Nutty dyons

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Abstract

We argue that the Einstein-Yang-Mills-Higgs theory presents nontrivial solutions with a NUT charge. These solutions approach asymptotically the Taub-NUT spacetime and generalize the known dyon black hole configurations. The main properties of the solutions and the differences with respect to the asymptotically flat case are discussed. We find that a nonabelian magnetic monopole placed in the field of gravitational dyon necessarily acquires an electric field, while the magnetic charge may take arbitrary values.

1 Introduction

A feature of certain gauge theories is that they admit classical solutions which are interpreted as representing magnetic monopoles. For nonabelian gauge fields interacting with a Higgs scalar, there exist even regular configurations with a finite mass, as proven by the famous t'Hooft-Polyakov solution [1]. Typically, the magnetic monopoles admit also electrically charged generalizations - so called dyons, the Julia-Zee solution [2] of the SU(2)-Higgs theory possibly being the best known case. These solutions admits also gravitating generalizations, both regular and black hole solutions being considered in the literature (see [3] for a general review of this topics). In SU(2)-Einstein-Yang-Mills-Higgs (EYMH) theory, a branch of globally regular gravitating dyons emerges smoothly from the corresponding flat space solutions. The nonabelian black hole solutions emerge from the globally regular configurations, when a finite regular event horizon radius is imposed [4, 5]. These solutions cease to exist beyond some maximal value of the coupling constant α (which is proportional to the ratio of the vector meson mass and Planck mass).

It has been speculated that such configurations might have played an important role in the early stages of the evolution of the Universe. Also, various analyses indicate that the monopole and dyon solutions are important in quantum theories.

Since general relativity shares many similarities with gauge theories, one may ask whether Einstein's equations present solutions that would be the gravitational analogous of the magnetic monopoles and dyons. The first example of such a solution was found in 1963 by Newman, Unti and Tamburino (NUT) [6, 7]. This metric has become renowned for being "a counterexample to almost anything" [8] and represents a generalization of the Schwarzschild vacuum solution [9] (see [10] for a simple derivation of this metric and historical review). It is usually interpreted as describing a gravitational dyon with both ordinary and magnetic mass ¹. The NUT charge which plays a dual role to ordinary mass, in the same way that electric and magnetic charges are dual within Maxwell theory [11]. By continuing the NUT solution through its horizon one arrives in the Taub universe [7], which may be interpreted as a homogeneous, non-isotropic cosmology with the spatial topology S^3 .

As discussed by many authors (see e.g. [13, 14]), the presence of magnetic-type mass (the NUT parameter n) introduces a "Dirac-string singularity" in the metric (but no curvature singularity). This can be removed by appropriate identifications and changes in the topology of the spacetime manifold, which imply a periodic time coordinate. Moreover, the metric is not asymptotically flat in the usual sense although it does obey the required fall-off conditions.

A large number of papers have been written investigating the properties of the gravitational analogs of magnetic monopoles [15, 16], the vacuum Taub-NUT solution being generalized in different directions. The

¹Note that the Taub-NUT spacetime plays also an important role outside general relativity. For example the asymptotic motion of monopoles in (super-)Yang-Mills theories corresponds to the geodesic motion in a Euclideanized Taub-NUT background [12]. However, these developments are outside the interest of this work.

corresponding configuration in the Einstein-Maxwell theory has has been found in 1964 by Brill [17]. This abelian solution has been generalized for the matter content of the low-energy string theory, a number of NUT-charged configurations being exhibited in the literature (see e.g. [18] for a recent example and a large set of references). A discussion of the nonabelian counterparts of the Brill solution is presented in [19]. These configurations generalize the well known SU(2)-Einstein-Yang-Mills hairy black hole solutions [20], presenting, as a new feature, a nontrivial electric potential. However, the "no global nonabelian charges" results found for asymptotically flat EYM static configurations [21] are still valid in this case, too.

Here we present arguments for the existence of NUT-charged generalizations of the known EYMH black hole solutions [4, 5]. Apart from the interesting question of finding the properties of a Yang-Mills-Higgs dyon in the field of a gravitational dyon, there are a number of other reasons to consider this type of solutions. In some supersymmetric theories, closure under duality forces us to consider NUT-charged solutions. Furthermore, dual mass solutions play an important role in Euclidean quantum gravity [22] and therefore cannot be discarded in spite of their causal pathologies. Also, by considering this type of asymptotics, one may hope to attain more general features of gravitating nonabelian dyons.

