Exotic duality and higher spins

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Based on 2012.11356 (JHEP) in collaboration with Victor Lekeu (Imperial College London & AEI Potsdam)

and work in collaboration with Victor Lekeu, Andrea Campoleoni and Zhenya Skvortsov (UMONS)

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Plan of talk

1) Introduction & motivations

2) Results on the double-dual graviton and higher duals of vector field

2.1) Double dualisation of a massless vector in $n=4$: $\square \longmapsto \begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix}$

2.2) Double dualisation of spin-2 $h_{ab} \sim \square\square$ in $n=5$ into $D \sim \begin{smallmatrix}\square & \square \\ \square & \square\end{smallmatrix}$

3) Higher (or exotic) dualisation of the 3D graviton and scalar $\rightarrow$ spin-3 gauge field

3.1) Higher dualisation of graviton in 3D $\quad \square\square \longmapsto \square\square\square$

3.2) Higher dualisation of scalar in 3D $\quad \bullet \longmapsto \square\square\square$

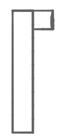
① Introduction – motivations

Duality in gravity $\rightsquigarrow$ Hull's works (circa 2000) on $\mathcal{N}_6 = (4,0)$ theory

and simultaneously West's e_{11} proposal [2001]

↳ In these works, a gauge field

$$C_{[n-3,1]} \rightsquigarrow C_{a_1 \dots a_{n-3},b} = C_{[a_1 \dots a_{n-3}],b} \quad \text{s.t.} \quad C_{[a_1 \dots a_{n-3},b]} \equiv 0$$

i.e. $C_{[n-2-1,1]} \sim$  of $GL(n)$ appears in Minkowski spacetime $\mathbb{R}^{1,n-1}$

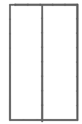
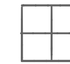
$n-3$

that propagates the d.o.f. of Fierz–Pauli's graviton h_{ab} s.t. $\eta^{bd} K_{ab,cd}(h) = 0$.

Hull's twisted on-shell duality, relating $K_{a_1 \dots a_{n-2},}{}^{b_1 b_2} := \partial_{[a_1} \partial^{[b_1} C_{a_2 \dots a_{n-2}],}{}^{b_2]}$

to $K_{ab,}{}^{cd} := -\frac{1}{2} \partial_{[a} \partial^{c} h^{d]}{}_{b]}$ via $\boxed{K_{[n-2,2]} = *_1 K_{[2,2]}}$.

In same work [Hull : 2001] a field

$D_{[n-3,\,n-3]} \sim$  of $Gl(n)$, $\xrightarrow[n=5]{}$ $D_{[2,2]} \sim$ 

$n-3 \quad n-3$

was introduced with field equations $\boxed{Tr^{\,n-3} K_{[n-2,\,n-2]}(D) = 0}$ $\xrightarrow[n=5]{}$ $Tr^2 K_{[3,3]} = 0$.

Twisted duality with h_{ab} : $\boxed{K_{[n-2,\,n-2]}(D) = *_1 *_2 K_{[2,2]}(h)}$ $\xrightarrow[n=5]{}$ $K_{[3,3]} = *_1 *_2 K_{[2,2]}(h)$.

EII $\quad d_1^{\dagger} K_{[2,2]} \equiv 0 \iff d_1 K_{[3,3]} = 0 \Leftrightarrow K_{[3,3]} = d_1 d_2 D_{[2,2]} =: K_{[3,3]}(D)$ BII

$\boxdot$ [Dubois Violette – Henneaux 2000]

But $\quad Tr\, K_{[2,2]} = 0$ (EI) $\longleftrightarrow$ $Tr \, *_1 *_2 K_{[3,3]} = 0 \Leftrightarrow Tr^2 K_{[3,3]} = 0$ (EI) EoM for D .

Field equations mapped to Field equations (and not on Bianchi identities) .

In presence of sources, Hull argued that $D_{[2,2]}$ couples to the usual T_{ab} stress tensor
confirming that $D_{[2,2]}$ is, on-shell, an avatar of h_{ab} .

