A computer assisted proof of the symmetry of solutions to a PDE

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The problem

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{PDE}$$

where Ω is an open bounded set in $\mathbb{R}^N$ and $p > 2$ (and $p < 2N/(N-2)$ if $N \geqslant 3$). and $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$.

Remarks:

- Elliptic, time independent.
- Trivial solution 0.
- Nonlinear, non-convex : infinitely many solutions.

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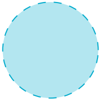
In this talk:



$\Omega = B(0,1)$

or

$\Omega = \,]-1,1[^2$

What is a symmetry?

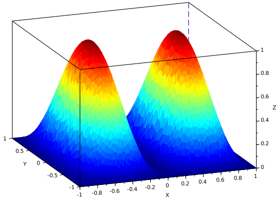
Let G be subgroup of $O(N)$ and $\sigma : G \to \{-1, 1\}$ be a group morphism.

We define an action of G on functions $u : \Omega \to \mathbb{R}$ by

$$gu(x) := \sigma(g)\, u(g^{-1}x), \qquad g \in G.$$

We say that G-symmetric if

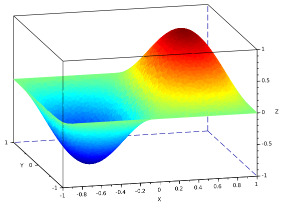
$$\forall g \in G, \quad gu = u.$$



$$g(x_1, x_2) := (-x_1, x_2)$$

⬅ $\sigma(g) = 1$

$\sigma(g) = -1$ ➡



Outline

1. Type of solutions
2. Asymptotic problem
3. Interval arithmetic
4. Computer assisted proof

Variational structure

$$\mathcal{E}_p : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \, dx - \frac{1}{p}\int_\Omega |u(x)|^p \, dx$$

where $H_0^1(\Omega)$ is the Sobolev space with zero Dirichlet boundary conditions, that is

$$H_0^1(\Omega) := \Big\{ u : \Omega \to \mathbb{R} \;\Big|\; u \in L^2(\Omega) \text{ and } \forall i = 1, \dots, N,\ \partial_i u \in L^2(\Omega), \text{ and } u = 0 \text{ on } \partial\Omega \Big\}.$$

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u is a solution to (PDE) $\quad\Leftrightarrow\quad \mathcal{E}_p'(u) = 0.$

where $\mathcal{E}_p'(u) : H_0^1(\Omega) \to \mathbb{R}$ is the Fréchet derivative of $\mathcal{E}_p$. It is a linear map given by

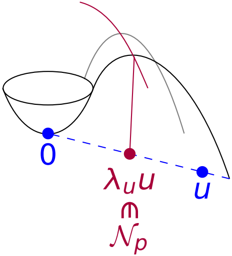
$$\mathcal{E}_p'(u)[v] = \int_\Omega \nabla u \nabla v - \int_\Omega |u|^{p-2} u v = \int_\Omega \bigl(-\Delta u - |u|^{p-2} u\bigr) v$$

Geometry & existence of a ground state

$$\mathcal{E}_p(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u(x)|^p \,\mathrm{d}x$$

has the property that

$$\forall u \neq 0,\ \exists! \lambda_u > 0, \quad \mathcal{E}_p(\lambda_u u) = \sup_{t \geqslant 0} \mathcal{E}_p(tu)$$



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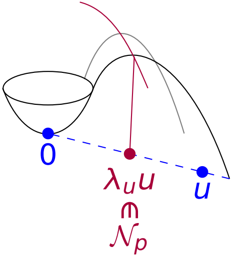

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Nehari manifold

$$\mathcal{N}_p := \left\{ u \in H^1_0(\Omega) \setminus \{0\} \;\middle|\; \mathcal{E}'_p(u)[u] = 0 \right\}$$

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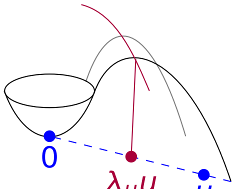