The paper is structured as follows: in the next Section we present the general framework and analyse the field equations and boundary conditions. In Section 3 we present our numerical results. We conclude with Section 4, where our results are summarized.

2 General framework and equations of motion

2.1 Action principle

The action for a gravitating non-Abelian SU(2) gauge field coupled to a triplet Higgs field with vanishing Higgs self-coupling is

$$S = \int \sqrt{-g} d^4 x \left(\frac{R}{16\pi G} - \frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) - \frac{1}{4} \text{Tr}(D_\mu \Phi D^\mu \Phi) \right), \tag{1}$$

with Newton's constant G. The field strength tensor is given by $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - ie[A_{\mu}, A_{\nu}]$, with $D_{\mu} = \partial_{\mu} - ie[A_{\mu},]$ being the covariant derivative and e the Yang-Mills coupling constant.

Varying the action (1) with respect to $g^{\mu\nu}$, A_{μ} and Φ we have the field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu},$$

$$\frac{1}{\sqrt{-g}}D_{\mu}(\sqrt{-g}F^{\mu\nu}) = \frac{1}{4}ie[\Phi, D^{\nu}\Phi],$$

$$\frac{1}{\sqrt{-g}}D_{\mu}(\sqrt{-g}D^{\mu}\Phi) = 0,$$
(2)

where the stress-energy tensor is

$$T_{\mu\nu} = 2Tr\{F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\} + Tr\{\frac{1}{2}D_{\mu}\Phi D_{\nu}\Phi - \frac{1}{4}g_{\mu\nu}D_{\alpha}\Phi D^{\alpha}\Phi\}.$$
(3)

2.2 Metric ansatz and symmetries

We consider NUT-charged spacetimes whose metric can be written locally in the form

$$ds^{2} = \frac{dr^{2}}{N(r)} + P^{2}(r)(d\theta^{2} + \sin^{2}\theta d\varphi^{2}) - N(r)\sigma^{2}(r)(dt + 4n\sin^{2}(\frac{\theta}{2})d\varphi)^{2},$$
(4)

the NUT parameter n being defined as usually in terms of the coefficient appearing in the differential $dt + 4n \sin^2(\theta/2) d\varphi$. Here θ and φ are the standard angles parametrizing an S^2 with ranges $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$.

Apart from the Killing vector $K_0 = \partial_t$, this line element possesses three more Killing vectors characterizing the NUT symmetries

$$K_{1} = \sin \varphi \partial_{\theta} + \cos \varphi \cot \theta \partial_{\varphi} + 2n \cos \varphi \tan \frac{\theta}{2} \partial_{t},$$

$$K_{2} = \cos \varphi \partial_{\theta} - \sin \varphi \cot \theta \partial_{\varphi} - 2n \sin \varphi \tan \frac{\theta}{2} \partial_{t},$$

$$K_{3} = \partial_{\varphi} - 2n \partial_{t}.$$
(5)

These Killing vectors form a subgroup with the same structure constants that are obeyed by spherically symmetric solutions $[K_a, K_b] = \epsilon_{abc} K_c$.

The $n \sin^2(\theta/2)$ term in the metric means that a small loop around the z-axis does not shrink to zero at $\theta = \pi$. This singularity can be regarded ar the analogue of a Dirac string in electrodynamics and is not related to the usual degeneracies of spherical coordinates on the two-sphere. This problem was first encountered in the vacuum NUT metric. One way to deal with this singularity has been proposed by Misner [8]. His argument holds also independently of the precise functional form of N and σ . In this construction, one considers one coordinate patch in which the string runs off to infinity along the north axis. A new coordinate system can then be found with the string running off to infinity along the south axis with $t' = t + 4n\varphi$, the string becoming an artifact resulting from a poor choice of coordinates. It is clear that the t coordinate is also periodic with period $8\pi n$ and essentially becomes an Euler angle coordinate on S^3 . Thus an observer with $(r, \theta, \varphi) = const$. follows a closed timelike curve. These lines cannot be removed by going to a covering space and there are no reasonable spacelike surface. One finds also that surfaces of constant radius have the topology of a three-sphere, in which there is a Hopf fibration of the S^1 of time over the spatial S^2 [8].

Therefore for n different from zero, the metric structure (4) generically shares the same troubles exhibited by the vacuum Taub-NUT gravitational field [23], and the solutions cannot be interpreted properly as black holes.

