Charged-rotating black holes and black strings in higher dimensional Einstein-Maxwell theory with a positive cosmological constant

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Abstract

We present arguments for the existence of charged, rotating black holes in $d = 2N+1$ dimensions, with $d \geq 5$ with a positive cosmological constant. These solutions posses both, a regular horizon and a cosmological horizon of spherical topology and have N equal-magnitude angular momenta. They approach asymptotically the de Sitter spacetime background. The counterpart equations for $d = 2N + 2$ are investigated, by assuming that the fields are independant of the extra dimension y, leading to black strings solutions. These solutions are regular at the event horizon. The asymptotic form of the metric is not the de Sitter form and exhibit a naked singularity at finite proper distance.

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I. INTRODUCTION

Recently, there was a lot of interest in black holes and black strings solutions is space-time with arbitrary dimensions and with a cosmological constant Λ. In the case of a negative cosmological constant, the interest of these solutions is related to the correspondance between gravitating fields in an AdS space-time and the conformal field theory on the boundary of the AdS space-time [1, 2].

Black-string solutions denote a string like generalisation of 4-D black hole solution of d-dimensional Einstein gravity characterized with an event horizon of topology $S_{d-3} \times S_1$ [3]. Black string for $d = 5$ and $\Lambda < 0$ they were first considered in [4] and then generalized to d dimensions in [5]. The charged counterparts of these solutions for the minimal Einstein-Maxwell (EM) gravity have been obtained recently [6]. For space-times of even dimensions $d \geq 6$, the S_{d-3} part of the metric can be deformed by using the ideas of [7] in such a way that some rotating black string solutions can be constructed while using normal differential equations only. Rotating counterparts of the black strings of [5], constructed along these lines, are also obtained in [6]. Another direction of investigation of these objects is non-uniform black strings, where a non trivial dependence on the extra-dimension is required [8]. Recently, $\Lambda = 0$ rotating nonuniform solutions have been considered as well [9].

Generalizing the solutions of Tangerlini [10] and of Myers-Perry [11], higher dimensional charged, rotating black holes were constructed in [12] with asymptotically flat space-time using the symmetries of [7]. Very recently, rotating black holes in Einstein-Maxwell (EM) theory were constructed in [13] for odd space-time dimensions $d \geq 5$ and $\Lambda < 0$.

The litterature investigating higher dimensional black holes and black strings with a positive cosmological constant is by far less abundant, although the problem desserves to be investigated for several reasons. Namely :

i) the recent experiments are rather consistent with a cosmological constant of the positive sign,

ii) as observed e.g. in [14], a positive cosmological constant can have important consequences on the physical properties of some classical solutions, these could play a role in inflation

iii) it is mathematically interesting to see whether the solutions available for $\Lambda < 0$ can be analytically continued for $\Lambda > 0$,

iv) it is numerically challenging since we expect a cosmological horizon to occur at an intermediate value of the radial variable of space-time,

v) connections between quantum gravity in DeSitter space-time and conformal field theory on the boundary exist, see e.g. [15],

vi) finally, such solutions would extend the pattern of already known solutions of Einstein equations in higher dimensional space-times.

Up to now, several charged and/or rotating black holes solutions of the Einstein equations coupled to an electromagnetic fields in space-times with $d \geq 4$ (and $\Lambda > 0$) are known [16], [17]; they are constructed with a Chern-Simons term which appears naturally in the bosonic sector of minimal five-dimensional supergravity. Vacuum solutions of the vacuum 5-dimensional Einstein gravity with $\Lambda > 0$ were obtained in [18], but a systematic study of (rotating) solutions of the minimal Einstein-Maxwell theory (with $\Lambda > 0$) has, to our knowledge, not yet been adressed. Let us mention that static black rings solutions have also been constructed for 5-d DeSitter supergravity theory [19]. Solutions of the Einstein-Maxwell-dilaton system were are considered in [20] with both signs of the cosmological constant.

Considering the minimal theories of Einstein and Einstein-Maxwell gravity with the same kinds of symmetries of the fields as in [13] and [6] but with a positive cosmological constant can lead to solutions presenting drastic differences with respect to the case of negative cosmological constant, although the equations to solve are basically similar. In this paper, we will reconsider the equations of [6, 13] for positive values of the cosmological constant and construct several families of solutions of these equations. The paper is organized as follow : Sect. 2 is devoted to rotating, charged black holes in odd dimensions. The ansatz, the equations, the boundary conditions and the numerical results are presented successively. Non-rotating and rotating black strings (in even dimensions) are described in Sect. 3, following a similar pattern. The results are summarized in Sect. 4.

II. BLACK HOLES

We consider the Einstein-Maxwell action with a positive cosmological constant Λ

$$I = \frac{1}{16\pi G}\int_M d^dx\sqrt{-g}(R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}) - \frac{1}{8\pi G}\int_{\partial M} d^{d-1}x\sqrt{-h}K, \tag{1}$$

in a $d-$dimensional spacetime. The last term in (1) is the Hawking-Gibbons surface term [21], which is required in order to have a well-defined variational principle. The factor K represents the trace of the extrinsic curvature for the boundary $\partial\mathcal{M}$ and h is the induced metric of the boundary. Along with many authors, we note $\Lambda = \pm(d-2)(d-1)/(2\ell^2)$, the sign $+$ will be considered throughout this paper.

A. The ansatz

To obtain rotating black hole solutions, representing charged $U(1)$ generalizations of the vacuum solutions discussed in [7], we consider space-times with odd dimensions and possessing $N = (d-1)/2$ commuting Killing vectors $\eta_k = \partial_{\varphi_k}$.

We use a parametrization for the metric, corresponding to a generalization of the Ansatz previously used for asymptotically flat solutions [12]

$$\begin{aligned} ds^2 = -b(r)dt^2 + \frac{dr^2}{f(r)} + g(r)\sum_{i=1}^{N-1}\left(\prod_{j=0}^{i-1}\cos^2\theta_j\right)d\theta_i^2 \\ +h(r)\sum_{k=1}^{N}\left(\prod_{l=0}^{k-1}\cos^2\theta_l\right)\sin^2\theta_k\left(d\varphi_k - w(r)dt\right)^2 \\ +p(r)\left\{\sum_{k=1}^{N}\left(\prod_{l=0}^{k-1}\cos^2\theta_l\right)\sin^2\theta_k d\varphi_k^2 - \left[\sum_{k=1}^{N}\left(\prod_{l=0}^{k-1}\cos^2\theta_l\right)\sin^2\theta_k d\varphi_k\right]^2\right\}, \end{aligned} \tag{2}$$

In the above formula $\theta_0 \equiv 0$ and $\theta_N \equiv \pi/2$ are assumed; the non trivial angles have $\theta_i \in [0, \pi/2]$ for $i = 1, \dots, N-1$, while $\varphi_k \in [0, 2\pi]$ for $k = 1, \dots, N$. The functions b, f, h, g, w depend on the variable r. The consistency of the ansatz imposes $p(r) = g(r) - h(r)$.

The parametrization of the U(1) potential, consistent with the symmetries of the line element (2) is

$$A_\mu dx^\mu = V(r)dt + a_\varphi(r) \sum_{k=1}^{N} \left(\prod_{l=0}^{k-1} \cos^2\theta_l \right) \sin^2\theta_k d\varphi_k \tag{3}$$

Here, the electric and magnetic potentials $V(r)$ and $a_\varphi(r)$ also depend on r. The Einstein-Maxwell equations lead to a consistent system of ordinary differential equations in the different radial functions.

