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'Additive difference' models without additivity and subtractivity ☆

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Abstract

This paper studies conjoint measurement models tolerating intransitivities that closely resemble Tversky's additive difference model while replacing additivity and subtractivity by mere decomposability requirements. We offer a complete axiomatic characterisation of these models without having recourse to unnecessary structural assumptions on the set of objects. This shows the pure consequences of several cancellation conditions that have often been used in the analysis of more traditional conjoint measurement models. Our models contain as particular cases many aggregation rules that have been proposed in the literature.

Keywords: Conjoint measurement; Nontransitive preferences; Additive difference model; Cancellation conditions

1. Introduction

This paper pursues the analysis of conjoint measurement models tolerating intransitivity initiated in Bouyssou and Pirlot (2002). We are therefore interested in numerical representations of a binary relation $\succsim$ on a product set $X = X_1 \times X_2 \times \cdots \times X_n$; the elements of X are vectors $x = (x_1, \ldots, x_i, \ldots, x_n), x_i \in X_i$ interpreted as alternatives evaluated on several attributes. The models that we study all admit a representation of the following type:

$$\text{(M–D)} \quad x \succsim y \Leftrightarrow F(\varphi_1(u_1(x_1), u_1(y_1)), \ldots, \varphi_n(u_n(x_n), u_n(y_n))) \geqslant 0, \tag{1}$$

where u_i are real-valued functions on X_i, φ_i are real-valued functions on $u_i(X_i) \times u_i(X_i)$ and F is a real-valued function on $U = \prod_{i=1}^{n} \varphi_i(u_i(X_i) \times u_i(X_i))$. We refer to model (1) as to model (M–D) for reasons that will become clear soon.

Variants of this model are obtained by combining additional properties of F and φ_i, e.g.

- the functions φ_i may be supposed to be nondecreasing (resp. nonincreasing) in their first (resp. second) argument;
- they may be skew-symmetric ($\varphi_i(v_i, w_i) = -\varphi_i(w_i, v_i)$, $v_i, w_i \in u_i(X_i)$);
- F may be supposed nondecreasing (resp. increasing) in all its arguments;
- F may be odd ($F(u) = -F(-u), u \in U \subseteq \mathbb{R}^n$).

These additional properties are motivated by the interpretation of the functions. Intuitively, if a preference can be represented in model (1), the preference of x over y can be explained as resulting from a positive balance, obtained through using function F, of the "differences of preference", represented by $\varphi_i(u_i(x_i), u_i(y_i))$, between x

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and y on each attribute. The balance is supposed to improve in favour of x if any of the differences of preference becomes more favourable to x as compared with y (hence it is natural to consider the case in which F is a nondecreasing function of its arguments $\varphi_i(u_i(x_i), u_i(y_i))$). In the same way, a "difference of preference" $\varphi_i(u_i(x_i), u_i(y_i))$ should not decrease either when the "position" of x on attribute i (represented by $u_i(x_i)$) improves or that of y on the same attribute (represented by $u_i(y_i)$) deteriorates (hence the case in which φ_i is nondecreasing in its first argument and nonincreasing in the second is natural). The same intuition leads to the hypothesis following which φ_i would be skew-symmetric (since $\varphi_i(u_i(x_i), u_i(y_i))$ could be interpreted as the opposite of the "difference of preference" represented by $\varphi_i(u_i(y_i), u_i(x_i))$). Finally, it is tempting to view the result of the comparison of x to y as the opposite of the result of the comparison of y to x, justifying, intuitively, the oddness of F. Despite the appealing character of this intuition, it should be observed that models in which only part of the above properties are fulfilled also deserve attention.

This paper will provide a fairly complete axiomatic analysis of model (M–D) and its variants. When compared to the models studied in Bouyssou and Pirlot (2002) (see model (M) defined by Eq. (6) below), model (M–D) adds the extra feature of "well-behaved" preferences on the components of the product set governed by the functions u_i's whereas they still encompass possibly nontransitive preference relations $\succsim$.

Referring to our model by the label (M–D) is a reminder of model (M) and its variants studied in Bouyssou and Pirlot (2002). It also evokes the fact that the easiest way to interpret model (M–D) is to relate it to Tversky's Additive Difference model (Tversky, 1969) in which:

$$x \succsim y \iff \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) \geqslant 0, \tag{2}$$

where Φ_i are increasing and odd real-valued functions. The ability of this model to capture nontransitive preference relations $\succsim$ together with well-behaved marginal preferences on each attribute and the "intra-dimensional" information processing strategy that it suggests have made it quite popular in Psychology (see, e.g. Aschenbrenner (1981) or Montgomery and Svenson, 1976). In line with the strategy followed in Bouyssou and Pirlot (2002), going from (2) to (M–D) amounts to replacing both the addition and the subtraction operations by mere decomposability[1] requirements, hence the title of this paper. Keeping in mind the analysis in Bouyssou and Pirlot (2002), this replacement will drastically simplify the analysis of the model while allowing to dispense with unnecessary[2] structural conditions on the set of objects. In fact, all axiomatic analyses of the additive difference model (2) known so far (Fishburn (1980) and Croon (1984) for the $n = 2$ case, Fishburn (1992) for $n \geqslant 3$, the work of Bouyssou (1986) in the $n = 2$ case being an exception) use unnecessary structural conditions on the set of objects, which, as in traditional models of conjoint measurement (see Krantz, Luce, Suppes, & Tversky, 1971, Chapter 9, Furkhen & Richter, 1991) interact with, necessary, cancellation conditions and therefore somewhat contribute to obscure their interpretation.

On a technical level, we follow the same strategy as in Bouyssou and Pirlot (2002), i.e., we investigate how far it is possible to go in terms of numerical representations without imposing any transitivity requirement on the preference relations and any unnecessary structural requirement on the set of objects. We refer to Bouyssou and Pirlot (2002) for a detailed motivation for such an approach. Let us simply mention here that in such a framework numerical representations are quite unlikely to possess any "nice" uniqueness properties. These representations are not studied here for their own sake and our results are not intended to give clues on how to build them. They are used as a framework allowing to understand the consequences of a number of requirements on $\succsim$.

It is useful to compare the models studied in this paper with more classical ones as well as with the one studied in Bouyssou and Pirlot (2002). The point of departure of nearly all conjoint measurement models is the additive utility model (Krantz et al., 1971; Debreu, 1960):

$$x \succsim y \iff \sum_{i=1}^{n} u_i(x_i) \geqslant \sum_{i=1}^{n} u_i(y_i) \tag{3}$$

which gives an *additive* representation of *transitive* preferences. This model has been generalised in two distinct directions. The first one keeps the transitivity aspect of (3) but relaxes additivity to a mere decomposability requirement. The desired representation is such that:

$$\begin{aligned} x \succsim y \iff & G(u_1(x_1), u_2(x_2), \ldots, u_n(x_n)) \\ & \geqslant G(u_1(y_1), u_2(y_2), \ldots, u_n(y_n)) \end{aligned} \tag{4}$$

with G increasing in all its arguments. Such models are amenable to a very simple axiomatic analysis that dispenses with unnecessary structural restrictions on X (see Krantz et al., 1971, Chapter 7). Obviously the

[1] This is another justification for the label (M–D): the letter D evokes the decomposability of the differences of preference.

[2] By "unnecessary structural conditions", we mean conditions on the X_i's that, when combined with the appropriate axioms on $\succsim$, would ensure the existence of a representation of $\succsim$ in a given model, without being necessary conditions for the existence of such a representation.

uniqueness results for (4) are much weaker than what can be obtained with (3).

Another generalisation of (3) consists in looking for *additive* representations of *nontransitive* preferences. This gives rise to models of the following type:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) \geqslant 0, \tag{5}$$

where the real-valued functions p_i, defined on $X_i \times X_i$, may have additional properties, e.g. be skew-symmetric. Such models have received much attention (see Bouyssou, 1986; Fishburn, 1990a, b, 1991b; Vind, 1991). Their additive nature however imposes either the use of a denumerable scheme of, hardly interpretable, axioms in the finite case (see, e.g. Fishburn, 1991a); a similar phenomenon occurs with the axiomatisation of the additive value model (3), see Fishburn (1970, p. 45) and Scott and Suppes (1958) or the use of (unnecessary) structural restrictions on the set of objects (see Vind (1991); Fishburn (1990b, 1991a)).

The nontransitive decomposable models studied in Bouyssou and Pirlot (2002) combine these two lines of generalisation. They are of the following type:

$$(\mathrm{M}) \quad x \succsim y \Leftrightarrow F(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)) \geqslant 0, \tag{6}$$

where F and p_i may have several additional properties (e.g. F odd and increasing in all its arguments and/or p_i skew-symmetric). We refer to it as to the (M) model.

The relations between these models can easily be understood using the following diagram (taken from Bouyssou & Pirlot, 2002):

Additive Transitive Model (3)	$\leftrightarrow$	Decomposable Transitive Model (4)
$\updownarrow$		$\updownarrow$
Additive Nontransitive Models (5)	$\leftrightarrow$	Decomposable Nontransitive Models (M) (6)

in which going from left to right amounts to replacing additivity by decomposability and going from top to bottom amounts to abandoning transitivity. We refer to Bouyssou and Pirlot (2002) for a detailed analysis of the relations between these various models.

The models at the bottom line of the above diagram say nothing on the properties of marginal preferences on each attribute. This is somewhat counter-intuitive since one would mainly expect intransitivity to occur only when information is aggregated. The additive difference model does not have this difficulty; in our diagram, it lies on the left column in between the fully transitive (3) and the fully nontransitive (5). Similarly, the models of type (M–D) studied in this paper lie in between (4) and (M) on the right column of the diagram, tolerating intransitivity but imposing well-behaved marginal preferences. This gives rise to the following picture of the models:

Additive Transitive Model (3)	$\leftrightarrow$	Transitive Decomposable Model (4)
$\updownarrow$		$\updownarrow$
Additive Difference (2)	$\leftrightarrow$	Models (M–D) (1)
$\updownarrow$		$\updownarrow$
Additive Nontransitive Models (5)	$\leftrightarrow$	Decomposable Nontransitive Models (M) (6)

in which as before going from left to right relaxes additivity and going from top to bottom relaxes transitivity.

Note that in Bouyssou and Pirlot (2004), we investigated another line of generalisation of model (4) that allows for intransitivity but does not generalize the additive difference model (2). More precisely, we study relations $\succsim$ on X that admit numerical representations of the type

$$x \succsim y \Leftrightarrow H(u_1(x_1), \ldots, u_n(x_n); u_1(y_1), \ldots, u_n(y_n)) \geqslant 0, \tag{7}$$

where H is a function of $2n$ arguments and may enjoy properties such as nondecreasingness (or increasingness) in its first n arguments and nonincreasingness (or decreasingness) in its last n arguments. It is remarkable that the axioms used in Bouyssou and Pirlot (2004) to characterise the variants of model (7) are precisely those that will be needed here, together with the axioms introduced in Bouyssou and Pirlot (2002) for model (M).

The rest of the paper is organized in five sections (numbered from 2 to 6) and an appendix. In Section 2, we introduce our notation and recall classical definitions. Section 3 shows that it is possible under very mild hypotheses, to go from model (M) to model (M–D) whenever F has no special properties. More precisely, for each of the special cases of model (M) studied in Bouyssou and Pirlot (2002), we show (Theorem 1) that $p_i(x_i, y_i)$ can always be substituted by $\varphi_i(u_i(x_i), u_i(y_i))$ (under a condition that essentially limits the cardinality of X_i, when X_i is not denumerable). In all the models considered in this section, φ_i and u_i are not supposed to enjoy any special property. We then introduce several variants.

Having in mind the weak orders on X_i^2 represented by the functions $p_i(x_i, y_i)$, we start Section 4 by recalling and adapting general results about weak orders on any Cartesian product $A \times A$. The axioms that will allow us to characterise all variants of model (M–D) considered here are then presented and studied.

The core of the paper is Section 5 in which an axiomatic characterisation of all our models is provided.

It is divided into four subsections. Sections 5.1 and 5.2 handle the case in which the sets X_i are finite or denumerable, while the nondenumerable case is left for Section 5.3. In Section 5.1, we characterise the models in which $\varphi_i(u_i(x_i), u_i(y_i))$ is nondecreasing in its first argument and nonincreasing in the second, for all i (Theorem 2); in Section 5.2, the case in which φ_i is increasing in its first argument and decreasing in the second is examined (Theorem 3). Both cases are dealt with for nondenumerable sets X_i in Section 5.3 (Theorems 4 and 5). The issues of the equivalence of models and the independence of axioms is examined systematically, in Sections 5.1.2, 5.2.2 and 5.3.2. The results obtained are discussed in Section 5.4. We comment in particular on the (non-)uniqueness of the representations in our various models and draw the attention on special representations that may be called *regular*. Some connections between our models and the additive difference model (2) and the additive conjoint measurement model (5) are also established in that subsection. Conclusions and perspectives for further research are briefly presented in Section 6. The more technical proofs are relegated in the appendix as well as eighteen examples mainly used for showing that our axioms are independent.

The reader who is less interested in the technicalities of the nondenumerable case may focus on Sections 5.1, 5.2 and 5.4. Contrary to the case of more classical models, it should be noticed that the nondenumerable case brings little new from a conceptual viewpoint. It mainly draws the attention on the monotonicity (strict or not) of the relation on "differences of preference" w.r.t. the "marginal traces".

2. Notation and definitions

A *binary relation* S on a set A is a subset of $A \times A$; we write aSb for $(a,b) \in S$. A binary relation S on A is said to be:

- *reflexive* if $[aSa]$,
- *irreflexive* if $[Not\, aSa]$,
- *complete* if $[aSb$ or $bSa]$,
- *symmetric* if $[aSb] \Rightarrow [bSa]$,
- *asymmetric* if $[aSb] \Rightarrow [Not\, bSa]$,
- *transitive* if $[aSb$ and $bSc] \Rightarrow [aSc]$,
- *Ferrers* if $[aSb$ and $cSd] \Rightarrow [aSd$ or $cSb]$,
- *semi-transitive* if $[aSb, bSc] \Rightarrow [aSd$ or $dSc]$,

for all $a, b, c, d \in A$. A binary relation is:

- a *weak order* (resp. an *equivalence*) if it is complete and transitive (resp. reflexive, symmetric and transitive),
- an *interval order* if it is complete and Ferrers (Fishburn, 1970),
- a *semi-order* if it is a semi-transitive interval order (Luce, 1956).

For more detail on relations in the context of preference analysis, the reader is referred to Fishburn (1985), Roubens and Vincke (1985), Pirlot and Vincke (1997).

If S is an equivalence on A, A/S will denote the set of equivalence classes of S on A.

A subset $B \subseteq A$ is *dense* in A w.r.t. a relation S if $\forall a, c \in A, aSc \Rightarrow [\exists b \in B$ such that $aSbSc]$. If S is a weak order on A, there is a numerical representation of S on the real numbers (i.e. $\exists f : A \to \mathbb{R}$ such that $aSb \Leftrightarrow f(a) \geqslant f(b)$) iff there is a finite or denumerable set B that is dense in A w.r.t. S. This condition for the existence of a numerical representation is called *order density* and will be referred to as such in the sequel.

In this paper $\succsim$ will always denote a binary relation on a set $X = \prod_{i=1}^n X_i$ with $n \geqslant 2$. Elements of X will be interpreted as alternatives evaluated on a set $N = \{1, 2, \ldots, n\}$ of attributes and $\succsim$ as an "at least as good as" preference relation between alternatives ($x \succsim y$ reads "x is at least as good as y"). We note $\succ$ (resp. $\sim$) the asymmetric (resp. symmetric) part of $\succsim$. A similar convention holds when $\succsim$ is starred, superscripted and/or subscripted.

For any nonempty subset J of the set of attributes N, we denote by X_J (resp. X_{-J}) the set $\prod_{i \in J} X_i$ (resp. $\prod_{i \notin J} X_i$). With customary abuse of notation, (x_J, y_{-J}) will denote the element $w \in X$ such that $w_i = x_i$ if $i \in J$ and $w_i = y_i$ otherwise. When $J = \{i\}$ we shall simply write X_{-i} and (x_i, y_{-i}).

Let J be a nonempty set of attributes. We define the following two binary relations on X_J:

$$x_J \succsim_J y_J \text{ iff } (x_J, z_{-J}) \succsim (y_J, z_{-J}), \quad \text{for all } z_{-J} \in X_{-J}, \tag{8}$$

$$x_J \succsim_J^{\circ} y_J \text{ iff } (x_J, z_{-J}) \succsim (y_J, z_{-J}), \quad \text{for some } z_{-J} \in X_{-J}, \tag{9}$$

where $x_J, y_J \in X_J$. We refer to $\succsim_J$ as the *marginal relation* or *marginal preference* induced on X_J by $\succsim$. When $J = \{i\}$ we write $\succsim_i$ instead of $\succsim_{\{i\}}$.

If, for all $x_J, y_J \in X_J$, $x_J \succsim_J^{\circ} y_J$ implies $x_J \succsim_J y_J$, we say that $\succsim$ is independent for J. If $\succsim$ is independent for all nonempty subsets of attributes, we say that $\succsim$ is *independent*. It is not difficult to see that a binary relation is independent if and only if it is independent for $N \backslash \{i\}$, for all $i \in N$ (see, e.g., Wakker, 1989). A relation is said to be *weakly independent* if it is independent for all subsets containing a single attribute; while independence implies weak independence, it is clear that the converse is not true (Wakker, 1989).

3. Intra-attribute decomposability

This section is divided into three subsections that play a preparatory role in the paper. We first show that all relations admit a representation in model (M–D) as soon as quite a natural cardinality condition is fulfilled. In Section 3.2, we adapt results about inter-attribute decomposability, previously obtained in Bouyssou and Pirlot (2002), to the context of (M–D) models. The final subsection lists the variants of the (M–D) model that will be analysed in the sequel and states some of their elementary properties.

3.1. Intra-attribute decomposition of model (M)

In a previous paper (Bouyssou & Pirlot 2002), we extensively studied model (M) and characterised several of its specialisations obtained by imposing additional requirements on F or the p_i's. A possible interpretation of these models is that the preference can be described as resulting from a description (by means of the functions p_i) of the differences between alternatives on each attribute separately; these single attribute descriptions are combined by means of a function F that carries all inter-attribute information. In that paper, we referred to this sort of decomposability of the preference as to *inter-criteria decomposability*. What we are examining here is the possibility of further decomposing model (M) by specifying a particular functional form $\varphi_i(u_i(x_i), u_i(y_i))$ for the functions $p_i(x_i, y_i)$; we call this further step "intra-attribute decomposition", since it intuitively amounts to analysing on each attribute the "difference of preference" possibly reflected by $p_i(x_i, y_i)$ as a function of "values" $u_i(x_i), u_i(y_i)$, respectively attached to x_i and y_i. Substituting $p_i(x_i, y_i)$ in model (M) with a function $\varphi_i(u_i(x_i), u_i(y_i))$ leads to model (M–D) presented in the introduction (u_i is a real-valued function defined on X_i and φ_i is a real-valued function defined on $u_i(X_i) \times u_i(X_i)$).

As already noted by Goldstein (1991), all binary relations satisfy model (M) at least when the cardinality of X_i does not exceed that of $\mathbb{R}$, the set of real numbers. The same holds for model (M-D); the functions u_i and φ_i can indeed be constructed as follows. Define the binary relations $\sim_i^*$ on X_i^2 and $\sim_i^\pm$ on X_i, letting for all $x_i, y_i, z_i, w_i \in X_i$,

$$
\begin{aligned}
&(x_i, y_i) \sim_i^* (z_i, w_i) \text{ iff} \\
&\quad [(x_i, a_{-i}) \succsim (y_i, b_{-i}) \Leftrightarrow (z_i, a_{-i}) \succsim (w_i, b_{-i}), \\
&\quad \text{for all } a_{-i}, b_{-i} \in X_{-i}]
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
&x_i \sim_i^\pm y_i \text{ iff} \\
&\quad [(x_i, a_{-i}) \succsim b \Leftrightarrow (y_i, a_{-i}) \succsim b, \quad \text{for all } a_{-i} \in X_{-i}, b \in X] \\
&\quad \text{and } [c \succsim (x_i, d_{-i}) \Leftrightarrow c \succsim (y_i, d_{-i}), \\
&\qquad \text{for all } c \in X, d_{-i} \in X_{-i}].
\end{aligned} \tag{11}
$$

It is clear that $\sim_i^*$ (resp. $\sim_i^\pm$) is an equivalence on the set X_i^2 (resp. X_i).

Call LCC_i (*Low Cardinality Condition*) the assertion stating that the set of equivalence classes $X_i/\sim_i^\pm$ of $\sim_i^\pm$ has at most the cardinality of $\mathbb{R}$. If LCC_i is satisfied for all $i = 1, \ldots n$, we say that $\succsim$ satisfies property LCC; LCC will trivially be fulfilled if for instance the cardinality of all X_i is at most that of $\mathbb{R}$. Under hypothesis LCC, which, obviously, is necessary for model (M–D), it is clear that there are real-valued functions u_i on X_i such that, for all $x_i, y_i \in X_i$:

$$x_i \sim_i^\pm y_i \Leftrightarrow u_i(x_i) = u_i(y_i). \tag{12}$$

Given a particular representation of $\succsim$ in model (M), define φ_i on $u_i(X_i) \times u_i(X_i)$ letting, for all $x_i, y_i \in X_i$,

$$\varphi_i(u_i(x_i), u_i(y_i)) = p_i(x_i, y_i). \tag{13}$$

The well-definedness of φ_i easily follows from the definitions of $\sim_i^*$ and $\sim_i^\pm$.