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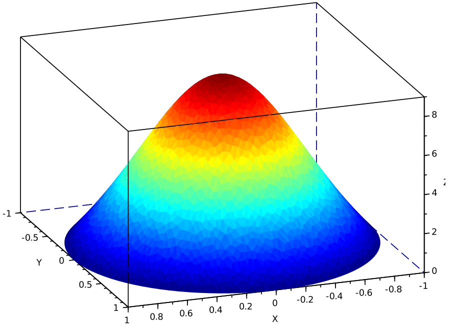
Solution

$$\text{minimize } 0 \neq u \mapsto \sup_{t\geqslant 0} \mathcal{E}_p(tu) \qquad \text{i.e.,} \qquad \begin{cases} \text{minimize } \mathcal{E}_p(u) \\ \text{s.t. } u \in \mathcal{N}_p \end{cases}$$

Computation of the ground states

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$



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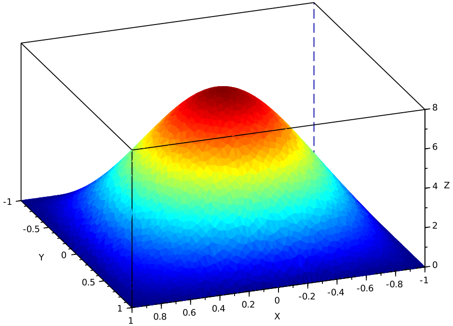


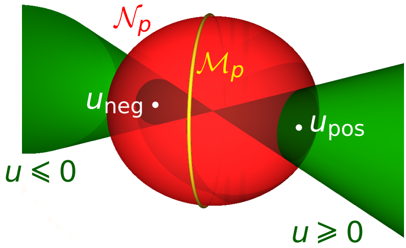
$\Omega =]-1,1[^2$

Geometry & existence of a least-energy sign-changing solution

Nodal Nehari "manifold"

$$\mathcal{M}_p := \left\{ u \in H^1_0(\Omega) \;\middle|\; u^+ \in \mathcal{N}_p \text{ and } u^- \in \mathcal{N}_p \right\}$$

where $u^+(x) := \max\{u(x), 0\}$
and $u^-(x) := \min\{u(x), 0\}$
(so $u = u^+ + u^-$).



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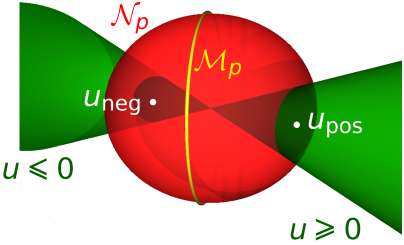
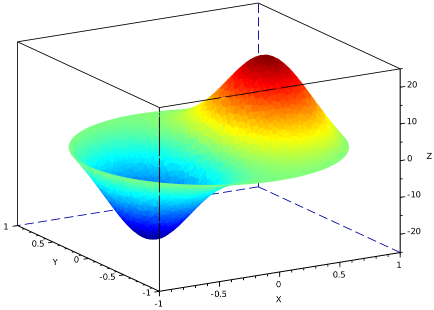


Solution

minimize $\boxed{u \mapsto \sup_{t,s \geqslant 0} \mathcal{E}_p(tu^+ + su^-)}$ i.e., $\begin{cases} \text{minimize } \mathcal{E}_p(u) \\ \text{s.t. } u \in \mathcal{M}_p \end{cases}$

Computation of the least-energy sign-changing solutions

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$



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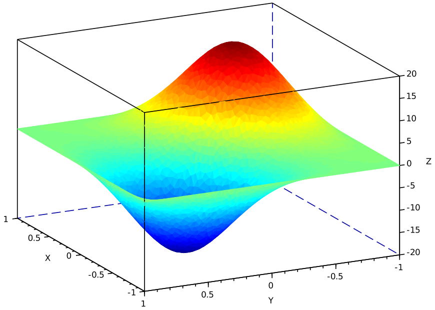

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Asymptotic problem $p \to 2$

$$(\mathrm{PDE})_p \begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \qquad (\mathrm{L}) \begin{cases} -\Delta u = \lambda_2 u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

If $(u_p)_{p>2}$ is a family of solutions to $(\mathrm{PDE})_p$, then, up to a subsequence,

$$\boxed{\lambda_2^{-1/(p-2)} u_p \xrightarrow[p\to 2]{} u_*}$$

where u_* is a solution to (L) where λ_2 is the second eigenvalue of $-\Delta$.

Theorem: for $p \approx 2$, u_p inherits the symmetries of u_*.

Eigenvalues

Recall that $\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$ has a solution $u \neq 0$ iff $\lambda = \lambda_k$ for some $k \in \mathbb{N}^{\geqslant 1}$ for a sequence $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \xrightarrow[n\to\infty]{} +\infty$.

