Meet Your Expectations With Guarantees: Beyond Worst-Case Synthesis in Quantitative Games

V. Bruyère (UMONS) E. Filiot (ULB)
M. Randour (UMONS-ULB) J.-F. Raskin (ULB)

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LSV seminar

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The talk in two slides (1/2)

- Verification and synthesis:
  - ▷ a reactive **system** to *control*,
  - ▷ an *interacting* **environment**,
  - ▷ a **specification** to *enforce*.

- Focus on *quantitative properties*.

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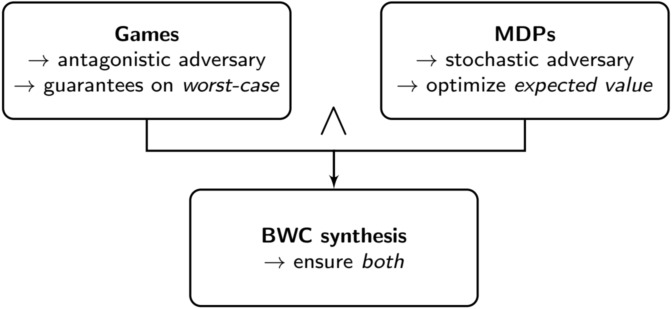
- Verification and synthesis:
  - ▷ a reactive **system** to *control*,
  - ▷ an *interacting* **environment**,
  - ▷ a **specification** to *enforce*.
- Focus on *quantitative properties*.
- Several ways to look at the interactions, and in particular, *the nature of the environment*.

The talk in two slides (2/2)

Games
→ antagonistic adversary
→ guarantees on *worst-case*

MDPs
→ stochastic adversary
→ optimize *expected value*

The talk in two slides (2/2)



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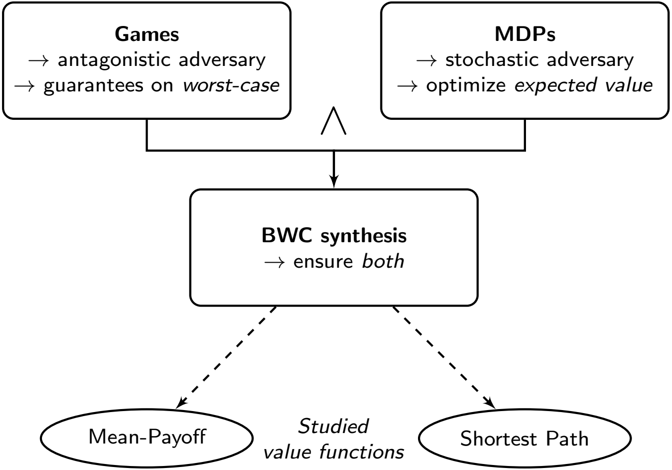

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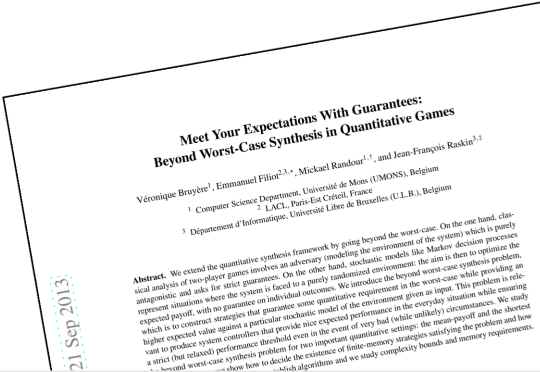
Full paper available on arXiv: abs/1309.5439



Meet Your Expectations With Guarantees:
Beyond Worst-Case Synthesis in Quantitative Games

Véronique Bruyère[1], Emmanuel Filiot[2,3,*], Mickael Randour[1,†], and Jean-François Raskin[3,‡]

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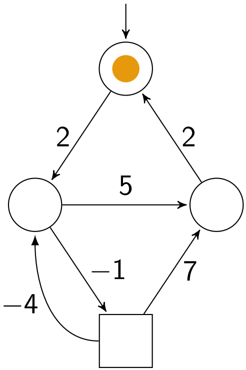
Abstract. We extend the quantitative synthesis framework by going beyond the worst-case. On the one hand, classical analysis of two-player games involves an adversary (modeling the environment of the system) which is purely antagonistic and asks for strict guarantees. On the other hand, stochastic models like Markov decision processes represent situations where the system is faced to a purely randomized environment: the aim is then to optimize the expected payoff, with no guarantee on individual outcomes. We introduce the beyond worst-case synthesis problem, which is to construct strategies that guarantee some quantitative requirement in the worst-case while providing an higher expected value against a particular stochastic model of the environment given as input. This problem is relevant to produce system controllers that provide nice expected performance in the everyday situation while ensuring a strict (but relaxed) performance threshold even in the event of very bad (while unlikely) circumstances. We study the beyond worst-case synthesis problem for two important quantitative settings: the mean-payoff and the shortest ... show how to decide the existence of finite-memory strategies satisfying the problem and how ... algorithms and we study complexity bounds and memory requirements.

21 Sep 2013

Quantitative games on graphs



- Graph $\mathcal{G} = (S, E, w)$ with $w\colon E \to \mathbb{Z}$
- Two-player *game* $G = (\mathcal{G}, S_1, S_2)$
  - ▷ $\mathcal{P}_1$ states = ◯
  - ▷ $\mathcal{P}_2$ states = □
- Plays have values
  - ▷ $f\colon \mathsf{Plays}(\mathcal{G}) \to \mathbb{R} \cup \{-\infty, \infty\}$
- Players follow *strategies*
  - ▷ $\lambda_i\colon \mathsf{Prefs}_i(G) \to \mathcal{D}(S)$
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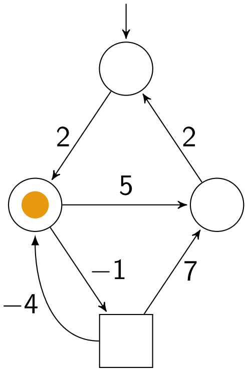

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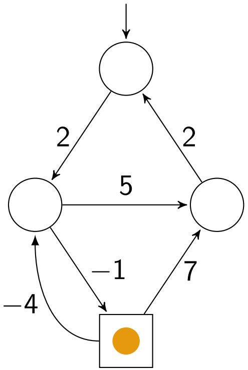

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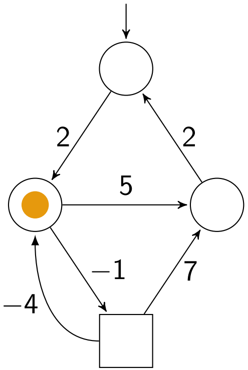