Since the intuition behind $\varphi_i(u_i(x_i), u_i(y_i))$ is the idea of a "difference of preference" between the "values" $u_i(x_i)$ and $u_i(y_i)$, it is natural to impose on φ_i monotonicity conditions that will bring it closer to an algebraic difference; we thus consider imposing on φ_i the following conditions:

Property 1: φ_i is nondecreasing in its first argument and nonincreasing in the second;

Property 1′: φ_i is increasing in its first argument and decreasing in the second.

We call (M–D1) (resp. (M–D1′)) model (M–D) with the additional property that φ_i satisfies Property 1 (resp. Property 1′). As we can see from Lemma 1 below, these requirements imposed on φ_i in the absence of any hypothesis on F do not restrict the generality of the model.

Lemma 1. *A relation $\succsim$ on X satisfies model* (M–D1) *or, equivalently, model* (M–D1′) *iff property LCC holds.*

Proof. We construct a representation of $\succsim$ according to model (M–D1′).

(a) Choose a function $u_i : X_i \to \mathbb{R}$, satisfying (12), which is possible in view of hypothesis LCC_i.

(b) Define a real-valued function φ_i on $u_i(X_i) \times u_i(X_i)$ verifying the following requirements:

- φ_i assigns different values to different classes of $\sim_i^*$;
- φ_i is increasing in its first argument and decreasing in the second.

Remark that the former condition will be fulfilled if φ_i separates all pairs (x_i, y_i) and (z_i, w_i) such that *Not* $[x_i \sim_i^\pm z_i]$ or *Not*$[y_i \sim_i^\pm w_i]$. Indeed, if $x_i \sim_i^\pm z_i$ and $y_i \sim_i^\pm w_i$, it is easily checked that $(x_i, y_i) \sim_i^* (z_i, w_i)$. Hence, if *Not*$[(x_i, y_i) \sim_i^* (z_i, w_i)]$, either *Not*$[x_i \sim_i^\pm z_i]$ or *Not*$[y_i \sim_i^\pm w_i]$ (or both) and $\varphi_i(u_i(x_i), u_i(y_i)) \neq \varphi_i(u_i(z_i), u_i(w_i))$.

In case X_i is at most denumerable, there is a straightforward way of building appropriate u_i's and φ_i's. Choose for u_i a function that separates the classes of $\sim_i^\pm$ and is valued in the set of positive integers $\mathbb{N}$; define φ_i by $\varphi_i(u_i(x_i), u_i(y_i)) = u_i(x_i) + \frac{1}{u_i(y_i)}$; it is readily checked that φ_i fulfills both conditions above.

The general case, under the *LCC* hypothesis, is a little more technical (and may be skipped by the uninterested reader). The function u_i may, without loss of generality, be chosen to be valued in the open $]0,1[$ interval. Each number $a \in]0,1[$ can be represented in binary notation as a sequence $(a_1, a_2, \ldots, a_k, \ldots)$ of binary digits 0 or 1. Using a binary representation[3] of the numbers of the $]0,1[$ interval, we define a function $f_1 :]0,1[\to]0,1[$ that maps any number $a \in]0,1[$ (with binary representation $(a_1, a_2, \ldots, a_k, \ldots)$) onto the number the binary representation of which is $(a_1, 0, a_2, 0, \ldots, a_k, 0, \ldots)$. This function is increasing and injective. Define similarly the increasing and injective function $f_2 :]0,1[\to]0,1[$ mapping the binary representation of $a \in]0,1[$ onto $(0, a_1, 0, a_2, \ldots, 0, a_k, \ldots)$. A function φ_i satisfying the required properties may be defined as $\varphi_i(u_i(x_i), u_i(y_i)) = f_1(u_i(x_i)) + f_2(1 - u_i(y_i))$. This function φ_i is clearly increasing with $u_i(x_i)$ and decreasing with $u_i(y_i)$. It also separates any pair $(u_i(x_i), u_i(y_i))$ from any pair $(u_i(z_i), u_i(w_i))$ as soon as $u_i(x_i) \neq u_i(z_i)$ or/and $u_i(y_i) \neq u_i(w_i)$.

(c) Finally, define F as follows:

$$F(\varphi_1(u_1(x_1), u_1(y_1)), \ldots \varphi_n(u_n(x_n), u_n(y_n))) = \begin{cases} 1 & \text{if } x \succsim y \\ -1 & \text{if } Not[x \succsim y]. \end{cases}$$

The latter function is well-defined, due to the property that φ_i distinguishes the equivalence classes of $\sim_i^*$: it never occurs that $x \succsim y$ and $Not[z \succsim w]$ while for all i, $\varphi_i(u_i(x_i), u_i(y_i)) = \varphi_i(u_i(z_i), u_i(w_i))$. The latter equalities indeed would imply that for all i, $(x_i, y_i) \sim_i^* (z_i, w_i)$, which in turn would imply that $x \succsim y$ iff $z \succsim w$. $\square$

As a corollary, we get that models (M–D), (M–D1) and (M–D1′) all are equivalent and impose no restriction on the relations (apart from necessary cardinality conditions). In order to get nontrivial models, we shall study the combinations of properties 1 and 1′ together with various properties of F and additional requirements on φ_i. The latter have been investigated in Bouyssou and Pirlot (2002) in the context of model (M); for the sake of completeness, we recall in the next subsection relevant definitions and results, adapting them to model (M–D).

3.2. Previous results on inter-attribute decomposable models

3.2.1. Models

Consider model (M). Requiring (M) together with $F(\mathbf{0}) \geqslant 0$ (where $\mathbf{0}$ denotes the vector of $\mathbb{R}^n$ all coordinates of which are equal to 0) and $p_i(x_i, x_i) = 0$, leads to a model labelled (M0) that is not much constrained since it encompasses all relations that are reflexive and independent:

$$\text{(M0)}\quad x \succsim y \Leftrightarrow F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geqslant 0 \quad \text{with } p_i(x_i, x_i) = 0 \quad \text{for all } x_i \in X_i \text{ and } F(\mathbf{0}) \geqslant 0. \tag{14}$$

Provided we suppose that *LCC* is in force, we may proceed as we did with model (M) in Section 3.1, i.e. defining functions u_i and substituting $p_i(x_i, y_i)$ with $\varphi_i(u_i(x_i), u_i(y_i))$. The constructed functions φ_i inherit the property of p_i, namely, $\varphi_i(u_i(x_i), u_i(x_i)) = 0$, leading to model (M0–D):

$$\text{(M0–D)}\quad x \succsim y \Leftrightarrow F(\varphi_1(u_1(x_1), u_1(y_1)), \ldots, \varphi_n(u_n(x_n), u_n(y_n))) \geqslant 0 \quad \text{with} \quad \varphi_i(u_i(x_i), u_i(x_i)) = 0 \text{ for all } x_i \in X_i \text{ and } F(\mathbf{0}) \geqslant 0. \tag{15}$$

In view of bringing model (M) "closer" to an addition operation, like in model (5), additional properties on F have been considered. A natural requirement is to impose that F be nondecreasing or increasing in all its arguments. This respectively leads to models (M1) and (M1′). An additional requirement is the skew symmetry of each function p_i, i.e. $p_i(x_i, y_i) = -p_i(y_i, x_i)$, for all $x_i, y_i \in X_i$. Adding this condition to (M1) and (M1′) leads to (M2) and (M2′). Going one step further in the direction of an addition operation, we add to models (M2) and (M2′) the requirement that F should be odd; this defines models (M3) and (M3′).The definition of these various models is recalled in Table 1.

These models combine in different ways the increasingness of F, its oddness and the skew symmetry of the functions p_i; defining functions u_i and substituting $p_i(x_i, y_i)$ with $\varphi_i(u_i(x_i), u_i(y_i))$ is again possible under the assumption that *LCC* holds. The properties of p_i are inherited by φ_i; the resulting models are denoted by suffixing their initial label by "–D".

3.2.2. Axioms

The characterisations of models (Mk), for $k = 0, 1, 2, 3$, and (Mk'), for $k = 1, 2, 3$, obtained in Bouyssou and Pirlot (2002) obviously remain true for the "suffixed" models (Mk–D) or (Mk'–D), provided *LCC* is in force. For studying these models, three conditions have proved useful. Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^n X_i$. This relation is said to satisfy:

$RC1_i$ if

$$\left.\begin{array}{r} (x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succsim (w_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} (x_i, c_{-i}) \succsim (y_i, d_{-i}) \\ \text{or} \\ (z_i, a_{-i}) \succsim (w_i, b_{-i}), \end{array}\right.$$

$RC2_i$ if

$$\left.\begin{array}{r} (x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (y_i, c_{-i}) \succsim (x_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} (z_i, a_{-i}) \succsim (w_i, b_{-i}) \\ \text{or} \\ (w_i, c_{-i}) \succsim (z_i, d_{-i}), \end{array}\right.$$

[3] Rational numbers have two binary representations; choose one way of representing each rational; using one particular representation or another as described in the rest of the proof may lead to different functions φ_i, but all fullfill the requirements.

Table 1
Model (M–D) and its variants

(M–D)	$x \succsim y \Leftrightarrow F(\varphi_1(u_1(x_1), u_1(y_1)), \ldots, \varphi_n(u_n(x_n), u_n(y_n))) \geqslant 0$
(M0–D)	(M–D) with $\varphi_i(u_i(x_i), u_i(x_i)) = 0$ and $F(\mathbf{0}) \geqslant 0$
(M1–D)	(M0–D) with F nondecreasing in all its arguments
(M1′–D)	(M0–D) with F increasing in all its arguments
(M2–D)	(M1–D) with φ_i skew-symmetric
(M2′–D)	(M1′–D) with φ_i skew symmetric
(M3–D)	(M2–D) with F odd
(M3′–D)	(M2′–D) with F odd

TC_i if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, b_{-i}) \succsim (w_i, a_{-i}) \\ \text{and} \\ (w_i, c_{-i}) \succsim (z_i, d_{-i})\end{array}\right\} \Rightarrow (x_i, c_{-i}) \succsim (y_i, d_{-i}),$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$.

We say that $\succsim$ satisfies $RC1$ (resp. $RC2$, TC) if it satisfies $RC1_i$ (resp. $RC2_i$, TC_i) for all $i \in N$; $RC12$ (resp. $RC12_i$) is short for $RC1$ and $RC2$ (resp. $RC1_i$ and $RC2_i$).

Condition $RC1_i$ (inteR-attribute Cancellation) suggests that $\succsim$ induces on X_i^2 a relation that compares "preference differences" in a well-behaved way: if (x_i, y_i) is a "larger preference difference" than (z_i, w_i) and $(z_i, c_{-i}) \succsim (w_i, d_{-i})$ then we should have $(x_i, c_{-i}) \succsim (y_i, d_{-i})$ and *vice versa*. This relation, which we denote by $\succsim_i^*$, is formally defined as

$$(x_i, y_i) \succsim_i^* (z_i, w_i) \text{ iff [for all } c_{-i}, d_{-i} \in X_{-i}, (z_i, c_{-i}) \succsim (w_i, d_{-i}) \Rightarrow (x_i, c_{-i}) \succsim (y_i, d_{-i})] \tag{16}$$

for all $x_i, y_i, z_i, w_i \in X_i$. Relation $\succsim_i^*$ is transitive by construction and $RC1_i$ exactly amounts to asking that it is complete, hence a weak order. The equivalence relation $\sim_i^*$ defined in (10) is the symmetric part of $\succsim_i^*$.

Condition $RC2_i$ suggests that the "preference difference" (x_i, y_i) is linked to the "opposite" preference difference (y_i, x_i). Again, $RC1_i$ and $RC2_i$ are equivalent to requiring that the relation $\succsim_i^{**}$, defined on X_i^2 by

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \text{ iff } [(x_i, y_i) \succsim_i^* (z_i, w_i) \text{ and } (w_i, z_i) \succsim_i^* (y_i, x_i)], \tag{17}$$

be complete (it is transitive by construction) and thus a weak order.

Condition TC_i (Triple Cancellation) is a classical cancellation condition that has been often used in the analysis of the additive value model (see e.g. Wakker (1989) or Bouyssou and Pirlot (2002), for interpretations).

No other condition is required in order to characterise models (M0), (M1), (M1′), (M2), (M2′), (M3) and (M3′) as long as the sets X_i are finite or denumerable. When the latter hypothesis is not fulfilled, restrictions must be imposed in order to ensure that either $\sim_i^*$, $\succsim_i^*$ or $\succsim_i^{**}$ have a numerical representation. These will be needed also for the characterisation of the suffixed models. Property LCC ensures that each equivalence class of $\sim_i^{\pm}$ can be unambiguously identified by a real number (which is realised by the functions u_i); we have seen in the proof of Lemma 1 that this implies that there are enough real numbers to label the equivalence classes of $\sim_i^*$; thus LCC, that is necessary for guaranteeing the existence of the u_i functions in the D-suffixed models, can substitute the (weaker) hypothesis used in the characterisation of the initial models (condition C^* in Bouyssou and Pirlot, 2002). The condition used for ensuring the representability of weak orders remains necessary. This condition can be formulated as follows.

We say that $\succsim$ satisfies OD_i^* if there is a finite or countably infinite subset of X_i^2 that is dense in X_i^2 for $\succsim_i^*$. In case $\succsim_i^*$ is a weak order, OD_i^* ensures that it has a numerical representation, i.e. there exists a real-valued function p_i on X_i^2 such that, for all $(x_i, y_i), (z_i, w_i) \in X_i^2$, $(x_i, y_i) \succsim_i^* (z_i, w_i)$ iff $p_i(x_i, y_i) \geqslant p_i(z_i, w_i)$. Condition OD^* is said to hold if condition OD_i^* holds for $i = 1, 2, \ldots, n$.

3.2.3. Results

The theorem below describes all "–D" suffixed models listed in Table 1.

Theorem 1. *Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^n X_i$. If X is at most denumerable, then:*

(1) *the relation $\succsim$ satisfies model* (M–D),
(2) *$\succsim$ satisfies model* (M0–D) *iff $\succsim$ is reflexive and independent*,
(3) *$\succsim$ satisfies model* (M1–D) *iff $\succsim$ satisfies model* (M1′–D) *iff $\succsim$ is reflexive, independent and satisfies $RC1$*,
(4) *$\succsim$ satisfies model* (M2–D) *iff $\succsim$ satisfies model* (M2′–D) *iff $\succsim$ is reflexive and satisfies $RC12$*,
(5) *$\succsim$ satisfies model* (M3–D) *iff $\succsim$ is complete and satisfies $RC12$*,
(6) *$\succsim$ satisfies model* ($M3'$–D) *iff $\succsim$ is complete and satisfies TC.*
(7) *If X is not denumerable, parts* (1) *and* (2) *remain valid iff the requirement that $\succsim$ satisfies condition LCC is added; parts* (3)–(6) *remain valid iff the requirement that $\succsim$ satisfies conditions LCC and OD^* is added.*

The above results constitute a straightforward adaptation of Theorems 1 and 2 in Bouyssou and Pirlot (2002); the characterisation of models (M) to (M3′) extends immediately to that of the corresponding "M–D" model if X is denumerable since we have seen that, in such a case, $p_i(x_i, y_i)$ decomposes without further condition into $\varphi_i(u_i(x_i), u_i(y_i))$. Part (7) deserves a word of explanation. Condition LCC obviously is necessary to guarantee the existence of u_i in all models and OD^* is necessary in all models in which F is required to be at least nondecreasing (the latter was shown in Bouyssou and Pirlot (2002, Theorem 2)). It should be noted that condition LCC may not be dispensed of, even in the presence of OD^*, in part (7) of the theorem, as shown by Example 18 in Appendix B. Bouyssou and Pirlot (2002) showed that the conditions used in this theorem are independent.

3.3. Variants of intra- and inter-attribute decomposable models

Lemma 1 shows that imposing monotonicity properties on φ_i without requirements on F does not lead to new models; in the same way, as we have seen in Theorem 1, the conditions previously considered in model (M) and imported in model (M–D) without imposing monotonicity properties on φ_i do not generate new models either (as long as the cardinality of X_i is not strictly larger than that of $\mathbb{R}$). It thus remains to examine, the possible effect of properties imposed correlatively both on the "inter" and the "intra" components of the model; doing this will achieve our main goal that is to study the variants of model (M–D) as was stated in the introduction.

For each of the eight models described in Table 1, we consider two specialisations in which property 1 (respectively 1′) is imposed on the functions φ_i. They are various instances of "nontransitive decomposable models" with which the intra-attribute decomposability requirements combine without implying however the full force of additivity and subtractivity. These variants will be identified by replacing the suffix "–D" either by "–D1" or by "$-$D1′" depending on the fact that property 1 or 1′ is respectively added. For each model in Table 1, we shall thus consider a version in which, for all $i = 1, \ldots, n$, $\varphi_i(u_i(x_i), u_i(y_i))$ is nondecreasing in $u_i(x_i)$ and nonincreasing in $u_i(y_i)$ (property 1) and a version in which it is increasing in $u_i(x_i)$ and decreasing in $u_i(y_i)$ (property 1′).

The –D1 or $-$D1′ variants of model (M–D) have been analysed in Section 3.1 and proven equivalent to the unconstrained model (M–D). The same is true for the –D1 or $-$D1′ variants of model (M0–D) that are equivalent to (M0–D1), because (M0) does not impose any monotonicity on F. We state this result in the following lemma; its proof—a slight modification of that of Lemma 1—is relegated in Appendix A.1.

Lemma 2. *A relation $\succsim$ on X satisfies model* (M0–D1) *or, equivalently, model* (M0–D1′) *iff it is reflexive, independent and satisfies property LCC. These conditions are independent.*

Remarks

(1) The preliminary study done so far leaves us with twelve models to analyse, namely, for $k = 1, 2, 3$, (Mk–D1), (Mk'–D1), (Mk–D1′) and (Mk'–D1′). Some of these will turn out to be equivalent; their characterisation requires axioms that will be introduced in Section 4.2 below. Fig. 1 shows the implications between those models; for the sake of readability, only direct implications are drawn. Note that we have also:
- (M$k-$D1) $\Rightarrow$ (Mk), for $k = 1, 2, 3$;
- (M$k'-$D1) $\Rightarrow$ (Mk'), for $k = 1, 2, 3$.

(2) It is interesting to observe and easy to prove that the various properties imposed on F, φ_i and u_i in our models induce properties of the marginal preferences $\succsim_J$, $J \subseteq N$, and links between $\succsim_J$ and $\succsim$ that become closer and closer to what is obtained with the additive value function model (3). For the reader's convenience, we recall in the next proposition three consequences that were established in Bouyssou and Pirlot (2002) and that we adapt to the "(M–D)" context. We add two new consequences that reveal possible effects of interaction between monotonicity conditions imposed both on F and φ_i.

Proposition 1. *Let $\succsim$ be a binary relation on $X = \prod_{i=1}^{n} X_i$.*

(1) *If $\succsim$ satisfies model* (M1–D) *or* (M1′–D) *then, for all $J \subseteq N$:*

$$[x_i \succ_i y_i, \text{ for all } i \in J] \Rightarrow Not[y_J \succsim_J x_J].$$

(2) *If $\succsim$ satisfies model* (M2–D) *or* (M2′–D) *then:*
- *$\succsim_i$ is complete,*
- *for all $J \subseteq N$, $[x_i \succ_i y_i$ for all $i \in J] \Rightarrow [x_J \succ_J y_J]$.*

(3) *If $\succsim$ satisfies model* (M3′–D) *then, for all $J \subseteq N$:*
- *$[x_i \succsim_i y_i$ for all $i \in J] \Rightarrow [x_J \succsim_J y_J]$,*
- *$[x_i \succsim_i y_i$ for all $i \in J, x_j \succ_j y_j$, for some $j \in J] \Rightarrow [x_J \succ_J y_J]$.*

(4) *If $\succsim$ satisfies model* (M1–D1) *then $\succsim_i$ is a semiorder.*

(5) *If $\succsim$ satisfies model* (M3′–D1′) *then $\succsim_i$ is a weak order.*

Proof. For the proof of parts (1)–(3), see Bouyssou and Pirlot (2002, proposition 1).

(4) We first prove that $\succsim_i$ has the Ferrers property, i.e., if $x_i \succsim_i y_i$ and $z_i \succsim_i w_i$, then at least one of the following holds: $z_i \succsim_i y_i$ or $x_i \succsim_i w_i$. From the premise, using obvious notation, we get $F(\varphi_i(u_i(x_i), u_i(y_i)), \mathbf{0}_{-i}) \geqslant 0$ and $F(\varphi_i(u_i(z_i), u_i(w_i)), \mathbf{0}_{-i}) \geqslant 0$. We have either $u_i(y_i) \geqslant u_i(w_i)$ or $u_i(y_i) < u_i(w_i)$. In the former case, due to the monotonicity properties of F and φ_i, we get $F(\varphi_i(u_i(x_i), u_i(w_i)), \mathbf{0}_{-i}) \geqslant 0$, hence $x_i \succsim_i w_i$; in the latter case, $F(\varphi_i(u_i(z_i), u_i(y_i)), \mathbf{0}_{-i}) \geqslant 0$ and thus $z_i \succsim_i y_i$. The Ferrers property of $\succsim_i$ is thus established. It is well-known

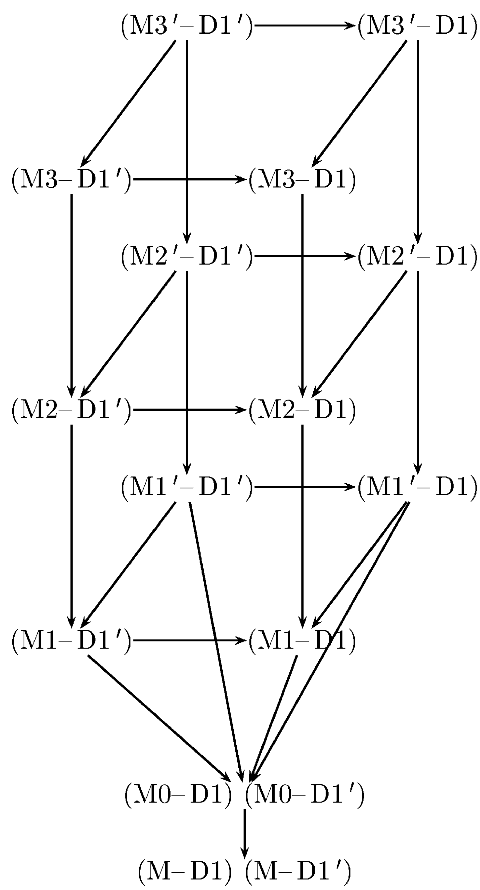

Fig. 1. Graph of implications.