2.3 Matter fields ansatz

While the Higgs field is given by the usual form

$$\Phi = \phi \tau_3, \tag{6}$$

the computation of the appropriate SU(2) connection compatible with the Killing symmetries (5) is a more involved task. This can be done by applying the standard rule for calculating the gauge potentials for any spacetime group [24, 25]. According to Forgacs and Manton, a gauge field admit a spacetime symmetry if the spacetime transformation of the potential can be compensated by a gauge transformation [24] $\mathcal{L}_{K_i}A_{\mu} = D_{\mu}W_i$, where \mathcal{L} stands for the Lie derivative.

Taking into account the symmetries of the line element (4) we find the general form

$$A = \frac{1}{2e} \Big\{ \Big(dt + 4n \sin^2(\frac{\theta}{2}) d\varphi \Big) u(r) \tau_3 + \nu(r) \tau_3 dr + \Big(\omega(r) \tau_1 + \tilde{\omega}(r) \tau_2 \Big) d\theta + \Big[\cos \theta \tau_3 + (\omega(r) \tau_2 - \tilde{\omega}(r) \tau_1) \sin \theta \Big] d\varphi \Big\}.$$

$$\tag{7}$$

This gauge connection remains invariant under a residual U(1) gauge symmetry which can be used to set $\nu = 0$. Also, because the variables ω and $\tilde{\omega}$ appear completely symmetrically in the EYMH system, the two amplitudes must be proportional and we can always set $\tilde{\omega} = 0$ (after a suitable gauge transformation). Thus, similar to the n = 0 case, the gauge potential is described by two functions $\omega(r)$ and u(r) which we shall refer to as magnetic and electric potential, respectively.

2.4 Field equations and known solutions

Within the above ansatz, the classical equations of motion can be derived from the following reduced action

$$S = \int dr dt \left[\frac{1}{8\pi G} \left(\sigma (1 - NP'^2 - PP'N') + 2P'(\sigma NP)' + \frac{n^2 \sigma^3 N}{P^2} \right) \right]$$
(8)

$$-\Big(\frac{1}{e^2}(N\sigma\omega'^2 + \frac{\sigma(\omega^2 - 1 + 2nu)^2}{2P^2} - \frac{P^2u'^2}{2\sigma^2} - \frac{\omega^2u^2}{\sigma N}) + \frac{1}{2}\sigma NP^2\phi'^2 + \sigma\omega^2\phi^2\Big)\Big],$$

where the prime denotes the derivative with respect to the radial coordinate r.

At this point, we fix the metric gauge by choosing $P(r) = \sqrt{r^2 + n^2}$, which allows a straightforward analysis of the relation with the abelian configurations.

Dimensionless quantities are obtained by considering the rescalings $r \to r/(\eta e)$, $\phi \to \phi \eta$, $n \to n/(\eta e)$, $u \to \eta e u$ (where η is the asymptotic magnitude of the Higgs field). As a result, the field equations depend only on the coupling constant $\alpha = \sqrt{4\pi G \eta}$.

The EYMH equations reduce to the following system of five non-linear differential equations

$$rN' = 1 - N + \frac{n^2 N}{P^2} (3\sigma^2 - 1) - 2\alpha^2 \left(N\omega'^2 + \frac{1}{2P^2} (\omega^2 - 1 + 2nu)^2 + \frac{P^2 u'^2}{2\sigma^2} + \frac{\omega^2 u^2}{\sigma^2 N} + \frac{1}{2} N P^2 \phi'^2 + \omega^2 \phi^2 \right),$$

$$\sigma' = \frac{n^2 \sigma (1 - \sigma^2)}{rP^2} + \frac{\alpha^2 \sigma}{r} (P^2 \phi'^2 + 2\omega'^2 + \frac{2\omega^2 u^2}{\sigma^2 N^2}),$$

$$(N\sigma\omega')' = \sigma\omega \left(\frac{(\omega^2 - 1 + 2nu)}{P^2} + \phi^2 - \frac{u^2}{\sigma^2 N} \right),$$

$$(N\sigma P^2 \phi')' = 2\sigma\omega^2 \phi,$$

$$\left(\frac{P^2 u'}{\sigma} \right)' = \frac{2\omega^2 u}{\sigma N} - \frac{2n\sigma}{P^2} (\omega^2 - 1 + 2nu).$$
(9)

Two explicit solution of the above equations are well known. The vacuum Taub-NUT one corresponds to

$$\omega(r) = \pm 1, \qquad u(r) = 0, \qquad \sigma(r) = 1, \quad \phi(r) = 1, \quad N(r) = 1 - \frac{2(Mr + n^2)}{r^2 + n^2}.$$
 (10)