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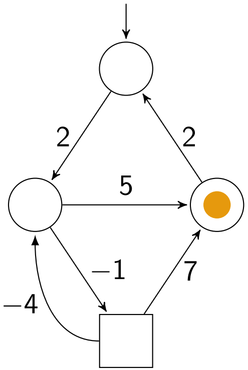

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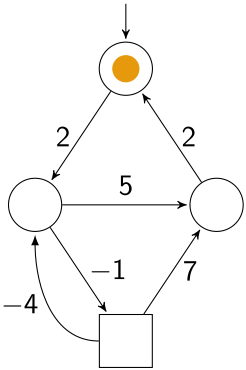

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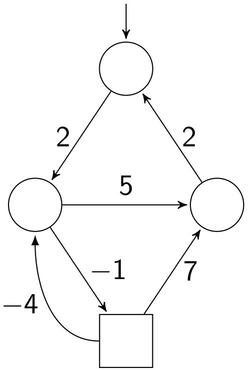

Then, $(2, 5, 2)^\omega$

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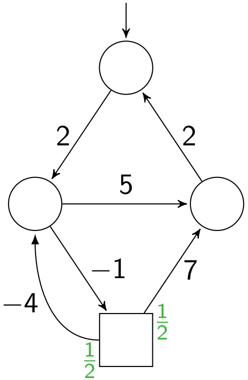
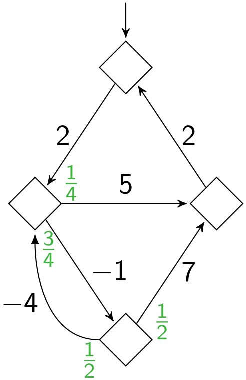
Markov decision processes



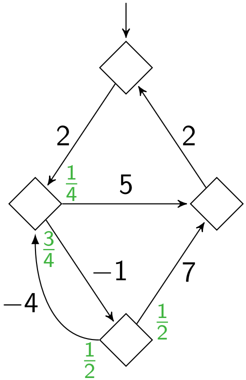
- MDP $P = (\mathcal{G}, S_1, S_\Delta, \Delta)$ with $\Delta \colon S_\Delta \to \mathcal{D}(S)$
  - ▷ $\mathcal{P}_1$ states = ◯
  - ▷ stochastic states = □
- MDP = game + strategy of $\mathcal{P}_2$
  - ▷ $P = G[\lambda_2]$

Markov chains



- MC $M = (\mathcal{G}, \delta)$ with $\delta \colon S \to \mathcal{D}(S)$
- MC = MDP + strategy of $\mathcal{P}_1$
  = game + both strategies
  - ▷ $M = P[\lambda_1] = G[\lambda_1, \lambda_2]$

Markov chains



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- MC = MDP + strategy of $\mathcal{P}_1$
  = game + both strategies
  - ▷ $M = P[\lambda_1] = G[\lambda_1, \lambda_2]$
- Event $\mathcal{A} \subseteq \text{Plays}(\mathcal{G})$
  - ▷ probability $\mathbb{P}^{M}_{s_{\text{init}}}(\mathcal{A})$
- Measurable $f \colon \text{Plays}(\mathcal{G}) \to \mathbb{R} \cup \{-\infty, \infty\}$
  - ▷ *expected value* $\mathbb{E}^{M}_{s_{\text{init}}}(f)$

Classical interpretations

- **System** trying to ensure a specification $= \mathcal{P}_1$
  - ▷ whatever the actions of its **environment**

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- The environment can be seen as
  - ▷ *antagonistic*
    - two-player game, *worst-case* threshold problem for $\mu \in \mathbb{Q}$
    - $\exists ?\, \lambda_1 \in \Lambda_1, \forall\, \lambda_2 \in \Lambda_2, \forall\, \pi \in \mathsf{Outs}_G(s_{\mathsf{init}}, \lambda_1, \lambda_2),\ f(\pi) \geq \mu$

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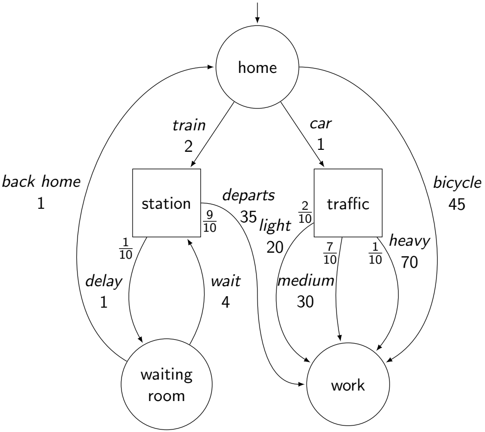
  - ▷ *fully stochastic*
    - MDP, *expected value* threshold problem for $\nu \in \mathbb{Q}$
    - $\exists ?\, \lambda_1 \in \Lambda_1,\ \mathbb{E}^{P[\lambda_1]}_{s_{\mathsf{init}}}(f) \geq \nu$

What if you want both?

In practice, we want both

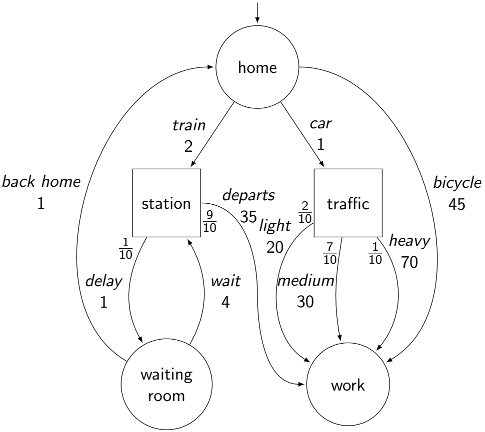
1. nice expected performance in the everyday situation,
2. strict (but relaxed) performance guarantees even in the event of very bad circumstances.

Example: going to work



- ▷ Weights = minutes
- ▷ Goal: *minimize our expected time* to reach "work"
- ▷ **But**, important meeting in one hour! Requires *strict guarantees* on the worst-case reaching time.

Example: going to work



- Optimal expectation strategy: take the car.
  - $\mathbb{E} = 33$, $\text{WC} = 71 > 60$.
- Optimal worst-case strategy: bicycle.
  - $\mathbb{E} = \text{WC} = 45 < 60$.