Indeed, from twisted-duality $\quad K_{[3,3]}(D) = *_1 *_2 K_{[2,2]}(h)$. (*)

$$K_{[3,3]}(D) = d_1 d_2 D_{[2,2]} = *_1 *_2 d_1 d_2 h_{[1,1]} \propto \eta_{[1,1]} d_1 d_2 h_{[1,1]} \qquad (\text{since } n=5)$$

$$\Leftrightarrow \quad d_1 d_2 \left(D_{[2,2]} - a\, \eta_{[1,1]} h_{[1,1]} \right) = 0 \qquad a \in \mathbb{R}_0$$

$$\Leftrightarrow \quad D_{[2,2]} = a\, \eta_{[1,1]} h_{[1,1]} + d^{\{2\}} \xi_{[2,1]} \qquad (**)$$

$\Rightarrow$ When the EoMs for h_{ab} Fierz-Pauli are satisfied (since twisted on-shell duality (*) was used),

then the $D_{[2,2]}$-field is conformally flat up to a gauge transformation (**)

[Marc H., Victor Lekeu and Amaury Leonard 1909.12706]

Therefore, inverting (**) to express $h_{[1,1]} = H_{[1,1]}(\mathrm{Tr}\, D, \xi)$ and plugging in

$S^{FP}[H_{ab}(\mathrm{Tr}\, D, \xi)]$ gives field equations equivalent to (**).

$\Rightarrow$ The traceless part of $D_{[2,2]}$ does not enter this action.

However, before [Marc-Victor-Amaury 2019]'s note on the double-dual graviton,

an action principle had been proposed for a $D_{[2,2]}$ gauge field (plus other fields)

that propagates the degrees of freedom of a single graviton:

N.B, Paul Cook, Mitya Ponomarev [2012]

"Off-shell Hodge dualities in linearized gravity and E_{11}"

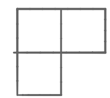
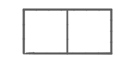
• In the paper 2012.11356 with Victor Lekeu, the first thing done was to

clarify the non-triviality of the action proposed earlier with P. Cook and D. Ponomarev.

② Results on the double-dual graviton and higher duals of vector field

2.1) Double dualisation of a massless vector [N.B., P. Sundell, P. West 2015]

Idea : A_b viewed as a $A_{[0,1]}$ bi-form

$A_{[0,1]} \xrightarrow[\text{double dualize}]{} C_{[n-0-2\,,\,n-1-2]} \underset{n=4}{=} C_{[2,1]} \sim$  or $n=3$: $C_{[1,1]} \sim$ 

- Starts from Maxwell and integrate by part : $S[A_\mu] = -\frac{1}{2}\int d^n x\, (\partial_a A_b\, \partial^a A^b - \partial_a A^a\, \partial_b A^b)$
- Parent action $S[Y^{\cdot\cdot|\cdot}, P_{\cdot|}{}^{\cdot}] = \int d^n x\, (P_{a|b}\, \partial_c Y^{ca|b} - \frac{1}{2} P_{a|b} P^{a|b} + \frac{1}{2} P^{a|}{}_a\, P_{b|}{}^b)$

. $\frac{\delta S[Y,P]}{\delta P_{a|b}} \approx 0 \iff \partial_c Y^{ca|b} - P^{a|b} + \eta^{ab} P^{c|}{}_c \approx 0 \xrightarrow{Tr} \partial_c Y^{ca|}{}_a + (n-1) P^{a|}{}_a \approx 0 \iff P^{a|}{}_a = \frac{1}{1-n}\partial_c Y^{ca|}{}_a$

$$P^{a|b} \approx \partial_c Y^{ca|b} - \eta^{ab}\frac{1}{n-1}\partial_c Y^{cd|}{}_d$$

substitute to get

$$S[P_{a|b}] = \int d^n x \left[\frac{1}{2}\partial_c Y^{ca|b}\, \partial_d Y^d{}_{a|b} - \frac{1}{2(n-1)}\partial_a Y^{ab|}{}_b \right]$$