(and easy to prove[4]) that the Ferrers property implies completeness provided the relation is reflexive, which is the case of $\succsim_i$ in (M1–D1).

The semi-transitivity property results from showing, in a similar manner, that $x_i \succsim_i y_i$ and $y_i \succsim_i z_i$ entail either $x_i \succsim_i w_i$ or $w_i \succsim_i z_i$, for any $w_i \in X_i$.

(5) Since we already know that $\succsim_i$ is a semi-order, it remains to prove that the marginal indifference $\sim_i$ is transitive[5]. Due to the skew-symmetry of φ_i and the increasingness of F in model (M3′), it is readily seen that $x_i \sim_i y_i$ if and only if $\varphi_i(u_i(x_i), u_i(y_i)) = 0$. In model (M3′–D1′), since φ_i is decreasing in its second argument and since $\varphi_i(u_i(x_i), u_i(x_i)) = 0$, we have $x_i \sim_i y_i$ if and only if $u_i(x_i) = u_i(y_i)$. From this, one clearly obtains that $x_i \sim_i y_i$ and $y_i \sim_i z_i$ imply $x_i \sim_i z_i$. $\square$

[4] Just apply the Ferrers property to derive aSb or bSa from aSa and bSb.

[5] The asymmetric parts of an interval order or a semi-order S is transitive (see Roubens & Vincke, 1985, p. 22) and a weak order is a complete relation the symmetric and asymmetric part of which are transitive (see Roubens & Vincke, 1985, p. 18).

Remarks

(1) Obviously, any property of $\succsim_i$, valid in a model, is inherited by any of the more constrained model (see the implications between models in Fig. 1). In particular, the semi-order property (Proposition 1.4) is valid in models (M2–D1) and (M3–D1).
(2) Pure (M) models, without intra-attribute decomposability, confer little structure to the marginal preferences $\succsim_i$. It is only with (M2) that $\succsim_i$ becomes a complete relation. On the contrary, in the intra-decomposable models, from (M1–D1) on, $\succsim_i$ is a semi-order.
(3) It is only in the more restrictive model (M3′–D1′) that $\succsim_i$ is a weak order. In such a model, two elements of X_i that are marginally indifferent must have equal u_i values, as results from the proof of Proposition 1.5.

4. Axioms

This section has two subsections. The first one states and proves an auxiliary result on relations defined on a Cartesian product of a set with itself. In the second subsection, we present the axioms that will help us to analyse the models introduced in Section 3.2; we prove some elementary consequences of these axioms.

4.1. Properties of weak orders on X_i^2

In view of setting down the axioms that govern intra-attribute decomposability in our models, we first pay attention to the weak order $\succsim^{p_i}$ on X_i^2 represented by the function $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$, i.e. $(x_i, y_i) \succsim^{p_i} (z_i, w_i) \Leftrightarrow p_i(x_i, y_i) \geqslant p_i(z_i, w_i)$. Note that p_i need not be a numerical representation of $\succsim_i^*$ or $\succsim_i^{**}$ (it may be "finer" in the sense that it may discriminate between pairs that are in the indifference relation $\sim_i^*$ or $\sim_i^{**}$) and hence, $\succsim^{p_i}$ is not necessarily $\succsim_i^*$ or $\succsim_i^{**}$ (but it must satisfy (32) or (33), see Lemma 5 in Section 5.2.1).

What will be of particular interest is linking properties of φ_i to those of $\succsim^{p_i}$. In order to reduce notational burden and since the following definitions and results are fairly general and may be interesting in their own, we formulate them in terms of a set A (instead of X_i) and a function f (instead of p_i).

To any binary relation $\succsim^A$ defined on a cartesian product A^2, can be associated the relations E and T defined on A letting, for all $a, b \in A$:

$$aEb \Leftrightarrow (a,c) \sim^A (b,c) \quad \text{and} \quad (c,b) \sim^A (c,a), \quad \text{for all } c \in A. \tag{18}$$

and

$$aTb \Leftrightarrow (a,c) \succsim^A (b,c) \quad \text{and} \quad (c,b) \succsim^A (c,a) \quad \text{for all } c \in A. \tag{19}$$

Relation T is usually called the *trace* of $\succsim^A$ and E is the symmetric part of T. Following mainly Monjardet (1984) and Doignon, Monjardet, Roubens, and Vincke (1988), we say that:

- $\succsim^A$ is *strongly linear* iff $[Not((b,c)\succsim^A(a,c))$ or $Not((c,a)\succsim^A(c,b))] \Rightarrow [(a,d)\succsim^A(b,d)$ and $(d,b)\succsim^A(d,a)]$,
- $\succsim^A$ is *strongly independent* iff $[(a,c)\succsim^A(b,c))$ or $(c,b)\succsim^A(c,a))] \Rightarrow [(a,d)\succsim^A(b,d)$ and $(d,b)\succsim^A(d,a)]$,
- $\succsim^A$ is *reversible* iff $[(a,b)\succsim^A(c,d) \Rightarrow (d,c)\succsim^A(b,a)]$,

for all $a,b,c,d\in A$.

We note a few simple and useful observations in the following lemma (its proof is left to the reader).

Lemma 3. *Let $\succsim^A$ be a relation on A^2, $\sim^A$, its symmetric part, T, its trace and E, the symmetric part of T. We have*:

(1) *If $\sim^A$ is an equivalence, then E is an equivalence.*
(2) *If $\succsim^A$ is transitive, then T is transitive.*
(3) *$\succsim^A$ is strongly linear iff T is complete.*

As an elementary consequence of these properties, we have that the trace T of a strongly linear weak order $\succsim^A$ is a weak order.

The following result studies the situation in which $\succsim^A$ is a weak order induced on A^2 by a function $f: A^2\to\mathbb{R}$. The case in which A is not denumerable raises technical problems of representability on the real numbers. In addition to the condition LCC_i introduced in Section 3.1 (the relation $\sim_i^\pm$ corresponds exactly to E), we need the classical order density condition (see Section 2) to ensure that the trace T is representable on $\mathbb{R}$.

Proposition 2. *Let $f: A^2\to\mathbb{R}$ and $\succsim^f$ be the weak order induced on A^2 by f, i.e. $(a,b)\succsim^f(c,d)$ iff $f(a,b)\geqslant f(c,d)$, for all $a,b,c,d\in A$.*

(1) *$\succsim^f$ is reversible iff there is a function f' such that $f'(a,b) = -f'(b,a)$ and $(a,b)\succsim^f(c,d)$ iff $f'(a,b)\geqslant f'(c,d)$.*
(2) *Suppose that A is at most denumerable. There are a function $u: A\to\mathbb{R}$ and a function $\varphi: u(A)\times u(A)\to\mathbb{R}$ such that $f(a,b) = \varphi(u(a),u(b))$. Furthermore,*
    (a) *the function φ can be taken to be nondecreasing in its first argument and nonincreasing in its second argument iff $\succsim^f$ is strongly linear;*
    (b) *the function φ can be taken to be increasing in its first argument and decreasing in its second argument iff $\succsim^f$ is strongly independent.*
(3) *In case A is not a denumerable set, there exist a function $u: A\to\mathbb{R}$ and a function $\varphi: u(A)\times u(A)\to\mathbb{R}$ such that $f(a,b) = \varphi(u(a),u(b))$ iff the number of equivalence classes of the relation E is not larger than the cardinality of $\mathbb{R}$. Properties* 2(a) *and* 2(b) *hold iff there is a finite or denumerable subset of A that is dense in A for T.*

Proof. 1) Sufficiency is obvious. We prove necessity. Suppose that $\succsim^f$ is reversible. Define $f'(a,b) = f(a,b) - f(b,a)$; f' obviously is skew-symmetric. We show that f' provides another representation of $\succsim^f$. Since $\succsim^f$ is reversible, we have $(a,b)\succsim^f(c,d)$ iff $(d,c)\succsim^f(b,a)$. Hence, $f(a,b)\geqslant f(c,d)$ and $f(d,c)\geqslant f(b,a)$ and finally, $f'(a,b)\geqslant f'(c,d)$. Conversely, if $f'(a,b)\geqslant f'(c,d)$, we have that $f(a,b)\geqslant f(c,d)$. Suppose, on the contrary, that $f(a,b)<f(c,d)$. Since $f'(a,b)\geqslant f'(c,d)$, it must be that $f(b,a)<f(d,c)$ implying $(d,c)\succsim^f(b,a)$ and, since $\succsim^f$ is reversible, $(a,b)\succsim^f(c,d)$, a contradiction.

2) The existence of u_i and φ_i has been established in Section 3.1, around (13); this proof transposes immediately for establishing the existence of u and φ ($\sim_i^*$ corresponds to $\sim^f$ and $\sim_i^\pm$ to E).

Part (2)(a) [$\Rightarrow$]. Suppose that $Not[(b,c)\succsim^f(a,c)]$ or $Not[(c,a)\succsim^f(c,b)]$, for some $a,b,c\in A$. This is equivalent to $f(b,c)<f(a,c)$ or $f(c,a)<f(c,b)$. Using the monotonicity properties of φ, we obtain from both inequalities that $u(a)>u(b)$ and that $\varphi(u(a),u(d))\geqslant \varphi(u(b),u(d))$ and $\varphi(u(d),u(b))\geqslant \varphi(u(d),u(a))$, for all $d\in A$. This establishes that $\succsim^f$ is strongly linear.

Part (2)(a) [$\Leftarrow$]. Since $\succsim^f$ is a strongly linear weak order, T is a weak order (Lemma 3, parts (2) and (3)). Let u be a numerical representation of T, i.e. aTb iff $u(a)\geqslant u(b)$; such a representation exists since A is finite or denumerable. Define φ by $\varphi(u(a),u(b)) = f(a,b)$. φ is well-defined since $u(c) = u(d)$ iff $c(T\cap T^{-1})d$, i.e. cEd; the reasoning made just after formula (13) thus holds. Moreover φ is nondecreasing in its fist argument and nonincreasing in the second. To prove the former, suppose that $u(a)>u(b)$; this implies aTb. We have for all $c\in A$, $(a,c)\succsim^f(b,c)$ and hence $f(a,c)\geqslant f(b,c)$. Nonincreasingness in the second argument is similarly proven.

Part (2)(*b*) [$\Rightarrow$]. Suppose, on the contrary, that $\succsim^f$ is not strongly independent. Among the four possible cases, we have, for instance, that $(a,c)\succsim^f(b,c)$ and $Not[(a,d)\succsim^f(b,d)]$, for some $a,b,c,d\in A$. This is tantamount to $\varphi(u(a),u(c))\geqslant\varphi(u(b),u(c))$ and $\varphi(u(a),u(d))<\varphi(u(b),u(d))$, which imply respectively, due to increasingness of φ in its first argument, $u(a)\geqslant u(b)$ and $u(a)<u(b)$, a contradiction. The other cases can be dealt with similarly.

Part (2)(*b*) [$\Leftarrow$]. We define u and φ as in part (2)(*a*). Let $a,b\in A$ be such that $u(a)>u(b)$. Since u is a numerical representation of T, we have aTb and $Not[bTa]$; strong independence implies that, for all $c\in A$, $Not[(b,c)\succsim^f(a,c)]$ and $Not[(c,a)\succsim^f(c,b)]$, i.e. $f(a,c)>f(b,c)$ and $f(c,b)>f(c,a)$. Suppose, for instance, that φ is not increasing in its first argument. This would imply that, for some $a, b, d \in A$, with $u(a) > u(b)$, $f(a,d) \leqslant f(b,d)$, a contradiction. A similar argument proves that φ is decreasing in its second argument.

3) In case A is not denumerable, the condition on E is clearly necessary and sufficient for being able to represent each equivalence class of that relation by a real number. The order density condition makes it possible to consider a numerical representation of the weak order T by means of a real-valued function u; this condition is thus sufficient. To show it is also necessary, it suffices to observe that any function u in a representation of $\succsim^f$ with φ monotonic is a representation of a weak order that is at least as fine as T. In other words, if aTb and $Not[bTa]$, then $u(a) > u(b)$. $\square$

Remarks

(1) Proposition 2 reformulates in our framework classical results that may essentially be found in Doignon et al. (1988), Tversky and Russo (1969) (see also Pirlot and Vincke (1997) for a synthesis). Take any numerical representation of $\succsim^A$. This representation may be seen as a valued relation on A^2. In the terminology of Doignon et al. (1988, Section 4.4) the valued relation obtained when $\succsim^A$ is strongly linear is a *coherently biordered* valued relation. The families of binary relations obtained by considering all the cuts of these valued relations have been well studied (Doignon et al., 1988). To our knowledge the valued relations obtained when replacing linearity by independence have received no particular name in the literature.

(2) Doignon et al. (1988) distinguish three less restrictive versions of linearity, namely, left-linearity, right-linearity and linearity. We do not investigate these notions for the sake of conciseness; the reader should note that distinguishing left and right linearity (or independence) has strong connections with a slightly more general model where $p_i(x_i, y_i)$ is decomposed as $\varphi_i(u_i(x_i), v_i(y_i))$ with u_i not necessarily equal to v_i. These variants can easily be dealt with using our methods.

(3) The results in Doignon et al. (1988) are expressed for finite sets. They extend, at least those we consider, to denumerable sets, without further condition. In view of obtaining the results in Section 5.3.1 below, we need further extension to nondenumerable sets and we obtain it under rather straightforward necessary and sufficient conditions, as shown in part (3) of Proposition 2.

(4) It is important to note that in case f has particular features—for instance if f vanishes on the diagonal ($f(a,a) = 0$, for all a) or f is skew-symmetric—these are inherited by φ. This will be of importance in our models when $f = p_i$ and $\succsim^f$ possibly is the relation $\succsim_i^*$ or the relation $\succsim_i^{**}$.

4.2. Axioms for intra-criteria decomposability

In view of Propositions 2.2 and 2.3, and the construction of numerical representations for models of type (M) (see Bouyssou & Pirlot, 2002), obtaining intra-decomposable models boils down to imposing linearity conditions on $\succsim_i^*$ and $\succsim_i^{**}$. In order to do so, we introduce a number of intrA-attribute Cancellation (AC) conditions and of Triple intrA-attribute Cancellation (TAC) conditions. We say that $\succsim$ satisfies:

$AC1_i$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim y\\ \text{and}\\ (z_i, c_{-i}) \succsim w\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i, a_{-i}) \succsim y\\ \text{or}\\ (x_i, c_{-i}) \succsim w,\end{array}\right.$$

$AC2_i$ if

$$\left.\begin{array}{r}x \succsim (y_i, b_{-i})\\ \text{and}\\ z \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}x \succsim (w_i, b_{-i})\\ \text{or}\\ z \succsim (y_i, d_{-i}),\end{array}\right.$$

$AC3_i$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim y\\ \text{and}\\ z \succsim (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(w_i, a_{-i}) \succsim y\\ \text{or}\\ z \succsim (w_i, d_{-i})\end{array}\right.$$

for all $x, y, z, w \in X$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$.

We say that $\succsim$ satisfies $AC1$ (resp. $AC2$, $AC3$) if it satisfies $AC1_i$ (resp. $AC2_i$, $AC3_i$), for $i = 1, 2, \ldots, n$. We shall also use $AC123$ (resp. $AC123_i$) as a short form for the conjunction of conditions $AC1$, $AC2$ and $AC3$ (resp. $AC1_i$, $AC2_i$ and $AC3_i$).

Condition $AC1_i$ suggests that the elements of X_i can be linearly ordered considering "upward dominance": if x_i "upward dominates" z_i then $(z_i, c_{-i}) \succsim w$ entails $(x_i, c_{-i}) \succsim w$. Condition $AC2_i$ has a similar interpretation considering now "downward dominance". More formally, let $\succsim_i^+$ (resp. $\succsim_i^-$) denote the left (resp. right) trace induced by $\succsim$ on X_i, i.e.

$$x_i \succsim_i^+ z_i \text{ iff } \forall c_{-i} \in X_{-i}, w \in X, \quad [(z_i, c_{-i}) \succsim w \Rightarrow (x_i, c_{-i}) \succsim w] \tag{20}$$

$$y_i \succsim_i^- w_i \text{ iff } \forall a_{-i} \in X_{-i}, z \in X, \quad [z \succsim (y_i, a_{-i}) \Rightarrow z \succsim (w_i, a_{-i})]. \tag{21}$$

It was shown in Bouyssou and Pirlot (2004, Lemma 3) that $AC1_i$ (resp. $AC2_i$) is equivalent to imposing that $\succsim_i^+$ (resp. $\succsim_i^-$) is a complete relation, hence a weak order (since it is transitive by definition).

Condition $AC3_i$ ensures that the linear arrangements of the elements of X_i obtained considering upward and downward dominance are not incompatible. In other terms, the trace $\succsim_i^\pm$ that is the intersection of $\succsim_i^+$

and $\succsim_i^-$, i.e.

$$x_i \succsim_i^{\pm} z_i \text{ iff } [x_i \succsim_i^+ z_i \text{ and } x_i \succsim_i^- z_i], \tag{22}$$

is also a complete relation, hence a weak order.

It is also quite important to note that $\succsim_i^{\pm}$ is also the trace of $\succsim_i^*$ and $\succsim_i^{**}$ (defined by formulae (16) and (17)). Indeed, we can easily check that we have:

$$\begin{aligned} x_i \succsim_i^{\pm} y_i \text{ iff } & \forall z_i \in X_i, (x_i, z_i) \succsim_i^* (y_i, z_i) \\ & \text{and } \forall w_i \in X_i, (w_i, y_i) \succsim_i^* (w_i, x_i). \end{aligned} \tag{23}$$

The latter expression implies that $\succsim_i^{\pm}$ is the trace both of $\succsim_i^*$ and $\succsim_i^{**}$. Remark that the relation $\sim_i^{\pm}$, defined in (11), is the symmetric part of $\succsim_i^{\pm}$.

The Triple intrA-attribute Cancellation (TAC) conditions read as follows. We say that $\succsim$ satisfies

$TAC1_i$ if

$$\left.\begin{array}{r} (x_i, a_{-i}) \succsim y \\ \text{and} \\ y \succsim (z_i, a_{-i}) \\ \text{and} \\ (z_i, b_{-i}) \succsim w \end{array}\right\} \Rightarrow (x_i, b_{-i}) \succsim w$$

$TAC2_i$ if

$$\left.\begin{array}{r} (x_i, a_{-i}) \succsim y \\ \text{and} \\ y \succsim (z_i, a_{-i}) \\ \text{and} \\ w \succsim (x_i, b_{-i}) \end{array}\right\} \Rightarrow w \succsim (z_i, b_{-i})$$

for all $y, w \in X$, all $x_i, z_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$.

We say that $\succsim$ satisfies $TAC1$ (resp. $TAC2$) if it satisfies $TAC1_i$ (resp. $TAC2_i$), for $i = 1, 2, \ldots, n$. We shall also use $TAC12$ (resp. $TAC12_i$) for the conjunction of conditions $TAC1$ and $TAC2$ (resp. $TAC1_i$ and $TAC2_i$).

The $TAC1_i$, $TAC2_i$ conditions are variants of the classical triple cancellation condition, like TC_i in Section 3.2. As soon as $\succsim$ is complete, $TAC1$ and $TAC2$ become powerful conditions (as was the case of TC in models (M)) that imply $AC123$; they will help to make sure, in certain models, that ties can be broken just by using "upward" or "downward dominance".

The above axioms and their consequences have been studied in detail in Bouyssou and Pirlot (2004). The following lemma recalls results that will be needed in the sequel and establishes new ones showing that some of the axioms are intimately related to strong linearity of $\succsim_i^*$ and $\succsim_i^{**}$.