Without fixing a metric gauge, the ansatz presented above leads to following reduced Lagrangean of the EM system, say $L = L_g - L_M$. The Einstein part L_g and Mawxell part L_M read respectively :

$$\begin{aligned} L_g = &(d-3)g^{\frac{(d-7)}{2}}\sqrt{\frac{bh}{f}}((d-1)g-h) + \frac{1}{2}\sqrt{fhb}g^{\frac{(d-3)}{2}}(\frac{b'}{b} + (d-3)\frac{g'}{g})(\frac{h'}{h} + (d-3)\frac{g'}{g}) \\ &-\frac{1}{4}(d-2)(d-3)\sqrt{fhb}g^{\frac{(d-7)}{2}}g'^2 + \frac{1}{2}g^{\frac{(d-3)}{2}}h\sqrt{\frac{fh}{b}}w'^2 - \frac{(d-2)(d-1)}{\ell^2}g^{\frac{(d-3)}{2}}\sqrt{\frac{bh}{f}} \end{aligned} \tag{4}$$

$$L_M = \frac{g^{\frac{(d-7)}{2}}}{\sqrt{bfh}}\left(2b\left(2(d-3)a_\varphi^2 h + fg^2 a_\varphi'^2\right) - 2fg^2h(wa_\varphi' + V')^2\right). \tag{5}$$

B. The equations

The EM equations obtained from the ansatz discussed above can be obtained in a standard way. For the numerical construction of the solutions, the "‘metric gauge"’ has to be fixed; we find it is convenient to fix it by imposing $g(r) = r^2$. With this gauge, the following field equations are found after an algebra :

$$\begin{aligned} &f' + \frac{f}{(d-2)}\Bigg(- \frac{rh}{2b}w'^2 + \frac{2r}{b}V'^2 + \frac{4rw}{b}a_\varphi' V' + \frac{h'}{h}(1 - \frac{rb'}{2b}) - 2r(\frac{1}{h} - \frac{w^2}{b})a_\varphi'^2 + \frac{b'}{b} \\ &+ \frac{(d-1)(d-4)}{r}\Bigg) + \frac{1}{(d-2)r^3}((3d-5)h + 4(d+1)a_\varphi^2 - (d-1)^2r^2) + \frac{(d-1)r}{\ell^2} = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} &b'' + \frac{1}{d-2}\Bigg(4(5-2d)wa_\varphi' V' + \frac{(d-3)}{2h}b'h' - \frac{2(d-3)b}{2rh}h' - 2(2d-5)V'^2 \\ &+\frac{1}{2}(3-2d)hw'^2 - 2(\frac{b}{h} + (2d-5)w^2)a_\varphi'^2 + (d-2)\left(\frac{b'f'}{2f} - \frac{b'^2}{2b}\right) + \frac{(d-3)^2}{r}b' \\ &-\frac{(d-3)b}{r^4 f}(12a_\varphi^2 + h) + \frac{(d-3)b}{r^2}\left(\frac{d-1}{f} + 4 - d\right) + \frac{(d-1)(d-2)b}{\ell^2 f}\Bigg) = 0, \end{aligned} \tag{7}$$

$$h'' + \frac{1}{(d-2)}\Bigg(\frac{(2d-5)h^2}{2b}w'^2 + \frac{2h}{b}V'^2 + \frac{4hw}{b}a'_\varphi V' - \frac{(d-2)h'}{2}(\frac{h'}{h} - \frac{f'}{f}) \tag{8}$$

$$+\frac{(d-3)}{2b}b'h' + \frac{(d-3)^2}{r}h' + 2(\frac{hw^2}{b} + 2d-5)a'^2_\varphi - \frac{(d-3)h}{rb}b' - \frac{(d-3)(2d-3)h^2}{r^4 f}$$

$$-\frac{12(d-3)a_\varphi^2 h}{r^4 f} - \frac{(d-3)(d-4)h}{r^2} + \frac{(d-1)h}{f}(\frac{d-2}{\ell^2} - \frac{d-3}{r^2})\Bigg) = 0,$$

$$w'' - \frac{4w}{h}a'^2_\varphi - \frac{4a'_\varphi V'}{h} + \frac{(d-3)w'}{r} + \frac{1}{2}\left(-\frac{b'}{b} + \frac{f'}{f} + \frac{3h'}{h}\right)w' = 0, \tag{9}$$

for the gravity part, and

$$V'' - \frac{w}{b}b'a'_\varphi + \frac{w}{h}a'_\varphi h' + \frac{1}{2}(\frac{2(d-3)}{r} - \frac{b'}{b} + \frac{f'}{f} + \frac{h'}{h})V'$$
$$+(1 + \frac{hw^2}{b})a'_\varphi w' + \frac{hw}{b}V'w' + \frac{2(d-3)a_\varphi hw}{r^4 f} = 0, \tag{10}$$

$$a''_\varphi + \frac{1}{2}(\frac{2(d-3)}{r} + \frac{b'}{b} + \frac{f'}{f} - \frac{h'}{h})a'_\varphi - \frac{h}{b}(wa'_\varphi + V')w' - \frac{2(d-3)a_\varphi h}{r^4 f} = 0. \tag{11}$$

for the U(1) potentials. It can easily be seen that the equations of motion present the first integral

$$g^{\frac{(d-3)}{2}}\sqrt{\frac{fh}{b}}(wa'_\varphi + V') = (d-3)q. \tag{12}$$

Thus, similar to the asymptotically flat case [12] case, the electric potential can be eliminated from the equations (10) by making use of the first integral (12). The cosmological constant parameter can be arbitrarily rescaled by a rescaling of the radial variable r and of the fields h and w. In this section, we use this arbitrariness to choose $r_h = 1$ without loosing generality.

Known solutions The vacuum black holes discussed in [7] are recovered for a vanishing gauge field and

$$f(r) = 1 - \frac{r^2}{\ell^2} - \frac{2M\Xi}{r^{d-3}} + \frac{2Ma^2}{r^{d-1}},\ \ h(r) = r^2(1 + \frac{2Ma^2}{r^{d-1}}),$$
$$w(r) = \frac{2Ma}{r^{d-3}h(r)},\ \ \ g(r) = r^2,\ \ \ b(r) = \frac{r^2 f(r)}{h(r)}, \tag{13}$$

where M and a are two constants related to the solution's mass and angular momentum and $\Xi = 1 + a^2/\ell^2$.

C. Constraint of regularity about the horizon

We are interested in black hole solutions, with an horizon located at $r = r_h$. The solutions can be expanded in the neighbourhood of the horizon in the same was as in the case of a negative cosmological constant [13], i.e.

$$b(r) = b_1(r - r_h) + O(r - r_h)^2, \quad f(r) = f_1(r - r_h) + O(r - r_h)^2,$$
$$h(r) = h_0 + h_1(r - r_h) + O(r - r_h)^2, w(r) = w_h + w_1(r - r_h) + O(r - r_h)^2,$$
$$a_\varphi(r) = a_0 + a_1(r - r_h) + O(r - r_h)^2, \quad V(r) = V_0 + V_1(r - r_h) + O(r - r_h)^2$$

Although it is not clear if the thermodynamical properties are well defined with the presence of the cosmlogical horizon, one may still define them in the standard way. The Hawking temperature and the event horizon area of these solutions can be obtained in a standard way, leading to

$$T_H = \frac{\sqrt{f_1 b_1}}{4\pi}, \quad A_H = V_{d-2} r_h^{d-2}. \tag{14}$$

Along with [13], we also write the mass and the angular velocity at the horizon defined by means of the appropriate Komar integrals :

$$M_H = \frac{V_{d-2}}{8\pi G_d}\sqrt{\frac{fhg^2}{b}}(b' - hww')|_{r=r_h} \quad , \quad J_H = \frac{V_{d-2}}{8\pi G_d} 2\sqrt{\frac{fg^2h^3}{b}}\ w'|_{r=r_h} \tag{15}$$

where V_{d-2} denotes the area of the $d-2$ dimensional sphere. These quantities can be easily evaluated from the numerical solutions.