The U(1) Brill solution [17] has the form

$$\omega(r) = 0, \quad u(r) = u_0 + \frac{nQ_m - Q_e r}{r^2 + n^2}, \quad \sigma(r) = 1, \quad \phi(r) = 1, \quad N(r) = 1 - \frac{2(Mr + n^2)}{r^2 + n^2} + \frac{\alpha^2(Q_e^2 + Q_m^2)}{4(r^2 + n^2)}. \quad (11)$$

and describes a gravitating abelian dyon with a mass M, electric charge Q_e and magnetic charge $Q_m \equiv 1 - 2u_0n$, u_0 being an arbitrary constant, corresponding to the asymptotic value of the electric potential.

It can be stressed that the Brill solution possesses two, one or zero horizons, according to the values of the free parameters $Q_e, M, u(\infty)$. In the same way as in the case of Reissner-Nordström solutions, the extremal Brill solution can be defined as the solutions with a degenerate horizon at $r = r_0$. This gives the following conditions, fixing M and r_0

$$r_0 = M$$
, $M^2 + n^2 - \frac{\alpha^2}{4}(Q_e^2 + Q_m^2) = 0.$ (12)

As we will see later, it is convenient to further specify the arbitrary constant $u(\infty)$ in such a way the $u(r_0) = 0$, this implying

$$\frac{1-Q_m}{2n} + \frac{nQ_m - MQ_e}{M^2 + n^2} = 0, (13)$$

which fixes Q_m and leaves Q_e as the only remaining free parameter. In the following we will refer to this solution as to the extremal Brill solution. As far as we could see, it is not possible to express M and Q_m in a closed form depending on (α, n, Q_e) , but the solution can be constructed numerically.

2.5 Boundary conditions

We want the metric (4) to describe a nonsingular, asymptotically NUT spacetime outside an horizon located at $r = r_h$. Here $N(r_h) = 0$ is only a coordinate singularity where all curvature invariants are finite. A nonsingular extension across this null surface can be found just as at the event horizon of a black hole. If the time is chosen to be periodic, as discussed above, this surface would not be a global event horizon, although it would still be an apparent horizon. The regularity assumption implies that all curvature invariants at $r = r_h$ are finite.

The corresponding expansion as $r \to r_h$ is

$$N(r) = N_{1}(r - r_{h}) + O(r - r_{h})^{2},$$

$$\sigma(r) = \sigma_{h} + \sigma_{1}(r - r_{h}) + O(r - r_{h})^{2},$$

$$\omega(r) = \omega_{h} + \omega_{1}(r - r_{h}) + O(r - r_{h})^{2},$$

$$u(r) = u_{1}(r - r_{h}) + u_{2}(r - r_{h})^{2} + O(r - r_{h})^{3},$$

$$\phi(r) = \phi_{h} + \phi_{1}(r - r_{h}) + O(r - r_{h})^{2},$$

(14)

where $P_h^2 = r_h^2 + n^2$ and

$$N_{1} = \frac{1}{r_{h}} \left(1 - 2\alpha^{2} \left(\frac{(\omega_{h}^{2} - 1)^{2}}{2P_{h}^{2}} + \frac{1}{2} \frac{u_{1}^{2} P_{h}^{2}}{\sigma_{h}^{2}} + \omega_{h} \phi_{h}^{2} \right), \quad \sigma_{1} = \frac{n^{2} \sigma_{h} (1 - \sigma_{h}^{2})}{r_{h} P_{h}^{2}} + \frac{\alpha^{2} \sigma_{h}}{r_{h}} (P_{h}^{2} \phi_{1}^{2} + 2\omega_{1}^{2} + \frac{2\omega_{1}^{2} u_{1}^{2}}{\sigma_{h}^{2} N_{1}^{2}}),$$
$$\omega_{1} = \frac{\omega_{h}}{N_{1}} \left(\frac{\omega_{h}^{2} - 1}{P_{h}^{2}} + \phi_{h}^{2} \right), \quad u_{2} = \frac{\sigma_{1} u_{1}}{2\sigma_{h}} - \frac{n \sigma_{h}^{2} (\omega_{h}^{2} - 1)}{P_{h}^{4}} + \frac{u_{1} \omega_{h}^{2}}{N_{1} P_{h}^{2}} - \frac{u_{1} r_{h}}{P_{h}^{2}}, \quad \phi_{1} = \frac{2\omega_{h}^{2} \phi_{h}}{N_{1} P_{h}^{2}}, \quad (15)$$

 σ_h , u_1 , ω_h , ϕ_h being arbitrary parameters.

