

Self-duality in Generalized Lorentz Superspaces

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Abstract

We extend the notion of self-duality to spaces built from a set of representations of the Lorentz group with bosonic or fermionic behaviour, not having the traditional spin-one upper-bound of super Minkowski space. The generalized derivative vector fields on such superspaces are assumed to form a superalgebra. Introducing corresponding gauge potentials and hence covariant derivatives and curvatures, we define generalized self-duality as the Lorentz covariant vanishing of certain irreducible parts of the curvatures.

1 Introduction

Self-duality (or antiself-duality) has been an important tool in the study of gauge theories and is well-known to have led to many profound and successful results. Some years ago [1], self-duality was generalized to spaces of higher dimensions by introducing a completely antisymmetric four-index tensor. Here, we generalize self-duality in another direction keeping, this time, Lorentz covariance.

We introduce, in a Lorentz covariant fashion, generalized superspaces with bosonic and fermionic coordinates $\{Y\}$. Corresponding to each coordinate, we define an operator X (generalizing the derivative) which transforms under the Lorentz group in the same way as the coordinate. These generalized derivatives are assumed to form a superalgebra and to act linearly on the coordinates. Then, by associating a gauge potential A to each of these generalized derivatives, we define, in a natural fashion, the covariant derivative $\mathcal{D} = X + A$ and the corresponding curvatures F . Generalized self-dualities are then defined in terms of the Lorentz covariant conditions of the vanishing of curvature components corresponding to subsets of specified Lorentz behaviour.

These generalized superspaces are modeled on standard superspace with coordinates $\{Y^{\alpha\dot{\alpha}}, Y^\alpha, Y^{\dot{\alpha}}\}$ (see the next section for notation) and super derivative operators $\{X_{\alpha\dot{\alpha}}, X_\alpha, X_{\dot{\alpha}}\}$. They provide, in particular, higher dimensional spaces having manifest four-dimensional Lorentz covariance.

2 Multi-index notation

We use the dotted ($\dot{\alpha} = 1, 2$) and undotted ($\alpha = 1, 2$) indices familiar for the Lorentz group. These can be lowered (or raised) using the antisymmetric $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ tensors with $\epsilon_{12} = 1$ (or $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$ with $\epsilon^{12} = -1$).

We define the multi-indices $[A]$, $[B]$, $[\dot{A}]$ and $[\dot{B}]$ as the sets of, respectively, $2a$, $2b$, $2\dot{a}$ and $2\dot{b}$ symmetrized indices (a, b, \dot{a} and \dot{b} being integers or half-integers),

$$\begin{aligned} [A] &= \alpha_1 \alpha_2 \dots \alpha_{2a} \quad , & [B] &= \beta_1 \beta_2 \dots \beta_{2b} \quad , \\ [\dot{A}] &= \dot{\alpha}_1 \dot{\alpha}_2 \dots \dot{\alpha}_{2\dot{a}} \quad , & [\dot{B}] &= \dot{\beta}_1 \dot{\beta}_2 \dots \dot{\beta}_{2\dot{b}} \quad . \end{aligned} \quad (1)$$

Using the ϵ 's we also define the multi-index tensors

$$\begin{aligned} \epsilon_{[s]} &= \epsilon_{\alpha_1 \beta_1} \epsilon_{\alpha_2 \beta_2} \dots \epsilon_{\alpha_s \beta_s} \\ \epsilon^{[\dot{t}]} &= \epsilon^{\dot{\alpha}_1 \dot{\beta}_1} \epsilon^{\dot{\alpha}_2 \dot{\beta}_2} \dots \epsilon^{\dot{\alpha}_t \dot{\beta}_t} \\ \epsilon^{[0]} &= \epsilon_{[0]} = 1 \quad . \end{aligned} \quad (2)$$

Finally, $S[A]$ denotes the symmetrization operator, i.e. the sum over all permutations of indices appearing in $[A]$.