For the solutions to be regular at the horizon r_h (or at the cosmological horizon r_c), the equation for h leads to the condition $\Gamma_1(x = r_h) = 0$ with

$$\begin{aligned}
\Gamma_1(x) \equiv \quad & 8b'h^2(12a^2 + 7h) + 4xb'hh'(12a^2 + 5h) \\
- \ & 32b'h^2x^2 + 8x^3b'h(f'h - 4h') \\
+ \ & 2hx^4(\frac{24}{\ell^2}b'h^2 - f'(4(a')^2hw^2 + 8a'hwV' + b'h' + 5h^2(w')^2 + 4h(V')^2)) \\
+ \ & h'x^5(-\frac{24}{\ell^2}b'h^2 - f'(4(a')^2hw^2 + 8a'hwV' - b'h' - h^2(w')^2 + 4h(V')^2))
\end{aligned} \tag{16}$$

where we posed $a_\phi \equiv a$. The value $f'(x_h)$ can be extracted from the equation for f, giving

$$f'(x_h) = \frac{4b_1h_0}{x_h^3}\frac{6x_h^4/\ell^2 + 8x_h^2 - 12a_0^2 - 5h_0}{8b_1h_0 + x_h(4h_0(V_1 + a_1w_0)^2 - b_1h_1 - h_0^2w_1^2)} \tag{17}$$

In the same way, the two Maxwell lead to a single condition $\Gamma_2(x = r_h) = 0$ with

$$\Gamma_2(x) \equiv 4ab'h + x^4f'(hw'V' + a'hww' - a'b') \tag{18}$$

D. The asymptotics and global charges

In this section, we follow the lines of [13] to present the global charges characterizing the solutions asymptotically. This uses a formalism developped namely in [22, 23] and also used in [14, 24] The metric functions have the following asymptotic behaviour in terms of three arbitrary constants α, β and $\hat{J}$

$$b(r) = -\frac{r^2}{\ell^2} + 1 + \frac{\alpha}{r^{d-3}} + O(1/r^{2d-6}), \quad f(r) = -\frac{r^2}{\ell^2} + 1 + \frac{\beta}{r^{d-3}} + O(1/r^{d-1}), \tag{19}$$
$$h(r) = r^2(1 + \frac{\beta - \alpha}{r^{d-1}} + O(1/r^{2d-4})), \quad w(r) = \frac{\hat{J}}{r^{d-1}} + O(1/r^{2d-4}),$$

The asymptotic expression of the gauge potential is similar to the asymptotically flat case

$$V(r) = -\frac{q}{r^{d-3}} + O(1/r^{2d-4}), \quad a_\varphi(r) = \frac{\hat{\mu}}{r^{d-3}} + O(1/r^{2d-4}). \tag{20}$$

The mass-energy of the solutions and angular momentum associated with an angular direction is

$$E = \frac{V_{d-2}}{16\pi G_d}(\beta - (d-1)\alpha), \quad J = \frac{V_{d-2}}{8\pi G_d}\hat{J} \ . \tag{21}$$

The above relations can be proven by using a background subtraction approach or the counterterm formalism [22, 23, 24].

The electric charge and the magnetic moment of the solutions are given by

$$Q = \frac{(d-3)V_{d-2}}{4\pi G_d}q, \quad \mu = \frac{(d-3)V_{d-2}}{4\pi G_d}\hat{\mu} \ . \tag{22}$$

The ansatz in $f, \ b, \ h$ has to advantage to present a direct connection with the closed form vacuum rotating solution.

E. Numerical results

The system of equations above does not admit, to our knowledge, explicit solutions for generic values of w_h and V_h. We therefore relied on a numerical method to construct solutions. We solved the equations in the case $d = 5$ and we hope that this case catches the qualitative properties of the pattern of the solutions; the numerical solver Colsys [25] was used throughout this paper.

The positive cosmological constant leads to the occurence of a cosmological horizon at $r = r_c$ (with $r_h < r_c < \infty$) where $f(r_c) = b(r_c) = 0$. This creates a difficulty since r_c constitutes an apparent singular point of the equations. In order to overcome this difficulty, we have solved the equations in two steps. In the first step, we supplemented the system by the trivial equation $d\ell^2/dx = 0$ and impose the conditions of two regular horizon at $r = r_h$ and $r = r_c$, fixing r_c, r_h by hand and solve the equations for $r \in [r_h, r_c]$. The appropriate set of twelfe boundary conditions at the two horizons then read

$$f = 0 \ , \ b = 0 \ , \ b' = 1 \ , \ w = w_h \ , V = 0 \ , \ a' = a'_h \ , \Gamma_1 = 0 \ , \ \Gamma_2 = 0 \ \ \text{for} \ \ r = r_h \tag{23}$$

$$f = 0 \ , \ b = 0 \ , \Gamma_1 = 0 \ , \ \Gamma_2 = 0 \ \ \text{for} \ \ r = r_c \tag{24}$$

The functions $\Gamma_{1,2}$ are defined above; here we take advantage of the arbitrary scale of the field $b(r)$ to impose $b'(r_h) = 1$ and of the arbitrary additive constant of the electric potential to assume $V(r_h) = 0$. The function $b(r)$ will be renormalized appropiately after the second step in order to set (19). The values w_h and a'_h are, a priori, arbitrary and control the total angular momentum and the (electric and magnetic) charges of the black hole.

The numerical value of ℓ^2 is determined by the first step, together with the values of all the fields at $r = r_c$. Use of these fields and of the value of ℓ^2 can then be used as a suitable set of Cauchy data to solve the equations for $r \in [r_c, \infty]$. The disatvantage of the method is that we cannot perform a systematic analysis of the solution for a fixed value of the cosmological constant. Fortunately, the numerical value of ℓ^2 depends only a little on $w(r_h)$ and $a(r_h)$ once r_h, r_c are fixed.

In the following, we present the results corresponding to $r_h = 1, r_c = 3$, this case corresponds to a large value of the coupling constant but allows one to analyze the effect of Λ on the solution. We hope that the results for this case capture the main features of the solutions for generic values of Λ The profiles of the solution corresponding to $w_h = 0.6, a'_h = 0.5$ are presented in Fig. 1. It corresponds to $1/\ell^2 = 0.0945$ and $q = -0.08538$. The smoothness of the profile at $r = r_c$ can be appreciated on the plot. It is also worth to point out that the numerical solutions approach the asymptotic behaviour (19) although all boundary conditions are imposed at $r = r_c$ for the second step of our construction. We supplemented Fig. 1 with the plot of the tt component of the metric $g_{tt} = b - hw^2$, showind that there is a small region around the event horizon where g_{tt} is negative, defining an ergoradius r_e where $g_{tt}(r_e) = 0$ (on the figure, $r_e \approx 1.17$).

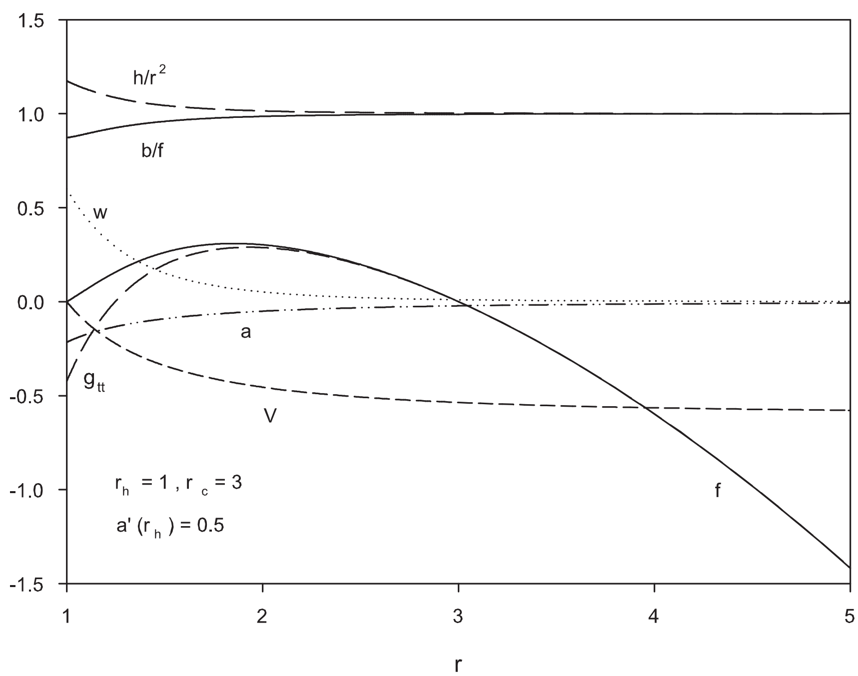

FIG. 1: The profiles of a generic solution correponding to $r_h = 1, r_c = 3$, $w_h = 0.6, a'_h = 0.5$