Lemma 4. *We have*:

(1) *Model* (M1–D1) *implies* $AC123$.
(2) *Model* (M3′–D1′) *implies* $TAC12$.
(3) $\succsim_i^+$ *is complete iff* $AC1_i$ *holds.*
(4) $\succsim_i^-$ *is complete iff* $AC2_i$ *holds.*
(5) $\succsim_i^{\pm}$ *is complete iff* $AC123_i$ *holds.*
(6) $AC123_i$ *iff* $\succsim_i^*$ *is strongly linear iff* $\succsim_i^{**}$ *is strongly linear.*
(7) *If* $\succsim$ *is complete*, $TAC12_i$ *implies* $AC123_i$ *and if one of the alternatives in the consequent of any of* $AC1_i$, $AC2_i$ *or* $AC3_i$ *is false, then the preference in the other branch of the alternative is strict.*
(8) *If* $\succsim$ *is complete*, $TAC1_i$ *is equivalent to the completeness of* $\succsim_i^+$ *and the following condition*:

$$[x \succsim y \text{ and } z_i \succ_i^+ x_i] \Rightarrow (z_i, x_{-i}) \succ y. \tag{24}$$

(9) *If* $\succsim$ *is complete*, $TAC2_i$ *is equivalent to the completeness of* $\succsim_i^-$ *and the following condition*:

$$[x \succsim y \text{ and } y_i \succ_i^- w_i] \Rightarrow x \succ (w_i, y_{-i}). \tag{25}$$

Proof. (1) The premise of $AC1_i$ yields in terms of model (M1–D1):

$$F(\varphi_i(u_i(x_i), u_i(y_i)), (\varphi_j(u_j(a_j), u_j(y_j)))_{j \neq i}) \geqslant 0$$

and

$$F(\varphi_i(u_i(z_i), u_i(w_i)), (\varphi_j(u_j(c_j), u_j(w_j)))_{j \neq i}) \geqslant 0.$$

Due to the monotonicity of F and φ_i, either $u_i(z_i) \geqslant u_i(x_i)$ and

$$F(\varphi_i(u_i(z_i), u_i(y_i)), (\varphi_j(u_j(a_j), u_j(y_j)))_{j \neq i)}) \geqslant 0,$$

or $u_i(x_i) > u_i(z_i)$ and

$$F(\varphi_i(u_i(x_i), u_i(w_i)), (\varphi_j(u_j(c_j), u_j(w_j)))_{j \neq i}) \geqslant 0,$$

which implies that $AC1_i$ is satisfied. The proof for $AC2_i$ and $AC3_i$ is similar.

(2) The premise of $TAC1_i$, interpreted in terms of model (M3′–D1′), yields three inequalities:

$$F(\varphi_i(u_i(x_i), u_i(y_i)), (\varphi_j(u_j(a_j), u_j(y_j)))_{j \neq i}) \geqslant 0, \tag{26}$$

$$F(\varphi_i(u_i(y_i), u_i(z_i)), (\varphi_j(u_j(y_j), u_j(a_j)))_{j \neq i}) \geqslant 0, \tag{27}$$

$$F(\varphi_i(u_i(z_i), u_i(w_i)), (\varphi_j(u_j(b_j), u_j(w_j)))_{j \neq i}) \geqslant 0. \tag{28}$$

Due to skew-symmetry of φ_i and oddness of F, Eq. (27) may be rewritten as:

$$F(\varphi_i(u_i(z_i), u_i(y_i)), (\varphi_j(u_j(a_j), u_j(y_j)))_{j \neq i}) \leqslant 0. \tag{29}$$

We deduce from Eqs. (26) and (29), using the increasingness of F (resp. φ_i) in its ith (resp. first) argument, that $u_i(x_i) \geqslant u_i(z_i)$; substituting $u_i(z_i)$ by $u_i(x_i)$ in Eq. (28) yields:

$$F(\varphi_i(u_i(x_i), u_i(w_i)), (\varphi_j(u_j(b_j), u_j(w_j)))_{j \neq i}) \geqslant 0,$$

which establishes $TAC1_i$. The proof for $TAC2_i$ is similar.

Parts (3)–(5) were respectively proven as Lemma 3, parts 1, 2 and 4 in Bouyssou and Pirlot (2004).

(6) Using (23), we observed above that $\succsim_i^{\pm}$ is not only the trace of $\succsim$ but also of both $\succsim_i^*$ and $\succsim_i^{**}$. Applying Lemma 3.3, with $A = X_i$ and $\succsim^A = \succsim_i^*$ or

$\succsim_i^{**}$, we get that $\succsim_i^*$ and $\succsim_i^{**}$ are strongly linear iff $\succsim_i^{\pm}$ is complete, which, in turn, is equivalent to $AC123_i$ (by part (5) of the present lemma).

(7) We prove that, if $\succsim$ is complete, $TAC1_i$ implies $AC1_i$ and $AC3_i$. Suppose that $AC1_i$ is violated so that $(x_i, a_{-i}) \succsim y$, $(z_i, b_{-i}) \succsim w$, $Not[(z_i, a_{-i}) \succsim y]$ and $Not[(x_i, b_{-i}) \succsim w]$, for some $x_i, z_i \in X_i$, $a_{-i}, b_{-i} \in X_{-i}$ and $y, w \in X$. Since $\succsim$ is complete, we know that $y \succsim (z_i, a_{-i})$. Using $TAC1_i$, $(x_i, a_{-i}) \succsim y$, $y \succsim (z_i, a_{-i})$ and $(z_i, b_{-i}) \succsim w$ imply $(x_i, b_{-i}) \succsim w$, a contradiction.

Similarly, suppose that $AC3_i$ is violated so that $(x_i, a_{-i}) \succsim y, w \succsim (x_i, b_{-i}), Not[(z_i, a_{-i}) \succsim y]$ and $Not[w \succsim (z_i, b_{-i})]$, for some $x_i, z_i \in X_i$, $a_{-i}, b_{-i} \in X_{-i}$ and $y, w \in X$. Since $\succsim$ is complete, we have: $(z_i, b_{-i}) \succsim w$. Using $TAC1_i$, $(z_i, b_{-i}) \succsim w$, $w \succsim (x_i, b_{-i})$ and $(x_i, a_{-i}) \succsim y$ imply $(z_i, a_{-i}) \succsim y$, a contradiction.

One proves similarly that $TAC2_i$ implies $AC2_i$ and $AC3_i$.

For proving the second part of the thesis, we need using $TAC1_i$ (resp. $TAC2_i$) for the statement concerned with $AC1_i$ (resp. $AC2_i$) and both $TAC1_i$ and $TAC2_i$ for the statement concerned with $AC3_i$. Let us prove the result for $AC1_i$ (the proof is similar in the two other cases). Suppose that the premise of $AC1_i$ is verified, i.e. $(x_i, a_{-i}) \succsim y$ and $(z_i, c_{-i}) \succsim w$, while the first alternative of the consequent is false, i.e. $Not[(z_i, a_{-i}) \succsim y]$; suppose eventually that the second branch of the alternative is not a strict preference, which means that $(x_i, c_{-i}) \sim w$. Applying $TAC1_i$ to the premise $[(z_i, c_{-i}) \succsim w$, $w \succsim (x_i, c_{-i})$ and $(x_i, a_{-i}) \succsim y]$ yields $(z_i, a_{-i}) \succsim y$, a contradiction. If, on the contrary, the second branch of the alternative is false, i.e. $Not[(x_i, c_{-i}) \succsim w]$, and supposing that the first branch of the alternative is $(z_i, a_{-i}) \sim y$, we get, applying $TAC1_i$ to $[(x_i, a_{-i}) \succsim y$, $y \succsim (z_i, a_{-i})$ and $(z_i, c_{-i}) \succsim w]$, the fact that $(x_i, c_{-i}) \succsim w$, a contradiction.

Parts (8) and (9) were respectively shown as Lemma 4, parts (4) and (5) in Bouyssou and Pirlot (2004).

5. Main results

We are now in a position to provide a characterisation of all intra- and inter-decomposable models defined in Section 3.3, using in particular the "AC" and "TAC" conditions introduced in the previous section. For ease of reading and in order to concentrate first on the core arguments, we start with the case in which the X_i's are at most denumerable, postponing to Section 5.3, the technicalities inherent to sets of arbitrary cardinality. In the denumerable case, we deal separately (respectively in Sections 5.1 and 5.2) with the "–D1" and the "$-$D1′" models, finally showing that all pairs of models differing only by –D1 or $-$D1′ are equivalent except for (M3′–D1) and (M3′–D1′). For the sake of completeness, we include in our theorems, results about models (M–D1), (M–D1′), (M0–D1) and (M0–D1′) that were already included in Lemmas 1 and 2.

5.1. *Nonstrictly monotonic decomposable models in the denumerable case*

In this section, we consider the models studied in Theorem 1, with the additional property that they admit a representation in which φ_i is nondecreasing in its first argument and nonincreasing in the second.

5.1.1. *Characterisation results*

The monotonicity property of φ (nondecreasing in its first argument and nonincreasing in the second) is obtained for all models (except for M and M0) as soon as conditions $AC123$ are added to the axioms stated in Theorem 1.

Theorem 2. *Let $\succsim$ be a binary relation on a finite or countably infinite set $X = \prod_{i=1}^n X_i$. Then*:

(1) *$\succsim$ satisfies model* (M–D1);
(2) *$\succsim$ satisfies model* (M0–D1) *iff $\succsim$ is reflexive and independent*;
(3) *$\succsim$ satisfies model* (M1′–D1) *iff $\succsim$ is reflexive, independent and satisfies $RC1$ and $AC123$*;
(4) *$\succsim$ satisfies model* (M2′–D1) *iff $\succsim$ is reflexive and satisfies $RC12$ and $AC123$*;
(5) *$\succsim$ satisfies model* (M3–D1) *iff $\succsim$ is complete and satisfies $RC12$ and $AC123$*;
(6) *$\succsim$ satisfies model* (M3′–D1) *iff $\succsim$ is complete and satisfies TC and $AC123$.*

Proof. Parts (1) and (2) are consequences of Lemmas 1 and 2. For all parts from (3) to (6), necessity results from Theorem 1 and Lemma 4.1. It remains to prove sufficiency.

(3) We have to recall how a reflexive, independent relation satisfying $RC1$ can be represented in model (M1′). Detailed justification of such a construction can be found in Bouyssou and Pirlot (2002). Due to $RC1$, $\succsim_i^*$ is a weak order on X_i^2; since X_i^2 is denumerable, we may choose for $p_i : X_i^2 \to \mathbb{R}$, a numerical representation of $\succsim_i^*$. Since $\succsim$ is independent, we have $(x_i, x_i) \sim_i^* (y_i, y_i)$, for all $x_i, y_i \in X_i$; we may thus impose that $p_i(x_i, x_i) = 0$ for all $x_i \in X_i$. We then define F for instance as

$$F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) = \begin{cases} \exp(\sum_{i=1}^n p_i(x_i, y_i)) & \text{if } x \succsim y, \\ -\exp(-\sum_{i=1}^n p_i(x_i, y_i)) & \text{otherwise.} \end{cases} \quad (30)$$

Under $AC123$, $\succsim_i^*$ is strongly linear (Lemma 4.6); by Proposition 2.2, there are functions u_i and φ_i such that the numerical representation p_i of $\succsim_i^*$ may be written as $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$ with φ_i nondecreasing in its first argument and nonincreasing in the second.

(4) The construction of F for a relation that satisfies model (M2′) is almost the same; the only difference lies in the fact that we may choose p_i a numerical representation of the weak order $\succsim_i^{**}$ (instead of $\succsim_i^{*}$) and in addition impose that $p_i(x_i, y_i) = -p_i(y_i, x_i)$ (*skew-symmetry*). The skew-symmetric p_i functions may then be decomposed as in the previous case since by Lemma 4, part 6, $\succsim_i^{**}$ is strongly linear.

(5) and (6) For models (M3) and (M3′), which are distinct, we have to modify slightly the definition of F. Take for p_i a skew-symmetric numerical representation of $\succsim_i^{**}$, like in model (M2′), and define F as follows:

$$F(p_1(x_1,y_1), p_2(x_2,y_2), \ldots, p_n(x_n,y_n)) = \begin{cases} \exp(\sum_{i=1}^n p_i(x_i,y_i)) & \text{if } x \succ y, \\ 0 & \text{if } x \sim y, \\ -\exp(-\sum_{i=1}^n p_i(x_i,y_i)) & \text{otherwise.} \end{cases} \quad (31)$$

F again is well-defined (see Bouyssou and Pirlot (2002) for details); it is odd in view of the definition of F and the fact that the relation $\succsim$ is complete. In model (M3), F is nondecreasing in all p_i but not necessarily strictly increasing; in this model we may not exclude indeed that $x \sim y, (z_i, w_i) \succ_i^{**} (x_i, y_i)$ and $(z_i, x_{-i}) \sim (w_i, y_{-i})$, for some $x, y \in X$ and $z_i, w_i \in X_i$. In model ($M3'$), when axiom TC is in force, such a situation never occurs and, with the same construction, F is strictly increasing. Due to Lemma 4, part (6), $\succsim_i^{**}$ is strongly linear and in both models, p_i may thus be decomposed as in case (3). □

5.1.2. Equivalence of models and independence of axioms

The equivalence of two pairs of models directly results from Theorem 2 and the previous results. We note them in the following corollary.

Corollary 1. *If X is at most denumerable,*

(1) *models* (M1–D1) *and* (M1′–D1) *are equivalent*;
(2) *models* (M2–D1) *and* (M2′–D1) *are equivalent*.

The proof is immediate since by Theorem 1 and Lemma 4.1, the weaker model (M1–D1) (resp. (M2–D1)) satisfies all the properties that characterise the stronger (M1′–D1) (resp. (M2′–D1)), according to Theorem 2.

In Appendix B we provide examples showing that none of the axioms characterising the models described in Theorem 2, parts (3)–(6) is a consequence of the others (for part (1) there is nothing to prove and proving the independence of the axioms for part (2) is left to the reader). Table 2 summarises the properties of the Examples 1–8 in Appendix B; properties that are fulfilled (resp. violated) by an example are encoded by "1" (resp. "0") in that table. The nonredundancy of the properties used for characterising the various models in Theorem 2 is established

Table 2
Properties of Examples 1–8 in Appendix B

	R	C	$RC1$	$RC2$	I	TC	$AC1$	$AC2$	$AC3$
Ex1	0	0	1	1	1	1	1	1	1
Ex2	1	1	1	1	1	0	1	1	1
Ex3	1	1	1	1	1	1	0	1	1
Ex4	1	1	1	1	1	1	1	0	1
Ex5	1	1	1	1	1	1	1	1	0
Ex5	1	1	1	1	1	1	1	1	1
Ex6	1	1	1	0	0	0	1	1	1
Ex7	1	1	1	0	1	0	1	1	1
Ex8	1	1	0	1	1	0	1	1	1

Meaning of the abbreviations: "R" for "reflexive"; "C" for "complete"; "I" for "independent".

- for part (3), by Examples 1, 6, 8, 3, 4, 5;
- for parts (4) and (5), by Examples 1, 8, 7, 3, 4, 5;
- for part (6), by Examples 1, 2, 3, 4, 5.

The order in which the examples are listed corresponds to the order in which the properties characterising the models appear in parts (3)–(6) of Theorem 2: for each model, each example violates the corresponding property in the characterisation of the model while it satisfies all the others.

5.2. Strictly monotonic decomposable models in the denumerable case

In this section we extend our analysis to "strictly monotonically" decomposable models, i.e. we deal with all models suffixed by $-$D1′.

5.2.1. Characterisation results

The following theorem shows that –D1 and $-$D1′ models cannot be distinguished except in the more constrained (M3′) case.

Theorem 3. *Let $\succsim$ be a binary relation on a finite or countably infinite set $X = \prod_{i=1}^n X_i$. Then:*

(1) *parts* (1)–(5) *of Theorem* 2 *remain true when $D1$ is substituted by* D1′ *in the labels of the models*;
(2) $\succsim$ *satisfies model* (M3′–D1′) *iff* $\succsim$ *is complete and satisfies TC and $TAC12$.*

Except for the first two models (corresponding to parts (1) and (2) of Theorem 2, which have been proved in Lemmas 1 and 2 respectively), the proof of Theorem 3 is rather technical. It develops the following idea. For each of the models characterised in Theorem 2, with the exception of the sixth one, we show that the functions φ_i that appear in the representation and are nondecreasing in their first argument and nonincreasing in their second, can be substituted by functions that are strictly increasing in their first argument and strictly decreasing in their second. The proof of the theorem relies on Lemmas 5 and 6 stated below; the proof of these lemmas is deferred to Appendix A.2 and A.3.

Since we are planning to transform the functions φ_i that appear in the representation of $\succsim$ in our models, we need knowing how much freedom we have for doing so. It is important to keep in mind that the functions p_i appearing in the various (Mk) and (Mk') models need not be a numerical representation of $\succsim_i^*$ (in model (M1) or (M1′)) or of $\succsim_i^{**}$ (in models (M2), (M2′), (M3) or (M3′)). Our first lemma states the precise (necessary and sufficient) conditions that p_i has to fulfil in the numerical representations of the various models.

Lemma 5.

(1) *Let $\succsim$ satisfy model* (M1) *or* (M1′). *A function $p_i : X_i^2 \to \mathbb{R}$, with $p_i(x_i, x_i) = 0$, for all $x_i \in X_i$, can be used in a representation of $\succsim$ according to model* (M1) *or* (M1′) *iff*

$$(z_i, w_i) \succ_i^* (x_i, y_i) \Rightarrow p_i(z_i, w_i) > p_i(x_i, y_i). \tag{32}$$

(2) *Let $\succsim$ satisfy model* (M2), (M2′) *or* (M3). *A function $p_i : X_i^2 \to \mathbb{R}$, with $p_i(x_i, y_i) = -p_i(y_i, x_i)$, for all $x_i, y_i \in X_i$, can be used in a representation of $\succsim$ according to model* (M2), (M2′) *or* (M3) *iff*

$$(z_i, w_i) \succ_i^{**} (x_i, y_i) \Rightarrow p_i(z_i, w_i) > p_i(x_i, y_i). \tag{33}$$

(3) *Let $\succsim$ satisfy model* (M3′). *A function $p_i : X_i^2 \to \mathbb{R}$, with $p_i(x_i, y_i) = -p_i(y_i, x_i)$, for all $x_i, y_i \in X_i$, can be used in a representation of $\succsim$ according to model* (M3′) *iff*

$$(z_i, w_i) \succ_i^{**} (x_i, y_i) \Rightarrow p_i(z_i, w_i) > p_i(x_i, y_i)$$

and

$$\left.\begin{array}{r} (z_i, w_i) \sim_i^{**} (x_i, y_i) \text{ and} \\ \exists a_{-i}, b_{-i} \in X_{-i} \\ \text{s.t. } (x_i, a_{-i}) \sim (y_i, b_{-i}) \end{array}\right\} \Rightarrow p_i(z_i, w_i) = p_i(x_i, y_i). \tag{34}$$

The next lemma states, in a fairly general framework, the conditions under which a function φ of two variables that is nondecreasing in its first argument and nonincreasing in the second can be transformed into a strictly monotonic function ψ while preserving the ordering induced by φ on its domain of definition. Consider a function $\varphi : U \times U \to \mathbb{R}$, with U, a subset of $\mathbb{R}$, and suppose that φ is nondecreasing in its first argument and nonincreasing in the second. There are two types of situations that may cause the lack of strict monotonicity of φ in its variables; we denote by S, the set of values r of φ for which either there are $a, b, c \in U$ such that:

$$\varphi(a, c) = \varphi(b, c) = r \quad \text{with } a > b \tag{35}$$

or there are $a, c, d \in U$ such that:

$$\varphi(a, c) = \varphi(a, d) = r \quad \text{with } c > d. \tag{36}$$

Clearly, φ is strictly monotonic iff S is empty. The role played by the set S is crucial as we can see in the next lemma.

Lemma 6. *Let U be a subset of the $]0, 1[$ interval and $\varphi : U \times U \to \mathbb{R}$ that vanishes on the diagonal ($\varphi(u, u) = 0$, for all $u \in U$) and is nondecreasing in its first argument and nonincreasing in the second.*

(1) *If S is at most denumerable, there exists a function $\psi : U \times U \to \mathbb{R}$ that vanishes on the diagonal, is increasing in its first argument and decreasing in the second and satisfies the following properties: for all $u, v, u', v' \in U$,*

$$[\varphi(u, v) > \varphi(u', v')] \Rightarrow [\psi(u, v)) > \psi(u', v')]. \tag{37}$$

and

$$[\varphi(u, v) = \varphi(u', v')] \Rightarrow [\psi(u, v)) = \psi(u', v')] \quad \text{iff } \varphi(u, v) \notin S. \tag{38}$$

If, in addition, φ is skew-symmetric, there exists a skew-symmetric ψ with the same properties as above.

(2) *If S is not denumerable, there is no function ψ that is increasing in its first argument, decreasing in the second and satisfies* (37).

We are now in a position to prove Theorem 3.

Proof of Theorem 3. (1) The assertion about models (M–D1′) and (M0–D1′) are established respectively by Lemmas 1 and 2.

Model (M1′–D1′). We know from Theorem 2.3 that the conditions are necessary and that they enable to build a representation of $\succsim$ within model (M1′–D1). Following the construction process outlined in the proof of Theorem 2.3, we have $(z_i, w_i) \succsim_i^* (x_i, y_i)$ iff $p_i(z_i, w_i) \geqslant p_i(x_i, y_i)$ and $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$, for all $x_i, y_i, z_i, w_i \in X_i$. Use Lemma 6.1 and substitute φ by a strictly monotonic function ψ_i $(U) = u_i(X_i)$ may w.l.o.g. be supposed to be included in the $]0, 1[$ interval and the set S associated with φ by (35) and (36) is denumerable, since X_i is at most denumerable. According to Eq. (37), the function $p_i'(x_i, y_i) = \psi_i(u_i(x_i), u_i(y_i))$ satisfies the necessary and sufficient condition (32) so that it can be used in a representation of $\succsim$ within model (M1′). Since $p_i'(x_i, y_i)$ decomposes (by definition) as a function $\psi_i(u_i(x_i), u_i(y_i))$ that is increasing in its first argument and decreasing in the second, we thus have a representation of $\succsim$ in model (M1′–D1′).