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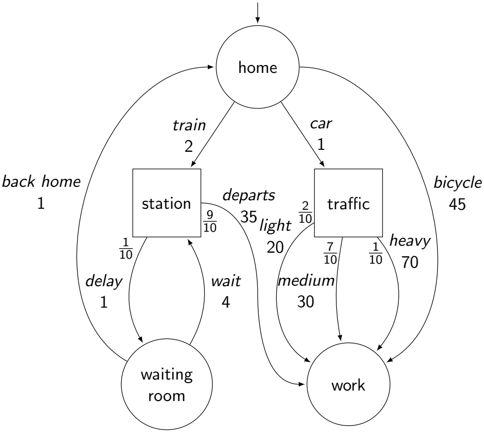

- ▷ Optimal expectation strategy: take the car.
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- ▷ Optimal worst-case strategy: bicycle.
  - $\mathbb{E} =$ WC $= 45 < 60$.
- ▷ **Sample BWC strategy**: try train up to 3 delays then switch to bicycle.
  - $\mathbb{E} \approx 37.56$, WC $= 59 < 60$.
  - Optimal $\mathbb{E}$ under WC constraint
  - Uses finite **memory**

Beyond worst-case synthesis

Formal definition

Given a game $G = (\mathcal{G}, S_1, S_2)$, with $\mathcal{G} = (S, E, w)$ its underlying graph, an initial state $s_{\mathsf{init}} \in S$, a finite-memory stochastic model $\lambda_2^{\mathsf{stoch}} \in \Lambda_2^F$ of the adversary, represented by a stochastic Moore machine, a measurable value function $f\colon \mathsf{Plays}(\mathcal{G}) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$, and two rational thresholds $\mu, \nu \in \mathbb{Q}$, the *beyond worst-case (BWC) problem* asks to decide if $\mathcal{P}_1$ has a finite-memory strategy $\lambda_1 \in \Lambda_1^F$ such that

$$\begin{cases} \forall\, \lambda_2 \in \Lambda_2,\ \forall\, \pi \in \mathsf{Outs}_G(s_{\mathsf{init}}, \lambda_1, \lambda_2),\ f(\pi) > \mu & (1) \\ \mathbb{E}_{s_{\mathsf{init}}}^{G[\lambda_1, \lambda_2^{\mathsf{stoch}}]}(f) > \nu & (2) \end{cases}$$

and the *BWC synthesis problem* asks to synthesize such a strategy if one exists.

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Notice the highlighted parts!

Mean-payoff value function

- $\mathrm{MP}(\pi) = \liminf_{n\to\infty} \left[\frac{1}{n} \cdot \sum_{i=0}^{i=n-1} w\big((s_i, s_{i+1})\big)\right]$

- Sample play $\pi = 2,\ -1,\ -4,\ 5,\ (2,\ 2,\ 5)^\omega$
  - ▷ $\mathrm{MP}(\pi) = 3$
  - ▷ long-run average weight $\rightsquigarrow$ *prefix-independent*

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Games: worst-case threshold problem [LL69, EM79, ZP96, Jur98, GS09]

Memoryless optimal strategies exist for both players and the problem is in $\mathsf{NP} \cap \mathsf{coNP}$.

MDPs: expected value threshold problem [Put94, FV97]

Memoryless optimal strategies exist and the problem is in P.

BWC MP problem: overview

Theorem (algorithm & complexity)

The BWC problem for the mean-payoff is in $\mathbf{NP} \cap \mathbf{coNP}$ *and at least as hard as deciding the winner in mean-payoff games.*

- Additional modeling power for free!

BWC MP problem: overview

Theorem (algorithm & complexity)

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▷ Additional modeling power for free!

Theorem (memory bounds)

Memory of **pseudo-polynomial** *size may be necessary and is always sufficient to satisfy the BWC problem for the mean-payoff: polynomial in the size of the game and the stochastic model, and polynomial in the weight and threshold values.*

Philosophy of the algorithm

- Classical worst-case and expected value results and algorithms as *nuts and bolts*
- *Screw them together* in an adequate way

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Three key ideas

1. To characterize the expected value, look at *end-components* (ECs)
2. *Winning ECs* vs. *losing ECs*: the latter must be avoided to preserve the worst-case requirement!
3. *Inside a WEC*, we have an interesting way to play. . .

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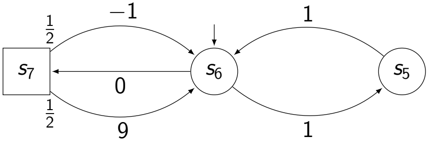
$\Longrightarrow$ **Let's go bottom-up, starting from an ideal case**

An ideal situation



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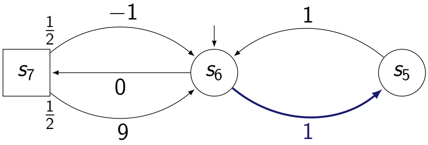
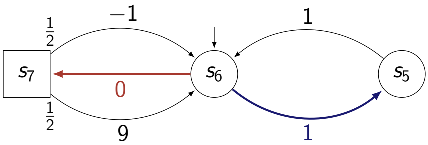


Game interpretation

- ▷ Worst-case threshold is $\mu = 0$
- ▷ **All** states are winning: memoryless optimal worst-case strategy $\lambda_1^{wc} \in \Lambda_1^{PM}(G)$, ensuring $\mu^* = 1 > 0$

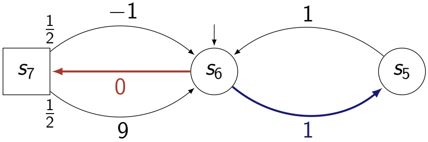
An ideal situation



MDP interpretation

- ▷ All states are reachable with probability one (even surely)
- ▷ The highest achievable expected value is the same in all states: $\nu^* = 2$
- ▷ Memoryless optimal expected value strategy $\lambda_1^e \in \Lambda_1^{PM}(P)$

A cornerstone of our approach



BWC problem: **what kind of threholds $(0, \nu)$ can we achieve?**

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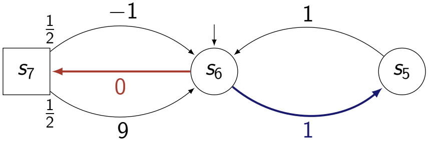


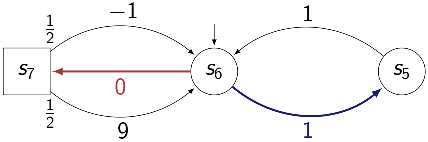
BWC problem: **what kind of threholds $(0, \nu)$ can we achieve?**

Key result

For all $\varepsilon > 0$, there exists a finite-memory strategy of $\mathcal{P}_1$ that satisfies the BWC problem for the thresholds pair $(0, \nu^* - \varepsilon)$.

▷ We can be **arbitrarily close to the optimal expectation** while ensuring the worst-case!

Combined strategy



We define $\lambda_1^{cmb} \in \Lambda_1^{PF}$ as follows, for some well-chosen $K, L \in \mathbb{N}$.

(a) Play λ_1^{e} for K steps and memorize $\mathsf{Sum} \in \mathbb{Z}$, the sum of weights encountered during these K steps.

(b) If $\mathsf{Sum} > 0$, then go to *(a)*.
Else, play λ_1^{wc} during L steps then go to *(a)*.