We manage to construct several branches of solutions for different values of a'_h and increasing the parameter w_h. For fixed a'_h, the solutions exist only for sufficiently large values of w_h, this is illustrated on Fig.2 where we plot the asymptotic charges M, J, Q, μ as functions of w_h for two values of a'_h; for $a'_h = 0.1$ (resp. $a'_h = 0.5$) the branch exist for $w_h > 0.1$ (resp. $w_h > 0.465$). We strongly suspect that another branch of solutions, with a larger mass exist, terminating at the same value of w_h but we cannot construct it at this moment. The numerical integration between the two horizons and the related constraints make the construction of an eventual second branch very tricky. We nevertheless observe that the asymptotic mass is positive for all values of the parameters that we have explored. The figure further suggests that, if a second branch exists, its asymptotic mass will be larger than the one of the first branch that we constructed. The magnetic moment is rather independant on the parameter w_h. Also, when we solved the equation for larger values of r_c, corresponding to smaller values of Λ, we observe that the asymptotic quantities plotted on Fig. 2 depend weakly on r_c.

The numerical values of the different fields at the horizon r_h are presented on Figs. 3 and

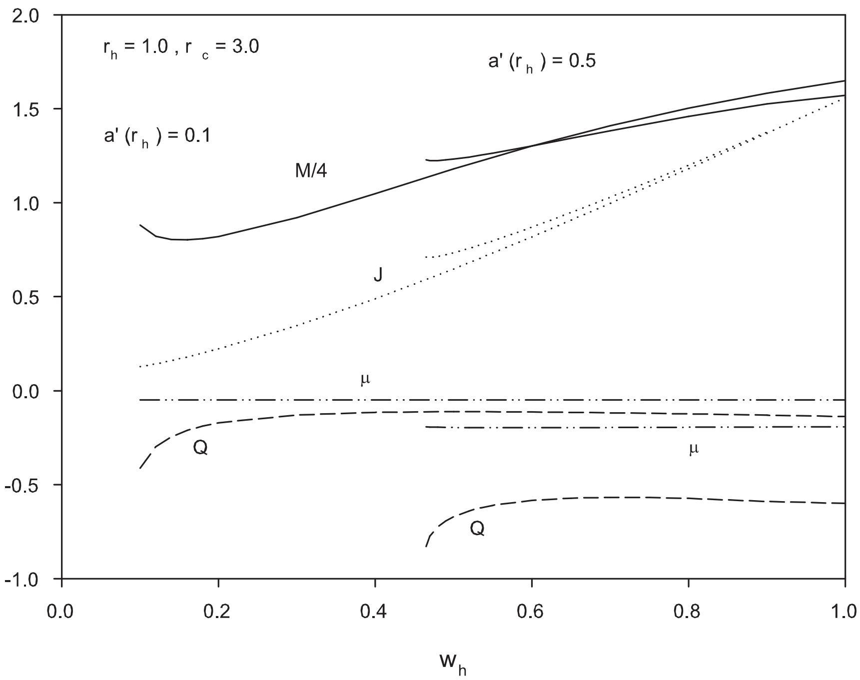

FIG. 2: The asymptotic charges M, J, Q, μ are presented as functions of w_h for $a'_h = 0.1$ and $a'_h = 0.5$

4 for the case $a'_h = 0.1$ and $a'_h = 0.5$ respectively. The value a_h is negative and depends weakly on w_h, e.g. we find $a_h \sim -0.049$ (resp. $a_h \sim -0.215$) for $a'_h = 0.1$ (resp. $a'_h = 0.5$); correspondingly we find $\ell^2 \sim 10.3$ and $\ell^2 \sim 10.6$ (ℓ^2 is indicated on Fig. 4 only).

The physical parameters characterizing the solutions at the horizon can then be computed from the numerical data. In particular, the natural normalisation of the function b (i.e. such that $b(r) \to -r^2/\ell^2$ for $r \to \infty$) which renders the space-time asymptoticall DeSitter is determined from the solution in the asymptotic region. On Fig. 5, we have superposed the Hawking Temperature T_H, the mass and angular momentum at the horizon as functions of the parameter w_h for the solutions obtained with $a'(x_h) = 0.1$ (corresponding to $1/\ell^2 = 0.097$) and existing for $w_h > 0.1$ and $a'(x_h) = 0.5$ (corresponding to $1/\ell^2 = 0.094$) and existing for $h_h > 0.462$). We see in particular that the Komar mass is positive at the even horizon. For the sake of completeness, the same quantities relative to the cosmological horizon r_c are presented on Fig. 6. These physical parameters depend only weakly of $a'(r_h)$ when the parameter w_h becomes sufficiently large. We note that the mass M_c is negative due to the fact that $b'(r_c)$ is negative and the rotation energy, although positive, is not enough to make the combination in 15 positive at $r = r_c$.

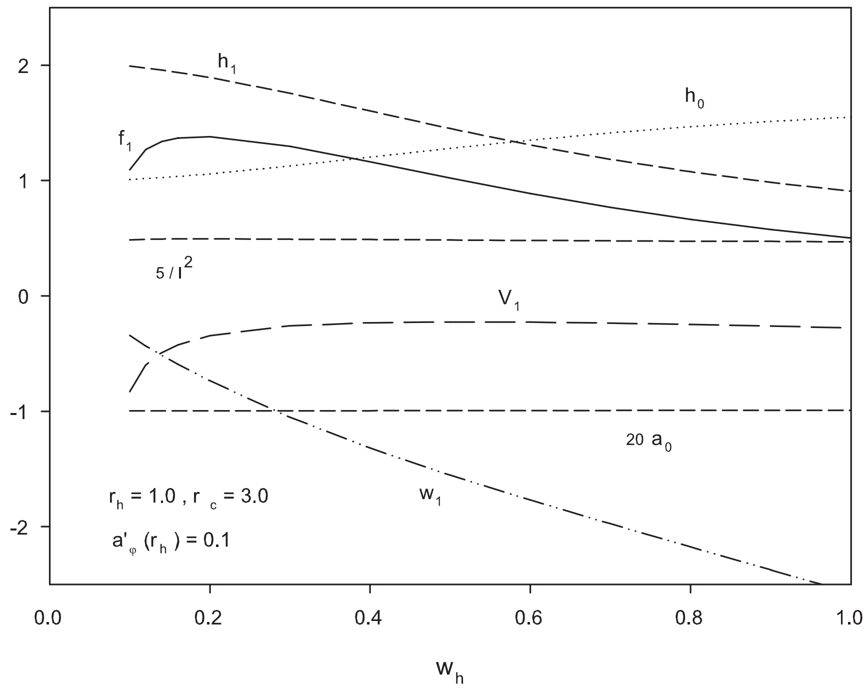

FIG. 3: Some numerical parameters characterizing the solution at the horizon as functions of w_h for $a'_h = 0.1$