3 Coordinates

The generalized coordinates $\{Y\}$ are a set of Lorentz tensors $Y_{[A]}^{[\dot{A}]}$ transforming as (a, \dot{a}) representations of the Lorentz group. These representations are called even (bosonic) if $2(a + \dot{a})$ is even and odd (fermionic) if $2(a + \dot{a})$ is odd. They commute or anticommute according to the obvious rules

$$\left[Y_{[A]}^{[\dot{A}]} , Y_{[B]}^{[\dot{B}]} \right]_{\eta} = 0 \quad , \quad (3)$$

where the graded bracket is defined with

$$\eta = (-1)^{4(a+\dot{a})(b+\dot{b})+1} \quad . \quad (4)$$

Note that the multiplicity of a given representation can be any non-negative integer. Traditional superspaces possess only a finite number of representations, each appearing possibly more than once. To simplify the present exposition, we shall not introduce a multiplicity index. In other words, we assume that each representation appears only once. The generalization to representations with multiplicities is essentially obvious and will be discussed in [2].

4 Generalized derivatives

Let us associate to the coordinates $\{Y\}$, generalized derivatives $\{X\}$ satisfying a Lorentz covariant superalgebra \mathcal{A} . Using the multi-indices (for $0 \leq s \leq \min(2a, 2b)$ and for $0 \leq t \leq \min(2\dot{a}, 2\dot{b})$)

$$\begin{aligned} [C(s)] &= \alpha_{s+1}\alpha_{s+2}\dots\alpha_{2a}\beta_{s+1}\beta_{s+2}\dots\beta_{2b} \quad , \\ [\dot{C}(t)] &= \dot{\alpha}_{t+1}\dot{\alpha}_{t+2}\dots\dot{\alpha}_{2\dot{a}}\dot{\beta}_{t+1}\dot{\beta}_{t+2}\dots\dot{\beta}_{2\dot{b}} \quad , \end{aligned} \quad (5)$$

the Lorentz-allowed supercommutation relations of the X 's are

$$\begin{aligned} \left[X_{[A]}^{[\dot{A}]} , X_{[B]}^{[\dot{B}]} \right]_{\eta} &= \sum_{s=0}^{\min(2a, 2b)} \sum_{t=0}^{\min(2\dot{a}, 2\dot{b})} t(a, \dot{a}; b, \dot{b}; a+b-s, \dot{a}+\dot{b}-t) \\ & S[A]S[\dot{A}]S[B]S[\dot{B}]\epsilon_{[s]}\epsilon^{[t]} X_{[C(s)]}^{[\dot{C}(t)]} \quad , \end{aligned} \quad (6)$$

where the $t(a, \dot{a}; b, \dot{b}; c, \dot{c})$'s depending on six integers or half-integers are the structure constants of the superalgebra and η is again given by (4). Obviously, the allowed values of the structure constants are restricted by (skew-)symmetry and by super Jacobi identities. A similar class of algebras has been considered previously by Fradkin and Vasiliev [3].

It is natural but not compulsory to identify the generators of the Lorentz group itself with the six generators in the set $\{X\}$ behaving as $(1, 0) \oplus (0, 1)$.

5 Action of the generalized derivatives on the coordinates

We take the elements of the algebra \mathcal{A} as acting on the space coordinates Y as generalized derivatives. This means that the supercommutation relations between the differential operators X and the coordinates Y are taken to be linear in the Y 's i.e. the X 's transform the coordinates at most linearly among themselves. The X 's together with the Y 's therefore combine to form an enlarged superalgebra. Let us write, using the by now familiar notation,

$$\begin{aligned} & \left[X_{[A]}^{[\dot{A}]} , Y_{[B]}^{[\dot{B}]} \right]_{\eta} \\ &= \sum_{s=0}^{\min(2a, 2b)} \sum_{t=0}^{\min(2\dot{a}, 2\dot{b})} u(a, \dot{a}; b, \dot{b}; a+b-s, \dot{a}+\dot{b}-t) \\ & \quad S[A]S[\dot{A}]S[B]S[\dot{B}]\epsilon_{[s]}\epsilon^{[t]} Y_{[C(s)]}^{[\dot{C}(t)]} \\ & \quad + c(a, \dot{a})\delta_{ab}\delta^{\dot{a}\dot{b}} S[A]S[\dot{A}]\epsilon_{[2a]}\epsilon^{[2\dot{a}]} \quad . \end{aligned} \quad (7)$$

The u 's are new structure constants. The central c -terms can be either zero or, if they are non-zero and if there are no multiplicities, can be set to 1 by a suitable renormal-

ization of the operators. The allowed values of the u 's and the c 's are restricted by the super-Jacobi identities for the combined superalgebra of the X 's and the Y 's.