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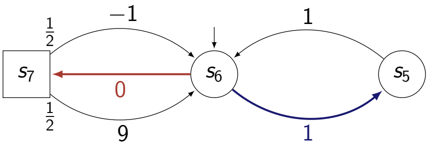

Intuitions

- *Phase (a)*: try to increase the expectation and approach the optimal one
- *Phase (b)*: compensate, if needed, losses that occured in *(a)*

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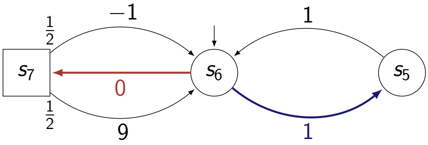

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Proving the strategy is up to the job requires some technical work, but let's review the *key ideas*

- $\exists\, K, L \in \mathbb{N}$ for any thresholds pair $(0, \nu^* - \varepsilon)$
- plays = sequences of periods starting with phase *(a)*

Combined strategy: worst-case requirement

Does any consistent outcome have a strictly positive MP?

- $\forall\, K$, $\exists\, L(K)$, linear in K, s.t. *(a)* + *(b)* has
  MP $\geq 1/(K + L) > 0$
  because $\mu^* = 1 > \mu = 0$
- Periods *(a)* induce MP $\geq 1/K$ (not followed by *(b)*)
- Weights are integers and period length bounded
  $\rightsquigarrow$ inequality remains strict for play

Combined strategy: expected value requirement

Can we ensure an ε-optimal expected value?

- When $K \to \infty$, $\mathbb{E}_{(a)} \to \nu^*$

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- As $K \to \infty$, we have $L(K) \to \infty$ (potentially bigger losses to compensate), which may prevent $\mathbb{E}_{(a)+(b)} \to \nu^*$
- But as $K \to \infty$, we also have $\mathbb{P}_{(b)} \to 0$: losses after period *(a)* are less probable
  - ▷ Intuition through a *Bernouilli process*

Bernouilli process

Assume our phase *(a)* is a simple fair coin tossing sequence with *heads* granting 1 and *tails* granting 0

- ▷ The expected MP is $1/2$ whatever the $\#$ of tosses
- ▷ Let $\varepsilon = 1/6$, what is the probability to witness an MP $> 1/2 - 1/6 = 1/3$ after K tosses?

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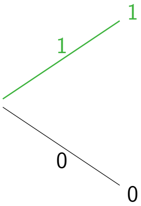

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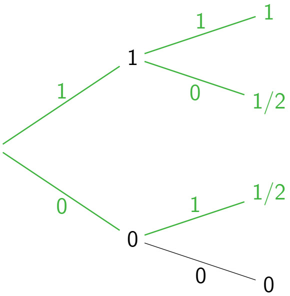

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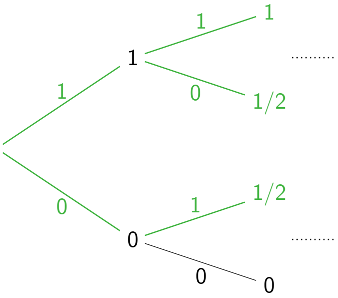
$$\vdots$$

for any $\varepsilon > 0$, when $K \to \infty$, it tends to one



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Using Chernoff bounds and Hoeffding's inequality for Markov chains [Tra09, GO02], we can bound the probability of being far from the optimal after K steps of *(a)* in our combined strategy

- $\mathbb{P}_{(b)}$ decreases exponentially while $L(K)$ only needs to increase polynomially
- The *overall contribution* of *(b)* tends to zero when $K \to \infty$
- Hence $\mathbb{E}_{(a)+(b)} \to \nu^*$ as claimed

The ideal case: wrap-up

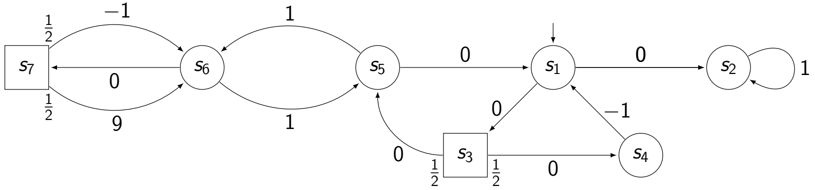
The combined strategy works in any subgame such that

1. it constitutes an EC in the MDP,
2. all states are worst-case winning in the subgame.

Such **winning ECs** (WECs) are the crux of BWC strategies in arbitrary games.

But to explain that, let's first zoom out and consider the big picture.

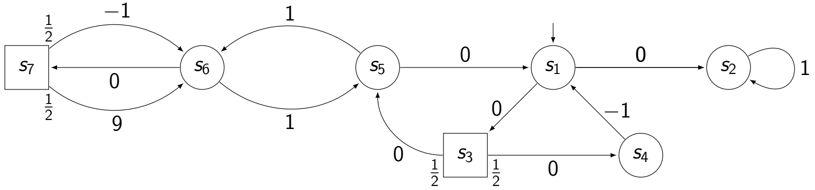
Zooming out



Arbitrary game, with ideal case as a subgame. We assume all states are worst-case winning.

- BWC strategies **must avoid** WC losing states at all times: an antagonistic adversary can force WC losing outcomes from there (due to prefix-independence)
- Some preprocessing can be done and in the remaining game, $\mathcal{P}_1$ has a **memoryless WC winning strategy** from all states

End-components: what they are



An **EC** of the MDP $P = G[\lambda_2^{\mathsf{stoch}}]$ is a subgraph in which $\mathcal{P}_1$ can ensure to stay despite stochastic states [dA97], i.e., a set $U \subseteq S$ s.t.

(i) $(U, E \cap (U \times U))$ is strongly connected,

(ii) $\forall\, s \in U \cap S_\Delta$, $\mathsf{Supp}(\Delta(s)) \subseteq U$, i.e., in stochastic states, all outgoing edges stay in U.