. From $\quad S[P_{a|b}] = \int d^n x \left[\frac{1}{2}\, \partial_c Y^{ca|b}\, \partial_d Y^{d}{}_{a|b} - \frac{1}{2(n-1)}\, \partial_a Y^{ab|}{}_b \right]$

invariant under $\quad \delta Y^{ab|}{}_c = \delta_c^{[a}\, \partial^{b]} \lambda + \partial_d \Upsilon^{abd|}{}_c \,,$

one decomposes $\quad Y^{ab|}{}_c = X^{ab|}{}_c + \delta_c^{[a} Z^{b]} \,, \qquad X^{ab|}{}_b = 0$

and

- Hodge-dualise in $n=4$: $\quad X^{ab|}{}_c \longleftrightarrow T_{ab|c} \sim \begin{array}{|c|c|}\hline a & c \\ \hline b \\ \cline{1-1}\end{array}$

with gauge transformations

$$\begin{cases} \delta \; \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array} = \overset{S}{\begin{array}{|c|c|}\hline & \\ \hline \partial \\ \cline{1-1}\end{array}} + \overset{A}{\begin{array}{|c|c|}\hline & \partial \\ \hline \\ \cline{1-1}\end{array}} \\ \delta Z_a = \partial_a \lambda + \partial^b A_{ab} \end{cases}$$

- Hodge-dualise in $n=3$: $\quad Y^{ab|}{}_c = \varepsilon^{abd}\, h_{dc} + 2\, \delta_c^{[a} Z^{b]} \,,$

where $\quad h_{\cdot\cdot} \sim \begin{array}{|c|c|}\hline \cdot & \cdot \\ \hline\end{array}\quad$ and $\quad \delta h_{ab} = 2\, \partial_{(a} \epsilon_{b)} \,, \quad \delta Z_a = \partial_a \lambda + \varepsilon_{abc}\, \partial^b \epsilon^c$

Ⓐ In dim. $n=3$,

$$\boxed{S[h_{ab}, Z_a] = \int d^3x \left[-\tfrac{1}{2}\partial_a h_{bc}\partial^a h^{bc} + \tfrac{1}{2}\partial_a h_{bc}\partial^b h^{ac} + \tfrac{1}{2}\varepsilon^{bcd}\partial^a h_{ab} F_{cd}(Z) + \tfrac{1}{4}F^{ab}(Z)F_{ab}(Z) \right]}$$

where $F_{ab}(Z) := 2\,\partial_{[a} Z_{b]}$ and copy the gauge transformation laws :

$$\delta h_{ab} = 2\,\partial_{(a}\epsilon_{b)} \quad , \quad \delta Z_a = \partial_a \lambda + \varepsilon_{abc}\partial^b \epsilon^c$$

One can dualise Z_a vector field into a scalar σ in $n=3$, then define

$$\phi := 2\sigma + \eta^{ab} h_{ab}$$ to obtain

$$S[\phi, h_{ab}] = \int d^3x \left[\mathcal{L}^{FP}(\partial h) + \tfrac{1}{2}\partial_a\phi\,\partial^a\phi + \partial_a\phi\,(\partial_b h^{ab} - \partial^a h) \right] \qquad (\triangle)$$

Field equations $\Rightarrow$ $\boxed{\mathrm{Tr}^2 K_{[2,2]}(h) = 0 \quad , \quad \Box\phi = 0 \quad \& \quad K_{ab}(h) = \partial_a\partial_b\phi}$

$\Rightarrow$ No doubling of physical d.o.f. !

(B) In dim. $n=4$, perform some change of field variables and dualize $Z_a \leftrightarrow \tilde{A}_a$

$$S[T_{ab|c}, \tilde{A}_a] = \int d^4x \left[\mathcal{L}^{Curt.}(T_{ab,c}) + \frac{1}{4} F^{ab}(\tilde{A}) F_{ab}(\tilde{A}) - \frac{1}{\sqrt{2}} \tilde{A}^a K''_a(T) \right]$$

where $K^{a[3]}{}_{b[2]} := 6\, \partial^{[a} \partial_{b} T^{aa],}{}_{b]}$ curvature, $K'' := Tr^2 K$.

The gauge invariances are the ones expected for a Curtright field and a vector.