The second eigenspace E_2 on the ball

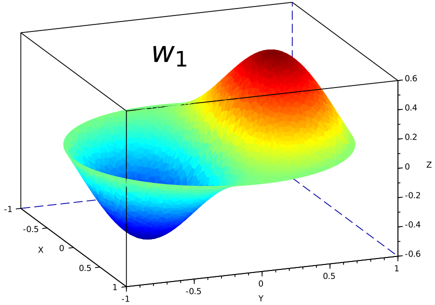
$$E_2 := \{u : \Omega \to \mathbb{R} \mid -\Delta u = \lambda_2 u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega\}$$

When $\Omega = B(0,1) \subseteq \mathbb{R}^2$,
$E_2 = \text{span}\{w_1, w_2\}$ where,
in polar coordinates (r, φ),

$$w_1(r\varphi) = J_1(\sqrt{\lambda_2} r) \sin(\varphi),$$

and

$$w_2(r\varphi) = J_1(\sqrt{\lambda_2} r) \cos(\varphi).$$



where J_ν are the Bessel functions of the first kind.

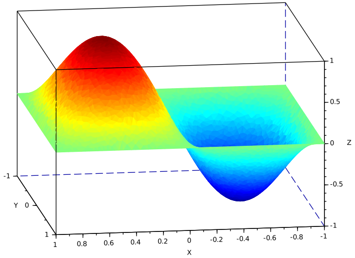
Theorem: For $p \approx 2$, u_p is anti-symmetric w.r.t. a diameter.

The second eigenspace E_2 on the square

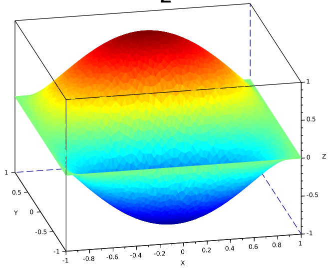
When $\Omega =]-1,1[^2$, $E_2 = \operatorname{span}\{v_1, v_2\}$ where

$$v_1(x,y) = \sin(\pi x)\cos(\tfrac{\pi}{2}y) \quad \text{and} \quad v_2(x,y) = \cos(\tfrac{\pi}{2}x)\sin(\pi y).$$

v_1

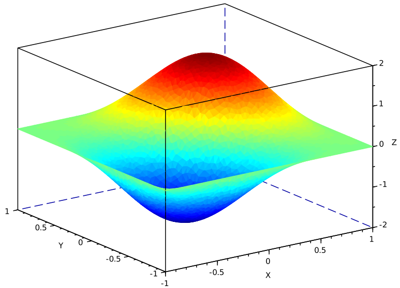


v_2

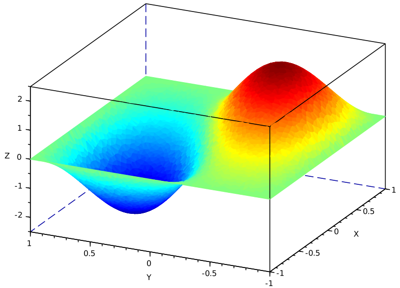


The second eigenspace E_2 on the square

$v_1 + v_2$



$v_1 - 2v_2$

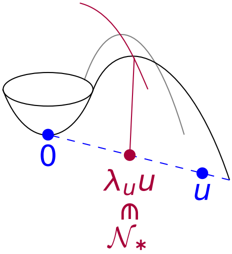


Question: What function is u_* in E_2?

Variational formulation (1/2)

Reduced functional

$$\mathcal{E}_* : E_2 \to \mathbb{R} : u \mapsto \int_\Omega u^2 - u^2 \log u^2$$



Reduced Nehari manifold

$$\mathcal{N}_* := \{u \in E_2 \setminus \{0\} \mid \mathcal{E}'_*(u)[u] = 0\}$$

Criteria: u_* is a solution to

minimize $u \mapsto \sup_{t \geqslant 0} \mathcal{E}_*(tu)$ i.e., $\begin{cases} \text{minimize } \mathcal{E}_*(u) \\ \text{s.t. } u \in \mathcal{N}_* \end{cases}$

Variational formulation (2/2)

If $\int_\Omega u^2 = 1$ (i.e., u is on the unit L^2-sphere),

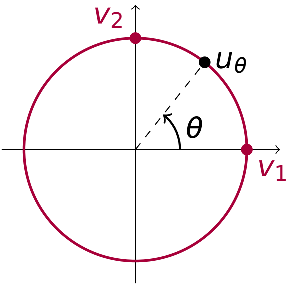
$$\boxed{S_*(u)} := \frac{1}{2}\log\Big(\sup_{t\geqslant 0}\mathcal{E}_*(tu)\Big) = -\int_\Omega u^2 \log|u|\,dx$$

We want to minimize S_* on the L^2-unit sphere of E_2.