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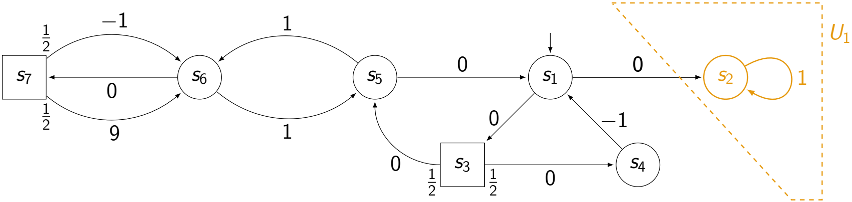

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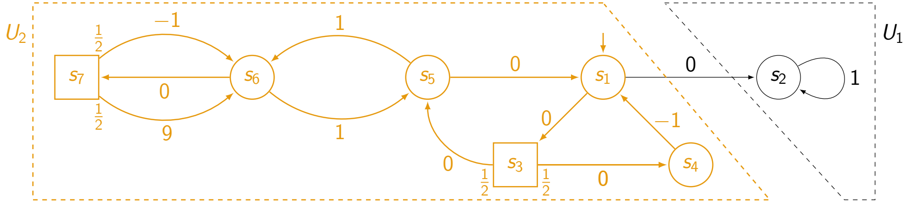

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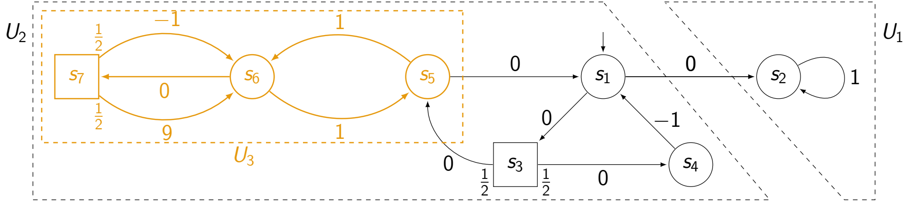

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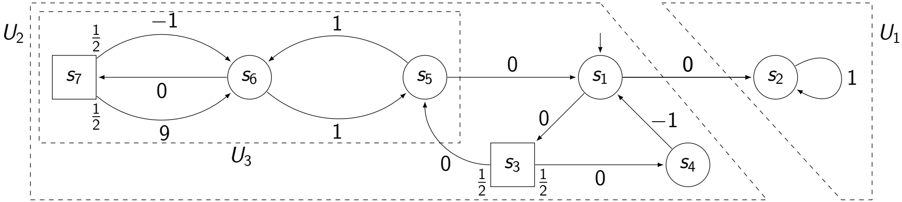

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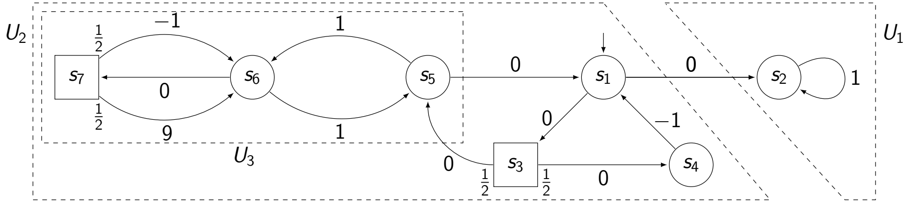

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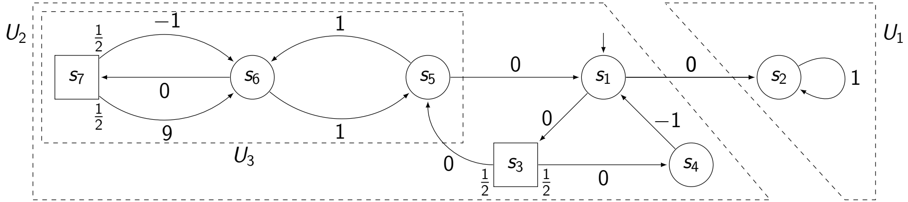
End-components: why we care



Lemma (Long-run appearance of ECs [CY95, dA97])

Let $\lambda_1 \in \Lambda_1(P)$ *be an* **arbitrary strategy** *of* $\mathcal{P}_1$*. Then, we have that*

$$\mathbb{P}^{P[\lambda_1]}_{s_{\mathsf{init}}}\left(\{\pi \in \mathsf{Outs}_{P[\lambda_1]}(s_{\mathsf{init}}) \mid \mathsf{Inf}(\pi) \in \mathcal{E}\}\right) = 1.$$

- ▷ By prefix-independence, only long-run behavior matters
- ▷ **The expectation on $P[\lambda_1]$ depends uniquely on ECs**

How to satisfy the BWC problem?

- *Expected value requirement*: reach ECs with the highest achievable expectations and stay in them
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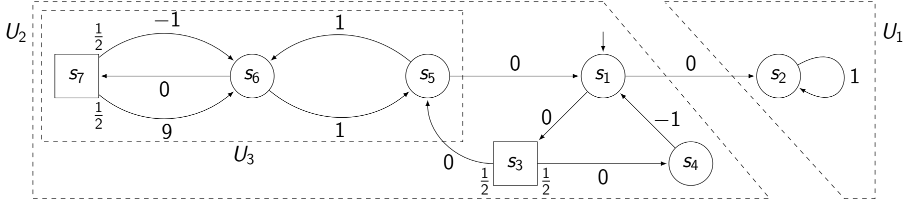
- *Expected value requirement*: reach ECs with the highest achievable expectations and stay in them
  - The optimal expected value is the same everywhere inside the EC [FV97], cf. ideal case
- *Worst-case requirement*: some ECs may need to be eventually avoided because risky!
  - The "ideal cases" are ECs but not all ECs are ideal cases...
  - Need to classify the ECs

Classification of ECs



▷ $U \in \mathcal{W}$, **the winning ECs**, if $\mathcal{P}_1$ can win in $G \upharpoonright U$, from all states:

$$\exists\, \lambda_1 \in \Lambda_1(G \upharpoonright U),\ \forall\, \lambda_2 \in \Lambda_2(G \upharpoonright U),\ \forall\, s \in U,\ \forall\, \pi \in \mathsf{Outs}_{(G \upharpoonright U)}(s, \lambda_1, \lambda_2),\ \mathsf{MP}(\pi) > 0$$

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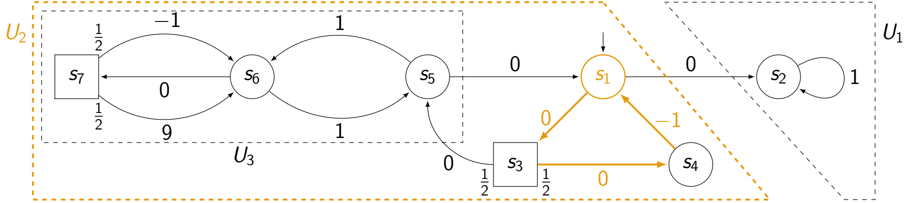

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- $\mathcal{W} = \{U_1, U_3, \{s_5, s_6\}, \{s_6, s_7\}\}$
- U_2 **losing**: from state s_1, $\mathcal{P}_2$ can force the outcome $\pi = (s_1 s_3 s_4)^\omega$ of $\mathrm{MP}(\pi) = -1/3 < 0$

Winning ECs: usefulness

Lemma (Long-run appearance of winning ECs)