- Field equations:

$$-\partial_a F^{ab}(\tilde{A}) + \frac{1}{\sqrt{2}} K''^{b} = 0 \qquad (1)$$

$$K'^{ab|}{}_c + \delta^{[a}_c K''^{b]} - \frac{1}{\sqrt{2}} \partial_c F^{ab} - \frac{1}{\sqrt{2}} \delta^{[a}_c \partial_d F^{b]d} = 0 \qquad (2)$$

Take the trace of (2), combine with (1) to get the EoMs and duality relation

$$\boxed{d^{\dagger} F_{[2]}(\tilde{A}) = 0 = Tr^2 K_{[3,2]}(T) \ \& \ Tr\, K_{[3,2]}(T) = \frac{1}{2} d_2 F_{[2,0]}(\tilde{A})} \Rightarrow \text{no doubling of d.o.f. !}$$

2.2) Double-dualisation of spin-2 in $n=5$.

The dualization procedure given in [N.B., P. Cook, D. Ponomarev 2012] for the double-dual graviton gives an action $S[D_{ab,cd}\,,\,Z^{ab|}{}_{c}] = \int d^5x\; \mathcal{L}(\partial D, \partial Z)$, $D \sim \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline\end{array}$

$Z^{ab|}{}_{c} = -Z^{ba|}{}_{c}$.

where $\mathcal{L}(\partial D, \partial Z) = \mathcal{L}(\partial Z) - \mathcal{L}(\partial D) + \mathcal{L}^{cross}$

features the complete $D_{ab,cd}$ field, including its traceless part.

The action is gauge invariant under

$$\begin{cases} \delta \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline\end{array} = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \partial \\ \hline\end{array} \\[2em] \delta Z^{mn|}{}_{a} = \lambda^{mn}{}_{a} + \partial^{[m}\xi^{n]}{}_{a} - \frac{1}{2}\,\delta^{[m}_{a}\,\partial_b\,\xi^{n]b} + \delta^{[m}_{a}\,\partial^{n]}\chi \end{cases}$$

where $\lambda_{abc} = \lambda_{[abc]}$ and $\xi^{a|b} \sim \boxed{a} \otimes \boxed{b}$

Perform a change of variable

$$Y^{ab|}{}_c := Z^{ab|}{}_c + \delta_c^{[a} Z^{b]}$$

to get

$$S[Y^{ab|}{}_c, D_{ab,cd}] = \int d^5x \Big[\mathcal{L}^{\text{curt.}}(Y_{ab|c}) - \mathcal{L}(\partial D)$$
$$+ \tfrac{1}{2}\epsilon_{abcde}\, \partial^e D^{cd,}{}_{mn} \Big(\overbrace{F^{abm|n}}^{3\partial^{[a}Y^{bm]|n}}(Y) - \tfrac{1}{2} F^{mna|b} \Big) \Big]$$

That propagates a *single* graviton.

Dualize $Y^{ab|}{}_c$ in $n=5$ $\left(\varepsilon_{abcde}\, \partial^d f^e{}_{|u}\, F^{abc|u}\right)$ for $f_{a|c} \sim h_{ab} + B_{[ab]}$ to get

$$S[D_{ab,cd}, f_{ab}] = \int \Big[\mathcal{L}^{FP}(h) - \mathcal{L}(\partial D) - \tfrac{3}{2}\, h_{ab}\, \tilde{K}'^{ab}(D) \Big]$$

where $\tilde{K}_{[2,2]}(D) := *_1 *_2 \underbrace{d_1 d_2 D_{[2,2]}}_{K_{[3,3]}}$. Field equations:

$$\boxed{\begin{gathered} \mathrm{Tr}\, K_{[2,2]}(h) = 0 = \mathrm{Tr}^2 K_{[3,3]}(D) \\ \& \quad K_{[2,2]}(h) \propto \mathrm{Tr}\, K_{[3,3]}(D) . \end{gathered}}$$

③ Higher (or exotic) dualisation of the 3D graviton and scalar

3.1) Higher dualisation of graviton in 3D

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \longmapsto \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array} \oplus \text{ more}$$

Spin-2 Fierz-Pauli $\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$ viewed as $h_{[0,1,1]}$ gauge field