Models (M2′–D1′) and (M3–D1′). The proof is similar to that for model (M1′–D1′) except that p_i is a skew-symmetric representation of $\succsim_i^{**}$; parts (4) and (5) of Theorem 2 are used together with Lemmas 6.1 and 5.2. (2) The conditions have already proven to be necessary (Theorem 1.2 and Lemma 4.2). Assuming that the axioms are satisfied implies that $\succsim$ has a representation in model (M3′–D1) with $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$ representing $\succsim_i^{**}$. By construction, φ_i, is nondecreasing in its first argument and nonincreasing in the second and skew-symmetric. Applying Lemma 6.1 yields a function ψ_i; letting $p_i'(x_i, y_i) = \psi_i(u_i(x_i), u_i(y_i))$, we have to check whether the additional condition (34) of Lemma 5.3 is fulfilled. Let $Y \subseteq X_i \times X_i$ be an equivalence class of the relation $\sim_i^{**}$ containing a pair (x_i, y_i) such that $\exists a_{-i}, b_{-i} \in X_{-i}$ with $(x_i, a_{-i}) \sim (y_i, b_{-i})$. We claim that Y contains neither pairs (x_i', y_i'), (x_i'', y_i') such that $u_i(x_i') > u_i(x_i'')$ nor pairs (x_i', y_i'), (x_i', y_i'') such that $u_i(y_i') > u_i(y_i'')$. This means that the value $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$ associated to all pairs in the class Y does not belong to the set S associated to φ_i. If true, all pairs in Y will be assigned the same number by ψ_i (according to (38)).

To prove the assertion, suppose, on the contrary, that there are pairs (x_i', y_i'), $(x_i'', y_i') \in Y$ such that $u_i(x_i') > u_i(x_i'')$ (the other case is treated similarly). Notice that since $(x_i, a_{-i}) \sim (y_i, b_{-i})$, for all (u_i, v_i) such that $(u_i, v_i) \sim_i^{**} (x_i, y_i)$, one has $(u_i, a_{-i}) \sim (v_i, b_{-i})$; we would thus have here $(x_i', a_{-i}) \sim (y_i', b_{-i})$ and $(x_i'', a_{-i}) \sim (y_i', b_{-i})$. We may assume that u_i represents the trace of $\succsim_i^{**}$ (as is done in the proof of Theorem 2); $u_i(x_i') > u_i(x_i'')$ consequently means that either $\exists w_i$ such that $(x_i', w_i) \succ_i^{**} (x_i'', w_i)$ or $\exists z_i$ such that $(z_i, x_i'') \succ_i^{**} (z_i, x_i')$, which in turn means respectively that $\exists c_{-i}, d_{-i} \in X_{-i}$ such that:

$$(x_i', c_{-i}) \succsim (w_i, d_{-i}) \quad \text{and} \quad Not\,[(x_i'', c_{-i}) \succsim (w_i, d_{-i})] \tag{39}$$

or

$$(z_i, c_{-i}) \succsim (x_i'', d_{-i}) \quad \text{and} \quad Not\,[(z_i, c_{-i}) \succsim (x_i', d_{-i})] \tag{40}$$

In case $(x_i', c_{-i}) \succsim (w_i, d_{-i})$ holds in (39), applying $TAC1_i$ to $(x_i'', a_{-i}) \succsim (y_i', b_{-i})$, $(y_i', b_{-i}) \succsim (x_i', a_{-i})$ and $(x_i', c_{-i}) \succsim (w_i, d_{-i})$ yields $(x_i'', c_{-i}) \succsim (w_i, d_{-i})$, contrary to (39).

Similarly, in case $(z_i, c_{-i}) \succsim (x_i'', d_{-i})$ holds in (40), applying $TAC2_i$ to $(x_i'', a_{-i}) \succsim (y_i', b_{-i})$, $(y_i', b_{-i}) \succsim (x_i', a_{-i})$ and $(z_i, c_{-i}) \succsim (x_i'', d_{-i})$ yields $(z_i, c_{-i}) \succsim (x_i', d_{-i})$, contrary to (40). □

5.2.2. Equivalence of models and independence of axioms

We list in the next corollary the equivalences of models that result from Theorems 2 and 3.

Corollary 2. *If X is at most denumerable, there are seven classes of distinct models, which are*:

(1) *models* (M–D1), (M–D1′), *that are equivalent*;
(2) *models* (M0–D1), (M0–D1′), *that are equivalent*;
(3) *models* (M1–D1), (M1–D1′), (M1′–D1) *and* (M1′–D1′), *that are equivalent*;
(4) *models* (M2–D1), (M2–D1′), (M2′–D1) *and* (M2′–D1′), *that are equivalent*;
(5) *model* (M3–D1) *and* (M3–D1′), *that are equivalent*;
(6) *model* (M3′–D1);
(7) *model* (M3′–D1′).

Proof. The equivalences of models listed above result from Corollary 1 and the fact that the characterisations of the first five models are the same in Theorems 2 and 3. The first two equivalences were already noted in Lemmas 1 and 2.

The distinctness of the seven classes of models can be shown by exhibiting appropriate examples. Since not all relations are reflexive and independent, the first two classes are distinct. Example 8 in Appendix A proves that there are models satisfying (M0–D1) and not (M1′–D1); by Example 7 we know that it is possible to satisfy (M1′–D1) without satisfying (M2′–D1). Example 9 verifies (M2′–D1) but not (M3–D1) and Example 10, (M3–D1) but not (M3′–D1). Example 13 shows that models (M3′–D1) and (M3′–D1′) are not equivalent since the relation in this example is complete and satisfies TC and $AC123$ but neither $TAC1$ nor $TAC2$; therefore it can be represented in model (M3′–D1) but not in model (M3′–D1′). The classical additive utility model (Eq. (3)) shows that the axioms characterising model (M3′–D1′) are not inconsistent. □

Independence of the axioms characterising models (M–D1′), (M0–D1′), (M1′–D1′), (M2′–D1′) and (M3–D1′) has been established in Section 5.1.2. Table 3 refers to examples showing that each of the axioms characterising model (M3′–D1′) is independent of the others.

Finally, in view of Bouyssou and Pirlot (2002) (where axioms $RC1$, $RC2$ and TC are studied) and of Bouyssou and Pirlot (2004) (where the scrutinized axioms are $AC1, AC2, AC3, TAC1$ and $TAC2$), it may be interesting to point out that there are no logical interactions between those two families of axioms. Example 11 in Appendix B shows that there are reflexive, independent and complete relations satisfying TC (and, hence, $RC1, RC2$) but none of $AC1$, $AC2$, $AC3$ (and a fortiori none of $TAC1$, $TAC2$). Conversely, as shown by Example 12, there are reflexive, independent and complete relations satisfying $TAC1$ and $TAC2$ (hence $AC1$, $AC2$ and $AC3$) but neither $RC1$ nor $RC2$ (and a fortiori not TC).

Table 3
Properties of Examples in Appendix B related with model (M3′–D1′)

	C	TC	$TAC1$	$TAC2$
Ex1	0	1	1	1
Ex12	1	0	1	1
Ex3	1	1	0	1
Ex4	1	1	1	0

"C" stands for "complete".

5.3. The nondenumerable case

Extending Theorems 2 and 3 to the case in which the X_i's are not supposed to be denumerable raises problems of numerical representability. Since the case of models (M–D1), (M–D1′), (M0–D1) and (M0–D1′) has been dealt with above using only *LCC* (Lemma 2), we concentrate on models at least as constrained as (M1–D1). Suppose that $\succsim$ satisfies the axioms for model (M1–D1) (or a more constrained model) as stated in Theorem 2; in case some (or all) of the X_i's are not denumerable, we observe that:

- the weak orders $\succsim_i^*$ (or $\succsim_i^{**}$) may not have a numerical representation;
- the weak orders $\succsim_i^{\pm}$ may not have a numerical representation and
- the functions φ_i (resp. u_i) that appear in the model may fail to be representations of $\succsim_i^*$ or $\succsim_i^{**}$ (resp. of $\succsim_i^{\pm}$, see (19)).

5.3.1. Characterisation results

We start by showing that the representability of $\succsim_i^*$ (or $\succsim_i^{**}$) and their traces is a necessary condition in the nondenumerable case. Theorem 1.7 indicates that models (Mk–D) and (Mk'–D), for $k = 1, 2, 3$, require property OD^* ensuring that $\succsim_i^*$ or $\succsim_i^{**}$ be representable on $\mathbb{R}$. OD^* is a fortiori necessary for all the models we consider, (M1–D1) and more constrained.

Lemma 5 states conditions that functions p_i must satisfy (and that are also sufficient) for being used in a representation of $\succsim$ in models (Mk) or (Mk'); from that, conditions on the functions φ_i can be derived. In the same spirit, the next lemma states a condition that u_i has to fulfil if used in model (M1–D1) or a more constrained one.

Lemma 7. *Let $\succsim$ satisfy model* (M1–D1) *or a more constrained one. If a function $u_i : X_i \to \mathbb{R}$ appears in a representation of $\succsim$ according to model* (M1–D1) *or a more constrained model, then, for all $x_i, y_i \in X_i$,*

$$x_i \succ_i^{\pm} y_i \Rightarrow u_i(x_i) > u_i(y_i) \tag{41}$$

Proof. Suppose, on the contrary, that for some $x_i, y_i \in X_i$, we have $x_i \succ_i^{\pm} y_i$ and $u_i(x_i) \leqslant u_i(y_i)$. From $x_i \succ_i^{\pm} y_i$ and using the completeness of both $\succsim_i^{\pm}$ and $\succsim_i^*$ in (M1–D1), we get that there is $z_i \in X_i$ such that $(x_i, z_i) \succ_i^* (y_i, z_i)$ or there is $w_i \in X_i$ such that $(w_i, y_i) \succ_i^* (w_i, x_i)$. In the former case, using Lemma 5.1, yields $\varphi_i(u_i(x_i), u_i(z_i)) > \varphi_i(u_i(y_i), u_i(z_i))$, which is not compatible with $u_i(x_i) \leqslant u_i(y_i)$ as long as φ_i is nondecreasing in its first argument. A similar contradiction can be derived from the other branch of the alternative. The same type of reasoning, using Lemma 5, enables to show the necessity of condition (41) for all models from (M1–D1) on. □

Condition (41) states that the weak order represented by u_i must be at least as fine as $\succsim_i^{\pm}$. Since an order finer (i.e. more discriminating) than an order that is not representable on the reals does not admit a numerical representation either, we have established that the following order-density condition is necessary. We say that $\succsim$ satisfies $OD_i^{\pm}$ if there is a finite or countably infinite subset of X_i that is dense in X_i for $\succsim_i^{\pm}$. Condition $OD^{\pm}$ is said to hold if condition $OD_i^{\pm}$ is in force for all $i \in N$.

Conditions OD^* and $OD^{\pm}$ are sufficient to extend the results of Theorem 2 to the uncountable case. Reconsidering the proof of parts (3)–(6) of Theorem 2, we see that the construction of a representation in the respective models can be worked out as soon as are available:

- a representation p_i of the weak order $\succsim_i^*$ (for all $i \in N$) in models (M1–D1) and (M1′–D1) or of $\succsim_i^{**}$ in model (M2–D1) and more constrained ones,
- and a representation of the trace of $\succsim_i^*$ (which, in view of (23) is also the trace of $\succsim_i^{**}$).

This is precisely what OD^* and $OD^{\pm}$ guarantee. Note that in models (M2–D1) and more constrained, OD^* implies that $\succsim_i^{**}$ is representable (see Bouyssou & Pirlot, 2002). We thus have the following extension of Theorem 2. The first two parts are consequences of Lemmas 1 and 2; they are stated here for the sake of completeness.

Theorem 4. *Let $\succsim$ be a binary relation on a product set $X = \prod_{i=1}^n X_i$. Then:*

1. *$\succsim$ satisfies model* (M–D1) *iff $\succsim$ satisfies property LCC;*
2. *$\succsim$ satisfies model* (M0–D1) *iff $\succsim$ is reflexive, independent and satisfies property LCC;*
3. *$\succsim$ satisfies model* (M1–D1) *or* (M1′–D1) *iff $\succsim$ is reflexive, independent and satisfies RC1, AC123, OD^* and $OD^{\pm}$;*
4. *$\succsim$ satisfies model* (M2–D1) *or* (M2′–D1) *iff $\succsim$ is reflexive and satisfies RC12, AC123, OD^* and $OD^{\pm}$;*
5. *$\succsim$ satisfies model* (M3–D1) *iff $\succsim$ is complete and satisfies RC12, AC123, OD^* and $OD^{\pm}$;*
6. *$\succsim$ satisfies model* (M3′–D1) *iff $\succsim$ is complete and satisfies TC, AC123, OD^* and $OD^{\pm}$.*

In order to extend Theorem 3, we need another axiom that is closely linked with the set S described in Eqs. (35), (36) and that will enable us to adapt the proof of Theorem 3, i.e. to modify function φ_i into a function ψ_i that is strictly monotonic in both its arguments. Let S_i^* (resp. S_i^{**}) denote the set of equivalence classes s of the relation $\succsim_i^*$ (resp. $\succsim_i^{**}$) that verify the following:

$$\exists (x_i, z_i), (y_i, z_i) \in s \quad \text{or } \exists (w_i, x_i), (w_i, y_i) \in s$$
$$\text{such that } Not[x_i \sim_i^{\pm} y_i]. \tag{42}$$

In view of Lemma 6 and the correspondence between S_i^* (or S_i^{**}) and the set S, it is no wonder that the cardinality of those sets does matter. We denote by Σ_i^* (resp. Σ_i^{**}), the property stating that S_i^* (resp. S_i^{**}) is denumerable; Σ^* (resp. Σ^{**}) stands for Σ_i^* (resp. Σ_i^{**}) holding for all $i \in N$. The necessity of Σ^* or Σ^{**} in the various models is established in the next lemma.

Lemma 8.

(1) [(M1–D1′), (M1′−D1′)] $\Rightarrow \Sigma^*$,
(2) [(M2–D1′), (M2′–D1′), (M3–D1′), (M3′−D1′)] $\Rightarrow \Sigma^{**} \Rightarrow \Sigma^*$.

Proof. (1) Let $\succsim$ belong to one of the models (Mk–D1′) or (Mk'–D1′) for $k = 1, 2, 3$. Since all these models are more constrained than (M1–D1′), $\succsim$ has a representation in the latter. Let F, φ_i, u_i, for $i \in N$, provide a representation of $\succsim$ in model (M1–D1′); φ_i is increasing in its first argument and decreasing in the second for all i. Let s denote an equivalence class of $\succsim_i^*$ containing a pair $(x_i, z_i), (y_i, z_i)$ with $x_i \succ_i^{\pm} y_i$. Suppose the set of classes such as s is not denumerable. Using Lemma 7 and the increasingness of φ_i in its first argument, $x_i \succ_i^{\pm} y_i$ entails $u_i(x_i) > u_i(y_i)$ and $\varphi_i(u_i(x_i), u_i(z_i)) > \varphi_i(u_i(y_i), u_i(z_i))$. Intervals $(\varphi_i(u_i(y_i), u_i(z_i))$, $\varphi_i(u_i(x_i), u_i(z_i)))$ corresponding to different classes s and s' are disjoint (in view of Lemma 5.1); they form a non-denumerable family of disjoint nonempty intervals of $\mathbb{R}$, which does not exist since each interval contains a distinct rational number. One similarly proves the denumerability of the set of equivalence classes s' of $\succsim_i^*$ containing a pair $(w_i, x_i), (w_i, y_i)$ with $x_i \succ_i^{\pm} y_i$ (using the decreasingness of φ_i in its second argument). This establishes that Σ^* holds in all models at least as constrained as (M1–D1′).

(2) Turning to Σ^{**}, consider $\succsim$, a relation that satisfies model (M2–D1′) or a more constrained model. Such a relation has a representation in (M2–D1′) using some functions F, φ_i, u_i, for $i \in N$, with φ_i increasing in its first argument and decreasing in the second. Reasoning as above but about equivalence classes s or s' of $\succsim_i^{**}$, we can prove that Σ^{**} must hold. Moreover, it is clear, in general, that Σ^{**} implies Σ^* since the equivalence classes of $\succsim_i^{**}$ are subdivisions of those of $\succsim_i^*$. □

The extension of Theorem 3 to sets of arbitrary cardinality is now at hand.

Theorem 5. *Let* $\succsim$ *be a binary relation on a set* $X = \prod_{i=1}^n X_i$. *Then*:

(1) $\succsim$ *satisfies model* (M–D1′) *iff* $\succsim$ *satisfies property* LCC;
(2) $\succsim$ *satisfies model* (M0–D1′) *iff* $\succsim$ *is reflexive, independent and satisfies property* LCC;
(3) $\succsim$ *satisfies model* (M1–D1′) *or* (M1′–D1′) *iff* $\succsim$ *is reflexive, independent and satisfies* $RC1$, $AC123$, OD^*, $OD^{\pm}$ *and* Σ^*;
(4) $\succsim$ *satisfies model* (M2–D1′) *or* (M2′–D1′) *iff* $\succsim$ *is reflexive and satisfies* $RC12$, $AC123$, OD^*, $OD^{\pm}$ *and* Σ^{**};
(5) $\succsim$ *satisfies model* (M3–D1′) *iff* $\succsim$ *is complete and satisfies* $RC12$, $AC123$, OD^*, $OD^{\pm}$ *and* Σ^{**};
(6) $\succsim$ *satisfies model* (M3′–D1′) *iff* $\succsim$ *is complete and satisfies* TC, $TAC12$, OD^*, $OD^{\pm}$ *and* Σ^{**}.

Proof. Parts (1) and (2) are consequences of Lemmas 1 and 2.

(3) In view of Lemma 8.1, it only remains to prove that the conditions are sufficient to guarantee the existence of a representation of $\succsim$ in model (M1′–D1′). Since the hypotheses of Theorem 4.3 are in force, we may construct a representation of $\succsim$ just as described in the proof of Theorem 4.3. According to that construction, $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$ is a representation of the weak order $\succsim_i^*$ and u_i is a representation of $\succsim_i^{\pm}$. Starting from that point, we may transform φ_i into a function ψ_i increasing in its first argument and decreasing in the second as done in the proof of Theorem 3.1. Such a transformation is made possible since Σ^* together with the fact that $\varphi_i(u_i(x_i), u_i(y_i))$ is a numerical representation of $\succsim_i^*$ imply that the set S_i, defined by (35) and (36), applied to φ_i instead of φ, is denumerable; the conditions required for applying Lemma 6.1 are thus fulfilled. The conclusion, i.e. the fact that the transformed representation is a representation of $\succsim$ in model (M1′–D1′), follows as in part 1 of Theorem 3.

(4)–(6) Necessity is a consequence of Lemma 8.2. The proof of sufficiency follows the same lines as in part (3) above; there is only one difference: $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$ is a representation of the weak order $\succsim_i^{**}$ (instead of $\succsim_i^*$) and hypothesis Σ^{**} is thus needed in order to transform φ_i into a function ψ_i that is increasing in its first argument and decreasing in the second. The proof that the transformed representation yields a representation of $\succsim$ in model (M2′–D1′) (resp. (M3–D1′), (M3′–D1′)) is the same as for part (4) (resp. 5, 6) of Theorem 4. □

5.3.2. *Independence of the axioms (final)*

The independence of LCC in models (M–D1′) and (M0–D1′) is obvious. Nonredundancy of the axioms has been established for the denumerable case in the previous sections; in that case, order density conditions as well as Σ^* and Σ^{**} are trivially fulfilled. None of these conditions can be dispensed of in the nondenumerable case. Examples 14–16 in Appendix B (see a summary of their properties in Table 4) establish the independence of these conditions in all the models. More specifically, for models (M1–D1′) or (M1′–D1′) in part (3) of Theorem 5, they respectively show that none of OD^*, $OD^{\pm}$ and Σ^* is redundant. For the models in parts (4)–(6) the same examples respectively show that none of OD^*, $OD^\pm$ and Σ^{**} is redundant. In case X is not a denumerable set, the equivalences of models in parts (3)–(5) of Corollary 2 break into two parts; we describe the resulting equivalences in the next corollary.

Corollary 3. *If X is not denumerable, there are ten classes of distinct models, which are*:

(1) *models* (M–D1) *and* (M–D1′), *that are equivalent*;
(2) *models* (M0–D1) *and* (M0–D1′), *that are equivalent*;
(3) *models* (M1–D1), (M1′–D1), *that are equivalent*;
(4) *models* (M1–D1′) *and* (M1′–D1′), *that are equivalent*;
(5) *models* (M2–D1), (M2′–D1), *that are equivalent*;
(6) *models* (M2–D1′) *and* (M2′–D1′), *that are equivalent*;
(7) *model* (M3–D1);
(8) *model* (M3–D1′);
(9) *model* (M3′–D1);
(10) *model* (M3′–D1′).

Proof. The equivalences stated in the corollary result from Theorems 4 and 5. In view of the proof of Corollary 2 and the examples used therein, we only have to justify that the subclasses of the equivalence classes that split in the nondenumerable case are distinct. A single example suffices to prove the latter. Example 16 is representable in model (M3′–D1) (and hence in models (M2′–D1) and (M1′–D1)) but in none of (M3′–D1′), (M2′–D1′) or (M1′–D1′) since the relation in the example satisfies neither Σ^* nor Σ^{**}. □

Table 4
Properties of Examples 14–16 in Appendix B

	R	C	I	TC	TAC	OD^*	$OD^\pm$	Σ^*	Σ^{**}
Ex14	1	1	1	1	1	0	1	1	1
Ex15	1	1	1	1	1	1	0	1	1
Ex16	1	1	1	1	1	1	1	0	0

Meaning of the abbreviations: "R" stands for "reflexive", "C" for "complete", "I" for "independent"; TAC stands for $TAC1$ and $TAC2$

5.4. Discussion of the results

We summarise in Table 5 the results obtained in Theorems 2–5. This table offers a synthetic view of all the models studied together with their characterisation. The axioms that appear in the columns headed by the label "Nondenumerable" have to be added to those characterising the corresponding models in the denumerable case in order to get a characterisation valid for the nondenumerable case.