III. BLACK STRINGS

Black strings solutions can also be constructed in the framework of the model (1) but we will limit our investigations to pure gravity in this case. One of the spacelike dimensions of space-time, say $z \equiv x_{d-1}$ plays a special role in a sense that the metric is the warped product of a $d-1$-dimensional black hole metric with the extra dimension. The corresponding horizon has the topology of $S_{d-3} \times S_1$ [3]. In the case of non-rotating, uniform black strings, the fields do not depend on the coordinate z and the metric takes the form

$$ds^2 = a(r)dz^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{d-3} - b(r)dt^2 \tag{25}$$

where $d\Omega^2_{d-3}$ denotes the metric on sphere S^{d-3}. Solutions of the corresponding Einstein equations are constructed in [5] for $\Lambda < 0$ and in [18] for $\Lambda > 0$. For d even, the part of the metric related to the $d-3$-dimensional sphere can be deformed according to the lines of Eq. (2) and rotating black strings can be constructed. The equations can be found in Eqs.(4.2) of the recent [6] by changing $\ell^2 \to -\ell^2$ but we will write them here for completenes. A convenient metric gauge choice in the numerical procedure is $h(r) = r^2$. In this gauge, the

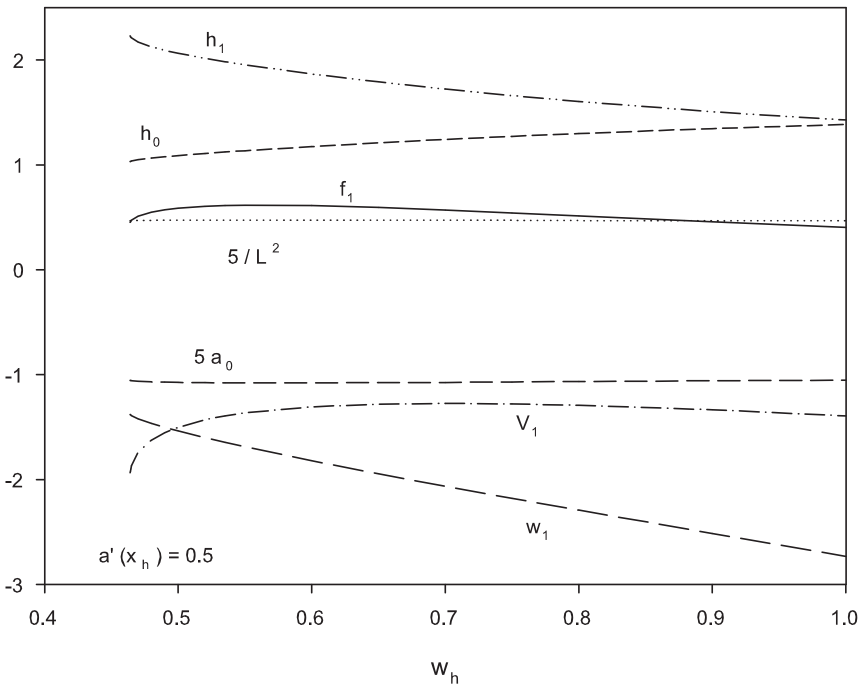

FIG. 4: Some numerical parameters characterizing the solution at the horizon as functions of w_h for $a'_h = 0.5$

field equations read :

$$\frac{a'}{a} = -\left[2\ell^2 fg\left(rgb' + b(2g + (d-4)rg')\right)\right]^{-1}\left[b\Big(4(d-1)(d-2)rg^2 - 4(d-4)\ell^2 g\big((d-2)r - fg'\big) + (d-4)r\ell^2(4r^2 + (d-5)fg'^2)\Big) + 2\ell^2 fg((d-4)rb'g' + g(2b' + r^3w'^2))\right],$$

$$f' = \frac{1}{d-2}\Bigg(\frac{(d-4)(2d-3)r^3}{g^2} - \frac{(d-4)(d-2)r}{g} - \frac{(d-1)(d-2)r}{\ell^2} + (\frac{a'}{a} + \frac{b'}{b})((d-4)\frac{rg'}{2g} - d + 3)f + \frac{rfa'b'}{2ab} - \frac{(d-3)(d-4)fg'}{g} + \frac{(5-2d)r^3 f}{2b}w'^2 + \frac{(d-5)(d-4)rf}{4g^2}g'^2\Bigg), \quad (26)$$

$$b'' = \frac{1}{d-2}\Bigg(\frac{(d-4)(d-5)b}{4g^2}g'^2 + \frac{(2d-3)r^2}{2}w'^2 - \frac{(d-3)(d-4)}{2g}b'g' + \frac{(d-4)b}{2ag}a'g' - \frac{(d-2)}{2f}b'f' + \frac{(d-2)b'^2}{2b} - \frac{(d-3)}{2a}a'b' + \frac{ba'}{ra} - \frac{(d-3)b'}{r} + \frac{(d-4)bg'}{rg} + \frac{(d-4)r^2 b}{fg^2} - \frac{(d-2)(d-4)b}{fg} - \frac{(d-1)(d-2)b}{\ell^2 f}\Bigg) \quad (27)$$

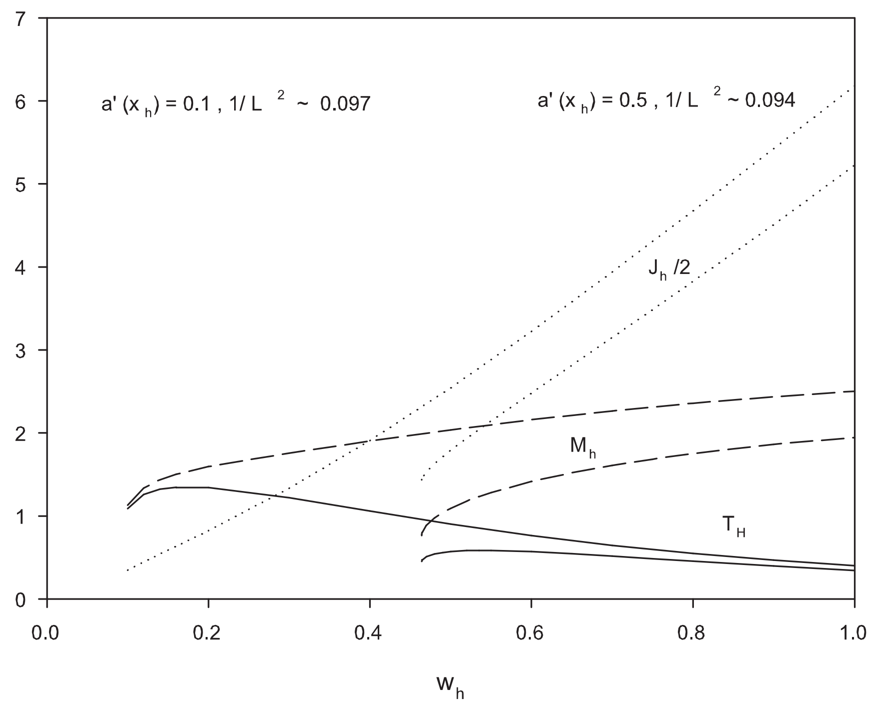

FIG. 5: The Hawking temperature at the event horizon T_H, the horizon mass M_h and the horizon angular momentum J_h are represented respectively in solid, dashed and dotted lines as functions of w_h for $a'_h = 0.1$ and $a'_h = 0.5$

$$g'' = \frac{1}{d-2}\Big(\frac{r^2 g}{2b}w'^2 - \frac{d^2-7d+4}{4g}g'^2 - (d-2)\frac{f'g'}{2f} + (\frac{a'}{a}+\frac{b'}{b})(\frac{g}{r}-g') + \frac{ga'b'}{2ab} - \frac{2g'}{r} + \frac{(4-3d)r^2}{fg} + \frac{d(d-2)}{f} - \frac{(d-1)(d-2)g}{\ell^2 f}\Big),$$

$$(g^{\frac{d-4}{2}}\sqrt{\frac{afh^3}{b}}w')' = 0\ .$$

The last equation in the relations above implies the existence of the first integral

$$w' = \alpha g^{-\frac{d-4}{2}}\sqrt{\frac{b}{afh^3}}, \tag{28}$$

where α is a constant controling the total angular momentum J of the solutions. For the numerical analysis, we fix the arbitrary rescaling of the radial variable by demanding $\ell^2 = 2000$.