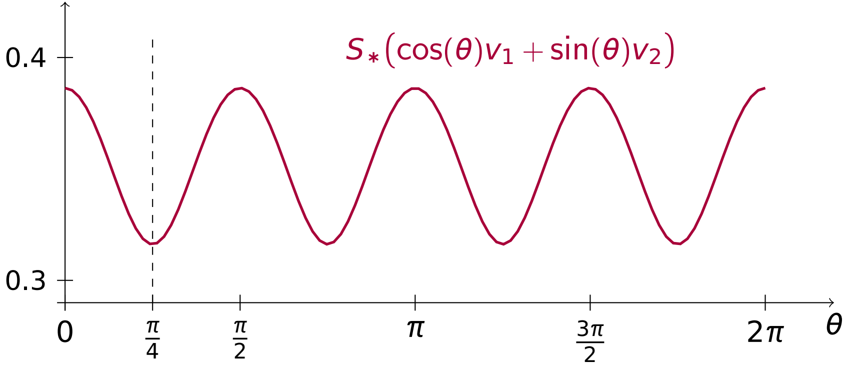
Because $|v_1|_{L^2} = 1$, $|v_2|_{L^2} = 1$ and $v_1 \perp v_2$ in L^2,

$$u_\theta := \cos\theta\, v_1 + \sin\theta\, v_2$$

parameterize the L^2-sphere of E_2.



Numerical simulation



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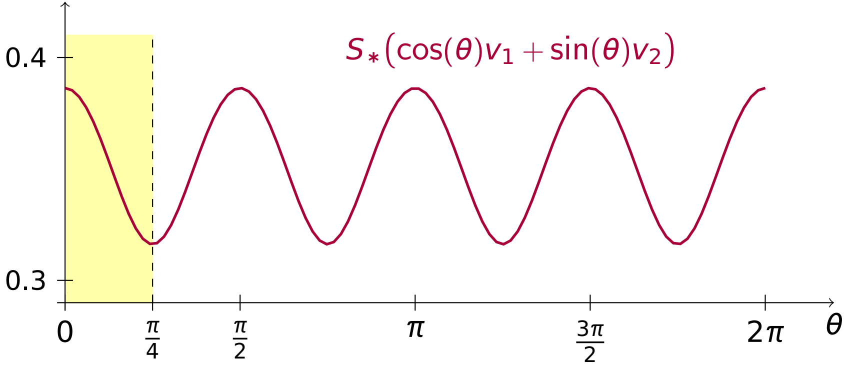

Because the problem is invariant by rotations of $\pi/2$ and axial symmetries and S_* is even, one has:

- S_* is $\pi/2$-periodic;
- $S_*(\frac{\pi}{4} - \theta) = S_*(\frac{\pi}{4} + \theta)$.

Express course on interval arithmetic (1/2)

Observation: floating point computations may be inaccurate due to rounding error.

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function

$$f(x,y) = 333.75\, y^6 + x^2(11x^2y^2 - y^6 - 121\, y^4 - 2) + 5.5\, y^8$$

In double precision, evaluating $f(77617, 33096)$ yields $-1.180592 \cdot 10^{21}$. The correct value is -2.