The analysis of the field equations as $r \to \infty$ gives the following expression in terms of the constants $c, u_0, Q_e, \tilde{\phi}_1, M$

$$N(r) \sim 1 - \frac{2M}{r} - \frac{2n^2 - \alpha^2 \left(\tilde{\phi}_1^2 + (1 - 2nu_0^2)^2 + Q_e^2\right)}{r^2} + \frac{M(2n^2 + \alpha^2 \tilde{\phi}_1^2)}{r^3} + \dots,$$

$$\sigma \sim 1 - \frac{\alpha^2 \tilde{\phi}_1^2}{2r^2} - \frac{4\alpha^2 \tilde{\phi}_1^2 M}{3r^3} + \dots, \quad \omega(r) \sim ce^{-\sqrt{1 - u_0^2}r} + \dots,$$

$$\phi \sim 1 - \frac{\tilde{\phi}_1}{r} + \frac{\tilde{\phi}_1 M}{r^2} + \dots, \quad u(r) \sim u_0 - \frac{Q_e}{r} + \frac{n(1 - 2nu_0)}{r^2} - \frac{Q_e(6n^2 + \alpha^2 \tilde{\phi}_1^2)}{6r^3} + \dots$$
(16)

Note that similar to the n = 0 asymptotically flat case, the magnitude of the electric potential at infinity cannot exceed that of the Higgs field, $|u_0| < 1^2$. The constant M appearing in the asymptotic expansion of the metric function N(r) can be interpreted as the total mass of solutions (this can be proven rigurously by applying the general formalism proposed in [28]). Note that M and n are unrelated on a classical level.

Also, no purely monopole solution can exist for a nonvanishing NUT charge (i.e. one cannot consistently set u = 0 unless $\omega = \pm 1$, in which case the vacuum Taub-NUT solution is recovered). Thus, a nonabelian magnetic monopole placed in the field of gravitational dyon necessarily acquires an electric field.

We close this section by remarking that the definition of the nonabelian charges is less clear for $n \neq 0$. Although we may still define a 't Hooft field strength tensor, in the absence of a nontrivial two-sphere at infinity on which to integrate, the only reasonable definition the nonabelian magnetic and electric charges is in terms of the asymptotic behavior of the gauge field. By analogy to the asymptotically flat case, Q_e and Q_m are defined from $F_{tr}^{(3)} \simeq Q_e/r^2$ and $F_{\theta\phi}^{(3)} \simeq Q_m \sin \theta$ (a similar problem occurs for an U(1) field [27]). Thus, since $Q_m = 1 - 2nu_0$, the usual quantization relation for the magnetic charge is lost for $n \neq 0$, which is a consequence of the pathological large scale structure of a NUT-charged spacetime.

²This depends on the asymptotic structure of the spacetime. For example, in an anti-de Sitter spacetime, u_0 may take arbitrary values [26].



Figure 1. The functions N(r), $\sigma(r)$ and u(r) are plotted for three typical solutions for the same values of (r_h, Q_e, α) .

3 Numerical results

Although an analytic or approximate solution appears to be intractable, we present in this section numerical arguments that the known EYMH black hole solutions can be extended to include a NUT parameter.

The equations of motion (9) have been solved for a large set of the parameters (α, n, Q_e, r_h) , looking for solutions interpolating between the asymptotics (14) and (16). NUT-charged solutions are found for any n = 0 EYMH dyonic black hole configuration by slowly increasing the parameter n (since the transformation $n \to -n$ leaves the field equations unchanged except for the sign of the electric potential, we consider here only positive values of n). As expected, these configurations have many features in common with the n = 0solutions discussed in [4]; they also present new features that we will pointed out in the discussion. Typical profiles for the metric functions N(r) and $\sigma(r)$ and for the electric potential u(r) are presented in Figure 1, for a dyonic black hole solution as well as for two NUT-charged solutions. The gauge function $\omega(r)$ and the Higgs scalar $\phi(r)$ interpolates monotonically between some constant values on the event horizon and zero respectively one at infinity, without presenting any local extremum (see Figure 3).