Curvature $K_{[2,2,1]}(h) \sim \begin{array}{|c|c|c|}\hline \ & \ & \partial \\ \hline \partial & \partial \\ \cline{1-2}\end{array} \xrightarrow[*_3]{} \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ & \ \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \partial & \partial & \partial \\ \hline\end{array} = K_{[2,2,2]}(\varphi)$

On-shell Twisted duality

$$*_3 K_{[2,2,1]}(h) = K_{ab|cd|u}(h)\, \varepsilon^{u}{}_{ef} \sim \begin{array}{|c|c|c|}\hline a & c & e \\ \hline b & d & f \\ \hline\end{array} \underset{TD}{=} K_{ab|cd|ef}(\varphi) := 8\, \partial_{[a} \partial_{[c} \partial_{[e} \varphi_{f]d]b]}$$

$\longrightarrow$ $\mathrm{Tr}\, K_{[2,2,1]}(h) = 0 = \mathrm{Tr}\, K_{[2,2,2]}$ (EI)

Since $n = 3$, (EI) $\Leftrightarrow K_{[2,2,1]}(h) = 0 = K_{[2,2,2]}(\varphi)$ Topological

To make that story <u>variational</u> :

1. $S_{FP}[h_{ab}] = \int d^3x \, [-\frac{1}{2}\partial_a h_{bc}\partial^a h^{bc} + \frac{1}{2}\partial_a h \, \partial^a h - \partial_a h^{ab}\partial_b h + \partial_a h^{ab}\partial^c h_{cb}]$

2. $S[{\color{blue}G_{a|bc}}, {\color{blue}D_{ab|}{}^{cd}}] = \int d^3x \, [-\frac{1}{2} G_{a|bc}G^{a|bc} + \frac{1}{2} G_{a|\bullet}{}^{\bullet} G^{a|\bullet}{}_{\bullet} - G_{\bullet|}{}^{\bullet b} G_{b|\bullet}{}^{\bullet} + G_{\bullet|}{}^{\bullet b} G^{\bullet|}{}_{ab}$

$+ G^{d|}{}_{bc}\partial^a {\color{blue}D_{ad|}{}^{bc}}]$

where $G_{a|bc} \sim \boxed{a} \otimes \boxed{b}\boxed{c}$, $D_{ad|}{}^{bc} \sim \begin{array}{|c|}\hline a \\ \hline d \\ \hline\end{array} \otimes \boxed{b}\boxed{c}$

3. Solve for $G_{a|bc}$ auxiliary field

$\hookrightarrow$ $S[D_{ab|}{}^{cd}] = \frac{1}{2}\int d^{\color{purple}3}x \, [\partial^a D_{ab|cd}\,\partial_e D^{eb|cd} - \partial^a D_{a b|c}{}^{b}\,\partial_d D^{de|c}{}_{e} - \partial^a D_{ab|c}{}^{c}\,\partial_e D^{eb|d}{}_{d}]$

4. Decompose $\tilde{D}_{a|bc} := -\frac{1}{2}\varepsilon_{amn} D^{mn|}{}_{bc} \sim \boxed{a}\otimes\boxed{b}\boxed{c} \xrightarrow[\downarrow\, so(3)]{} \boxed{}\boxed{}\boxed{} \oplus \boxed{}\boxed{} \oplus 2\times\boxed{}$

5. Dualise one vector into scalar $\bullet$, perform some field redefinitions to get ...

$$S[\varphi_{abc}, h_{ab}] = \frac{1}{2}\int d^3x \Big[-\partial_a\varphi_{bcd}\,\partial^a\varphi^{bcd} + \partial^a\varphi^b\,\partial^c\varphi_{abc} + \partial_a\varphi^{abc}\,\partial^d\varphi_{bcd}$$
$$- \tfrac{1}{7}\,\partial_a\varphi_b\,\partial^a\varphi^b - \tfrac{31}{28}\,\partial_a\varphi^a\,\partial^b\varphi_b$$
$$+ \tfrac{1}{2}\,\partial_a h_{bc}\,\partial^a h^{bc} + \tfrac{1}{14}\,\partial_a h\,\partial^a h - \tfrac{3}{7}\,\partial^a h_{ab}\,\partial_c h^{bc} - \tfrac{1}{7}\,\partial^a h\,\partial_c h_a{}^c$$
$$+ \tfrac{10}{7}\,\varepsilon_{apq}\,\partial^b h_b{}^a\,\partial^p\varphi^q - 2\,\varepsilon_{apq}\,\partial^b h^{ac}\,\partial^p\varphi^q{}_{bc}\Big] \qquad (3.32)$$