For later use, we give here the expression of the Kretschmann invariant with the metric (25). The expression of this quantity turn out to be particularly simple :

$$K = \frac{d-3}{r^4}(r^2(f')^2 + 2(d-4)(f-1)^2) \tag{29}$$

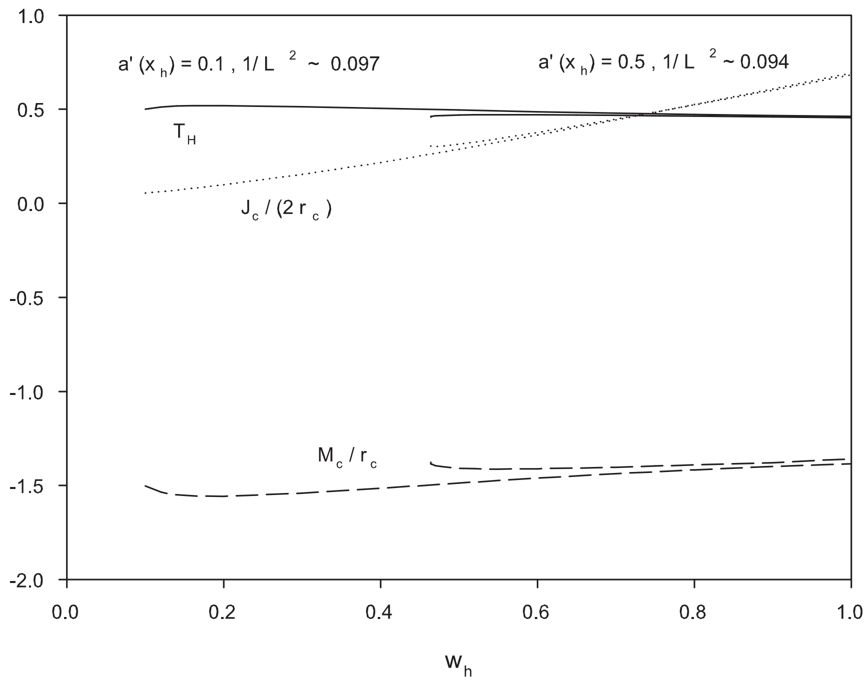

FIG. 6: The Hawking temperature at the cosmological horizon T_H, the horizon mass M_c and the horizon angular momentum J_c are represented respectively in solid, dashed and dotted lines as functions of w_h for $a'_h = 0.1$ and $a'_h = 0.5$

A. Boundary Conditions and asymptotic behaviour

In order to construct black string solutions with the equations above, we have to impose the appropriate boundary conditions at the horizon $r = r_h$ which we require to be regular. For this purpose, we set

$$f(r_h) = 0 \ , \ b(r_h) = 0 \ , \ a(r_h) = 1 \ , \ b'(r_h) = 1 \ , \ w(r_h) = w_h \tag{30}$$

plus another condition ensuring that the equation for g is regular at the horizon that we do not write here (it is Eq.(4.5) of [6]). The first condition above is necessary to produce a "'black"' object, the second one is necessary for regularity of the equation for b. The third and fourth conditions fix the arbitrary scales of the functions a and b. Finally the last condition involves an arbitrary parameter w_h which control the angular velocity of the black string at the horizon. The Horizon mass and angular velocity of the solution can, again, be determined with suitable Komar integrals, leading to

$$M_h = \sqrt{\frac{f}{bhg^2}}(b' - hww') \quad , \quad J = 2\sqrt{\frac{afg^2h^3}{b}}w'|_{r=r_h} \tag{31}$$

The complete specification of the boundary values needs three extra conditions which have to be looked for in the asymptotic region. Examining the asymptotic behaviour compatible with the classical equations reveals at least two possibilities. The first one corresponds to the metric of a De Sitter space-time

$$a(r) = -\frac{r^2}{\ell^2} \quad , \quad b(r) = -\frac{r^2}{\ell^2} \quad , \quad f(r) = -\frac{r^2}{\ell^2} \quad , \quad g(r) = r^2 \quad , \quad w(r) = J(\frac{l}{r})^{d-2} \tag{32}$$

more details about the corresponding asymptotic expansion of this solution are given in [6]. Correspondingly, the Kretschmann invariant approached a constant asymptotically. However (32) is not the only possibility, there exist a second one where the fields decay power likely according to

$$a(r) = A(\frac{r^2}{\ell^2})^{\alpha/2} \quad , \quad b(r) = B(\frac{r^2}{\ell^2})^{\beta/2} \quad , \quad f(r) = F(\frac{r^2}{\ell^2})^{\phi/2} \quad , \quad g(r) = r^2 \quad , \quad w(r) = \Omega\, \ell^4(\frac{\ell^2}{r^2})^{\omega/2} \tag{33}$$

with

$$\alpha = \beta = -2(d-3) - \sqrt{2(d-2)(d-3)}\,, \quad \phi = 2(d-2) + 2\sqrt{2(d-2)(d-3)}\ , \quad \omega = 2 + \frac{\phi}{4} \tag{34}$$

This leads to a singularity of the Kretschmann invariant in the limit $r \to \infty$ for $d > 3$. More details about the asymptotic expansion of (33) are presented in the next section.

B. Non rotating solution

In the case $w = 0$, the equation for w is trivial and the equation for g is satisfied by $g = r^2$. We will show that the remaining equations can be written in a decoupled form. We consider both $\Lambda > 0$ and $\Lambda < 0$; to this use, we define $\epsilon = -\Lambda/|\Lambda|$.

First, let us define

$$A(r) = \frac{ra'}{a} \quad B(r) = \frac{rb'}{b} \tag{35}$$

in terms of which the equations for f, a, b rewrite :

$$rf' = 2(d-4)\,(k-f) - 2(d-1)\frac{r^2}{\epsilon l^2} - f(r)\,(A+B) \tag{36}$$

$$rB'f = (d-1)(2-B)\frac{r^2}{\epsilon l^2} - (d-4)kB = 0 \tag{37}$$

$$(AB + 2(d-3)(A+B) - 2(d-3)(d-4))\,f = 2k(d-3)(d-4) - 2(d-1)(d-2)\frac{r^2}{\epsilon l^2} \tag{38}$$

Solving the third equation for $A = A(B, f)$, substituting $A(B, f)$ in the first equation and solving for $B = B(f, f')$ and finaly evaluating the second equation with $A = A(B(f, f'), f),\ B = B(f, f')$ leads to the following decoupled equations :

$$r^2 f f'' = -r f f' + (r f')^2 + 2\left((d-1) - (d-4)\frac{r^2}{\epsilon l^2}\right)^2 \tag{39}$$
$$- \; 2(d-4)\left((d-1) - (d-4)\frac{r^2}{\epsilon l^2}\right) f - 3\left(k(d-1) - (d-4)\frac{r^2}{\epsilon l^2}\right) r f'$$

$$B_\pm = -\tilde{B}(r,d,l) \tag{40}$$
$$\pm \sqrt{\tilde{B}(r,d,l)^2 - 2(d-3)(d-4)\left(1 - \frac{1}{f}\right) - 2(d-3)\frac{r f'}{f} - 2(d-1)(d-4)\frac{r^2}{\epsilon l^2}\frac{1}{f}}$$

where $\tilde{B}(r,d,l) \equiv \left((d-4) + \frac{1}{2}\frac{rf'}{f} + (d-4)\frac{1}{f} - (d-1)\frac{1}{f}\frac{r^2}{\epsilon l^2}\right)$

$$A = \frac{-2(d-1)(d-2)\frac{r^2}{\epsilon l^2} + 2(d-3)(d-4)f + 2(d-3)fB}{2(d-3)f + fB} \tag{41}$$

Note that the decoupling is still valid in the more general case considered in [5] where topological black holes are investigated as well (i.e. space-times where the spherical part $d\omega_{d-3}^2$ in (25) is replaced by the metric of an hyperbolic or flat manifold with the same dimensions)

C. Asymptotics

We have not been able to solve (40) explicitely, but we can study the asymptotic behavour of f with this equation. In fact, at first order, it appears that

$$f(r) \approx F_0 r^\phi \tag{42}$$

for a constant F_0 and for any $\phi \in \{2\} \cup]4, \rightarrow$, so we cannot conclude on the asymptotic behavour of f at this point. In fact, it is not possible to fix analytically ϕ, but it is possible to give a description of the asymptotic behavour of the metric functions in terms of only one parameter.