The domain of existence of the nonabelian nutty dyons can be determined in the space of parameters. If we fix the electric charge Q_e of the solution, then there likely exist a volume \mathcal{V}_Q in the parameter space of (α, n, r_h) inside which nonabelian solutions exist and on the side of which they become singular and/or bifurcate into abelian solution of the type of the Brill solution. For n = 0 the domain of the (α, r_h) plane where nonabelian solutions exist was determined in [5] for $Q_e = 0$ and in [4] for $Q_e \neq 0$.

The determination of \mathcal{V}_Q is of course a huge task. In this letter, we will not attempt to determine the shape of \mathcal{V}_Q accurately but rather attempt to sketch it by analyzing the pattern of solutions on some generic lines in the space of parameters. For definiteness we set $Q_e = 0.2$ in our numerical analysis, although nontrivial solutions have been found also for other values of the electric charge.

3.1 *n* varying

First, we have integrated the system of equations (9) with fixed values for α , r_h and Q_e and increased the NUT charge n. Our values here are $\alpha = 1.0$, $r_h = 0.2$ and $Q_e = 0.2$ corresponding to a generic values for



Figure 2. The values of the parameters M, $\sigma(r_h)$, N_m , N_M , $\phi(r_h)$, $\omega(r_h)$ and $u(\infty) = u_0$ are shown as a function of n for solutions with $r_h = 0.2$, $Q_e = 0.2$ and two different values of the coupling constant α .

the parameters (the corresponding n = 0 gravitating dyon was constructed in [4]). As far as the function u(r) is concerned, there exists a main difference between the case n = 0 and $n \neq 0$. Indeed in the case n = 0 this function behave asymptotically like $u(r) \sim u_0 + Q_e/r + O(1/r^4)$ while in the presence of a NUT charge the behaviour is instead $u(r) \sim u_0 + Q_e/r + K/r^2$, where, as seen from (16), the constant K increases with n. Thus, when n becomes large, it becomes more difficult to construct numerical solutions with a good enough accuracy ³, for a given value of the electric charge Q_e .

The effect of increasing n apparently depend strongly of the value α . For α small (typically $\alpha \leq 1$) the pattern can be summarized by the following points : (i) No local extrema of N(r) are found for small enough

 $^{^{3}}$ To integrate the equations, we used the differential equation solver COLSYS which involves a Newton-Raphson method [29].



Figure 3. The metric functions N(r), $\sigma(r)$ and the matter functions $\omega(r)$, $\phi(r)$ and u(r) are shown as a function of r for fixed values of (r_h, Q_e, n) and three different values of α . The functions N(r) and u(r) of the corresponding extremal Brill solution (with $\omega(r) = 0$, h(r) = 1) are also exhibited.

values of n. For larger n, the function N(r) develops a local maximum and also a local minimum, say N_M and N_m at some intermediate, n-depending values of r. For n large enough, we have $N_{max} > 1$. No local minimum of N persist for large enough n, the minimum of N(r) ($N_m = 1$) being attained as $r \to \infty$. (ii) The second metric function $\sigma(r)$ still remains monotonically increasing but the value $\sigma(r_h)$ diminishes when n increases. (iii) The asymptotic value $u(\infty)$ also decreases for increasing n. With the values choosen, we have $u(\infty) \approx 0.189$ for n = 0; we find $u(\infty) = 0$ for $n \approx 0.25$ and negative values for larger n.

These effects are illustrated on Figure 2a. On this Figure we have set 0 < n < 2 but we noticed no significant change of the behaviour for larger values of n.

For larger values of α (typically $\alpha \geq 2$) the situation is quite different, namely: (i) The function N(r) posseses both a local minimum and a local maximum. (ii) The values $w(r_h)$ and $\phi(r_h)$ increase with n and approach respectively zero and one, suggesting that the solution approaches an Abelian Brill solution. These results are summarize on Figure 2b for $\alpha = 2$. However, due to numerical difficulties, we could not determine properly the value of n where the bifurcation occur. The statement of a bifurcation into a Brill solution is confirmed in the next subsection where α is varying.

Nevertheless, it seems that there are two possible patterns for $n \to \infty$: for values of α smaller than a critical value $\hat{\alpha}$, solutions with large values of n seem to occur, while for $\alpha > \hat{\alpha}$, the solutions bifurcate into a Brill solution (for $Q_e = 0.2$ we find $\hat{\alpha} \sim 1.5$). The occurence of these two patterns is reminiscent to the case of n = 0 gravitating dyons.

Note also that, as shown in these plots, the mass parameter M takes negative values for large enough values of n. This is not a surprise, since something similar happens already for the U(1) Brill solution (11).