Let $\lambda_1^f \in \Lambda_1^F$ *be a* **finite-memory** *strategy of* $\mathcal{P}_1$ *that* **satisfies** *the BWC problem for thresholds* $(0, \nu) \in \mathbb{Q}^2$*. Then, we have that*

$$\mathbb{P}_{s_{\mathsf{init}}}^{P[\lambda_1^f]}\left(\left\{\pi \in \mathsf{Outs}_{P[\lambda_1^f]}(s_{\mathsf{init}}) \mid \mathsf{Inf}(\pi) \in \mathcal{W}\right\}\right) = 1.$$

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▷ A good finite-memory strategy for the BWC problem should *maximize* the expected value achievable through *winning* ECs

Winning ECs: computation

- Deciding if an EC is winning or not is in NP $\cap$ coNP (worst-case threshold problem)
- $|\mathcal{E}| \leq 2^{|S|} \rightsquigarrow$ exponential # of ECs

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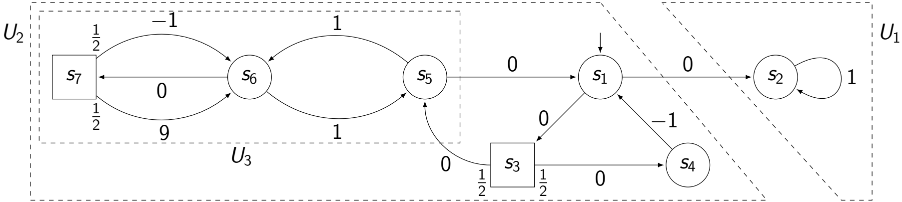

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But,

- ▷ possible to define a recursive algorithm computing the **maximal winning ECs**, such that $|\mathcal{U}_{\mathrm{W}}| \leq |S|$, in NP $\cap$ coNP.
- ▷ Uses polynomial number of of calls to
  - max. EC decomp. of sub-MDPs (each in $\mathcal{O}(|S|^2)$ [CH12]),
  - worst-case threshold problem (NP $\cap$ coNP).
- ▷ Critical complexity gain for the algorithm solving the BWC problem!

A natural way towards WECs

So we know we should only use WECs and we know how to play ε-optimally inside a WEC. *What remains to settle?*

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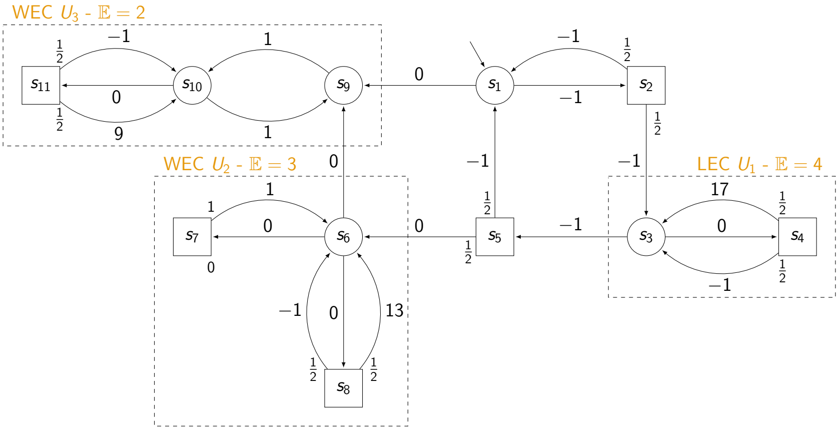
So we know we should only use WECs and we know how to play ε-optimally inside a WEC. *What remains to settle?*

- ▷ Determine **which** WECs to reach and **how**!
- ▷ Key idea: define a **global strategy** that will go towards the highest valued WECs and avoid LECs

Global strategy via modified MDP



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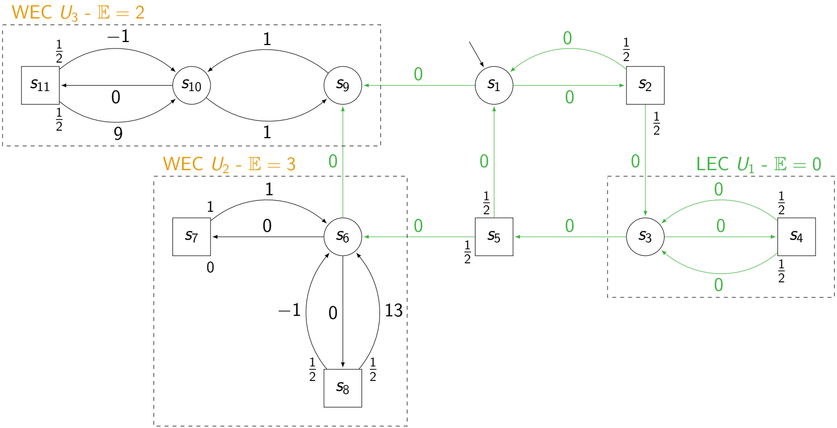
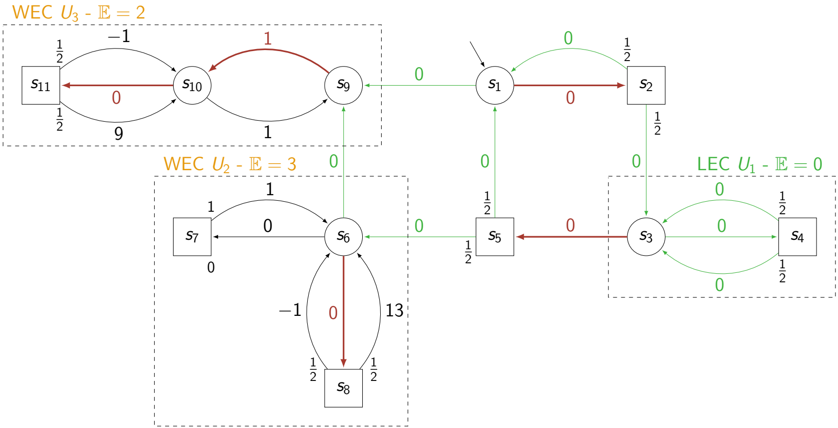


1. Modify weights:

$$\forall\, e = (s_1, s_2) \in E,\ w'(e) := \begin{cases} w(e) \text{ if } \exists\, U \in \mathcal{U}_{\mathrm{w}} \text{ s.t. } \{s_1, s_2\} \subseteq U, \\ 0 \text{ otherwise.} \end{cases}$$

Global strategy via modified MDP



2 Memoryless optimal expectation strategy λ_1^e on P'

- ▷ the probability to be in a good WEC (here, U_2) after N steps tends to one when $N \to \infty$

Global strategy via modified MDP

3. $\lambda_1^{glb} \in \Lambda_1^{PF}(G)$:
   (a) Play $\lambda_1^{e} \in \Lambda_1^{PM}(G)$ for N steps.
   (b) Let $s \in S$ be the reached state.
      (b.1) If $s \in U \in \mathcal{U}_{\mathrm{w}}$, play corresponding $\lambda_1^{cmb} \in \Lambda_1^{PF}(G)$ forever.
      (b.2) Else play $\lambda_1^{wc} \in \Lambda_1^{PM}(G)$ forever.