The relations between models are shown in graphical form in Fig. 2, which represents the same information as Corollary 3; this figure both shows which models are equivalent and which classes are contained in others. This picture is valid for the most general nondenumerable case. It simplifies in the denumerable case as indicated by Corollary 2: all models (M1) and (M1′) are equivalent as well as all models (M2) and (M2′); at the upper level, three distinct classes remain: one formed by (M3–D1) and (M3–D1′) and two classes each containing a single model, namely (M3′–D1) and (M3′–D1′).

5.4.1. Relationship between $\succsim_i^\pm$ and $\succsim_i^*$ or $\succsim_i^{**}$

The inter-relations between $\succsim_i^*$ or $\succsim_i^{**}$ and $\succsim$ have been investigated in Bouyssou and Pirlot (2004) (see Lemma 3, p. 689). Similarly, those between $\succsim_i^\pm$ and $\succsim$ are studied in Bouyssou and Pirlot (2004) (see Lemmas 2 and 4). We have the opportunity to examine here the relationships between $\succsim_i^\pm$ and $\succsim_i^*$ or $\succsim_i^{**}$. We observed through (23), that $\succsim_i^\pm$ is the trace of $\succsim_i^*$ and $\succsim_i^{**}$, which amounts saying that $\succsim_i^*$ and $\succsim_i^{**}$ respond positively to $\succsim_i^\pm$, i.e. $x_i \succsim_i^\pm y_i \Rightarrow (x_i, z_i) \succsim_i^* (y_i, z_i)$ and $(z_i, y_i) \succsim_i^* (z_i, x_i), \forall z_i$ (and similarly for $\succsim_i^{**}$). The "response" may however fail to be "strictly positive" even in the more constrained model (M3′–D1′); it may happen indeed that $x_i \succ_i^\pm y_i$ and for some

Table 5
Summary of the results in Theorems 2–5

	Denumerable		Nondenumerable	
	–D1	–D1′	–D1	–D1′
M			LCC	
M0	refl., indep.		LCC	
M1, M1′	refl., indep. $RC1$, $AC123$		OD^*, $OD^\pm$	OD^*, $OD^\pm$ Σ^*
M2, M2′	refl., $RC12$, $AC123$		OD^*, $OD^\pm$	OD^*, $OD^\pm$ Σ^{**}
M3	compl., $RC12$, $AC123$		OD^*, $OD^\pm$	OD^*, $OD^\pm$ Σ^{**}
M3′	compl., TC $AC123$	compl., TC $TAC12$	OD^*, $OD^\pm$ Σ^{**}	

Meaning of the abbreviations: "refl." for "reflexive", "indep." for "independent", "compl." for "complete".

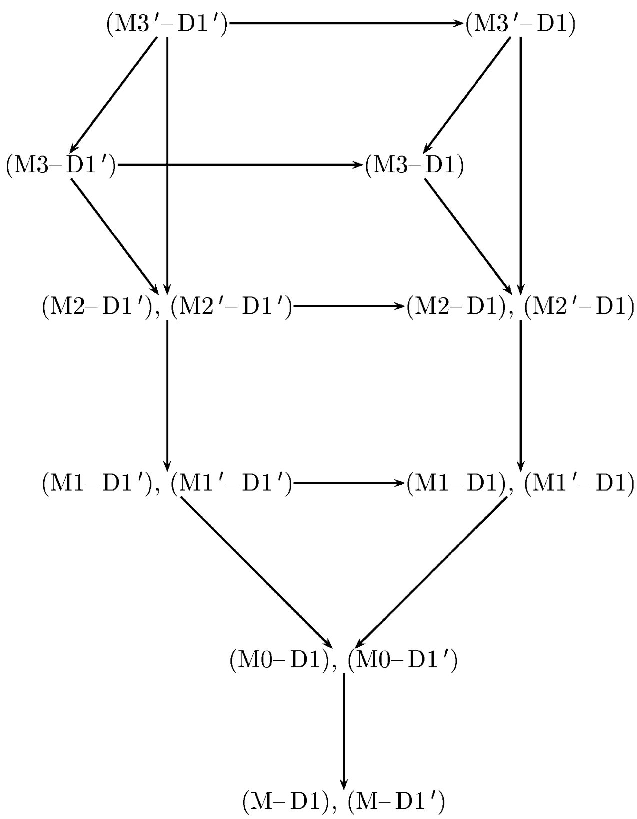

Fig. 2. Distinct models and implications in the nondenumerable case.

z_i, $(x_i, z_i) \sim_i^{**} (y_i, z_i)$ or for some w_i, $(w_i, y_i) \sim_i^{**} (w_i, x_i)$ (this is the case for a denumerable set of equivalence classes in Example 17). In this respect, the set S_i^* (resp. S_i^{**}) defined by formula (42) plays a crucial role. The response of $\succsim_i^*$ (resp. $\succsim_i^{**}$) is strictly positive if and only if the set S_i^* (resp. S_i^{**}) is empty. If this is not the case, as long as S_i^* (resp. S_i^{**}) is finite or denumerable, the response is not always strictly positive, but there is a representation in a model of type $-$D1′; in case S_i^* (resp. S_i^{**}) is not denumerable, representing $\succsim$ in such a model is no longer possible.

5.4.2. Uniqueness issues and regular representations

In this section, we only consider models from (M1–D1) and more constrained; in all these models, $\succsim_i^{\pm}$ and $\succsim_i^*$ (and $\succsim_i^{**}$ in models (M2–D1) and more constrained) are weak orders for all i. As noted in Bouyssou and Pirlot (2002), uniqueness results for the numerical representations of these models are very weak. Lemmas 5 and 7 give however indications on necessary conditions that φ_i and u_i have to fulfil if used in a numerical representation of one of our models. These conditions amount saying that φ_i represents a relation that is at least as fine as $\succsim_i^*$ (or $\succsim_i^{**}$ for models from (M2–D1) and more constrained); similarly, u_i must represent a relation that is finer than $\succsim_i^{\pm}$. The discussion in Section 5.4.1 has shown that it is not possible in all models to have representations in which:

- u_i is a numerical representation of the weak order $\succsim_i^{\pm}$ and
- $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$ is a numerical representation of the weak order $\succsim_i^*$ (for models (M1–D1) and more constrained) or of the weak order $\succsim_i^{**}$ (for models (M2–D1) and more constrained).

We call *regular* a representation in which this is the case (see Roberts, 1979 about regularisation of a scale of measurement; see also the considerations on regularity in relation with uniqueness of the representation in Bouyssou & Pirlot, 2002, Remark 4, p. 695).

In the proofs of Theorems 2 and 4, we built regular representations for the models (M1′–D1), (M2′–D1), (M3′–D1), which proves that regular representations always exist (even in the nondenumerable case) for all our $-$D1 models. Is it the case for the $-$D1′ models? In the proofs of Theorems 3 and 5 where the $-$D1′ models are studied, we start from regular representations in the corresponding $-$D1 model and change the functions φ_i into functions ψ_i that are increasing in their first argument and decreasing in the second. These alterations of the representations of $\succsim_i^*$ (or $\succsim_i^{**}$) respect conditions (37) and (38) of Lemma 6. If the representation with u_i and φ_i is regular, all pairs in any equivalence class of $\succsim_i^*$ (or $\succsim_i^{**}$) are associated the same value by φ_i. Due to (38), this is still the case after φ_i has been transformed into ψ_i unless the equivalence class belongs to S_i^* (or S_i^{**}). Hence, if S_i^* (or S_i^{**}) is empty, there is a regular representation in the $-$D1′ model. Thus, starting from a regular representation in a $-$D1 model, we have proven that a sufficient condition for a regular representation in the corresponding $-$D1′ model to exist is that S_i^* (or S_i^{**}) be empty. This condition is also clearly necessary. We thus have proven the following proposition.

Proposition 3.

(1) *A relation* $\succsim$ *that satisfies the hypotheses of any model* (Mk–D1) *or* (Mk′–D1) *for* $k = 1, 2$ *or* 3, *has a regular representation in that model*;
(2) *A relation* $\succsim$ *that satisfies the hypotheses of model* (M1–D1′) *or* (M1′–D1′) *has a regular representation in that model iff* S_i^* *is empty for all* $i \in N$;
(3) *A relation* $\succsim$ *that satisfies the hypotheses of model* (M2–D1′), (M2′–D1′), (M3–D1′) *or* (M3′–D1′) *has a regular representation in that model iff* S_i^{**} *is empty for all* $i \in N$.

As a direct consequence of the above proposition, we get a condition under which $\succsim_i^*$ is not only strongly linear but also *strongly independent* (see Section 4.1 for the definition of strong independence). This result is formalised in the following corollary.

Corollary 4.

(1) *If* $\succsim$ *satisfies model* (M1–D1′) *or* (M1′–D1′), $\succsim_i^*$ *is strongly independent iff* $S_i^* = \emptyset$ *for all* $i \in N$.
(2) *If* $\succsim$ *satisfies model* (M2–D1′), (M2′–D1′), (M3–D1′) *or* (M3′–D1′), $\succsim_i^{**}$ *is strongly independent iff* $S_i^{**} = \emptyset$ *for all* $i \in N$.

Proof. (1) Let $\succsim$ be a relation that can be represented in model (M1–D1′) (resp. (M1′–D1′)). According to Proposition 3, part (2), $S_i^* = \emptyset$ is a necessary and sufficient condition for such a relation $\succsim$ to admit a representation in model (M1–D1′) (resp. (M1′–D1′)) with $\psi_i(u_i(x_i), u_i(y_i))$ a representation of $\succsim_i^*$ that is increasing in its first argument and decreasing in the second and u_i a representation of $\succsim_i^{\pm}$. In view of Proposition 2, part 2(b), this is equivalent to saying that $\succsim_i^*$ is strongly independent .

(2) A similar result holds for $\succsim_i^{**}$ iff $S_i^{**} = \emptyset$ in model (M2–D1′) and more constrained –D1′ models. This establishes part (2). □

5.4.3. Variants left aside

For the sake of conciseness, not all variants of intra-attribute decomposable models have been investigated here. For instance, instead of using Eq. (13), we might have chosen to decompose p_i as $p_i(x_i, y_i) = \varphi_i(u_i(x_i), v_i(y_i))$, with a function u_i possibly different from the function v_i. In models (M–D) and (M0–D), this apparently more general decomposition has no incidence but, when combined with monotonicity properties of φ_i, the decomposition leads to models in which the "difference of preference" p_i may be understood via two possibly different linear orderings of X_i (for instance, those represented by u_i and v_i, respectively). It is rather straightforward—we leave it to the reader—to adapt the reasonings we made in the case $u_i = v_i$ to the case in which $u_i \neq v_i$ (mainly omitting $AC3$).

5.4.4. Relationships with models studied in the literature

Our intention, as stated in the introduction, was to develop the axiomatisation of models (M–D) in order to come as close as possible to Tversky's additive difference model (2), without making use of unnecessary structural assumptions or hardly interpretable conditions.

Let $\succsim$ be representable in model (2), i.e. $x \succsim y$ iff $\sum_{i=1}^n \Phi_i(u_i(x_i) - u_i(y_i)) \geqslant 0$, for some functions u_i and some functions Φ_i that are increasing and odd. Such a representation is a particular case of a representation of $\succsim$ in model (M3′–D1′) with, e.g. $F(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^n \Phi_i(\alpha_i)$ and $\varphi_i(u_i(x_i), u_i(y_i)) = u_i(x_i) - u_i(y_i)$. F is indeed increasing in all its arguments and odd and φ_i increasing in its first argument, decreasing in the second and skew-symmetric. A relation that is representable in model (2) is thus complete and satisfies TC, $TAC12$, OD^*, $OD^{\pm}$ and Σ^{**} (Theorem 5, part (6)). For a relation $\succsim$ that satisfies model (2), u_i is necessarily a numerical representation of the marginal preference $\succsim_i$ since $[(x_i, a_{-i}) \succsim (y_i, a_{-i}), \forall a_{-i} \in X_i]$ iff $[\Phi_i(u_i(x_i) - u_i(y_i)) \geqslant 0]$ iff $u_i(x_i) - u_i(y_i) \geqslant 0]$. We know (Bouyssou & Pirlot, 2004, Lemma 4, part (3)) that for a complete relation $\succsim$ that satisfies $TAC12$, the marginal preference $\succsim_i$ and the marginal trace $\succsim_i^{\pm}$ are identical. Thus u_i also represents $\succsim_i^{\pm}$. It is not hard to convince oneself that $u_i(x_i) - u_i(y_i)$ is a numerical representation of a relation on X_i^2 that is at least as fine as $\succsim_i^{**}$ (as is true in general of the functions φ_i involved in the representation of a relation that belongs to model (M3′–D1′)). It cannot be excluded for a relation belonging to model (2) that the relation represented by $u_i(x_i) - u_i(y_i)$ be strictly finer than $\succsim_i^{**}$. It is even possible that no regular representation of $\succsim$ exist, i.e. that one cannot find a representation of $\succsim$ in model (2) in which $u_i(x_i) - u_i(y_i)$ is a numerical representation of $\succsim_i^{**}$ (this is the case if $S_i^{**} \neq \emptyset$, as stated in Proposition 3.3).

Of all the models studied in this paper, (M3′–D1′) is the one closest to model (2) and the latter is a special case of the former. Coming closer to Tversky's model without using unnecessary and noninterpretable additional conditions is an interesting challenge for a further study.

Another type of model alluded to in the introduction is the nontransitive additive preference model (5). Model (5) is a particular case of model (M1′) (since $F(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^n \alpha_i$ is increasing in all its arguments) and of model (M3′), as soon as the p_i functions are assumed to be skew-symmetric. We assume in the sequel that the p_i's are skew-symmetric, which thus implies that $\succsim$ fulfills TC and OD^*. If $\succsim$ verifies model (5), the function $p_i(x_i, y_i)$ can be decomposed, as we shall see, into a function $\varphi_i(u_i(x_i), u_i(y_i))$ that is nondecreasing in its first argument and nonincreasing in the second, as soon as $\succsim$ verifies $AC123$ and $OD^{\pm}$. In such a case, we have:

$$x \succsim y \;\Leftrightarrow\; \sum_{i=1}^n \varphi_i(u_i(x_i), u_i(y_i)) \geqslant 0. \tag{43}$$

To prove this assertion, it suffices to apply the strategy of proof of Theorem 2, part (6) and Theorem 4, part (6). In the denumerable case (Theorem 2, part (6)), we started with a representation of $\succsim$ in model (M3′), where p_i represents $\succsim_i^{**}$; since we know that $\succsim_i^{**}$ is strongly linear for all i as soon as $AC123$ is in force, we get, by applying Lemma 4, part (6), a decomposition of $p_i(x_i, y_i)$ into $\varphi_i(u_i(x_i), u_i(y_i))$; in this representation, φ_i is nondecreasing in its first argument and nonincreasing in the second and u_i is a representation of $\succsim_i^{\pm}$. In the nondenumerable case, the same can be done provided $OD^{\pm}$ holds.

The only additional difficulty that appears when starting with a representation in model (5) is that the p_i's do not necessarily represent $\succsim_i^{**}$. It is easily shown that the relation on X_i^2 represented by $p_i(x_i, y_i)$ is at least as fine as $\succsim_i^{**}$, i.e. $[p_i(x_i, y_i) \geqslant p_i(z_i, w_i)] \Rightarrow [(x_i, y_i) \succsim_i^{**} (z_i, w_i)]$. In other words, using the completeness of $\succsim_i^{**}$, $[(x_i, y_i) \succ_i^{**} (z_i, w_i)] \Rightarrow [p_i(x_i, y_i) > p_i(z_i, w_i)]$. If p_i fails to be a representation of $\succsim_i^{**}$, it is because it assigns distinct values to some equivalent pairs $(x_i, y_i) \sim_i^{**} (z_i, w_i)$. Since such pairs are perfectly substitutable without any change in the preference $\succsim$, we may well transform p_i into a representation p_i' of $\succsim_i^{**}$ just by selecting a particular representative pair in each equivalence class of $\succsim_i^{**}$ and assigning to all pairs in the same class, the value assigned by p_i to the selected pair. In other terms, letting (z_i, w_i) be the pair selected in an equivalence class of $\succsim_i^{**}$, we define $p_i'(x_i, y_i) = p_i(z_i, w_i)$ for all $(x_i, y_i) \sim_i^{**} (z_i, w_i)$. We have $x \succsim y \Leftrightarrow \sum_{i=1}^n p_i(x_i, y_i) \geqslant 0 \Leftrightarrow \sum_{i=1}^n p_i'(x_i, y_i) \geqslant 0$. When this regularisation has been done, we know that the p_i''s are representations of the strongly linear relation $\succsim_i^{**}$ (if $AC123$ holds) and can thus be decomposed into $\varphi_i(u_i(x_i), u_i(y_i))$. In the nondenumerable case, $OD^{\pm}$ is needed for guaranteeing the existence of a representation u_i of $\succsim_i^{\pm}$. If we additionally impose that $\succsim$ satisfies $TAC12$, the marginal preferences $\succsim_i$ and the marginal traces $\succsim_i^{\pm}$ are identical (Bouyssou & Pirlot, 2004, Lemma 4, part (3)). Hence u_i also represents $\succsim_i^{\pm}$; one cannot guarantee in that case, even when imposing Σ^{**}, that φ_i can be transformed into a function ψ_i increasing in its first argument and decreasing in the second, as is done in Theorem 5, and still yielding an additive representation. In other words, we do not know the conditions that guarantee the existence of a representation of $\succsim$ as in model (43) with φ_i increasing in its first argument and decreasing in the second.

6. Conclusion

Our objective of characterising variants of model (M–D) using a limited number of cancellation axioms without any structural condition on the set of objects has been achieved. The present work has focused on further decomposition of the relations on "difference of preferences" that are central in our previous study (Bouyssou & Pirlot, 2002). Conditions that allow for decomposing these in terms of well-behaved marginal traces on each dimension have been obtained; this helps clarify the inter-relations between marginal traces and differences of preference (the relationships between preference and marginal traces as well as between marginal traces and marginal preferences have been studied in Bouyssou and Pirlot (2004) without the "mediation" of "differences of preference"). It is remarkable that, at the level of generality we place ourselves, there is no synergy between the axioms that permit a decomposition in terms of differences of preference (the models studied in Bouyssou and Pirlot (2002) and the axioms that permit a further decomposition of the differences of preference in terms of marginal traces; in other words these blocks of axioms are independent. The resulting model offers a framework that enables us to understand some fundamental features of a large variety of preference models.

The line of research initiated in Bouyssou and Pirlot (2002) has also proved useful here. The axioms that are used:

- appear to have a clear interpretation;
- could be subjected to empirical tests without theoretical difficulty.

Some models have been left aside, for instance those dropping only partially the additivity and subtractivity requirement of the additive difference model, such as:

$$x \succsim y \text{ iff } F([u_i(x_i) - u_i(y_i)]) \geqslant 0,$$

with F nondecreasing (or increasing) in its n arguments. Their analysis requires a different approach (in order to capture subtraction).

What was said in Bouyssou and Pirlot (2002) on the ability of models of type (M) to contain as particular cases most rules for the comparison of multidimensional objects remains valid here. All of these rules make indeed use of marginal preferences on each dimension. In particular, the various models studied in this paper were shown in Greco, Matarazzo, and Słowiński (1999a, b) to have close connections with preference models representable by decision rules extracted from rough sets approximations.

Future research on the topics introduced in this paper will include:

- the specialisation of our results to the case in which X is an homogeneous Cartesian product which includes the important case of decision under uncertainty;
- the study of additional conditions allowing to specify a precise functional form for F and φ_i;
- the generalisation of results to aggregation methods leading to valued preference relation (see Bouyssou & Pirlot, 1999; Bouyssou, Pirlot & Vincke, 1997; Pirlot & Vincke, 1997).

Appendix A. Proofs

A.1. Proof of Lemma 2

By Theorem 1.3, we know that model (M0–D1) implies that $\succsim$ is reflexive and independent. The necessity of hypothesis LCC is also clear since it determines the existence of appropriate functions u_i.

We show that it is possible to build a representation of $\succsim$ in model (M0–D1) given that $\succsim$ is reflexive, independent and satisfies LCC. The proof differs from that of Lemma 1, in the general not necessarily denumerable case, only in the construction of φ_i. In order to get $\varphi_i(u_i(x_i), u_i(x_i)) = 0$, for all $x_i \in X_i$, we build upon the construction of φ_i proposed in the proof of Lemma 1. Let $\varphi_i{}'(u_i(x_i), u_i(y_i)) = f_1(u_i(x_i)) + f_2(1 - u_i(y_i))$, with f_1 and f_2 as defined in the proof of Lemma 1 (we have just renamed as $\varphi_i{}'$, the function called φ_i in the proof of Lemma 1). Let $g{:}]0,1[\to]0,1[$ be the function that maps its argument $a \in]0,1[$ onto a number b, with $b \in]0,1[$; g works on the binary representation $(a_1, a_2, \ldots, a_{2k-1}, a_{2k} \ldots)$ of a, building the ternary representation $(b_1, \ldots, b_k, \ldots)$ of b as follows:

$$b_k = \begin{cases} 0 & \text{if} \quad (a_{2k-1}, a_{2k}) = (0,0), \\ 1 & \text{if} \quad (a_{2k-1}, a_{2k}) = (0,1) \text{ or } (1,0), \\ 2 & \text{if} \quad (a_{2k-1}, a_{2k}) = (1,1) \end{cases}$$

for $k = 1, 2, \ldots$

We define $\varphi_i(u_i(x_i), u_i(y_i)) = g(\varphi_i{}'(u_i(x_i), u_i(y_i))) - \frac{1}{2}$. The function φ_i takes its values in the $]-\frac{1}{2}, \frac{1}{2}[$ interval. It is not hard to convince oneself that, for all $x_i, y_i, z_i \in X_i$, $u_i(x_i) > u_i(y_i)$ implies $\varphi_i(u_i(x_i), u_i(y_i)) > \varphi_i(u_i(z_i), u_i(y_i))$; clearly, φ_i is also decreasing in its second argument. We observe in addition that, for all $x_i \in X_i$, $f_1(u_i(x_i)) + f_2(1 - u_i(x_i))$ is a number a, the binary representation of which is such that $(a_{2k-1}, a_{2k}) = (0,1)$ or $(1,0)$ for all k; g maps such a number onto the number with ternary representation $(1, 1, \ldots, 1, \ldots)$, i.e. onto $\frac{1}{2}$, which proves that $\varphi_i(u_i(x_i), u_i(x_i)) = 0$. With the same definition of F as in the proof of Lemma 1, observe that $F(\mathbf{0}) = 1 \geqslant 0$ as required.