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Basic idea: Compute an interval $[\underline{z}, \overline{z}]$ containing the true value:

$$f(x, y) \in [\underline{z}, \overline{z}],$$

the rounding of each endpoint taking care of rounding errors.
➠ guaranteed bounds

Express course on interval arithmetic (2/2)

Extend operations to intervals:

$$[\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$$

$$[\underline{x}, \overline{x}] \cdot [\underline{y}, \overline{y}] = \left[\min\{\underline{x}\,\underline{y}, \underline{x}\,\overline{y}, \overline{x}\,\underline{y}, \overline{x}\,\overline{y}\}, \max\{\underline{x}\,\underline{y}, \underline{x}\,\overline{y}, \overline{x}\,\underline{y}, \overline{x}\,\overline{y}\}\right]$$

$$\sin, \cos, \dots$$

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$$\sin, \cos, \dots$$

Fundamental property: Let $x \mapsto f(x)$ be a function and $I \mapsto \mathbf{f}(I)$ an interval extension of f. That means:

$$\forall I \text{ interval}, \qquad \forall x \in I,\ f(x) \in \mathbf{f}(I)$$

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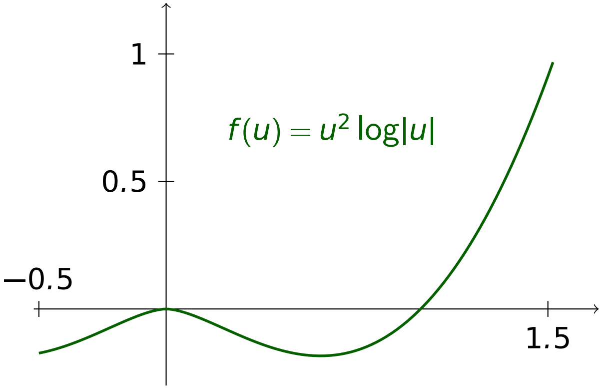
Dependency problem:

- $[\underline{x},\overline{x}]-[\underline{x},\overline{x}]=[\underline{x}-\overline{x},\overline{x}-\underline{x}] \supseteq [0,0]$ but $\neq$ (unless $\underline{x}=\overline{x}$).
- $\big([\underline{x},\overline{x}]\big)^2 \subseteq [\underline{x},\overline{x}]\cdot[\underline{x},\overline{x}]$ but in general $\neq$.
- etc.

Evaluation of basic functions

Recall that $S_*(u) = -\int_\Omega f(u)\,\mathrm{d}x$ where $f(u) := u^2 \log|u|$.



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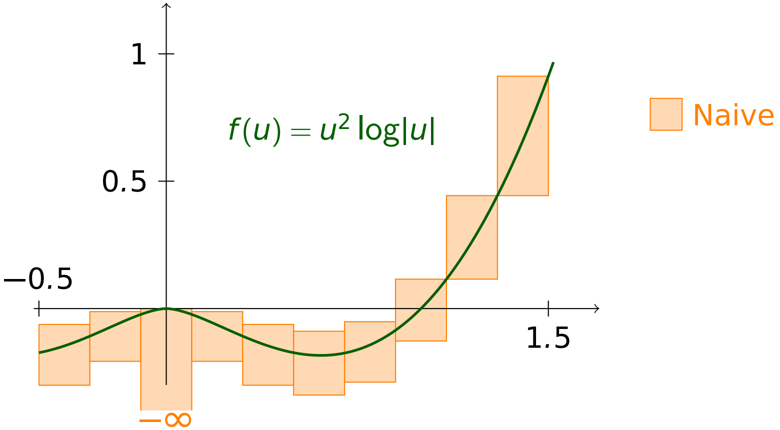

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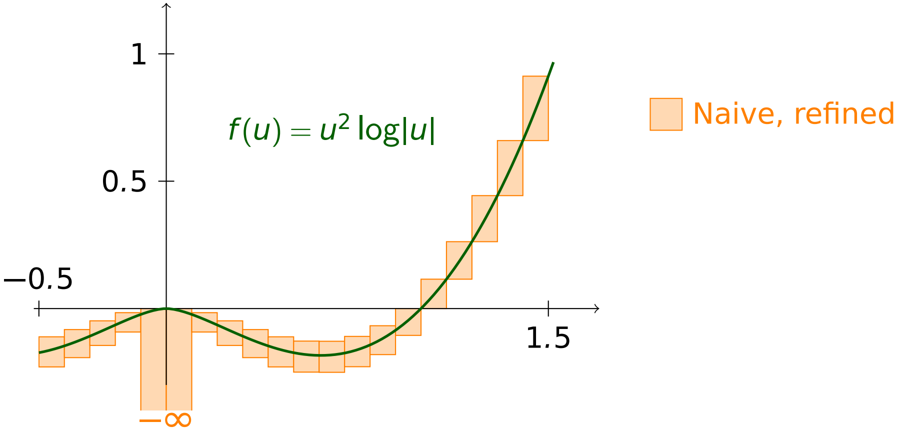