- λ_1^{wc} exists everywhere as WC losing states have been removed
- Parameter $N \in \mathbb{N}$ can be chosen so that overall expectation is arbitrarily close to optimal in P', or equivalently, optimal for BWC strategies in P
- Our algorithm computes this optimal value ν^* and answers YES iff $\nu^* > \nu \rightsquigarrow$ it is *correct* and *complete*

BWC MP problem: bounds

- *Complexity*
  - algorithm in NP $\cap$ coNP (P if MP games proved in P)
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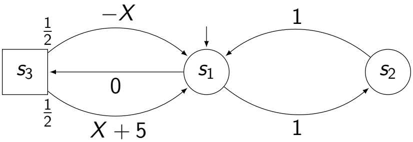

- *Memory*
  - ▷ pseudo-polynomial upper bound via global strategy
  - ▷ matching lower bound via family $(G(X))_{X \in \mathbb{N}_0}$ requiring polynomial memory in $W = X + 5$ to satisfy the BWC problem for thresholds $(0, \nu \in \,]1, 5/4[)$
    - ⇝ need to use (s_1, s_3) infinitely often for $\mathbb{E}$ but need pseudo-poly. memory to counteract $-X$ for the WC requirement

Shortest path - truncated sum

- Assume strictly positive integer weights, $w\colon E \rightarrow \mathbb{N}_0$
- Let $T \subseteq S$ be a *target set* that $\mathcal{P}_1$ wants to reach with a path of bounded value (cf. introductory example)
  - ▷ inequalities are reversed, $\nu < \mu$
- $\mathsf{TS}_T(\pi = s_0 s_1 s_2 \ldots) = \sum_{i=0}^{n-1} w((s_i, s_{i+1}))$, with n the first index such that $s_n \in T$, and $\mathsf{TS}_T(\pi) = \infty$ if $\forall\, n,\ s_n \notin T$

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Games: worst-case threshold problem

Memoryless optimal strategies as cycles are to be avoided, and the problem is in P, solvable using attractors and computation of the worst cost.

MDPs: expected value threshold problem [BT91, dA99]

Memoryless optimal strategies exist and the problem is in P.

BWC SP problem: overview

Theorem (algorithm)

The BWC problem for the shortest path can be solved in **pseudo-polynomial** *time: polynomial in the size of the game graph, the Moore machine for the stochastic model of the adversary and the encoding of the expected value threshold, and polynomial in the value of the worst-case threshold.*

Theorem (memory bounds)

Pseudo-polynomial *memory may be necessary and is always sufficient to satisfy the BWC problem for the shortest path.*

Theorem (complexity lower bound)

The BWC problem for the shortest path is **NP-hard**.

Key difference with MP case

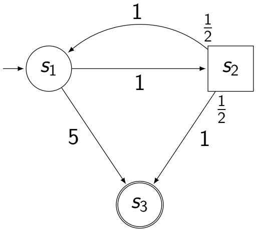
Useful observation

The set of all worst-case winning strategies for the shortest path can be represented through a finite game.

Sequential approach solving the BWC problem:

1. represent all WC winning strategies,
2. optimize the expected value within those strategies.

Pseudo-polynomial algorithm: sketch



1. Start from $G = (\mathcal{G}, S_1, S_2)$, $\mathcal{G} = (S, E, w)$, $T = \{s_3\}$, $\mathcal{M}(\lambda_2^{\mathsf{stoch}})$, $\mu = 8$, and $\nu \in \mathbb{Q}$

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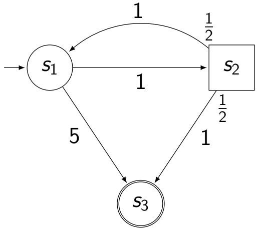

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2. Build G' by unfolding $\mathcal{G}$, tracking the current sum *up to the worst-case threshold* μ, and integrating it in the states of $\mathcal{G}'$.

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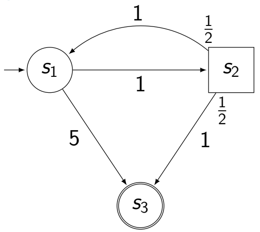

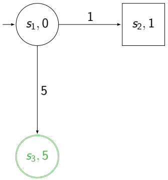

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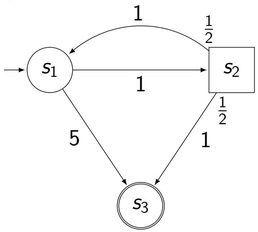

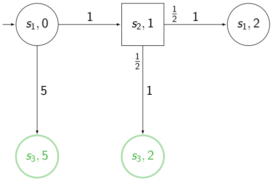

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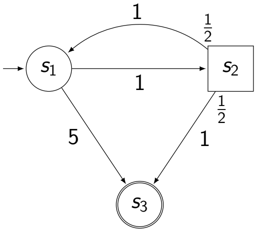

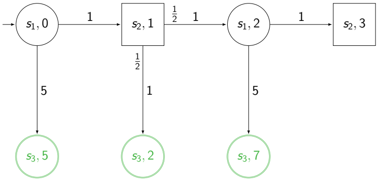

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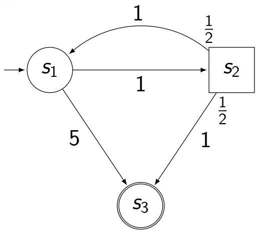

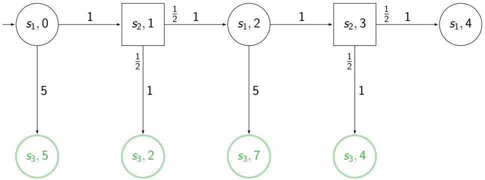

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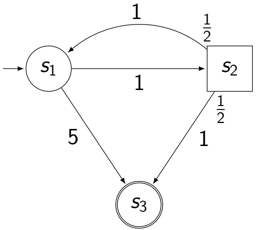

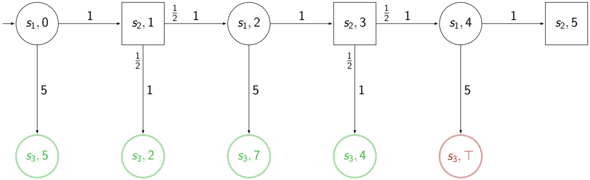

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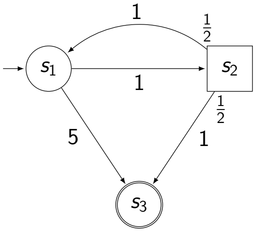