It is easy to verify that the constructed representation is well-defined (see Bouyssou and Pirlot (2002) for more details). The proof of the independence of the three axioms characterising the model is left to the reader. □

A.2. Proof of Lemma 5

(1) Necessity. Assume p_i is used in a representation according to model (M1) (or (M1′)) and suppose there exist x_i, y_i, z_i, w_i such that $(z_i, w_i) \succ_i^* (x_i, y_i)$ and $p_i(z_i, w_i) \leqslant p_i(x_i, y_i)$. There would then exist $a_{-i}, b_{-i} \in X_{-i}$ such that $Not[(x_i, a_{-i}) \succsim (y_i, b_{-i})]$ and $(z_i, a_{-i}) \succsim (w_i, b_{-i})$. A representation as in model (M1′) implies that $F(p_i(x_i, y_i), (p_j(a_j, b_j))_{j \neq i}) < 0$ and $F(p_i(z_i, w_i), (p_j(a_j, b_j))_{j \neq i}) \geqslant 0$, which contradicts the nondecreasingness of F.

Sufficiency: This results from the fact that the construction described by (30) does lead to a representation of $\succsim$ in a model of type (M1′) as soon as p_i verifies condition (32). The proof is identical to that of Theorem 1.3 in Bouyssouand Pirlot (2002).

(2) Necessity. Suppose there exist x_i, y_i, z_i, w_i such that $(z_i, w_i) \succ_i^{**} (x_i, y_i)$ and $p_i(z_i, w_i) \leqslant p_i(x_i, y_i)$. Then either

$$(z_i, w_i) \succ_i^* (x_i, y_i) \quad \text{with } p_i(z_i, w_i) \leqslant p_i(x_i, y_i)$$
$$\text{or } (y_i, x_i) \succ_i^* (w_i, z_i) \quad \text{with } p_i(y_i, x_i) \geqslant p_i(w_i, z_i).$$

In either case, an argument similar to that used in the proof of the necessity, in part (1), leads to the conclusion.

Sufficiency: In model (M2′), it is proved like for model (M1′). Proving it for model (M3) is slightly more delicate since the case $x \sim y$ must be distinguished from $x \succ y$; the proof can be done however using the same arguments as in Theorem 1.5 of Bouyssou and Pirlot (2002).

(3) Necessity. The same argument as for (M2′) and (M3) shows that condition (33) must be fulfilled. Suppose that condition (34) is violated. One would then have $(z_i, w_i) \sim_i^{**} (x_i, y_i)$, $(x_i, a_{-i}) \sim (y_i, b_{-i})$ for some $a_{-i}, b_{-i} \in X_{-i}$ and $p_i(z_i, w_i) \neq p_i(x_i, y_i)$. Since F is strictly increasing, $F(p_i(z_i, w_i), (p_j(a_j, b_j))_{j \neq i}) \neq 0$ while $(z_i, a_{-i}) \sim (w_i, b_{-i})$, a contradiction.

Sufficiency: Well-definedness of F is shown as for (M3). For proving increasingness, suppose $p_i(z_i, w_i) > p_i(x_i, y_i)$. This implies that $(z_i, w_i) \succsim_i^{**} (x_i, y_i)$. If $x \succ y$, Lemma 3.3 of Bouyssou and Pirlot (2002) says that $(z_i, x_{-i}) \succ (w_i, y_{-i})$ and the conclusion follows from the definition of F. If $x \sim y$, $F((p_j(x_j, y_j))_{j=1,\ldots,n}) = 0$. Consider two cases. If $(z_i, w_i) \succ_i^{**} (x_i, y_i)$, Lemma 3.5 of Bouyssou and Pirlot (2002) implies that $(z_i, x_{-i}) \succ (w_i, y_{-i})$ and hence F strictly increases since $F(p_i(z_i, w_i), (p_j(x_j, y_j))_{j \neq i}) > 0$. The second case is when $(z_i, w_i) \sim_i^{**} (x_i, y_i)$; then, by Lemma 3.4 of Bouyssou and Pirlot (2002), $(z_i, x_{-i}) \sim (w_i, y_{-i})$; this case is excluded by condition (34). Finally, the case when $Not[x \succsim y]$ is dealt with like for model (M3). □

A.3. Proof of Lemma 6

(1) Assuming that S is denumerable, we can modify φ in order to eliminate all situations described either by Eq. (35) or Eq. (36). This can be done by transforming $\varphi(u, v)$ for all $(u, v) \in \varphi^{-1}(r)$, $r \in S$, into

$$\alpha + \varphi(u, v) + \beta(u - v),$$

where α and β are arbitrary positive coefficients. After such a transformation, Eqs. (35) and (36) no longer hold within $\varphi^{-1}(r)$. Applying such a transformation to the whole domain $U \times U$ does not solve our problem since, in general, it does not preserve the ordering induced on $U \times U$ by φ; indeed, if $\varphi(u, v) = r \in S$ and $\varphi(u', v')$ is either larger or smaller than r, we must arrange that it remains so after the transformation. The idea is to make α and β depend on (u, v) in order that for all $r \in S$,

- a small interval rather than a single value r is reserved for representing the pairs $(u, v) \in \varphi^{-1}(r)$;
- these intervals are disjoint;
- the other values of φ (not in S) are transformed avoiding to let them fall into these intervals and preserving the order induced on $U \times U$ by φ.

Consider separately the positive part S^+ ($r > 0$) and the negative part S^- ($r < 0$) of S (the case $r = 0$ is treated apart) and number their respective elements in arbitrary order:

$r_1^+, r_2^+, \ldots$ for the elements of S^+,
$r_1^-, r_2^-, \ldots$ for the elements of S^-.

Note that it is not in general possible to number these elements in increasing (or decreasing) order of their value since S may have accumulation points or even be dense in $\mathbb{R}$.

For each u, v in $U \times U$ such that $\varphi(u, v) > 0$, we define $\psi(u, v)$ as follows:

$$\begin{cases} \varphi(u,v) + 1 + \sum_{k: r_k^+ < \varphi(u,v)} (1/2)^k & \text{if } \varphi(u,v) \notin S, \\ r_i^+ + 1 + \sum_{k: r_k^+ < r_i^+} (1/2)^k & \\ \quad + (1/2)^{i+1}(1 + u - v) & \text{if } \varphi(u,v) = r_i^+. \end{cases} \tag{A.1}$$

For each u, v in $U \times U$ such that $\varphi(u, v) < 0$, we define $\psi(u, v)$ as follows:

$$\begin{cases} \varphi(u,v) - 1 - \sum_{k: r_k^- > \varphi(u,v)} (1/2)^k & \text{if } \varphi(u,v) \notin S, \\ r_i^- - 1 - \sum_{k: r_k^- > r_i^-} (1/2)^k & \\ \quad - (1/2)^{i+1}(1 - u + v) & \text{if } \varphi(u,v) = r_i^-. \end{cases} \tag{A.2}$$

The class of pairs such that $\varphi(u, v) = 0$ requires particular attention since it contains the diagonal $\{(u, u); u \in U\}$ where $\psi(u, u)$ must be kept equal to 0. To fulfil this requirement, we define, for (u, v) such that $\varphi(u, v) = 0$, $\psi(u, v) = u - v$; the image by ψ of the pairs (u, v) such that $\varphi(u, v) = 0$ all lie in the $]-1, 1[$ interval.

A picture of the transformation of φ into ψ is shown in Fig. 3. The function ψ is now fully described. It vanishes on the diagonal (u, u), for all $u \in U$; it is strictly monotonic on $\varphi^{-1}(r)$, for all $r \in S$; to each value of φ corresponds a single value of ψ except for the values of φ belonging to S; hence (38) is satisfied. In order to show that ψ is strictly monotonic everywhere on $U \times U$, we have to prove that for all $u, v, u', v' \in U$, it satisfies (37).

Let us check the property for positive values of φ. The negative case is treated symmetrically; the case in which $\varphi(u, v) > 0$ and $\varphi(u', v') < 0$ is trivial since ψ keeps the sign of φ, the case in which $\varphi(u, v) = 0$ (resp. > 0) and $\varphi(u', v') < 0$ (resp. $= 0$) is dealt with observing that when $\varphi = 0$, ψ belongs to the interval $]-1, 1[$ and if $\varphi > 0$ (resp. $\varphi < 0$), then $\psi > 1$ (resp. $\psi < -1$).

In the cases in which neither $\varphi(u, v)$ nor $\varphi(u', v')$ belong to S, the result comes from the fact that the transformation applied both to $\varphi(u, v)$ and $\varphi(u', v')$, i.e.

$$\psi(u,v) = \varphi(u,v) + 1 + \sum_{k: r_k^+ < \varphi(u,v)} (1/2)^k$$

is an increasing function of $\varphi(u, v)$. In case $\varphi(u, v) = r_i^+$ and $\varphi(u', v') \notin S$, we have:

$$\begin{aligned} \psi(u',v') &= \varphi(u',v') + 1 + \sum_{k: r_k^+ < \varphi(u,v)} (1/2)^k \\ &< r_i^+ + 1 + \sum_{k: r_k^+ < r_i^+} (1/2)^k \\ &< \psi(u,v) \end{aligned}$$

since $1 + u - v > 0$. The remaining two cases are similar.

Note that the definition of ψ ensures that ψ is skew-symmetric as soon as φ has this property.

(2) Suppose that S is not denumerable; we show that a function ψ that is increasing in its first argument and decreasing in the second and satisfies (37) may not exist. For each $r \in S$, select two pairs (u_r, v_r) and (u_r', v_r') such that either Eq. (35) is fulfilled (with $(u_r, v_r) = (a, c)$ and $(u_r', v_r') = (b, c)$) or Eq. (36) is fulfilled (with $(u_r, v_r) = (a, d)$ and $(u_r', v_r') = (a, c)$). Suppose that there exists ψ such that for all $r \in S$, $\psi(u_r, v_r) > \psi(u_r', v_r')$. The intervals $]\psi(u_r', v_r'), \psi(u_r, v_r)[$ with $r \in S$, would form a nondenumerable family of disjoint nonempty open intervals of $\mathbb{R}$, which does not exist since $\mathbb{Q}$ is dense in $\mathbb{R}$. $\square$

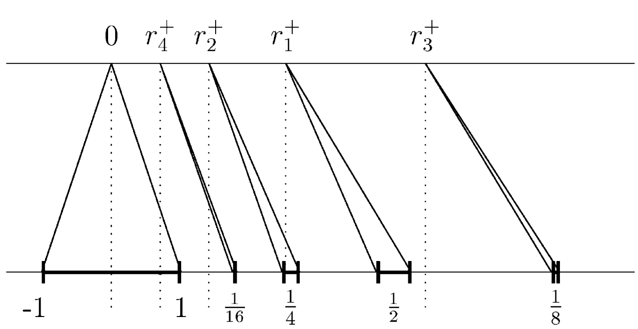

Fig. 3. Transformation of φ into ψ.

Appendix B. Examples

This section puts together the descriptions of 18 examples that are used in the main text, mostly for showing the independence of the axioms. The Examples from 1 to 13 serve for the case in which X is denumerable; the remaining ones illustrate the non-denumerable case. Some properties of the examples are summarised in Tables 2–4.

Example 1. Let X be any product set with X_i nonempty and at most denumerable, for all $i = 1, \ldots, n$. Let $\succsim$ be the empty relation on X. Obviously $\succsim$ is neither complete nor reflexive and conditions $RC1$, $RC2$, TC, $AC1$, $AC2$, $AC3$ are trivially satisfied as well as independence; axioms $TAC1$ and $TAC2$ are not contradicted either.

Example 2. Let $X = \{a,b,c\} \times \{d,e,f\}$; $x \succsim y$ iff $F(p_1(x_1,y_1), p_2(x_2,y_2)) \geqslant 0$ with

$$F(\alpha,\beta) = \begin{cases} \alpha+\beta & \text{if } |\alpha+\beta|>2, \\ 0 & \text{otherwise} \end{cases}$$

and p_1 and p_2 given in the following tables:

p_1	a	b	c
a	0	−2	−1
b	2	0	1
c	1	−1	0

p_2	d	e	f
d	0	0	−2
e	0	0	−2
f	2	2	0

F is odd and nondecreasing and p_1, p_2 are skew-symmetric; hence $\succsim$ is complete, satisfies $RC1$, $RC2$ and is independent. TC is violated since $(c,d) \succsim (a,f)$, $(a,e) \succsim (c,d)$, $(a,d) \succsim (b,e)$ but $Not[(a,d) \succsim (b,f)]$. It is easily checked that $AC1$, $AC2$ and $AC3$ hold with $b \succ_1^{\pm} c \succ_1^{\pm} a$ and $f \succ_2^{\pm} [d,e]$; $TAC1$ and $TAC2$ are not in force.

Example 3. Let $X = \{a,b,c\} \times \{d,e\}$; $x \succsim y$ iff $F(p_1(x_1,y_1), p_2(x_2,y_2)) \geqslant 0$ with p_1 and p_2 given in the following tables:

p_1	a	b	c
a	0	2	−1
b	−2	0	−1
c	1	1	0

p_2	d	e
d	0	2
e	−2	0

and F such that:

F	−2	0	2
−2	−41	−21	0
−1	−31	−9	10
0	−19	0	19
1	−10	9	31
2	0	21	41

F is odd and increasing in its two arguments and p_1, p_2 are skew-symmetric implying that $\succsim$ is complete, satisfies TC and hence satisfies $RC1$, $RC2$ and is independent. It is easy to check that we have: $c \succ_1^- a$, $a \succ_1^- b$, $c \succ_1^- b$, $c \succ_1^+ b$, $a \succ_1^+ b$, $Not[c \succsim_1^+ a]$, $Not[a \succsim_1^+ c]$, $d \succ_2^{\pm} e$. Hence $AC2$ and $AC3$ hold but $AC1_1$ is violated (while $AC1_2$ holds). One verifies indeed that we have $(c,d) \succsim (c,d)$ and $(a,e) \succsim (b,d)$ but neither $(a,d) \succsim (c,d)$ nor $(c,e) \succsim (b,d)$. $TAC1_1$ is therefore not in force since $\succsim_1^+$ is incomplete (Lemma 4.8). One easily verifies, using condition (25), that $TAC2$ holds. It suffices to check that, for all (x_1,x_2), $(y_1,y_2) \in X_1 \times X_2$, with $(x_1,x_2) \sim (y_1,y_2)$, the indifference between (x_1,x_2) and (y_1,y_2) becomes strict preference as soon as y_1 (resp. y_2) is substituted by z_1 (resp. z_2) such that $y_1 \succ_1^- z_1$ (resp. $y_2 \succ_2^- z_2$).

Example 4. This example is defined as the previous one (example 3) except that p_1 becomes $-p_1$. The effect of this modification is to interchange the roles of $AC1_1$ and $AC2_1$ since the value associated to the pair (y_1,x_1) is the value that was formerly associated to (x_1,y_1) in Example 3. The relation $\succsim$ is complete and verifies TC, $RC1$, $RC2$ and independence. We have: $b \succ_1^+ a \succ_1^+ c$, $b \succ_1^- c$, $b \succ_1^- a$, $Not[a \succsim_1^- c]$, $Not[c \succsim_1^- a]$, $d \succ_2^{\pm} e$. Hence $AC1$ and $AC3$ hold but $AC2_1$ is violated (while $AC2_2$ holds). One verifies indeed that $(c,d) \succsim (c,d)$ and $(b,e) \succsim (a,d)$ but neither $(c,d) \succsim (a,d)$ nor $(b,e) \succsim (c,d)$. Hence $\succsim_1^-$ is not complete and $TAC2_1$ is violated (Lemma 4.9). One verifies as in Example 3 that $TAC1$ holds.

Example 5. Let $X = \{a,b,c,d\} \times \{w,x,y,z\}$; $\succsim$ is defined as in Example 3 with the same table for F and p_1, p_2 given in the following tables:

p_1	a	b	c	d
a	0	1	2	2
b	−1	0	1	0
c	−2	−1	0	−2
d	−2	0	2	0

p_2	w	x	y	z
w	0	2	2	2
x	−2	0	2	2
y	−2	−2	0	2
z	−2	−2	−2	0

Since F is odd and increasing and p_1, p_2 are skew-symmetric, we know that $\succsim$ is complete and verifies TC (and hence $RC1$, $RC2$ and independence). It can be checked that we have: $w \succ_2^{\pm} x \succ_2^{\pm} y \succ_2^{\pm} z$; $a \succ_1^+ d \succ_1^+ b \succ_1^+ c$; $a \succ_1^- b \succ_1^- d \succ_1^- c$. Hence $AC1$ and $AC2$ hold but $AC3$ is violated since neither $d \succ_1^{\pm} b$ nor $b \succ_1^{\pm} d$.

Example 6. Let $X = \{a,b\} \times \{z,w\}$; $x \succsim y$ iff $p_1(x_1,y_1) + p_2(x_2,y_2) \geqslant 0$ with p_1 and p_2 given by the following tables:

p_1	a	b
a	−1	1
b	−1	1

p_2	z	w
z	1	0
w	1	1

$\succsim$ is clearly complete: the only two pairs missing in $\succsim$ are $Not[(a,z) \succsim (a,w)]$ and $Not[(b,z) \succsim (a,w)]$. Relation $\succsim$ satisfies $RC1$ (by construction) but violates $RC2$ because it is not independent: $[(b,z) \succsim (b,w)]$ but $Not[(a,z) \succsim (a,w)]$. Relation $\succsim$ satisfies $AC1$, $AC2$, $AC3$ with $w \succsim_2^{\pm} z$ and $a \succ_1^- b$, $a \sim_1^+ b$.

Example 7. Let $X = X_1 \times X_2$, with $X_1 = \{a,b,c,d\}$ and $X_2 = \{w,x,y,z\}$. For all $x,y \in X$, $x \succsim y$ iff $F(p_1(x_1;y_1); p_1(x_2;y_2)) \geqslant 0$, with

$$F(p_1,p_2) = \begin{cases} p_1+p_2 & \text{if } |p_1+p_2|>2, \\ 0 & \text{otherwise.} \end{cases}$$

Let p_1 and p_2 be defined by the following tables:

p_1	a	b	c	d
a	0	1	3	3
b	−3	0	0	3
c	−4	0	0	1
d	−4	−2	−2	0

p_2	w	x	y	z
w	0	0	2	4
x	0	0	2	4
y	−2	−2	0	4
z	−4	−4	−4	0

Since F is nondecreasing in both its arguments, $RC1$ holds. The relation $\succsim$ is independent and reflexive since $p_1(u,u) = p_2(v,v) = 0$ for all $u \in X_1$ and $v \in X_2$ and $F(0,0) = 0$. We have $a \succsim_1^{\pm} b \succsim_1^{\pm} c \succsim_1^{\pm} d$ and $[w,x] \succsim_2^{\pm} y \succsim_2^{\pm} z$; $\succsim$ thus satisfies $AC123$.

$RC2_1$ does not hold since one can verify that $(b,z) \succsim (d,y)$, $Not[(a,z) \succsim (b,y)]$, $(d,x) \succsim (b,x)$ and $Not[(b,x) \succsim (a,x)]$.

Example 8. The following example appears as Example 4 in Bouyssou and Pirlot (2004). Let $X = X_1 \times X_2$ with $X_1 = \{x_1, y_1, z_1\}$ and $X_2 = \{x_2, y_2, z_2\}$. Consider the reflexive binary relation $\succsim$ identical to the complete order: $(x_1,x_2) \succ (x_1,y_2) \succ (y_1,x_2) \succ (x_1,z_2) \succ (y_1,y_2) \succ (z_1,x_2) \succ (y_1,z_2) \succ (z_1,y_2) \succ (z_1,z_2)$, except that $(y_1,y_2) \sim (x_1,z_2)$ and $(z_1,x_2) \sim (y_1,y_2)$. It is shown in Bouyssou and Pirlot (2004) that this relation is complete, independent and satisfies $TAC12$ (it is a nontransitive semi-order). The relation $\succsim$ does not verify $RC1_1$ since $\succsim_1^*$ is not a complete relation; we have indeed neither $(z_1,y_1) \succsim_1^* (y_1,x_1)$ nor $(y_1,x_1) \succsim_1^* (z_1,y_1)$ since $(z_1,x_2) \succsim (y_1,y_2)$, $(y_1,y_2) \succsim (x_1,z_2)$ but $Not[(y_1,x_2) \succsim (x_1,y_2)]$ and $Not[(z_1,x_2) \succsim (y_1,z_2)]$. The incomplete (yet transitive) relation $\succsim_1^*$ is the following:

$$(x_1, z_1)$$
$$\downarrow$$
$$(x_1, y_1) \leftrightarrow (y_1, z_1)$$
$$\downarrow$$
$$(x_1, x_1) \leftrightarrow (y_1, y_1) \leftrightarrow (z_1, z_1)$$
$$\downarrow$$
$$(y_1, x_1); (z_1, y_1)$$
$$\downarrow$$
$$(z_1, x_1)$$

(the pointing down arrows represent $\succ_1^*$; the left-right arrows represent $\sim_1^*$; the nonrepresented pairs of $\succ_1^*$ and $\sim_1^*$ obtain by transitive closure of the diagram). Note that $(y_1,x_1), (z_1,y_1)$ are not joined by a left–right arrow since they are incomparable, as proven above. Property $RC1_2$ is violated; the same example implies that neither $(x_2,y_2) \succsim_2^* (y_2,z_2)$ nor $(y_2,z_2) \succsim_2^* (x_2,y_2)$. One easily checks $\succsim$ satisfies $RC2_1$ using the equivalence $RC2_i$ iff $[\forall a,b,c,d \in X_i, Not[(a,b) \succsim_i^* (c,d)] \Rightarrow (b,a) \succsim_i^* (d,c)]$ (see Bouyssou & Pirlot, 2002, Lemma 1, part (2)). One similarly proves that $RC2_2$ holds and $\succsim$ thus satisfies $RC2$. Property TC does not hold since $RC1$ is violated.