3.2 α varying

We now discuss the behaviour of the solutions for a varying α and the other parameters fixed. In absence of a NUT charge it is know [5, 4] that nonabelian dyonic black hole exist for $r_h \in [0, \sqrt{3 + 4Q_e^2}/2]$. For fixed Q_e and r_h and increasing α they bifurcates into an extremal Reissner-Nordström solution at $\alpha \sim \alpha_c$. The value α_c depends of course on r_h and Q_e . For $r_h \ll 1$ the value $\alpha_c \approx 1.4$ is found numerically and depends weakly on Q_e . For $r_h \sim \sqrt{3 + 4Q_e^2}/2$ we have $\alpha_c \approx \sqrt{(3 + 4Q_e^2)/(1 + Q_e^2)/2}$.

For n > 0 we see (e.g. on Figure 3) that the local maximum characterizing the function N(r) of a nutty solution (at least for large enough values of the NUT charge n) progressively disappears in favor of a local



Figure 4. The profiles of the functions N(r), $\sigma(r)$ and u(r) are represented for $Q_e = 0.2$, n = 0.5, $\alpha = 1$ and three different values of r_h .

minimum when α increases. This minimum appears far outside the event horizon $r = r_h$ and becomes deeper. In fact, the minimal value N_m approaches zero when α tends to a critical value, say $\alpha_c(n, Q_e, r_h)$. If we denote by r_m the value of the radial variable where $N(r_m) = 0$ (with $r_m > r_h$) our numerical results strongly indicate that the non abelian solution converges into an extremal Brill solution on the interval $r \in [r_m, \infty)$ for $\alpha \to \alpha_c$.

Indeed, the matter functions's profiles u, w, ϕ and the metric functions σ , N all approaches the profiles of the corresponding extremal Brill solution with the same α_c , Q_e , n. This result is illustrated on Figure 3 for n = 1, $r_h = 0.2$ and $Q_e = 0.2$; in this case, we find $\alpha_c \approx 2.35$ but we believe that the result holds for generic values of (n, r_h, Q_e) . The determination of the critical value $\alpha_c(n, r_h, Q_e)$ is not aimed in this letter. However, it seems that the value α_c depends weakly of n, for example we find $\alpha_c \approx 2.22$ for $n \in [3, 4]$.

Nevertheless we can conclude that nutty dyons exist on a finite interval of α and bifurcate into extremal Brill solutions for $\alpha = \alpha_c$.

3.3 r_h varying

In the case of gravitating dyonic black holes with event horizon r_h , the solutions approach the corresponding regular gravitating solution on the interval $]r_h, \infty[$ when the limit $r_h \to 0$ is considered. It is therefore a natural question to investigate how nutty-dyons behave in the same limit.

Considering this problem for a few generic values of (α, n) we reach the conclusion that, in the limit $r_h \to 0$, the nutty dyon becomes singular at r = 0 because the value $\sigma(r_h)$ tends to zero. This situation is illustrated on Figure 4 where the functions $N(r), \sigma(r)$ and u(r) are plotted for three different values of r_h and $\alpha = 1$, n = 0.5.

Remarkably, this Figure reveals that the functions $\sigma(r)$ and u(r) are rather independent of r_h (it is also true for $w(r), \phi(r)$ which are not represented) while the function N(r) indeed involves non trivially with r_h . Note also that for the metric gauge choice $P(r) = \sqrt{r^2 + n^2}$, the area of two-sphere $d\Omega^2 = P^2(r)(d\theta^2 + \sin^2\theta d\varphi^2)$ does not vanish at r = 0. However, by choosing a Schwarzschild gauge choice P(r) = r, a straightforward analysis of the corresponding field equations (which can easily be derived from (8)) implies that it is not possible to take a consistent set of boundary conditions at r = 0 without introducing a curvature singularity at that point. Therefore, no globally regular EYMH solutions are found for $n \neq 0$.

The determination of the domain of nutty dyons for fixed (α, n, Q_e) and increasing the horizon radius r_h is very likely an involved problem. Already in the case n = 0, discussed in [4] the numerical analysis reveals

several (up to three) branches of solutions on some definite intervals of the parameter r_h . We believe that similar patterns could occur for n > 0 but their analysis is out of the scope of this letter.

4 Further remarks

In this work we have analysed the basic properties of gravitating YMH system in the presence of a NUT charge. We have found that despite the existence of a number of similarities to the n = 0 case (for example the presence of a maximal value of the coupling constant α), the NUT-charged solutions exhibits some new qualitative features.