that is invariant under

$$\delta\varphi_{abc} = 3\,\partial_{(a}\widehat{\xi}_{bc)} - \tfrac{2}{3}\,\varepsilon_{(a}{}^{pq}\,\eta_{bc)}\partial_p\epsilon_q\ , \qquad (3.33)$$
$$\delta h_{ab} = 2\,\partial_{(a}\epsilon_{b)} + 2\,\varepsilon_{pq(a}\partial^p\widehat{\xi}^q{}_{b)}\ . \qquad (3.34)$$

Recall that this is a topological theory.

From a general result [Grigoriev – Mkrtchyan – Skvortsov 2005], we know that

it can be put in Chern – Simons form [To appear]

3.2) Double exotic dualisation of vector in 3D : $\square \longmapsto \square\!\square\!\square \oplus$ more

Start from the first exotic dualisation of Maxwell's theory in $n = 3$

$$S[h_{ab}, Z_a] = \int d^3x \left[-\tfrac{1}{2}\partial_a h_{bc}\partial^a h^{bc} + \tfrac{1}{2}\partial_a h_{bc}\partial^b h^{ac} + \tfrac{1}{2}\varepsilon^{bcd}\partial^a h_{ab} F_{cd}(Z) + \tfrac{1}{4}F^{ab}(Z)F_{ab}(Z) \right]$$

where $\quad \delta h_{ab} = 2\,\partial_{(a}\epsilon_{b)} \quad , \quad \delta Z_a = \partial_a \lambda + \varepsilon_{abc}\partial^b \epsilon^c$

and dualise $h_{ab} \sim \boxed{a|b}$ in empty column.

After some steps (parent action, daughter action, $so(3)$ decomposition, field redef, dualisation, field redef.), one gets an action $S[\phi_{abc}, h_{ab}, A_a]$ invariant under entangled gauge transformations :

$$\boxed{\begin{aligned} \delta\phi_{abc} &= 3\,\partial_{(a}\hat{\xi}_{bc)} - \tfrac{2}{3}\varepsilon_{(a}{}^{mn}\eta_{bc)}\partial_m \epsilon_n \\ \delta h_{ab} &= 2\,\partial_{(a}\epsilon_{b)} + 2\,\varepsilon_{mn(a}\partial^m \hat{\xi}^n{}_{b)} \\ \delta A_a &= \partial_a \lambda + \varepsilon_{abc}\partial^b \epsilon^c \end{aligned}}$$

④ Outlook

- In the topological case in 3D, pursuing the higher (exotic) dualisations to produce higher (undecomposable) spin fields, we expect an infinite spectrum of fields of rank $\in \{2, 3, 4, \dots\}$.

↳ It cannot match the well-known Blencowe model, since here we have an undecomposable structure at the level of gauge transformations.

- In the propagating case in 3D, pursuing the higher (exotic) dualisations to produce higher (undecomposable) spin fields, we expect an infinite spectrum of fields of rank $\in \{1, 2, 3, \dots\}$.

- In the case of double exotic dual of a scalar field in $n = 3$, the action

$$S[h_{ab}, \sigma] = \int d^3x \left[-\frac{1}{2} \partial_a h_{bc} \partial^a h^{bc} + \partial_a h^{ab} \partial^c h_{bc} + 2 \partial_a \sigma \left(\partial^a \sigma + \partial_b h^{ab} \right) \right]$$

is obtained in [2012.11356], invariant under

$$\delta h_{ab} = 2 \partial_{(a} \epsilon_{b)} \quad , \quad \delta \sigma = - \partial^a \epsilon_a \quad .$$

↳ We now have found a direct relation with the spin-2 triplet system

as studied and discussed in

"Maxwell-like Lagrangians for higher-spins", [A. Campoleoni & D. Francia, 1206.5877]

- In the higher-spin cases, other undecomposable, finite-dimensional representations are expected.