We have to consider two cases : $\phi = 2$ and $\phi > 4$. In the case $\epsilon = +1$, where the cosmological constant is negative, asymptotics obey $\phi = 2$ [5]. In the case $\epsilon = -1$, there are strong evidence that $\phi = 2$ leads to singular solutions [18]. Since this paper focus on the case $\epsilon = -1$, we will assume $\phi > 4$, which is equivalent to assume that the argument in five dimensions generalises to d dimensions.

It turns out that the asymptotic behavour of the metric functions is given by a one parameter family of functions:

$$f \rightarrow F_0 r^\phi \ , \ \phi = -\frac{\beta^2 + 2(d-4)\beta + 2(d-3)(d-4)}{\beta + 2(d-3)} \tag{43}$$

$$A(r) \to \alpha(\beta) = -\frac{2(d-3)(d-4) + 2(d-3)\beta}{\beta + 2(d-3)} \tag{44}$$

$$B(r) \to \beta \tag{45}$$

Since these behavour are compatible with the equations of motion for each value of β such that $\phi > 4$, it is not possible to fix β without more asomptions, but there are strong numerical evidences for $a(r) = b(r)$ at first order in the asymptotic region. This means that the value of β chosen by the system is a fixed point of $\alpha(\beta)$ are exactly the values (34). Note the particular relation between the exponents :

$$\alpha + \beta + \phi + 2(d-4) = 0 \tag{46}$$

We have constructed the higher order corrections for the function $f(r)$ and obtained

$$f(r) = F_0 r^{\phi} - \frac{d-1}{d-3+\beta}\frac{r^2}{l^2} + \frac{d-4}{d-4+\beta} \tag{47}$$

with ϕ discussed previously. The reparametrisation of F_0 used in (33) figure out the peculiar dependance of the solution under rescaling of ℓ and r

D. Energy of the solution

The energy of the solution is given by the following quantity [21]

$$E = -2M_P^{d-2}\int_{S_t^{\infty}} N\left(K_{d-2} - K_{0,d-2}\right) \tag{48}$$

where M_P is the Planck Mass, N being the full spacetime lapse function.

The trace extrinsic curvature of the border at fixed time, $K_{(d-2)}$ is well defined, although space-time is asymptotically singular :

$$K_{(d-2)} = \sqrt{f}\left(\frac{(d-3) + \frac{\beta}{2}}{r}\right) \tag{49}$$

This quantity, once integrated according to (48), is not divergent. That's the reason why we won't consider a reference background. Moreover, it is not clear wich background to use with such an asymptotic.

We can now compute the energy of the solutions which exhibits an asymptotic behavour as described in the previous section. By (46), the energy is simply given by

$$E = M_P^{d-2}\mathcal{A}_{d-3}L\sqrt{2F_0(d-2)(d-3)} \tag{50}$$

where L is the length of the extradimension in the y direction and $\mathcal{A}_{d-3}$ is the area of the unit $d-3$ sphere. This energy is finite and depend only on F_0 wich depends essentially on the dimension of space-time and of the cosmological constant, further results will be given in Sect. 3.7.

E. Geodesic Equations and Curvature Invariant

We computed the Kreshman curvature invariant (29): and it turns out that the spacetime is asymptotically singular, with the asymptotic behavour of previous section.
Moreover, the geodesic equation at fixed θ_i (the angular sector of the metric),

$$\dot{r}^2 = \frac{\mathcal{E}^2}{b} - \frac{\mathcal{Z}^2}{a} - m^2 \tag{51}$$

where $\mathcal{E}, \mathcal{Z}$ are conserved quantities along the geodesic, implies that this singularity can be reached in a finite proper time for an observer of mass m from outside the horizon. A similar result holds for null motion.

F. Numerical Results

We solved numerically equations (35), (36) and (37) with the boundary conditions for many values of the horizon r_h. It turns out that in the regular case, the asumption $a = b$ holds everywhere, independently of the number of dimensions. This can be understood since a and b both play a "spectator role" in the metric. Nothing explicitely depends on z, neither on t in the metric, so in a sense, a and b play the same role, at least in the regular case, since the initial values for a and b are the same On fig.6, we present the evolution of the ratio rf'/f for non rotating black string solutions and for several values of d and $r_h = 0.5$. The figure clearly demonstrates that the power law configuration (33) is approached. The exact values of the exponents coincide with our numerical values within the numerical accuracy required for the numerics, i.e. typically 10^{-8}.
However, nothing garanties that this asumption will still hold true in the black string case, because the boundary conditions break this "symetry". We checked numerically the validity of the asumption $a = b$ in the asymptotic region and it turned out that it is still verified. Moreover, we noticed that the ratio a/b behaves like $1/r_h$ for the normalisation of b we used. This factor can be absorbed in a rescalling of t. It is reasonable to think that a natural rescaling of t is such that $a/b \to C$, with C independant of r_h. This implies that $b'(r_h) = f'(r_h)$ wich reminds the 4-dimensionnal Schwarzchild (with or without cosmological constant).

We also investigated the solution in the interior region, i.e. for $r \leq r_h$, to check whether there exist a second horizon which would "'hide the asymptotic singularity"', but it turns out that this is not the case. We have a naked singularities both, at $r = 0$ and at $r = \infty$. We are tempted to interpret this solution as an hypercylindre with an horizon at some equator by matching the origin with the point at infinity by means of an appropriate system of coordinates, still to be found, but that we expect to exist.

G. Rotating black strings

In absence of explicit solutions, we have integrated numerically the equations for rotating black strings for $r \in [r_h, \infty]$ for several values of r_h and w_h by trying to interpolate between the local behaviour (30) and one of the possible asymptotic behaviours (32) or (33). Here we solved the equations for $d = 6$ but the results obtained for non rotating black string for $d > 6$