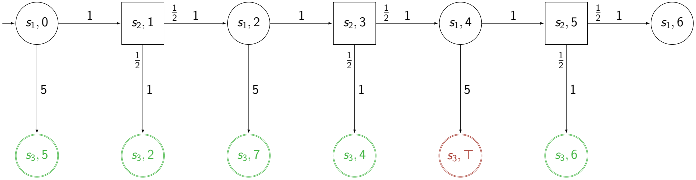

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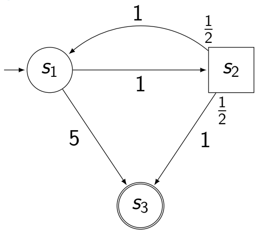
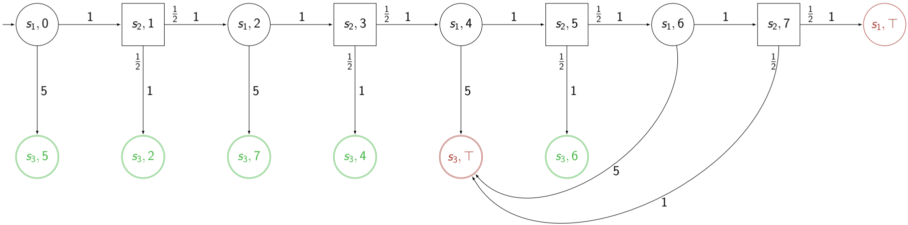
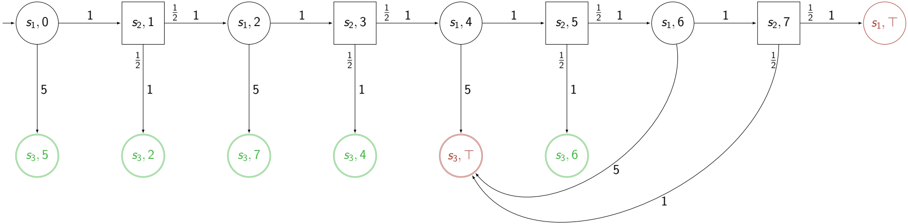




Pseudo-polynomial algorithm: sketch

3 Compute R, the attractor of T with cost $< \mu = 8$

4 Consider $G_\mu = G' \downharpoonright R$



Pseudo-polynomial algorithm: sketch

3. Compute R, the attractor of T with cost $< \mu = 8$
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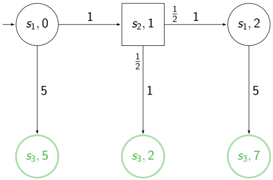
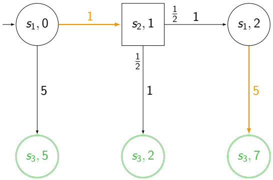


Pseudo-polynomial algorithm: sketch

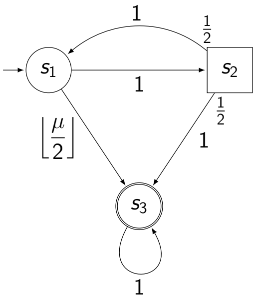
5. Consider $P = G_\mu \otimes \mathcal{M}(\lambda_2^{\text{stoch}})$
6. Compute memoryless optimal expectation strategy
7. If $\nu^* < \nu$, answer YES, otherwise answer NO



Here, $\nu^* = 9/2$

Memory bounds

- ▷ Upper bound provided by synthesized strategy
- ▷ Lower bound given by family of games $(G(\mu))_{\mu \in \{13+k\cdot 4 | k \in \mathbb{N}\}}$ requiring memory linear in μ
  - $\rightsquigarrow$ play (s_1, s_2) exactly $\left\lfloor \frac{\mu}{4} \right\rfloor$ times and then switch to (s_1, s_3) to minimize expected value while ensuring the worst-case



Complexity lower bound: NP-hardness

- Truly-polynomial algorithm very unlikely. . .
- Reduction from the K^{th} **largest subset problem**
  - ▷ commonly thought to be outside NP as natural certificates are larger than polynomial [JK78, GJ79]

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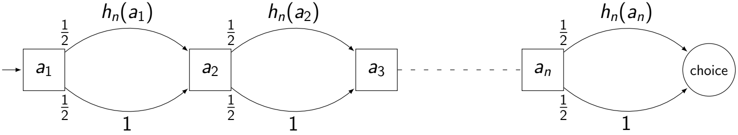
K^{th} largest subset problem

Given a finite set A, a size function $h \colon A \to \mathbb{N}_0$ assigning strictly positive integer values to elements of A, and two naturals $K, L \in \mathbb{N}$, decide if there exist K distinct subsets $C_i \subseteq A$, $1 \leq i \leq K$, such that $h(C_i) = \sum_{a \in C_i} h(a) \leq L$ for all K subsets.

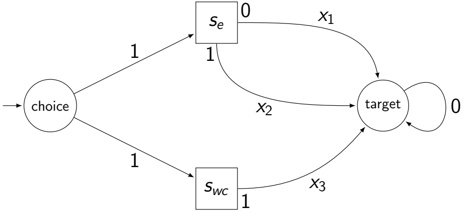
- Build a game composed of *two gadgets*

Random subset selection gadget



- Stochastically generates paths representing subsets of A: an element is selected in the subset if the upper edge is taken when leaving the corresponding state
- **All subsets are equiprobable**

Choice gadget



- s_e leads to lower expected values but may be dangerous for the worst-case requirement
- s_{wc} is always safe but induces an higher expected cost

Crux of the reduction

There exist (non-trivial) values for thresholds and weights s.t.

(i) an optimal (i.e., minimizing the expectation while guaranteeing a given worst-case threshold) strategy for $\mathcal{P}_1$ consists in choosing state s_e only when the randomly generated subset $C \subseteq A$ satisfies $h(C) \leq L$;

(ii) this strategy satisfies the BWC problem *if and only if* there exist K distinct subsets that verify this bound.

In a nutshell

- BWC framework combines worst-case and expected value requirements
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- BWC framework combines worst-case and expected value requirements
  - ▷ a natural wish in many practical applications
  - ▷ few existing theoretical support
- Mean-payoff: additional modeling power for no complexity cost (decision-wise)
- Shortest path: harder than the worst-case, pseudo-polynomial with NP-hardness result
- In both cases, pseudo-polynomial memory is both sufficient and necessary
  - ▷ but strategies have natural representations based on states of the game and simple integer counters

Beyond BWC synthesis?

Possible future works include

- study of other quantitative objectives,
- extension of our results to more general settings (multi-dimension [CDHR10, CRR12], decidable classes of games with imperfect information [DDG+10], etc),
- application of the BWC problem to various practical cases.

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Thanks!

Do not hesitate to discuss with us!

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