Example 9. Let $X = \mathbb{Q}^2$ and, for all $x,y \in X$, $x \succsim y \Leftrightarrow F(p_1(x_1,y_1),\ p_2(x_2,y_2)) \geqslant 0$, with $p_i(x_i,y_i) = \frac{2}{\Pi}\arctan(x_i - y_i)$ and $F(p_1,p_2) = p_1 + p_2 + p_1 p_2$.

A variant of this example, where $X = \mathbb{R}^2$ instead of $\mathbb{Q}$, was shown to satisfy $AC123$ in Bouyssou and Pirlot (2002, Example 3). It is easily checked that $\succsim$ satisfies model (M2′–D1) since all functions p_i are skew symmetric, increasing in their first argument and decreasing in the second and F is increasing in all its arguments (since the latter take their values in the $]-1,1[$ interval). The relation $\succsim$ is not complete (taking (x,y) such that $p_1(x_1,y_1) = 1/4$ and $p_2(x_2,y_2) = -1/4$, we have neither $(x_1,x_2) \succsim (y_1,y_2)$ nor $(y_1,y_2) \succsim (x_1,x_2)$). Hence $\succsim$ cannot be represented in model (M3–D1). Note that the above properties also hold (or not) in case $X = \mathbb{R}^2$.

Example 10. Let $X = \mathbb{Q}^2$ and, for all $x,y \in X$, $x \succsim y \Leftrightarrow F(p_1(x_1,y_1),\ p_2(x_2,y_2)) \geqslant 0$, with $p_i(x_i,y_i) = x_i - y_i$ and

$$F(p_1,p_2) = \begin{cases} p_1 + p_2 & \text{if } |p_1 + p_2| \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

A variant of this example (with $X = \mathbb{R}^2$ instead of $\mathbb{Q}^2$), was shown to satisfy $AC123$ in Bouyssou and Pirlot (2002, Example 4). By construction, $\succsim$ has a representation in (M3–D1). Simple examples show that $\succsim$ violates TC so that it cannot be represented in model (M3′–D1). One shows similarly that neither $TAC1$ nor $TAC2$ holds. All the above properties are also valid (or not) in case $X = \mathbb{R}^2$.

Example 11. Let $X = \{a,b,c,d\} \times \{0,1\}$; $x \succsim y$ iff $p_1(x_1,y_1) + p_2(x_2,y_2) \geqslant 0$ with $p_2(x_2,y_2) = x_2 - y_2$ and p_1 given by the following table:

p_1	a	b	c	d
a	0	1	−1	−1
b	−1	0	1	1
c	1	−1	0	1
d	1	−1	−1	0

Since p_1 and p_2 are skew-symmetric and F odd and increasing in its two arguments, $\succsim$ satisfies $RC12$ and TC. It satisfies none of $AC1$, $AC2$, $AC3$. It satisfies neither $AC1_1$ nor $AC2_1$ since, for any $x_2 \in X_2$, $(a,x_2) \succsim b,x_2)$, $(c,x_2) \succsim (d,x_2)$, $Not((c,x_2) \succsim (b,x_2))$ and $Not((a,x_2) \succsim (d,x_2))$. It does not satisfy $AC3_1$, since for any $x_2 \in X_2$, $(a,x_2) \succsim (b,x_2)$, $(d,x_2) \succsim (a,x_2)$, $Not((c,x_2) \succsim (b,x_2))$ and $Not((d,x_2) \succsim (c,x_2))$.

Example 12. Let $X = X_1 \times X_2 \times X_3 = \mathbb{Q}_+^3$ (where $\mathbb{Q}_+$ denotes the set of positive rational numbers) and for all $x,y \in \mathbb{Q}_+^3$, $x \succsim y$ iff $F(x_1,x_2,x_3) \geqslant F(y_1,y_2,y_3)$, with $F(x_1,x_2,x_3) = (x+y) \times z$. This relation is a weak order and hence is complete. Since F is increasing in its three arguments, $\succsim$ satisfies $AC123$ and $TAC12$.

It satisfies neither $RC1$ nor $RC2$. To show the former, consider the following sets of elements belonging to X_1, X_2 and X_3:

X_1	X_2	X_3
$a = 0.1$	$i = 5$	$\alpha = 1$
$b = 0.1$	$j = 0.1$	$\beta = 5$
$c = 5$	$k = 0.1$	$\gamma = 5$
$d = 5$	$l = 5$	$\delta = 1$.

It is easy to verify that $(a,i,\alpha) \succsim (b,j,\beta)$ and $Not[(c,i,\alpha) \succsim (d,j,\beta)]$, which implies that $Not[(c,d) \succsim_1^* (a,b)]$. We have similarly, $(c,k,\gamma) \succsim (d,l,\delta)$ and $Not[(a,k,\gamma) \succsim (b,l,\delta)]$, which implies that $Not[(a,b) \succsim_1^* (c,d)]$. Hence $\succsim_1^*$ is incomplete and $RC1_1$ does not hold.

To show that $RC2_1$ is also violated, one verifies that $(b,l,\delta) \succsim (a,k,\gamma)$ and $Not[(d,l,\delta) \succsim (c,k,\gamma)]$, which implies $Not[(d,c) \succsim_1^* (b,a)]$. This together with the previously obtained $Not[(c,d) \succsim_1^* (a,b)]$ invalidates $RC2_1$. As a consequence, TC does not hold either. Note that the above properties also hold (or not) in case $X = \mathbb{R}_+^3$.

Example 13. Let $X = \mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rational numbers; for $x, y \in X$, say that $x \succsim y$ iff $p_1(x_1,y_1) + p_2(x_2,y_2) \geqslant 0$ with

$$p_i(x_i,y_i) = \begin{cases} x_i - y_i & \text{if } |x_i - y_i| > 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily checked that $\succsim$ is complete and satisfies TC as well as $AC1$, $AC2$, $AC3$; it does not satisfy $TAC1$ since we have $(0,0) \succsim (1,0)$, $(1,0) \succsim (2,0)$, $(2,0) \succsim (0,2)$ and $Not[(0,0) \succsim (0,2)]$, contrary to $TAC1_1$. One similarly shows that $TAC2$ fails to be true since we have $(-2,0) \succsim (-1,1)$, $(-1,1) \succsim (0,0)$, $(0,0) \succsim (-2,2)$ and $Not[(0,0) \succsim (0,2)]$, contrary to $TAC2_1$.

Example 14. Let $X_1 = X_2 = \mathbb{R}$ and $X = X_1 \times X_2 = \mathbb{R}^2$. For $x = (x_1,x_2), y = (y_1,y_2) \in X$, we say that $x \succsim y$ iff $x_1 - y_1 > y_2 - x_2$ or $[x_1 - y_1 = y_2 - x_2$ and $\Delta(x_1,y_1) \geqslant 0]$, with

$$\Delta(x_1,y_1) = \begin{cases} 1 & \text{if } (x_1 > 0 \text{ and } y_1 \leqslant 0) \\ & \quad \text{or } (x_1 = 0 \text{ and } y_1 < 0), \\ 0 & \text{if } x_1, y_1 > 0 \text{ or} \\ & \quad x_1 = y_1 = 0 \text{ or } x_1, y_1 < 0, \\ -1 & \text{if } (y_1 > 0 \text{ and } x_1 \leqslant 0) \text{ or} \\ & \quad (y_1 = 0 \text{ and } x_1 < 0). \end{cases}$$

In other words, the objects are ranked in order of decreasing value of the sum of their coordinates $(x_1 + x_2)$; if x and y are tied, the tie is possibly broken when the sign of x_1 is greater than the sign of y_1 (the sign of a real number r being 1 if $r > 0$, 0 if $r = 0$ and -1 if $r < 0$. It is easy to check that $\succsim$ is complete and verifies TC; we have $\succsim_i^{**} = \succsim_i^*$ for $i = 1,2$; $(x_1,y_1) \succsim_1^* (z_1,w_1)$ iff $x_1 - y_1 > z_1 - w_1$ or $[x_1 - y_1 = z_1 - w_1$ and $\Delta(x_1,y_1) \geqslant \Delta(z_1,w_1)]$; $(x_2,y_2) \succsim_2^* (z_2,w_2)$ iff $x_2 - y_2 \geqslant z_2 - w_2$. Clearly, the weak order $\succsim_2^*$ admits a representation on the reals, while $\succsim_1^*$ does not; OD_1^* is not verified. Relations $\succsim_1^{\pm}$ and $\succsim_2^{\pm}$ are the usual ordering on $\mathbb{R}$; $AC1$, $AC2$, $AC3$ and $OD^{\pm}$ are thus satisfied. So are $TAC12$ since (24) and (25) hold. Property Σ_2^{**} is clearly satisfied and the same holds for Σ_1^{**} since $(x_1,z_1) \sim_1^{**} (y_1,z_1)$ implies $x_1 = y_1$.

Example 15. Let $X_1 = (\mathbb{R}_+ \cup \{0\}) \times \{0,1\}$, where $\mathbb{R}_+$ denotes the set of positive real numbers, and $X_2 = \mathbb{R}$; $X = X_1 \times X_2$. If x denotes an element of X, its first coordinate $x_1 \in X_1$ has itself two components that we denote respectively $x_1{}'$ $(\in \mathbb{R}^+)$ and x_1'' $(\in \{0,1\})$. For $x, y \in X$, we say that $x \succsim y$ iff $p_1(x_1,y_1) + p_2(x_2,y_2) \geqslant 0$ with $p_2(x_2,y_2) = x_2 - y_2$ and

$$p_1(x_1,y_1) = \begin{cases} 2 & \text{if } x_1{}' > y_1{}' = 0 \text{ and } x_1'' = 1, \\ 1 & \text{if } x_1{}' > y_1{}' \neq 0 \\ & \quad \text{or } [x_1{}' > y_1{}' = 0 \text{ and } x_1'' = 0], \\ 0 & \text{if } x_1{}' = y_1{}', \\ -1 & \text{if } y_1{}' > x_1{}' \neq 0 \\ & \quad \text{or } [y_1{}' > x_1{}' = 0 \text{ and } y_1'' = 0], \\ -2 & \text{if } y_1{}' > x_1{}' = 0 \text{ and } y_1'' = 1. \end{cases}$$

$\succsim$ is reflexive, independent and complete; it satisfies TC (and hence $RC1$ and $RC2$). It satisfies $AC1_1$ and $AC2_1$: $x_1 \succ_1^{\pm} y_1$ iff $x_1{}' > y_1{}'$ or $[x_1{}' = y_1{}' \neq 0$ and $x_1'' = 1$ and $y_1'' = 0]$; $x_1 \sim_1^{\pm} y_1$ iff $x_1 = y_1$ or $[x_1{}' = y_1{}' = 0]$; $\succsim_1^{\pm}$ is complete and, hence, $AC3_1$ is satisfied. Clearly, $\succsim$ does not satisfy $OD_1^{\pm}$. Property $AC123_2$ as well as $OD_2^{\pm}$ are obviously in force. On the contrary, $TAC1_1$ is not in force since taking for instance $x_1 = (3,0)$, $y_1 = (2,0)$ and $z_1 = (1,0)$, we have $(y_1,0) \sim (z_1,1)$ and $(x_1,0) \sim (z_1,1)$; we also have $x_1 \succ_1^{+} y_1$, which, in view of (24), is not compatible with $TAC1_1$. Finally, since $\succsim_1^{**}$ has only five equivalence classes, condition Σ_1^{**} is trivially fulfilled; this is also the case of Σ_2^{**}.

Example 16. Let $X_1 = \{x_1 = (x_{11},x_{12}) : x_{11} \in \mathbb{R}; x_{12} = 0$ if $x_{11} \neq 0$ and $x_{12} \in \{0,1\}$ if $x_{11} = 0\}$; $X_2 = \mathbb{Q}$, the set of rational numbers. Define, for all $x, y \in X$,

$$(x_1,x_2) \succsim (y_1,y_2) \quad \text{iff} \begin{cases} x_{11} - y_{11} > y_2 - x_2 & \text{or} \\ \begin{cases} x_{11} - y_{11} = y_2 - x_2 & \text{and} \\ x_{12} - y_{12} \geqslant 0. \end{cases} \end{cases}$$

This relation is complete, independent, satisfies TC, $TAC12$, OD^* and $OD^{\pm}$ but not Σ^*. Completeness and independence are straightforward. The relation $\succsim_1^*$

is as follows:

$$(x_1, y_1) \succsim_1^* (z_1, w_1)$$
$$\text{iff} \begin{cases} x_{11} - y_{11} > z_{11} - w_{11} \quad \text{or} \\ x_{11} - y_{11} = z_{11} - w_{11} \notin \mathbb{Q} \quad \text{or} \\ \begin{cases} x_{11} - y_{11} = z_{11} - w_{11} \in \mathbb{Q} \quad \text{and} \\ x_{12} - y_{12} \geqslant z_{12} - w_{12}. \end{cases} \end{cases}$$

This relation is complete and equal to $\succsim_1^{**}$, hence $\succsim$ satisfies $RC12_1$. It also verifies OD_1^* because the union of the three sets $\{(x_1, y_1) \in X_1^2$ such that $x_{11} - y_{11} \in \mathbb{Q}$ and $x_{12} - y_{12} = k\}$ for $k = -1, 0, 1$ forms a denumerable set that is dense in X_1^2 for the weak order $\succsim_1^{**}$. Using Lemma 9 below—a useful counterpart, for TC, of Lemma 4, parts (5) and (6), that concerns $TAC12$—one sees that $\succsim_1^{**}$ satisfies TC_1 since $(x_1, x_2) \sim (y_1, y_2)$ and $(z_1, w_1) \succ_1^{**} (x_1, y_1)$ implies $(z_1, x_2) \succ (w_1, y_2)$.

Since the relation $\succsim_2^{**}$ can be represented by the function $p_2(x_2, y_2) = x_2 - y_2$; it is easy to see that it is complete and that $\succsim$ satisfies $RC12_2$, TC_2 and OD_2^*. We hence get that $\succsim$ is complete and satisfies TC and OD^*.

The relation $\succsim_1^{\pm}$ is as follows:

$$x_1 \succsim_1^{\pm} y_1 \text{ iff} \begin{cases} x_{11} > y_{11} \quad \text{or} \\ (x_{11} = y_{11} = 0 \text{ and } x_{12} \geqslant y_{12}) \end{cases}$$

Since the additional condition in the case in which $x_{11} = y_{11} = 0$ applies only when $x_{11} = 0$, this does not raise any problem for the existence of a numerical representation of $\succsim_1^{\pm}$ and $OD_1^{\pm}$ holds. This relation is obviously complete and, thus, $\succsim$ verifies $AC123_1$. One easily checks, using conditions (24) and (25) that $TAC12_1$ is in force. Since $\succsim_2^{\pm}$ is just the natural order on $\mathbb{Q}$, one can show without difficulty that $\succsim_2^{\pm}$ enjoys the same properties as $\succsim_1^{\pm}$.

Finally, Σ^* and Σ^{**} do not hold. Let $r \in \mathbb{R} \backslash \mathbb{Q}$. In the equivalence class of $\sim_1^* (= \sim_1^{**})$ defined by $\{(x_1, y_1)$ such that $x_{11} - y_{11} = r\}$, the following two pairs can be found: $((0,0), (-r, 0))$ and $((0,1), (-r, 0))$; we have: $(0,0) \succ_1^{\pm} (0,1)$. The set $S_i^* (= S_i^{**}$, in this example) thus contains every equivalence class associated with an irrational number r and this set is not denumerable, in violation of Σ^* and Σ^{**}.

Lemma A.1. *If $\succsim$ is complete, TC_i is equivalent to $RC12_i$ and the following condition*:

$$[(x \succsim y \text{ and } ((z_i, w_i) \succ_i^{**} (x_i, y_i)) \Rightarrow ((z_i, x_{-i}) \succsim (w_i, y_{-i}))] \tag{A.3}$$

Proof. It is shown in Bouyssou and Pirlot (2002, Lemma 2, part (4)), that if $\succsim$ is complete, TC_i implies $RC12_i$. In Lemma 3, part (5) of the same paper, one proves that under the same completeness hypothesis, TC_i implies condition (A.3). The only thing that remains to be proven is thus the indirect part of the lemma. Suppose to the contrary that TC_i does not hold, i.e. that there are $x_i, y_i, z_i, w_i \in X_i$ and $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$ such that:

$$(x_i, a_{-i}) \succsim (y_i, b_{-i}) \tag{A.4}$$

$$\text{and } (z_i, b_{-i}) \succsim (w_i, a_{-i}) \tag{A.5}$$

$$\text{and } (w_i, c_{-i}) \succsim (z_i, d_{-i}) \tag{A.6}$$

$$\text{and } Not[(x_i, c_{-i}) \succsim (y_i, d_{-i})]. \tag{A.7}$$

Since $\succsim$ satisfies $RC12_i$, $\succsim_i^{**}$ is complete. In view of the latter, (A.6) and (A.7) yield $(w_i, z_i) \succ_i^{**} (x_i, y_i)$ and consequently $(y_i, x_i) \succ_i^{**} (z_i, w_i)$. Applying (A.3) to the latter together with (A.5) yields $(y_i, b_{-i}) \succ (x_i, a_{-i})$ contradicting (A.4). □

Example 17. Modify the set on which the relation in Example 16 is defined without changing the definition of $\succsim$ itself. Let $X_1 = \{x_1 = (x_{11}, x_{12}) : x_{11} \in \mathbb{Q}; x_{12} = 0$ if $x_{11} \neq 0$ and $x_{12} \in \{0, 1\}$ if $x_{11} = 0\}$; $X_2 = \mathbb{Z}$, the set of signed integers. It is straightforward to adapt the definitions of $\succsim_i^{**}$ and $\succsim_i^{\pm}$; all properties that were satisfied by $\succsim$ in Example 16 remain valid here; Σ^{**} is now valid since $\mathbb{Q} \backslash \mathbb{Z}$ is a denumerable set. For each $r \in \mathbb{Q} \backslash \mathbb{Z}$, the equivalence class of $\sim_1^{**}$ defined by $\{(x_1, y_1)$ such that $x_{11} - y_{11} = r\}$ contains the two pairs $((0,0), (-r, 0))$ and $((0,1), (-r, 0))$ with $(0,0) \succ_1^{\pm} (0,1)$. The set S_i^{**} is denumerable and $\succsim$ has a representation in model (M3′–D1′).

Example 18. Let $\succsim$ be the relation "is not included in" defined on $X = 2^{\mathbb{R}}$, the set of subsets of $\mathbb{R}$. In this example, $n = 1$ and $X_1 = X$. The cardinality of X is strictly larger than that of $\mathbb{R}$. We have $x \succsim y$ iff $F(p_1(x_1, y_1)) = p_1(x_1, y_1) \geqslant 0$, with

$$p_1(x_1, y_1) = \begin{cases} 1 & \text{if } x_1 \supsetneq y_1, \\ -1 & \text{if } x_1 \subsetneq y_1, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, $p_1(x_1, y_1)$ is nonnegative iff the subset of $\mathbb{R}$ that is labeled by x_1 is not strictly included in the subset labeled by y_1. It is clear that $\succsim$ satisfies the (M3′) model. The equivalence relation $\sim_1^{**}$ has only three classes, namely the set of pairs (x_1, y_1) such that $x_1 \supset y_1$, the pairs such that $x_1 \subset y_1$ and all the other pairs; OD^* holds. On the contrary, relation $\sim_1^{\pm}$ is quite discriminant since $x_1 \sim_1^{\pm} y_1$ iff $x_1 = y_1$ (x_1 is not included in exactly the same subsets of $\mathbb{R}$ as y_1). As a consequence, there are as many classes of the relation $\sim_1^{\pm}$ than there are elements in X, LCC is violated and $\sim_1^{\pm}$ is not representable by a function $u_1 : X_1 \to \mathbb{R}$.

One can easily build a more typical example where $\succsim$ is a relation on a product of two (or more) sets and retains the properties of the relation above. Consider, e.g. the relation $\succsim$ defined on $X = X_1 \times X_2$, with X_1, X_2 two copies of the set $2^{\mathbb{R}}$. Define $\succsim$ by $x \succsim y$ iff

$p_1(x_1, y_1) + p_2(x_2, y_2) \geqslant 0$; p_1 and p_2 in the latter expression are two copies respectively defined on the set X_1 and X_2 of the function p_1 introduced in the one–dimensional case. As in that case, $\succsim$ can be represented in model (M3′); it satisfies OD^* but not LCC.

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