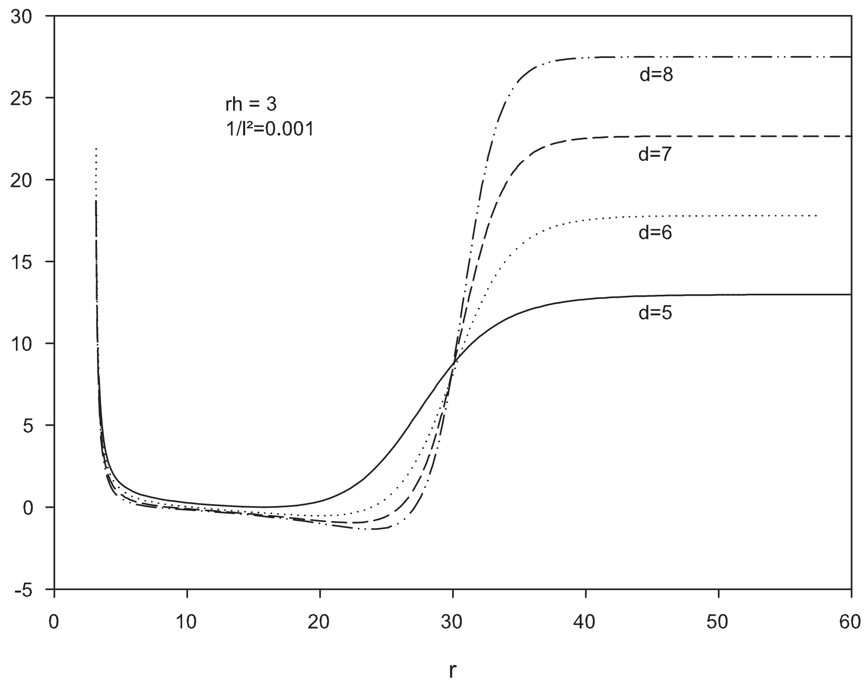

FIG. 7: The ratio xf'/f is given for several values of d

strongly suggest that rotating solutions exist also for higher dimensions. As can be expected from the non rotating case, our numerical results strongly suggest that the rotating solutions behaving regularly at the event horizon naturally evolve into the asymptotics determined by (33). The profiles of a rotating black string corresponding to $r_h = 0.5, w_h = 0.5$ is presented of Fig. 8. The metric functions profile are shown on this figure. We supplemented $g_{tt} = b - r^2\omega^2$, showing that there is a small ergoregion about the horizon where $g_{tt} < 0$ for $r_h \leq r \leq r_e$ where r_e denotes the ergo-horizon (on the figure $r_e \approx 0.537$). For this solution we further found $F \approx 1.125$, $A \approx 0.0741$, $B \approx 0.0186$ and $\Omega \approx 0.0073$ for the parameters defined in (33).

We also studied the dependance of different parameters characterizing the solution as functions of r_h and w_h. One main feature is that the values of F, A, B depend weakly of these parameters (e.g. for $w_h = 0.5, r_h = 0.8$ we find $F \approx 1.125, A \approx 0.0738, B \approx 0.0260$). The dependance of Ω on w_h is more sensible. On fig. 9 we have superposed Ω, f_1, w'_h and the ergo-horizon r_e as functions of w_h for $r_h = 0.5$. The qualititive behaviour of these parameters remains similar for different values of r_h. We noticed that, while increasing w_h, the function $w(r)$ becomes very peaked at the horizon, with a large derivative $w'(x_h)$. This renders the numerical integration tricky in this region but we have not detected a signal of a second branch of rotating black string.

Before finishing this section, we would like to mention that an analytic solution of the Lagrangian under consideration is available in the case $d = 4$ [26], see also [27] where this solution is interpreted in the context of gravitating cosmic strings. This solution was obtained with a different parametrisation of the metric, the correspondic space-time becomes periodic in the radial variable; the Kretschmann scalar possesses singularities, so this solution is clearly not of the DeSitter type asymptotically.

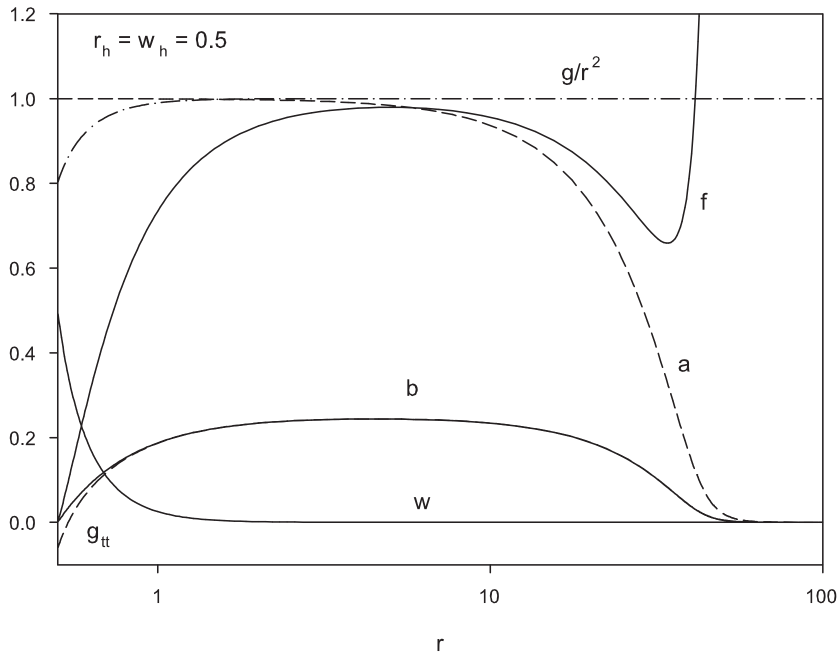

FIG. 8: The profiles of a rotating black string for $r_h = w_h = 0.5$

IV. SUMMARY

In this paper, we have studied the Einstein-Maxwell equations in space-times of arbitrary dimensions d and with a positive cosmological constant. By using appropriate ansatzes for the metric and the U(1)-fields, the equations can be transformed into a system of ordinary differential equations for odd values of d in the case of black holes and even d for black strings. Imposing a consistent set of boundary conditions, we solved these equation numerically, constructed several families of rotating black holes and black string solutions. For both cases, the conditions of an event horizon are imposed ar $r = r_h$. The main difference between black holes and black strings resides in the way space-times reaches its asymptotic form. In the case of black holes, space-time becomes asymptotically DeSitter, after crossing a regular cosmological horizon. In the case of black strings, our numerical results strongly suggest that the metric fields evolve asymptotically according to some power of the radial variable r with well specified non integer exponants depending on d. The evaluation of the Kretschmann invariant reveals that space-time become singular in the limit $r \to \infty$ for $d \geq 5$. In order to confirm these numerical results by an analytic argument, we manage to put the equations of [4, 5] in a decoupled form which makes slighly easier to construct the next to leading order of the asymptotic evolution of the fields. Althought we were not able to give an exact solutions to this decoupled form, we found interesting informations on the behavour of the solutions. We also showed that the energy of this kind of solution is finite.

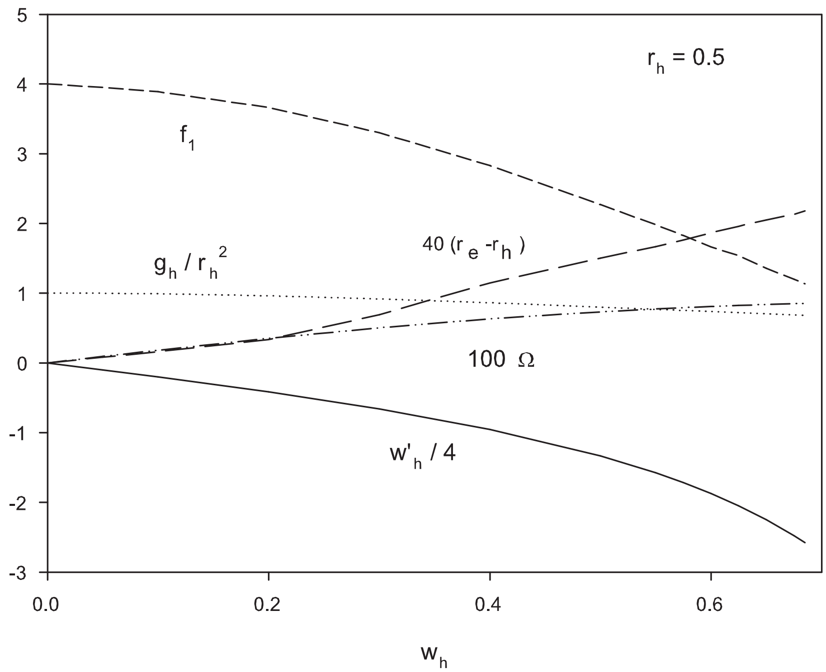

FIG. 9: The values of g, f', w' at the horizon, of the ergo radius and the parameter Ω are presented as functions of w_h for $r_h = 0.5$

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