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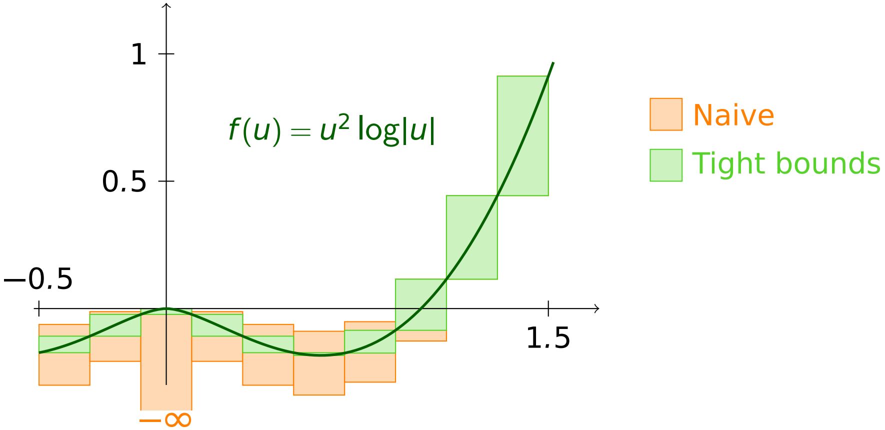

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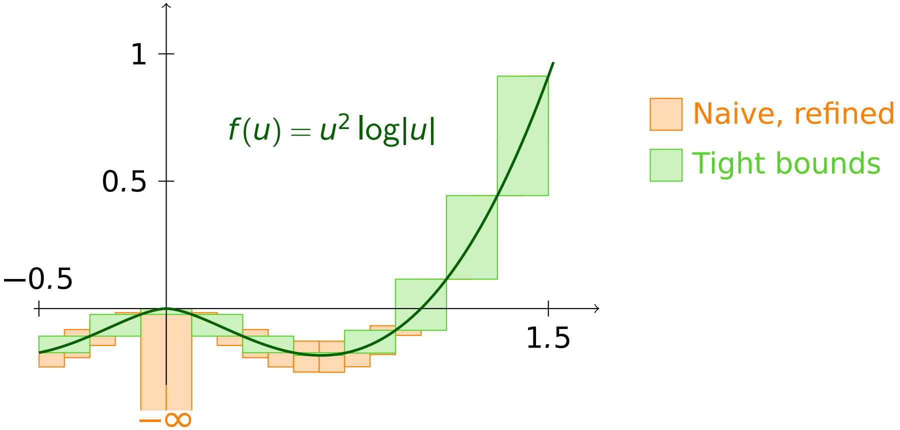

Adaptive integration

Compute $S_*(u) = -\int_\Omega u^2 \log|u| \, \mathrm{d}x$ where $u = \cos\theta \, v_1 + \sin\theta \, v_2$.

Basic scheme: partition Ω in a union of "small" P and estimate each integral with

$$\frac{1}{|P|} \int_P g(x) \, \mathrm{d}x \in g(P).$$

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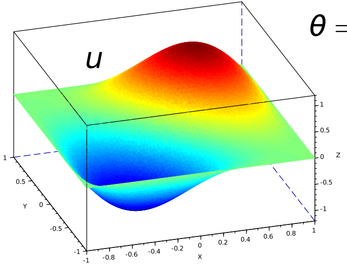
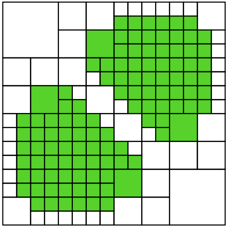
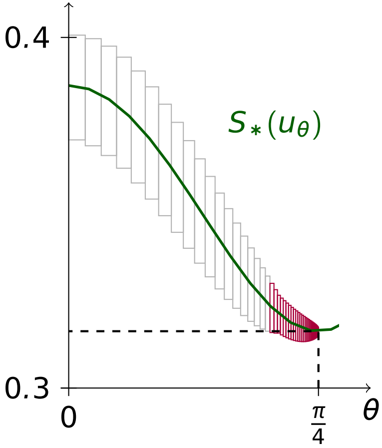
Higher order schemes: require some regularity ($u \in \mathcal{C}^2$).



