $|\varphi_1|$ is decreasing with respect to r.*
- *If $\kappa \in]j_{1,n}j_{1,n+1}, j_{0,n+1}j_{0,n+2}[$, for some $n \geq 1$, the first eigenvalue is simple and given by $\lambda_1(\kappa) = \alpha_{0,1}^2(\kappa) + \kappa^2/\alpha_{0,1}^2(\kappa)$ and the eigenfunctions are radial and have $n + 1$ nodal regions.*
- *If $\kappa \in]j_{0,n+1}j_{0,n+2}, j_{1,n+1}j_{1,n+2}[$, for some $n \geq 0$, the first eigenvalue is given by $\lambda_1(\kappa) = \alpha_{1,1}^2(\kappa) + \kappa^2/\alpha_{1,1}^2(\kappa)$ and the eigenfunctions φ_1 have the form*

  $$R_{1,1}(r)(c_1 \cos\theta + c_2 \sin\theta), \qquad c_1, c_2 \in \mathbb{R}.$$

  Moreover the function $R_{1,1}$ has n simple zeros in $]0, 1[$, i.e., φ_1 has $2(n + 1)$ nodal regions.

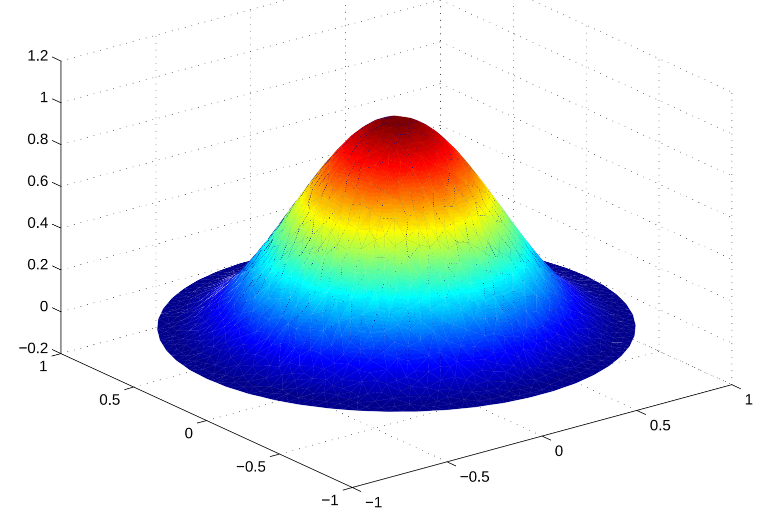


Figure 2: *Graph of φ_1 for $\kappa \in [0, j_{0,1}j_{0,2}[$.*

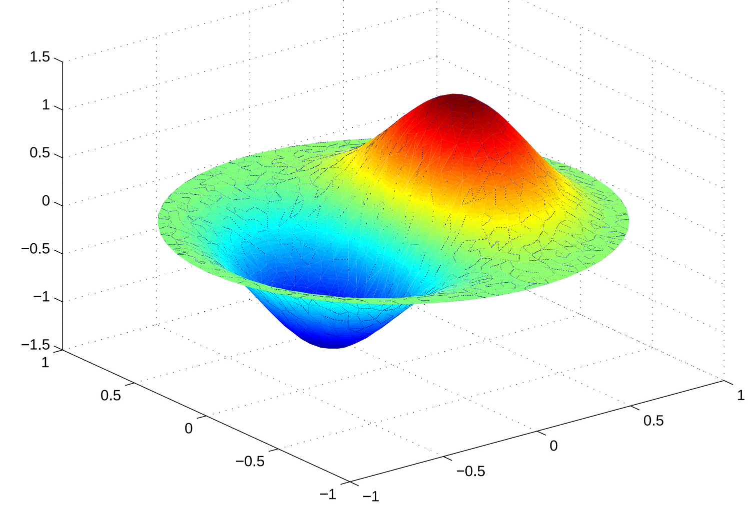


Figure 3: *Graph of φ_1 for $\kappa \in]j_{0,1}j_{0,2}, j_{1,1}j_{1,2}[$.*

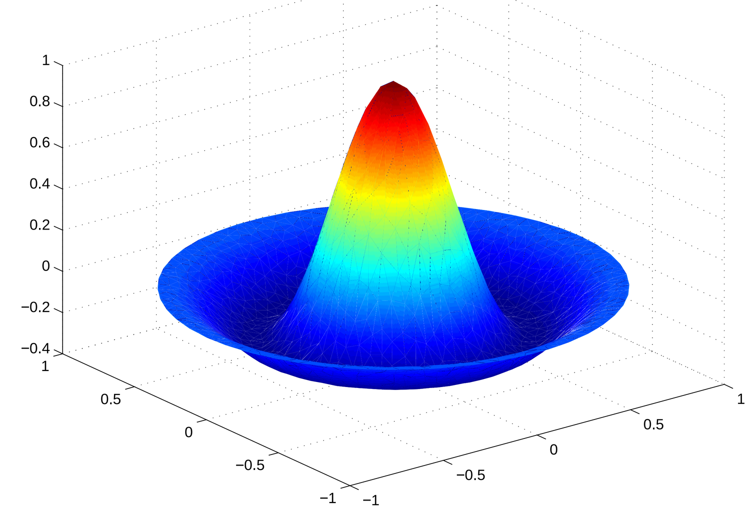


Figure 4: *Graph of φ_1 for $\kappa \in]j_{1,1}j_{1,2}, j_{0,2}j_{0,3}[$.*

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Matinée des Chercheurs 2015. 10 March 2015, Mons