6 Gauge fields

To every operator X we associate a generalized gauge potential A having the same Lorentz transformation properties and depending on the variables Y . These gauge potentials take values in the algebra of some matrix group whose N generators are denoted λ_k , $k = 1, \dots, N$, in some representation defined and normalized so that

$$\begin{aligned} [\lambda_k, \lambda_l] &= f_{kl}^m \lambda_m \quad , \\ \text{tr}(\lambda_k \lambda_l) &= \delta_{kl} \quad , \end{aligned} \tag{8}$$

where f_{kl}^m are the structure constants. We have

$$\begin{aligned} A_{[A]}^{[\dot{A}]} &= \sum_{k=1}^N A_{[A],k}^{[\dot{A}]} \lambda_k \quad , \\ A_{[A],k}^{[\dot{A}]} &= \text{tr} \left(A_{[A]}^{[\dot{A}]} \lambda_k \right) \quad . \end{aligned} \tag{9}$$

One can then naturally define the generalized covariant derivatives \mathcal{D} by the matrix

$$\mathcal{D}_{[A]}^{[\dot{A}]} = X_{[A]}^{[\dot{A}]} + A_{[A]}^{[\dot{A}]} \quad . \tag{10}$$

It is then also natural to define the generalized (in general reducible) curvature gauge fields \hat{F} , as matrices of functions of the coordinates (corresponding to the sets $[A]$, $[\dot{A}]$ and $[B]$, $[\dot{B}]$), by the equations

$$\begin{aligned} \hat{F}_{[A][B]}^{[\dot{A}][\dot{B}]} &= \left[\mathcal{D}_{[A]}^{[\dot{A}]} , \mathcal{D}_{[B]}^{[\dot{B}]} \right]_{\eta} \\ &\quad - \sum_{s=0}^{\min(2a, 2b)} \sum_{t=0}^{\min(2\dot{a}, 2\dot{b})} t(a, \dot{a} ; b, \dot{b} ; a + b - s, \dot{a} + \dot{b} - t) \\ &\quad \quad \quad S[A]S[\dot{A}]S[B]S[\dot{B}]\epsilon_{[s]}\epsilon^{[t]}\mathcal{D}_{[C(s)]}^{[\dot{C}(t)]} \quad . \end{aligned} \tag{11}$$

On the right hand side, the second term has been subtracted so as to yield gauge fields free of differential operators in a gauge-covariant manner. The gauge fields \hat{F} take values in the algebra of the underlying Lie group.

The behaviour of the gauge fields is not irreducible under the action of the Lorentz group. They decompose according to

$$\hat{F}_{[A][B]}^{[\dot{A}][\dot{B}]} = \sum_{s=0}^{\min(2a, 2b)} \sum_{t=0}^{\min(2\dot{a}, 2\dot{b})} S[A]S[B]S[\dot{A}]S[\dot{B}]\epsilon_{[s]}\epsilon^{[t]}F_{[C(s)]}^{[\dot{C}(t)]} \quad . \tag{12}$$

The irreducible components $F_{[C(s)]}^{[\dot{C}(t)]}$ transforming according to the $(a+b-s, \dot{a}+\dot{b}-\dot{t})$ Lorentz representations may be projected out by contracting the curvatures \hat{F} with the inverses of the generalized epsilon tensors $\epsilon_{[s]}$ and $\epsilon_{[\dot{t}]}$ in (3),

$$\begin{aligned}\epsilon^{[s]} &= \epsilon^{\beta_1\alpha_1}\epsilon^{\beta_2\alpha_2}\dots\epsilon^{\beta_s\alpha_s} \\ \epsilon_{[\dot{t}]} &= \epsilon^{\dot{\beta}_1\dot{\alpha}_1}\epsilon^{\dot{\beta}_2\dot{\alpha}_2}\dots\epsilon^{\dot{\beta}_t\dot{\alpha}_t} \\ \epsilon^{[s]}\epsilon_{[s]} &= 2^s\end{aligned}\tag{13}$$