$\theta = \pi/4$



Asymptotic problem on $\Omega =]{-1}, 1[^2$



Determine a small interval I such that $\pi/4 \in I$ and

$$\forall \theta \in [0, \pi/4] \setminus I, \quad \mathcal{E}_*(\theta) > E_*(\pi/4)$$

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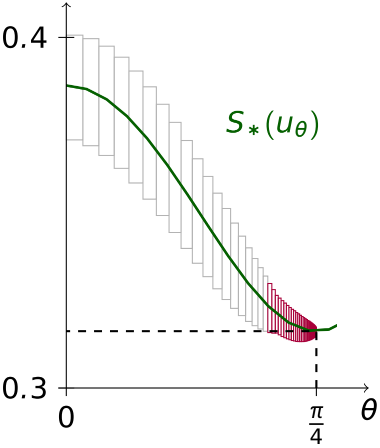

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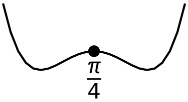
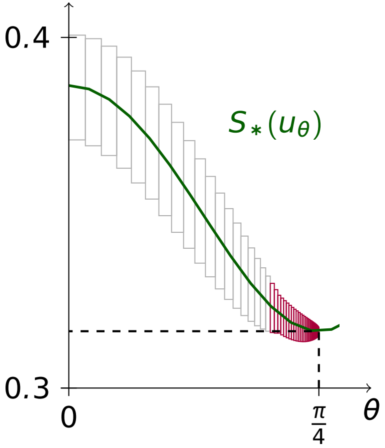


Asymptotic problem on $\Omega =]-1,1[^2$



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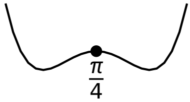
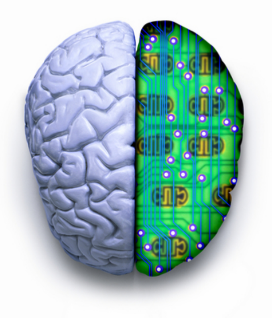


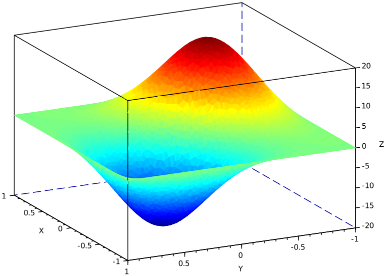
Solution: Show that

$$\forall \theta \in I, \quad \partial_\theta^2\big(S_*(u_\theta)\big) > 0 .$$

Thank you for your attention!



$\vdash$



Computing the second derivative

Recall that:

$$S_*(u) = -\int_\Omega u^2 \log|u| \,\mathrm{d}x$$

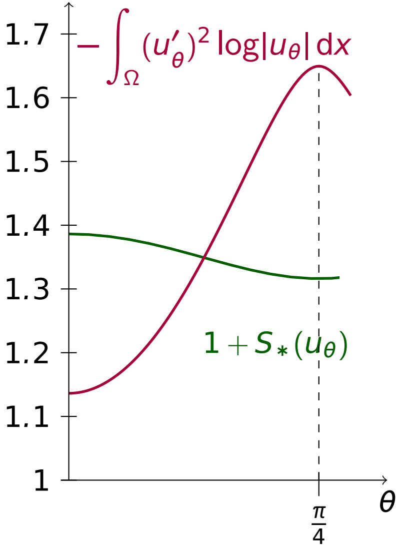
Let $u_\theta = \cos\theta\, v1 + \sin\theta\, v_2$ and $u'_\theta := \partial_\theta u_\theta$. Taking into account that $\int u_\theta^2 = 1$ and $\int (u'_\theta)^2 = 1$, one computes

$$\partial_\theta^2 \big(S_*(u_\theta)\big) = 2\Big(-1 - S_*(u_\theta) - \int_\Omega (u'_\theta)^2 \log|u_\theta| \,\mathrm{d}x\Big).$$

Thus

$$\partial_\theta^2 \big(S_*(u_\theta)\big) > 0 \quad \Leftrightarrow \quad -\int_\Omega (u'_\theta)^2 \log|u_\theta| \,\mathrm{d}x > 1 + S_*(u_\theta).$$