The static nature of a n = 0 spherically symmetric gravitating nonabelian solution implies that it can only produce a "gravitoelectric" field. There both nonabelian monopole and dyon black hole solutions are possible to exist, with a well defined zero event horizon radius limit. For a nozero NUT charge, the existence of the cross metric term $g_{\varphi t}$ shows that the solutions have also a "gravitomagnetic" field. The term $g_{\varphi t}$ does not produce an ergoregion but it will induce an effect similar to the dragging of inertial frames [30]. In this case we have found that only nonabelian dyons are possible to exist and the usual magnetic charge quantization relation is lost. The total mass of these solutions may be negative and the configurations do not survive in the limit of zero event horizon radius.

A discussion of possible generalizations of this work should start with the radially excited nutty dyons, for which the gauge function $\omega(r)$ possesses nodes. These configurations are very likely to exist, continuing for n > 0 the excited configurations discussed in [4]. Also, in our analysis, to simplify the general picture, we set the Higgs potential $V(\phi)$ to zero. We expect to find the same qualitative results for a nonvanishing scalar potential (at least if the parameters are not to large). It would be a challenge to construct axially symmetric NUT-charged dyons (the corresponding n = 0 monopoles are discussed in [31]). Such dyon solutions would present a nonvanishing angular momentum, generalizing the abelian Kerr-Newman-NUT configurations (a set of asymptotically flat rotating solutions have been considered recently in [32]).

Similar to the case n = 0, the solutions discussed in this work can also be generalized by including a more general matter content. However, we expect that these more general configurations will present the same generic properties discussed in this work. This may be important, since there are many indications that the NUT charge is an important ingredient in low energy string theory [27], conclusion enhanced by the discovery of "duality" transformations which relate superficially very different configurations. In many situations, if the NUT charge is not included in the study, some symmetries of the system remain unnoticed (see e.g. [33] for such an example). Therefore, we may expect the NUT charge to play a crucial role in the duality properties of a (supersymmetric-) theory presenting gravitating nonabelian dyons.

Unfortunately, the pathology of closed timelike curves is not special to the vacuum Taub-NUT solution but afflicts all solutions of Einstein equations solutions with "dual" mass in general [28]. This condition emerges only from the asymptotic form of the fields, and is completely insensitive to the precise details of the nature of the source, or the precise nature of the theory of gravity at short distances where general relativity may be expected to break down [23]. This acausal behavior precludes the nutty dyons solutions discussed in this paper having a role classically and implies a number of pathological properties of these configurations.

Nevertheless, there are various features suggesting that the Euclidean version of NUT-charged solutions play an important role in quantum gravity [22]. For example, the entropy of such solutions generically do not obey the simple "quarter-area law". As usual, a positive-definite metric is found by considering in (4) the analytical continuation $t \to it$, $n \to i n$, which gives $P^2(r) = r^2 - n^2$. In this case, the absence of conical singularities at the root r_h of the function N(r) imposes a periodicity in the Euclidean time coordinate

$$\beta = \frac{4\pi}{N'(r_h)\sigma(r_h)},\tag{17}$$

which should be equal with the one to remove the Dirac string $\beta = 8\pi n$. In the usual approach, the solution's parameters must be restricted such that the fixed point set of the Killing vector ∂_t is regular at the radial position $r = r_h$. We find in this way two types of regular solutions, "bolts" (with arbitrary $r_h = r_b > n$) or

"nuts" $(r_h = n)$, depending on whether the fixed point set is of dimension two or zero (see [13] for a discussion of these solutions in the vacuum case and [34] for a recent generalization with anti-de Sitter asymptotics).

We expect that the Euclidean nutty dyons will present some new features as compared to the Lorentzian counterparts. For example, globally regular solutions may exist in this case, since $r = r_h$ corresponds to the origin of the coordinate system (note also that the SU(2) Yang-Mills system is known to present self-dual solutions in the background of a vacuum Taub-NUT instanton [35]). In the absence of closed form solutions, the properties of these non-self dual EYMH solutions cannot be predicted directly from those of the Lorentzian configurations. However, similar to the Lorentzian case, they can be studied in a systematic way, by using both analytical and numerical arguments. For example, the magnitude of the electric potential at infinity of the Euclidean solutions, is not restricted. Also, the condition $\beta = 8\pi n$ implies $N'(r_h)\sigma(r_h) = 2n$ and introduces a supplementary constraint on the matter functions as $r \to r_h$. A study of such solutions may be important in a quantum gravity context.

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