and symmetrizing over the remaining multi-indices $[C(s)]$ and $[\dot{C}(t)]$ in (5):

$$F_{[C(s)]}^{[\dot{C}(t)]} = \kappa(s, t)S[C(s)]S[\dot{C}(t)]\epsilon^{[s]}\epsilon_{[\dot{t}]}\hat{F}_{[A][B]}^{[\dot{A}][\dot{B}]},\tag{14}$$

where $\kappa(s, t)$ are combinatorial factors. The gauge algebra components of the irreducible fields F may be extracted by taking the traces as in (9).

7 Generalized self-duality

Once a model has been defined by the choice of

- a Lie algebra,
- a coherent set of coordinates Y and of vector fields X (i.e. having chosen the t , u and c parameters satisfying the super Jacobi identities),

we define a set of generalized self-duality equations as the vanishing of a subset of the irreducible components (14) of the gauge fields.

When Y and X are reduced to belong to the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group only and the $X_{\alpha\dot{\alpha}}$ are simply the (commuting among themselves) derivatives with respect to the usual Minkowski coordinates $Y^{\alpha\dot{\alpha}}$, the irreducible components of the field \hat{F} defined above are $F_{\alpha\beta}$ and $F_{\dot{\alpha}\dot{\beta}}$. They transform respectively as the (1,0) and (0,1) representations of the Lorentz group. The usual self-duality equation is then equivalent to the vanishing of the (0,1) components,

$$F^{\dot{\alpha}\dot{\beta}} = 0 \quad ,\tag{15}$$

with the self-dual (1,0) components remaining (and vice-versa for anti-self-duality).

The simplest extension to our generalized superspaces is the vanishing of all curvature representations except for those transforming according to the $(a, 0)$ representations, $\{F_{\alpha_1\dots\alpha_{2a}}\}$, containing no dotted indices. In other words, the conditions

$$\{F_{\alpha_1\dots\alpha_{2a}}^{\dot{\alpha}_1\dots\dot{\alpha}_{2b}} = 0 ; 1 \leq 2b \leq \dot{t} \quad , \quad 0 \leq 2a \leq s\},\tag{16}$$

for any fixed choice of s, \dot{t} . These systems are solvable in the sense that the traditional self-duality conditions (15) are. They provide an infinite hierarchy of self-dual systems incorporating, for the choice of X 's and Y 's corresponding to standard superspace, the supersymmetric self-duality conditions (see for example [4]).

8 Conclusion and outlook

By successively defining supercoordinates, a superalgebra of superderivatives acting linearly on them and finally superpotentials in a manifestly Lorentz covariant manner, we have given a natural definition of super gauge fields on our generalized superspaces. The vanishing of well-chosen irreducible components of the curvature superfields is then a natural generalization of self-duality or antiself-duality. The procedure outlined here will be given in more detail in a forthcoming paper [2] where moreover the connections with other existing generalizations will be made clearer and where further explicit examples will be given.

There are many obvious generalizations of this approach. Let us mention two. One could, for example, try to quantum-deform the theory by allowing the coordinates to fulfil the commutation relations of a quantum space, the vector fields to satisfy a quantum algebra and act on the coordinates in a quantum way, with the underlying gauge algebra being also q-deformed. Another generalisation is to alter the Lorentz metric and work with spinors in spaces (euclidean or otherwise) of arbitrary dimension.

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References

- [1] E. Corrigan, C. Devchand, D.B. Fairlie and J. Nuyts, Nucl. Phys. **B214** (1983) 452–464.
- [2] C. Devchand and J. Nuyts, *Completely solvable systems on superspaces of dimension greater than $(4 | 4)$ generalizing self-duality*, (to be published).
- [3] E.S. Fradkin and M.A. Vasiliev, Ann.Phys. **177** (1987) 63.
- [4] C. Devchand and V. Ogievetsky, Nucl. Phys. **B481** (1996) 188.