Positiveness test for the second derivative



$$\forall u \in \mathbb{R}, \; -\log|u| \geqslant g(u)$$

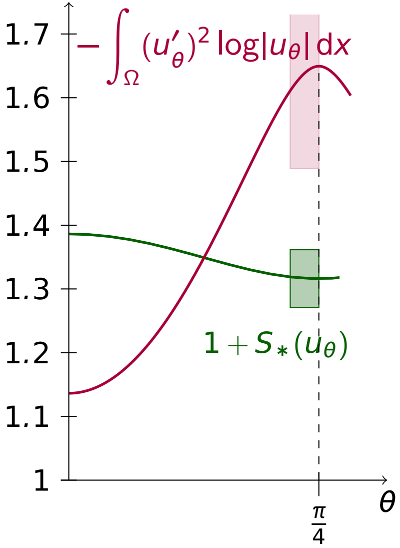
$$\Downarrow$$

$$-\int_\Omega (u_\theta')^2 \log|u_\theta| \, dx \geqslant \int_\Omega (u_\theta')^2 \, g(u_\theta) \, dx$$

The map g is just a truncation of $-\log$:

$$g(u) := \min\{-\log|u|, 1000\}.$$

Positiveness test for the second derivative



$$\forall u \in \mathbb{R},\ -\log|u| \geqslant g(u)$$
$$\Downarrow$$
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$$\geqslant \int_\Omega (u_\theta')^2 g(u_\theta) \,\mathrm{d}x$$

The map g is just a truncation of $-\log$:

$$g(u) := \min\{-\log|u|, 1000\}.$$

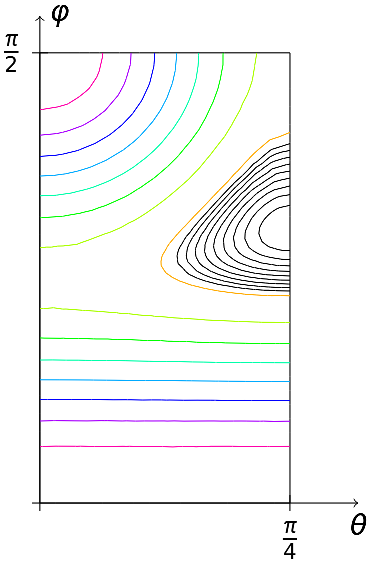
The 3D case

On $\Omega =]-1,1[^3$, $E_2 = \operatorname{span}\{v_1, v_2, v_3\}$ where

$$\begin{aligned}
v_1(x,y,z) &:= \sin(\pi x)\cos\Big(\frac{\pi}{2}y\Big)\cos\Big(\frac{\pi}{2}z\Big)\\
v_2(x,y,z) &:= \cos\Big(\frac{\pi}{2}x\Big)\sin(\pi y)\cos\Big(\frac{\pi}{2}z\Big)\\
v_3(x,y,z) &:= \cos\Big(\frac{\pi}{2}x\Big)\cos\Big(\frac{\pi}{2}y\Big)\sin(\pi z)
\end{aligned}$$

Let $u_{\theta,\varphi} := (\cos\theta v_1 + \sin\theta v_2)\sin\varphi + \cos\varphi v_3$.

The 3D case: minimizers



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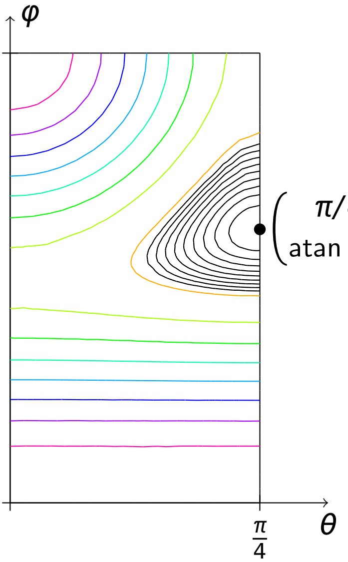

$$\begin{pmatrix} \pi/4 \\ \operatorname{atan}\sqrt{2} \end{pmatrix}$$

$$u_{\theta,\varphi} := (\cos\theta\, v_1 + \sin\theta\, v_2)\sin\varphi + \cos\varphi\, v_3$$

The minimum seems to be achieved for

$$(\theta,\varphi) = \left(\frac{\pi}{4}, \operatorname{atan}\sqrt{2}\right)$$

i.e., for

$$v_1 + v_2 + v_3.$$

The 3D case: minimizers

The zero set of $v_1 + v_2 + v_3$ is pictured below.